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NUMERICAL DESIGN OF FEEDBACK SYSTEMS WITH BACKLASH FOR INPUTS RESTRICTED IN MAGNITUDE AND SLOPE

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วิทยานิพนธ์ฉบับนี้พัฒนาวิธีการเชิงปฏิบัติสำหรับออกแบบระบบป้อนกลับเป็นหนึ่งซึ่งมีพลานต์ที่เป็นเชิงเส้นไม่แปรเปลี่ยนตามเวลา (ซึ่งมีพารามิเตอร์แบบก้อนหรือกระจาย) ต่อเชื่อมกับระยะคลอนและตัวควบคุม ปัญหาที่พิจารณาคือออกแบบตัวควบคุมเพื่อรับประกันว่าสัญญาณคลาดเคลื่อนและสัญญาณขาออกของตัวควบคุมอยู่ภายในขอบเขตที่กำหนดตลอดเวลาสำหรับสัญญาณขาเข้าทุกสัญญาณที่สอดคล้องกับเงื่อนไขขอบเขตที่กำหนดให้ วิทยานิพนธ์ฉบับบนี้ประกอบด้วยสองส่วนหลัก ส่วนแรกสืบสวนเสถียรภาพแบบสัญญาณขาเข้ามีขอบเขต/สัญญาณขาออกมีขอบเขตของระบบภายใต้สัญญาณขาเข้าสองจำพวกที่แตกต่างกัน โดยที่จำพวกแรกอธิบายลักษณะของประเภทของสัญญาณชั่วครู่ประเภทหนึ่ง ในขณะที่จำพวกที่สองเหมาะสมสำหรับอธิบายลักษณะของสัญญาณคงอยู่ ในส่วนที่สอง เนื่องจากเกณฑ์การออกแบบต้นกำเนิดไม่สามารถนำมาใช้คำนวณได้ เราจึงได้ประดิษฐ์เงื่อนไขเชิงปฏิบัติที่อยู่ในรูปของอสมการที่สามารถหาคำตอบได้ด้วยวิธีเชิงเลขเพื่อใช้หาตัวควบคุมในกรณีที่พารามิเตอร์ของพลานต์คงที่และในกรณีที่พารามิเตอร์ของพลานต์มีความไม่แน่นอนตามลำดับ ในสาระสำคัญระยะคลอนถูกแทนด้วยอัตราขยายคงที่และสัญญาณรบกวนที่มีขอบเขตซึ่งส่งผลให้ได้ระบบเชิงเส้นช่วย โดยการประยุกต์ใช้ทฤษฎีบทจุดตรึงแบบหลายค่าเราได้พิสูจน์ว่าถ้าตัวควบคุมตัวหนึ่งสอดคล้องกับอสมการเหล่านั้นสำหรับระบบเชิงเส้นที่เกี่ยวข้อง จะได้ว่าตัวควบคุมตัวนั้นก็จะเป็นคำตอบของปัญหาการออกแบบต้นกำเนิดด้วย ตัวอย่างเชิงเลขหลายตัวอย่างได้ถูกนำเสนอเพื่อที่จะแสดงให้เห็นถึงประโยชน์ของวิธีการที่พัฒนาขึ้นในที่นี้

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สาขาวิชา วิศวกรรมไฟฟ้า ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....

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KEYWORDS : NONLINEAR SYSTEMS / INPUT-OUTPUT STABILITY / BACKLASH / CONTROL SYSTEMS DESIGN / PRINCIPLE OF MATCHING/ METHOD OF INEQUALITIES

HOANG HAI NGUYEN : NUMERICAL DESIGN OF FEEDBACK SYSTEMS WITH BACKLASH FOR INPUTS RESTRICTED IN MAGNITUDE AND SLOPE. ADVISOR : ASST. PROF. SUCHIN ARUNSAWATWONG, Ph.D., 51 pp.

This thesis develops a practical method for designing unity feedback systems where a linear time-invariant (either lumped- or distributed- parameter) plant is in cascade connection with a backlash and a controller. The problem considered is to design a controller so as to ensure that the error and the controller output stay within prescribed bounds for all time and for all inputs satisfying given bounding conditions. The thesis comprises two main parts. Part 1 investigates the BIBO stability of the system subject to two different classes of inputs; the first class characterizes a type of transient signals while the other class is appropriate for characterizing persistent signals. In Part 2, since the original design criteria are computationally intractable, we derive a practical condition in the form of inequalities that can be solved by numerical methods so as to determine the controller for two cases in which the plant is fixed and in which the plant parameters have uncertainties, respectively. In essence, the backlash is replaced with a constant gain and a bounded disturbance, thereby resulting in an auxiliary linear system. By applying the multi-valued version of the fixed-point theorem, we have shown that if a controller satisfies such inequalities for the associated linear system, then it is also a solution of the original design problem. Numerical examples are provided in order to illustrate the usefulness of the method developed here.

Department : ....Electrical Engineering........	Student's Signature ....
Field of Study : ....Electrical Engineering........	Advisor's Signature ....
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Contents

List of Figures

Figure Page

Figure Page

CHAPTER I

INTRODUCTION

1.1 Introduction

Nowadays, with powerful computing facilities, all tedious numerical computations are performed by computers. Consequently, designers are allowed to concentrate on the design problems so that they are formulated in an accurate and realistic manner. Along this direction, Zakian [36–39]was prompted to develop a useful design framework which consists of the principle of matching (PoM) and the method of inequalities (MoI).

The method of inequalities [38, 39] suggests that the constrains and design specifications of a control system are expressed as a set of inequalities, that is,

$$\phi_i(\mathbf{p}) \leq \varepsilon_i, \quad i = 1, 2, ..., m, \tag{1.1}$$

where $\phi_i(\mathbf{p})$ is a performance measure, each constant ε_i denotes the largest value of the function $\phi_i(\mathbf{p})$ that the system can tolerate and $\mathbf{p} \in \mathbb{R}^n$, as usual, denotes the design parameter. The solution of the problem is any value $\mathbf{p}$ that satisfies (1.1). In practice, (1.1) are solved numerically by using search algorithms (see, for example, [38] and the references therein). Throughout this work, the search algorithm called moving boundaries process (MBP) ([38, 39]) is used.

The principle of matching [36, 37] considers the system in relation to the environment in which it operates. The environment generates the input f for the system and the input f is only known to the extent that it belongs to a set $\mathcal{P}$, called the possible set. The set of all inputs that the system can tolerate is called tolerable set $\mathcal{T}$. The environment-system couple is matched if

$$\mathcal{P} \subseteq \mathcal{T}. \tag{1.2}$$

The couple is said to be well matched if it is matched and the possible set $\mathcal{P}$ is close to the tolerable set $\mathcal{T}$ in some sense.

In other words, the method of inequalities suggests that the design problem should be stated in a form of the conjunction of inequalities while the principle of matching suggests what kind of inequalities should be chosen in order to make the formulation of the design problem more accurate and realistic. The framework has been used for designing control systems and been investigated by many researchers (see, for example, [5, 18, 20–22]. Readers are referred to [38] for a more complete list of references on this). In this connection, this thesis focuses on the principle of matching.

According to the principle of matching [36–38], a chief design objective is to guarantee that a response v of the system under consideration stays within a prescribed bound in the presence of all

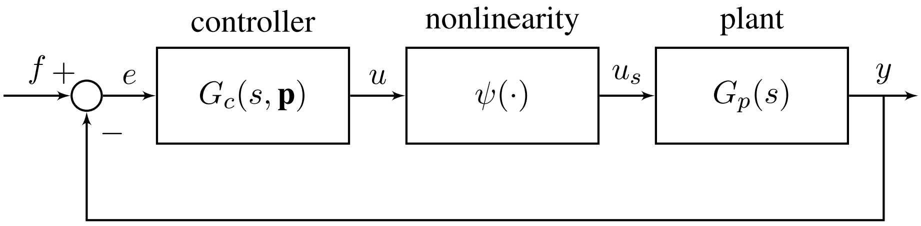

Figure 1.1: Feedback system with nonlinearity

possible inputs (that is, inputs that can happen or are likely to happen in practice). In this regard, the design criterion can be expressed as

$$|v(f,t)| \leq \varepsilon, \ \ \forall \ f \ \ \forall t \in \mathbb{R} \tag{1.3}$$

where $v(f,t)$ is the value of v at time t in response to a possible input f, and ε is the largest value of $|v(f,t)|$ that the system can tolerate. The criterion (1.3) is often utilized to monitor the performance of control systems in practice and has been investigated by many authors (for example, [20–22,31,32,35] and the references therein). See [37,38] and also [5,31] for details on this.

Specifically, Mai et al. [20–22] (see also [18]) have recently developed a method for designing a feedback control system that consists of a memoryless, sector-bounded nonlinearity and linear time-invariant convolution plants (see Fig. 1.1) to ensure that the error e and the controller output u stay within prescribed bounds for all time and for all possible inputs having bounded magnitude and bounded slope. It should be noted that the above-mentioned articles do not consider the case in which the nonlinearity $\psi(\cdot)$ does not satisfy a sector bound condition. Nor are they applicable to the case of $\psi(\cdot)$ being a backlash, which can be found in many practical applications (see [15, 23] and also the references therein).

Consider the feedback control system displayed in Fig. 1.1, which is described by

$$\left.\begin{aligned} u &= g_c * e \\ e &= f - u_s * g_p = f - \psi(u) * g_p \end{aligned}\right\} \tag{1.4}$$

where ψ is a backlash, g_p is the impulse response of the plant and has Laplace transform equal to $G_p(s)$, g_c is the impulse response of the controller and has Laplace transform equal to $G_c(s,\mathbf{p})$, and $\mathbf{p} \in \mathbb{R}^n$ denotes a design parameter vector. As usual, the asterisk denotes the convolution; that is

$$(g_c * e)(t) = \int_0^t g_c(t-\tau)e(\tau)d\tau, \quad t \geq 0. \tag{1.5}$$

Suppose that the input f is known only to the extent that it belongs to a *possible set* $\mathcal{P}$ (that is, the set of possible inputs).

In this work, we consider the possible set $\mathcal{P}$ consisting of continuous signals with bounded magnitude and bounded slope, and it can be defined in many ways. In general, $\mathcal{P} \subset \mathbb{L}_\infty$, where $\mathbb{L}_\infty \triangleq \ \{x : \mathbb{R}_+ \!\to \mathbb{R} | \ |x(t)| < \infty \ \ \forall t \geq 0\}$. When the possible inputs are continuous bounded functions, using the set $\mathbb{L}_\infty$ as the set of possible inputs may not be appropriate because $\mathbb{L}_\infty$ contains signals having stepwise discontinuities. For further discussion on this, see [37, 38].

The problem considered here is to determine a design parameter $\mathbf{p}$, which characterizes the controller transfer function $G_c(s, \mathbf{p})$, such that the following design criteria are satisfied.

$$\hat{e} \le E_{max} \text{ and } \hat{u} \le U_{max} \tag{1.6}$$

where the bounds E_{max} and U_{max} are given, and

$$\hat{e} \triangleq \sup_{f \in \mathcal{P}} \|e\|_\infty \text{ and } \hat{u} \triangleq \sup_{f \in \mathcal{P}} \|u\|_\infty. \tag{1.7}$$

The performance measures $\hat{e}$ and $\hat{u}$ are sometimes called the peak error and the peak controller output, respectively, for the possible set $\mathcal{P}$. Obviously, $\hat{e}$ and $\hat{u}$ depend upon $\mathbf{p}$.

It is worth noting that the design criteria (1.6) are used to ensure that the error e and the controller output u stay within their respective bounds E_{max} and U_{max} for all time and for all $f \in \mathcal{P}$. Observe that (1.6) are equivalent to

$$\left.\begin{aligned} |e(f,t)| \le E_{max}, \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \\ |u(f,t)| \le U_{max}, \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \end{aligned}\right\}. \tag{1.8}$$

Following previous work ([31, 32, 38, 39] and also the references therein), it is readily appreciated that in solving the inequalities (1.6) by numerical methods, a search algorithm needs to start from a *stability point*, that is, a point $\mathbf{p}$ for which

$$\hat{e}(\mathbf{p}) < \infty \text{ and } \hat{u}(\mathbf{p}) < \infty. \tag{1.9}$$

Therefore, it is necessary to establish a practical condition for determining stability points of the system (1.4) where $\psi(\cdot)$ is a backlash. Once such a point is obtained, design criteria (1.6) can be solved by numerical methods, provided that the peak outputs $\hat{e}$ and $\hat{u}$ can be computed in practice.

The objective of the thesis is to develop a systematic and practical method for designing a controller the feedback control system (1.4) where the nonlinearity is a backlash so as to guarantee the criteria (1.6) are satisfied. In order to find such a controller, we need to solve the problem of finding a stability point first. In other words, we need to establish practical conditions for determining stability points of the system.

1.2 Literature Review

This section will briefly describe the series of work done by Mai et al. [18–22]. Consider the system (1.4) where the nonlinearity $\psi(\cdot)$ is memoryless and sector bounded.

1. Mai et al. [19], based on the Popov criterion, have established a practical inequality for determining stability points by numerical methods. The key idea of this work is that, instead of using

the Popov plot, the convex hull of the Popov plot is used, which is more suitable for automatic computation.

2. Mai et al. [20, 21] have developed a method for designing the system so as to ensure that the error and the controller output stay within prescribed bounds for all time and for all inputs having bounded magnitude and bounded slope. The design formulation is based on the principle of matching, thereby explicitly considering the peak error and the peak controller output for such a possible set. The original design inequalities are replaced with the surrogate design criteria that are in keeping with the method of inequalities. The sufficient conditions for the satisfaction of the original design criteria are proved by using Schauder fixed point theorem (see, for example, [40]). By replacing the nonlinearity with a constant gain and an equivalent disturbance, the original nonlinear system becomes a linear system with two inputs. Therefore, the associated performance measures are readily obtainable by known methods.

3. Mai et al. [22] have extended the work [20, 21] to the case that the plant is an uncertain linear time-invariant convolution system. First, the Schauder fixed point theorem is used to show that a design solution for an uncertain linear system obtained by replacing the nonlinearity with a gain and a bounded disturbance (if exists) is also a solution for the original nonlinear problem. Then, by extending Zakian's theory of majorants and applying it to the so-obtained linear problem, the design inequalities that can readily be solved in practice are derived.

1.3 Scope of thesis

1. Provide stability conditions and then develop a computational inequality for determining stability points with inputs restricted in magnitude and slope for the system (1.4).

2. Develop a practical method for designing the system subjected to inputs satisfying bounding conditions on their magnitude and slope to ensure the condition (1.6).

3. Design controllers for some systems whose the linear plants are possibly described by non-rational transfer functions.

1.4 Methodology

1. Stability points can be obtained by using the stability results developed above.

2. By using the decomposition technique used in [18, 20–22, 25], the backlash can be replaced by a constant gain and an equivalent disturbance, thus the nominal system used during the design process becomes linear with two inputs.

3. Develop sufficient conditions for ensuring (1.6) in terms of inequalities that are in keeping with the MoI.

1.5 Expected Outcomes

1. A practical inequality for determining stability points of feedback control systems with a backlash.
2. A practical method for designing the system (1.4).
3. Numerical examples showing the usefulness of the developed method.

1.6 Thesis Outline

The organization of the thesis is as follows. Chapter 2 presents the input-output stability of a nonlinear feedback system where a linear convolution subsystem is in cascade with a hysteresis or a backlash subject to bounded inputs whose slopes belonging to the set $\mathbb{N}_2$ (which will be defined in details in Chapter 2). Based on this, a practical method for determining stabilizing controllers by numerical methods is developed. However, the input set considered in Chapter 2 is not suitable to characterize persistent inputs (that is, signals that vary persistently for all time). For that reason, Chapter 3 considers the input-output stability of the control feedback system where a linear convolution subsystem is in cascade with a backlash subject to a inputs having bounded magnitude, and the stability conditions established in Chapter 3 are used to guarantee the finiteness of the error and the controller output in the following chapters. Chapter 4 develops a practical numerical method for designing nonlinear feedback control systems with backlash under inputs whose magnitude and slope are bounded. Chapter 5 develops a method to design a robust controller for nonlinear feedback systems where the an uncertain linear time-invariant plant is in cascade connection with an uncertain backlash and a controller subject to bounded inputs. Conclusions and future works are given in Chapter 6.

CHAPTER II

STABILITY CONDITIONS FOR FEEDBACK SYSTEMS WITH HYSTERESIS SUBJECT TO BOUNDED INPUTS WHOSE SLOPES BELONGING TO THE SET $\mathbb{N}_2$

This chapter considers the input-output stability of feedback systems made up of a time-invariant linear element and a hysteresis element. From previous work, it is appreciated that for any input such that the two norm of the product of its slope and an increasing exponential function is finite, if the linear subsystem satisfies a Popov inequality for a sector bound and if the slope of the hysteresis lies within the same sector bound, then the outputs of the system are ensured to be finite for all time. Based on this, we develop a practical method for determining stabilizing controllers by numerical methods. To illustrate the usefulness and the potential of the method, three numerical examples are provided in which the plants are described by rational and non-rational transfer functions.

2.1 Introduction

Many researchers have been prompted to investigate the input-output stability of the feedback control system shown in Fig. 2.1, where $G(s)$ is a linear plant and $\psi(\cdot)$ is memoryless, sector-bounded nonlinearity. Following previous work ([9, 11, 12, 18, 19]), it is readily appreciated that the Popov criterion can be used to ensure that the system is stable in the sense that the outputs e_1, e_2, y_2 are bounded for any nonlinearity $\psi(\cdot)$ in a given sector bound whenever the inputs u_1, u_2 have bounded magnitude and bounded slope.

In this connection, Mai et al. [19] has developed a practical inequality that is equivalent to the Popov criterion for ensuring the input-output stability of the system. This inequality provides a readily computable test for checking the stability of the system and also can be used to determine stabilizing controllers for the system. In addition, the inequality has been used to design feedback control systems in conjunction with the design theories developed in [20–22].

It should be noted that the above-mentioned results are not applicable to the case of ψ being a hysteresis or backlash, which is a nonlinear element that does not satisfy a sector-bound condition and has memory. A hysteresis can be found in many practical applications, for example, structural or mechanical control systems (see [23, 24] and also the references therein).

This work considers the input-output stability of the feedback control system in Fig. 2.1 where $G(s)$ is the transfer function which can be non-rational and $\psi(\cdot)$ is a hysteresis. In [16, 17], it has been shown that whenever the inputs u_1, u_2 belong to a space of functions such that the two norm of the product of the slope of any function belonging to the space with an increasing exponential function is finite, if the linear subsystem satisfies a Popov inequality for a sector bound and if the slope of the hysteresis lies within the same sector bound then we can ensure that the magnitudes of the outputs e_1, e_2, y_1, y_2 are finite for all time and that there is no sustained oscillation in the system. It should be noted that the input spaces considered here is different from the input spaces investigated in [19–22].

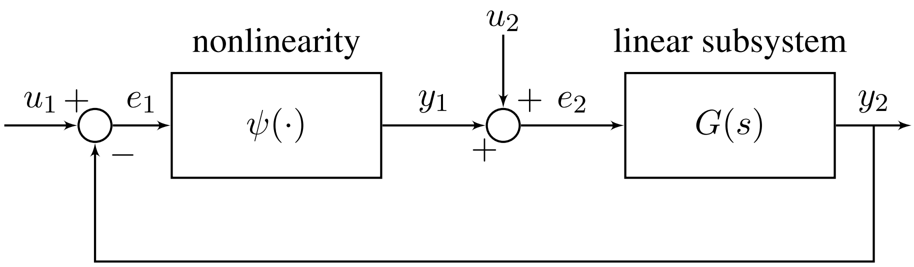

Figure 2.1: Feedback control system with nonlinearity

In the same spirit as [19], we develop a practical condition based on the Popov-type stability condition in [17] for obtaining stability points in the form of an inequality which is suitable for solution by numerical methods. Following the method of inequalities ([39], and also [38]), it is readily appreciated that this condition is a useful criterion for stabilizing feedback control systems with a hysteresis. The key idea used in this work is that the stability results obtained from the Popov plot is the same as the results obtained from the convex-hull of the Popov plot; however, the convex-hull is much easier to deal with.

2.2 System Description

The system in Fig. 2.1 is described by the following equations.

$$\left.\begin{aligned} e_1(t) &= u_1(t) - y_2(t) \\ e_2(t) &= u_2(t) + y_1(t) \\ y_1(t) &= \psi(e_1) \\ y_2(t) &= (g * e_2)(t) \end{aligned}\right\}, \quad \forall t \geq 0. \tag{2.1}$$

Suppose that the system is at rest for $t \leq 0$. The notation $\psi(\cdot)$ in equation (2.1) expresses the input-output characteristic of the hysteresis element in the system.

In order to give detailed descriptions of the system, the following definitions will be used. For each real $p \in [1, \infty)$, the normed space $\mathbb{L}_p$ is defined as

$$\mathbb{L}_p \triangleq \{x : [0, \infty) \to \mathbb{R} \mid \int_0^\infty |x(t)|^p dt < \infty\}.$$

We also define the space

$$\mathbb{N}_2[0, \infty) = \{x(\cdot) \mid \exists \sigma > 0 \text{ so that } e^{\sigma t} x(t) \in \mathbb{L}_2\}.$$

The relationship of $\mathbb{N}_2$ with $\mathbb{L}_1$ and $\mathbb{L}_2$ can be seen from the properties of space $\mathbb{N}_2$ shown in the following propositions.

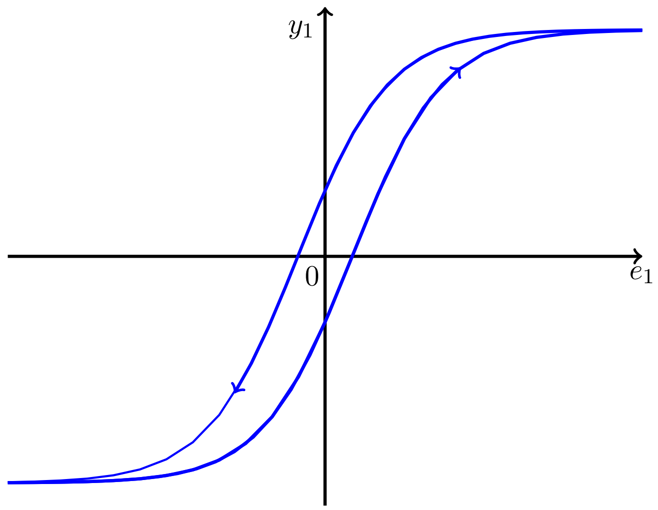

Figure 2.2: Example of a hysteresis element

Proposition 2.1 ([16, 17]). $\mathbb{N}_2 \subset \mathbb{L}_1 \cap \mathbb{L}_2$.

Proposition 2.2 ([16, 17]). *Let $\dot{x}$ denote the derivative of function x. If $\dot{x} \in \mathbb{N}_2$, then $x \in \mathbb{L}_\infty$ and there exists a finite constant A such that $x \to A$ when $t \to \infty$.*

Assumption 2.1. *The slopes of the inputs u_1, u_2 belong to $\mathbb{N}_2$.*

From Proposition 2.2, it can be seen that if u_1 and u_2 satisfy Assumption 2.1, then their magnitudes are bounded and they go to finite constants as $t \to \infty$.

Let $\mathcal{A}$ denote the set of generalized functions which have the form

$$g(t) = \begin{cases} g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.2}$$

where $\delta(\cdot)$ denotes the Dirac delta function, $0 \leq t_0 < t_1 < t_2 ...$ are constants, $\sum_{i=0}^{\infty} |g_i| < \infty$ and $g_a \in \mathbb{L}_1$. The norm $|| \cdot ||_{\mathcal{A}}$ of a function $g(\cdot)$ in $\mathcal{A}$ is defined as follows.

$$||g(\cdot)||_{\mathcal{A}} = \int_0^\infty |g_a(t)| dt + \sum_{i=0}^{\infty} |g_i|. \tag{2.3}$$

For a given $\sigma > 0$, the set $\mathcal{A}_\sigma$ consists of all functions that take the form

$$g(t) = \begin{cases} g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases} \tag{2.4}$$

in which $\delta(\cdot)$ denotes the Dirac delta function, $0 \leq t_0 < t_1 < t_2 ...$ are constants, $\sum_{i=0}^{\infty} |g_i| e^{\sigma t_i} < \infty$ and $\int_0^\infty |g_a| e^{\sigma t_i} dt < \infty$. It is easy to see that for any $\sigma > 0$, $\mathcal{A}_\sigma$ is a subset of $\mathcal{A}$. Now, we define the set $\mathcal{A}_-$ as

$$\mathcal{A}_- = \bigcup_{\sigma > 0} \mathcal{A}_\sigma. \tag{2.5}$$

With this definition, $\mathcal{A}_-$ consists of all functions g that belong to $\mathcal{A}_\sigma$ for any $\sigma > 0$. Moreover, the set $\mathcal{A}_-$ is a proper subset of $\mathcal{A}$ and is closed under convolution. The details of the sets $\mathcal{A}, \mathcal{A}_\sigma$ and $\mathcal{A}_-$ can be found in [11], [30].

Assumption 2.2. *The linear subsystem $G(s)$ is time-invariant and non-anticipative with zero initial conditions. The input e_2 and output y_2 are related by the convolution integral*

$$y_2(t) = (g * e_2)(t) = \int_0^t g(t-\tau)e_2(\tau)d\tau, \;\; t \geq 0 \tag{2.6}$$

where g is the impulse response of G(s) and g can be decomposed as $g(t) = c + g_1(t), \;\; t \geq 0$, in which c is a finite real constant and $g_1 \in \mathcal{A}_{-}$.

Note that this description of the linear part consists not only rational systems but also time-delay or distributed-parameter systems. Moreover, the linear subsystem is allowed to have one pole at the origin.

Assumption 2.3. *Let e_1 and y_1 denote the input and the output of the hysteresis, and let $\dot{e}_1, \dot{y}_1$ denote their first derivatives respectively. If $\dot{e}_1 = 0$, then $\dot{y}_1 = 0$. If $\dot{e}_1 \neq 0$, then there exists a real constant k such that*

$$0 \leq \dot{y}_1(t)\dot{e}_1(t) \leq k\dot{e}_1^2(t) \quad \forall t \geq 0. \tag{2.7}$$

Moreover, for all e_1 having a second derivative, it follows that

$$\int_0^t \dot{y}_1(\tau)\ddot{e}_1(\tau)\, d\tau \leq 0, \quad \forall t \geq 0. \tag{2.8}$$

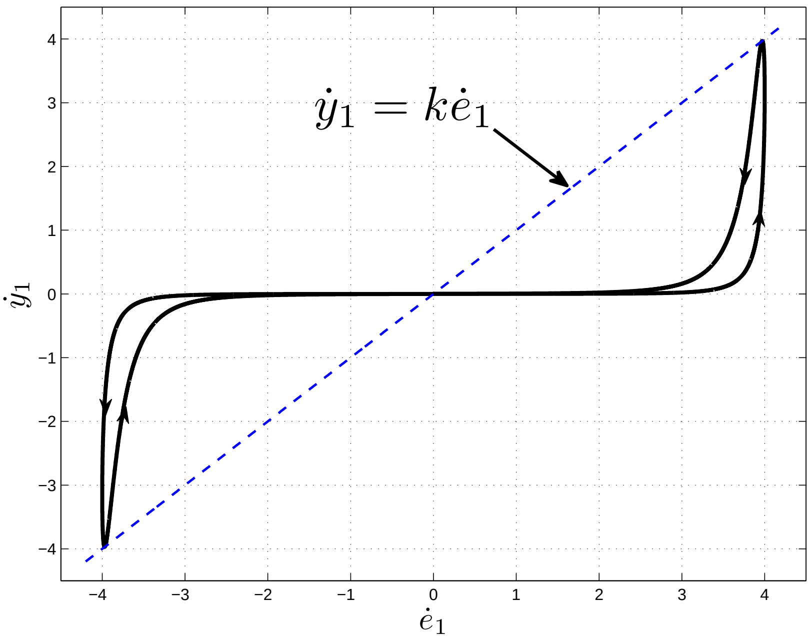

Figure 2.3: The plot $(\dot{e}_1, \dot{y}_1)$ of the hysteresis shown in Fig. 2.2

Assumption 2.3 can be explained graphically as follows. Inequality (2.7) implies that the curve $(\dot{e}_1, \dot{y}_1)$ lies inside the sector $[0, k]$ (see Fig. 2.3), which means that the slope of the hysteresis belongs to the sector $[0, k]$. Furthermore, (2.8) implies that if $\dot{e}_1$ and $\dot{y}_1$ follow the curve representing the graph of the derivative of the input and the derivative of the output, the curve rotates in a counterclockwise direction with increasing time. Backlash and some hysteresis satisfy this condition (see [16], [17]). See [16] for more details.

2.3 Stability Condition

This section presents a Popov-type stability condition ([16, 17]) to check the stability of the system in Fig. 2.1. The concept of stability considered here is that for any input whose slope belongs to $\mathbb{N}_2$, the magnitude of the outputs are bounded and there is no sustained oscillation within the system.

Theorem 2.1 ([16], [17]). *Let Assumptions 2.2 and 2.3 hold. Suppose that the derivatives $\dot{u}_1, \dot{u}_2$ of the inputs u_1, u_2 belong to $\mathbb{N}_2$. The derivatives $\dot{y}_1, \dot{y}_2, \dot{e}_1, \dot{e}_2$ of the outputs y_1, y_2, e_1, e_2 belong to $\mathbb{N}_2$ for any hysteresis whose slope lies in the interval $[0, k]$ if there exist real numbers $q < 0$ and $0 < \beta \ll 1$ such that*

$$\text{Re}\,[(1 + qj\omega)G(j\omega)] + \frac{1}{k} \geq \beta > 0, \quad \forall \omega \geq 0. \tag{2.9}$$

Consequently, the outputs e_1, e_2, y_1, y_2 are bounded and go to finite constants when $t \to \infty$.

The graphical interpretation of condition (2.9) is that if we can find a straight line (which is the Popov line, see Fig. 2.4) with a negative slope $1/q$, which passes through the point $K \triangleq (-1/k, 0)$ and divides the planes into 2 halves, such that the Popov plot $\{\text{Re}G(j\omega) + j\omega\text{Im}G(j\omega) : \omega \in [0, \infty)\}$ lies strictly to the right half of the Popov line, then the system satisfying Assumptions 2 and 3 is stable in the sense that the outputs are bounded and go to finite constants when $t \to \infty$ for any input whose slope belongs to $\mathbb{N}_2$.

For a given linear subsystem $G(s)$, condition (2.9) allows us to determine the maximum slope, denoted as k_m, of the hysteresis element. If the slope of the hysteresis, which lies in the sector $[0, k]$, satisfies the inequality

$$k < k_m \tag{2.10}$$

then the system is stable.

The stability condition (2.9) in Theorem 2.1 is a Popov-type inequality and takes the same form as the stability conditions presented in [1, 9, 11, 12, 19]. However, it should be noted that inequality (2.9) requires the sign of q (in other words, the slope $1/q$) to be negative while for the others the sign of q can be arbitrary.

2.4 Practical Design Inequality

We compute the value of k_m by finding the point $K_m \triangleq (-1/k_m, 0)$ on the negative side of the real axis such that K_m is the nearest point to the origin through which we are able to plot a Popov line satisfying condition (2.9). Using the Popov plot and using the convex hull of the Popov plot to determine K_m are both possible. However, using the convex hull is much more convenient because although Popov plots may have complex shapes, the convex hull of the Popov plot always have simple shapes, and therefore it makes inequality (2.10) more readily computable. In order to use the convex hull of Popov plot to compute k_m, the following propositions are useful.

Proposition 2.3 ([18]). *The Popov plot lies to the right of the Popov line if and only if so does its convex hull.*

The Proposition 2.3 assures that the results achieved from Popov plot and from its convex hull are the same.

Proposition 2.4 ([18]). *Let g be the impulse response of the transfer function $G(s)$. If g can be decomposed as $g(t) = c + g_1(t),\ \ t \geq 0$, in which $g_1, \dot{g}_1 \in \mathcal{A}$ and $|c| < \infty$, then the Popov plot of $G(s)$ lies in the finite plane.*

Proposition 2.4 states that the Popov plot exists in a finite plane and therefore the convex hull is always obtainable.

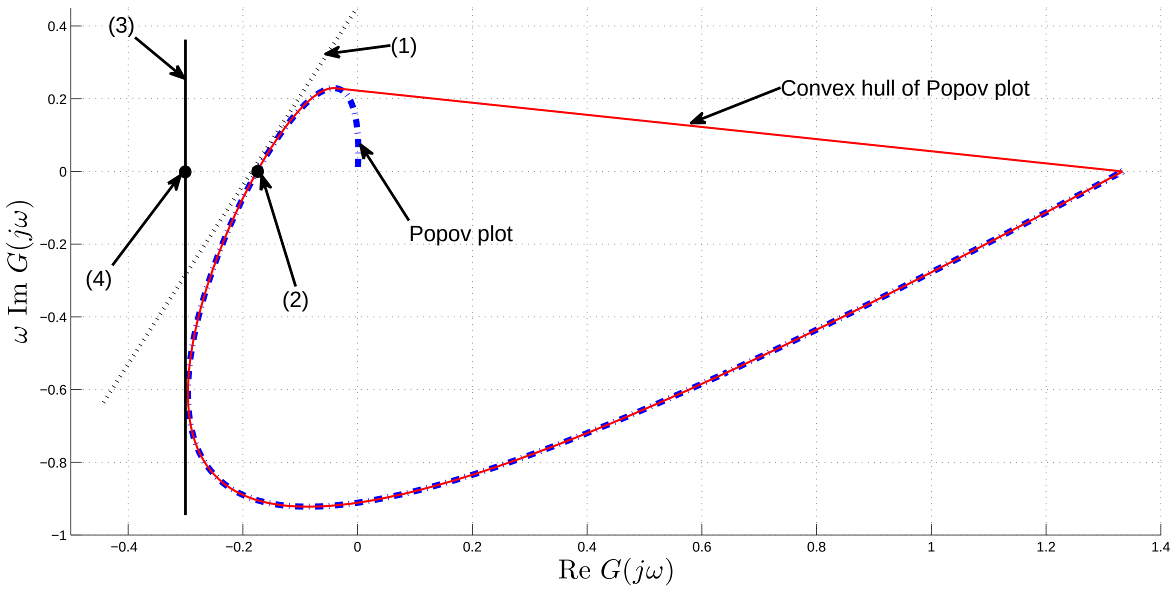

Figure 2.4: Popov plot and its convex hull: (1) is the Popov line and (2) is the corresponding point $(-1/k_m, 0)$ where ψ is sector-bounded and memoryless; (3) is the Popov line and (4) is the corresponding point $(-1/k_m, 0)$ where ψ is a hysteresis.

Since Theorem 2.1 requires that q be negative, it is necessary to develop a new algorithm for determining k_m, which is different from the one presented in [19].

Let Ω denote the convex hull of the Popov plot, the algorithm for determining k_m is outlined as follows.

input: Popov plot

output: k_m

begin

compute Ω;

$P = \{(x, y) \mid (x, y) \in \Omega, x < 0, y = 0\}$;

if $P = \{\}$,

$k_m = \infty$;

else

$\hat{x} = \min_{(x,y)\in P} x$;

calculate the slope q of the point $(\hat{x}, \hat{y}) \in P$;

if $q < 0$,

$k_m = -1/x$;

else

$x^* = \min_{(x,y)\in \Omega} x$;

$k_m = -1/x^*$;

end

end

end

The above algorithm can be explained as follows. First, we look for the point in Ω that is furthest to the left on the negative real axis. If such point does not exist, then K_m is at the origin (i.e., $k_m = \infty$). If such point exists, then we need to check the slope of the convex hull at that point. If the slope at that point is negative, then that point is K_m. If not, then K_m is the point $(x^*, 0)$, where x^* is the real part of the furthest point to the left of the convex hull.

Let $\mathbf{p} \in \mathbb{R}^n$ be a vector of design parameters in the linear subsystem $G(s, \mathbf{p})$, and define $\phi(\mathbf{p}) \triangleq k - k_m(\mathbf{p})$. Inequality (2.10) can be expressed as

$$\phi(\mathbf{p}) \leq -\gamma, \quad 0 < \gamma \ll 1 \tag{2.11}$$

Since $\phi(\mathbf{p})$ is readily computable in practice, it follows that inequality (2.11) provides a practical and useful condition to determine a stabilizing controller for the system. This inequality is always soluble by numerical methods and hence is in keeping with the method of inequalities (see [38], [39]).

2.5 Design of Stabilizing Controller

Consider the system in Fig. 2.5 where $G_p(s)$ and $G_c(s, \mathbf{p})$ are, respectively, the transfer functions of the plant and of the controller with design parameter $\mathbf{p} \in \mathbb{R}^n$. The inputs u_1, u_2 of the system are assumed to satisfy the condition that their derivatives $\dot{u}_1, \dot{u}_2$ belong to $\mathbb{N}_2$. Given the hysteresis $\psi(\cdot)$ with the slope lying in the interval $[0, k]$, we need to design a controller to stabilize the system.

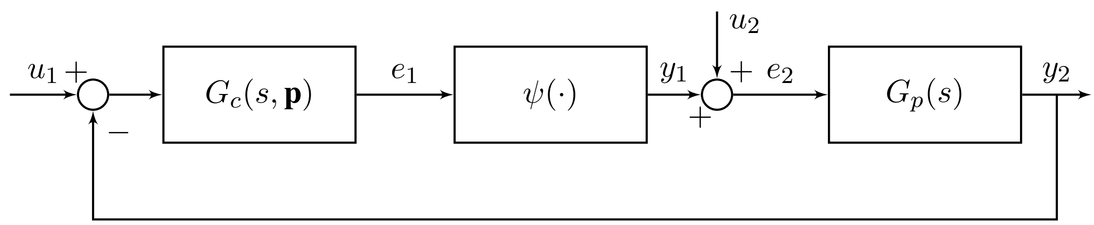

Figure 2.5: Controller design system

Define the composite transfer function

$$G(s) \triangleq G_c(s, \mathbf{p}) G_p(s).$$

Let g_c, g_p and g denote, respectively, the impulse response of the transfer functions $G_c(s), G_p(s)$ and $G(s)$. Then the following assumptions are required.

Assumption 2.4. *The impulse response g_c belongs to $\mathcal{A}_-$.*

Assumption 2.5. *The impulse response g can be decomposed as $g(t) = c + g_1(t),\ \ t \geq 0$, in which $|c| < \infty$, and $g_1 \in \mathcal{A}_-$, $\dot{g}_1 \in \mathcal{A}$.*

To enable one to use the results stated in Sections 3.2 and 3.3, arrange the block diagram as shown in Fig. 2.6, so that the system in Fig. 2.5 is transformed into the equivalent system in Fig. 2.1, in which the linear subsystem has the transfer function $G(s, \mathbf{p})$. The inputs of the equivalent closed-loop system in Fig. 2.6 are $\tilde{u}_1, u_2$, where $\tilde{u}_1 = g_c * u_1$. With the assumption that $\dot{u}_1, \dot{u}_2 \in \mathbb{N}_2$, we can show that the derivative of $\tilde{u}_1$ also belongs to $\mathbb{N}_2$ as follows.

$$\dot{\tilde{u}}_1 = \dot{\overline{g_c * u_1}} = g_c * \dot{u}_1 \tag{2.12}$$

Using the assumptions $g_c \in \mathcal{A}_-$ and $\dot{u}_1 \in \mathbb{N}_2$, it follows that $\dot{\tilde{u}}_1 \in \mathbb{N}_2$ (see Proposition 7.1 in Appendix for detail).

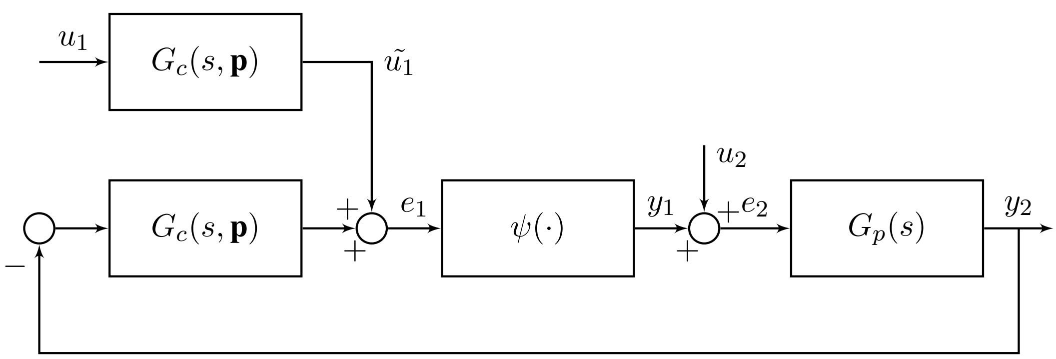

Figure 2.6: Equivalent closed-loop system

From the above, it is easy to see that inequality (2.11) can be used to obtain stabilizing controllers for the system.

2.6 Numerical Examples

In this section, three examples are given to illustrate how to design stabilizing controllers using the method in the previous part. Note that in the following, the search algorithm called the moving-boundaries-process (MBP) will be used for solving inequalities. For the detail of the algorithm, readers are referred to [38, 39] and also references therein.

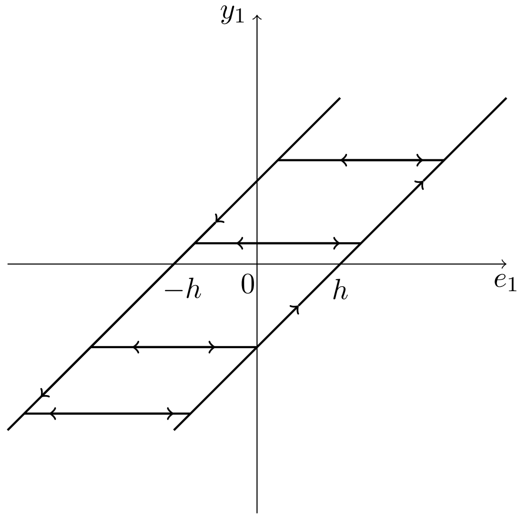

Figure 2.7: The friction-driven hysteresis model of backlash

2.6.1 Example 1: Rational system with backlash

Consider the system shown in Fig. 2.5 in which the transfer function of the plant is

$$G_p(s) = \frac{48}{(1+s)(1+0.1s)} \tag{2.13}$$

The nonlinearity in the system is a backlash which is assumed to satisfy the friction-driven hysteresis model (see Fig. 2.7).

$$\dot{y}_1 = \begin{cases} k\dot{e}_1 & \text{if } \dot{e}_1 \geq 0 \text{ and } y_1 = k(e_1 - h) \\ k\dot{e}_1 & \text{if } \dot{e}_1 \leq 0 \text{ and } y_1 = k(e_1 + h) \\ 0 & \text{elsewhere} \end{cases} \tag{2.14}$$

where e_1, y_1 denote the input and output of the backlash respectively, h is the backlash width and k is the gear ratio or the slope of the backlash. The model (2.14) is widely used (see, for example, [6, 15, 23]).

Suppose that we want to design a stabilizing controller $G_c(s, \mathbf{p})$ for the case in which the backlash has a unity slope ($k = 1$) and $h = 0.4$. The structure of the controller is chosen as

$$G_c(s, \mathbf{p}) = \frac{s + p_1}{s + p_2} \tag{2.15}$$

where $\mathbf{p}= [p_1,\ p_2]^T$ is the design parameter. Among all the possible controllers satisfying the requirement, the one with simple structure is usually preferred; therefore, designers should begin with a simple one first.

Inequality (2.11) is solved by the moving boundaries process. From the starting point $\mathbf{p}_0 = [1,\ 1]^T$, a stability point $\mathbf{p} = [0.35,\ \ 13.1]^T$ is located. The simulation results are displayed in Fig. 2.8. It can be seen that when $G_c(s) = 1$, the system has a sustained oscillation. On the other hand, when the stabilizing controller (2.15) is used, the output y_2 is bounded and goes to a finite constant.

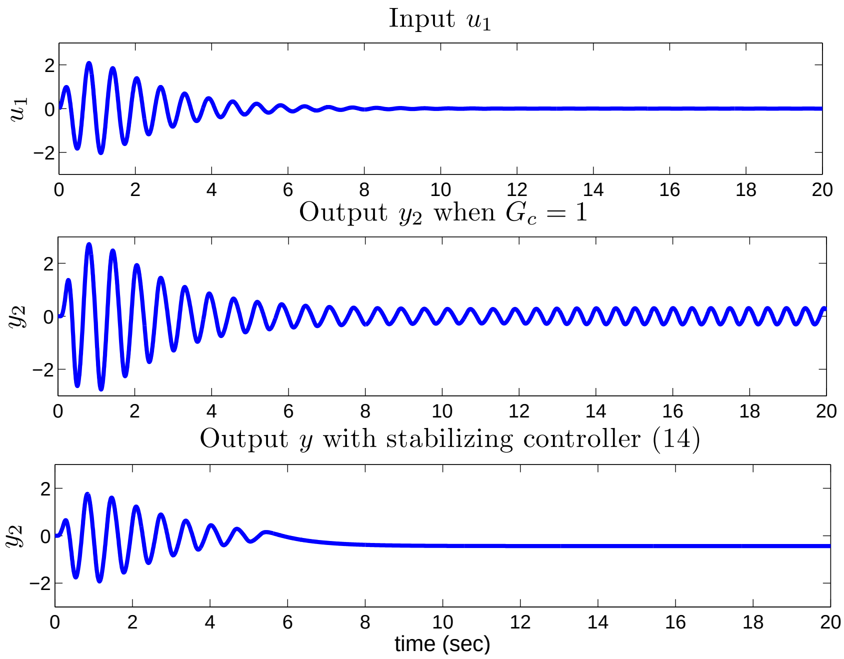

Figure 2.8: Simulation results of Example 2.6.1

2.6.2 Example 2: Time-delay system with backlash

Consider the system shown in Fig. 2.5 in which the plant is a time-delay system whose transfer function is

$$G_p(s) = \frac{48e^{-0.1s}}{(1+s)(1+0.1s)}. \tag{2.16}$$

The nonlinearity in the system is a backlash which is assumed to satisfy the friction-driven hysteresis model as shown in equation (2.14).

Suppose that we want to design a stabilizing controller $G_c(s, \mathbf{p})$ in case the backlash has a unity slope ($k = 1$) and $h = 0.4$. The structure of the controller is chosen as

$$G_c(s, \mathbf{p}) = \frac{s + p_1}{s + p_2} \tag{2.17}$$

where $\mathbf{p}= [p_1,\ p_2]^T$ is the design parameter. Using the moving boundaries process, from the starting point $\mathbf{p}_0 = [1,\ \ 1]^T$, a stability point $\mathbf{p} = [0.35,\ \ 37.4]^T$ is located. The simulation results are displayed in Fig. 2.9. It can be seen that when $G_c(s) = 1$, the output y_2 blows up as $t \to \infty$. On the other hand, when the stabilizing controller (2.17) is used, the output y_2 is bounded and goes to a finite constant.

2.6.3 Example 3: Heat conduction system with hysteresis

Consider the system shown in Fig. 2.5 in which the nonlinearity is a hysteresis whose slope is known only to the extent that lies in the interval $[0, 1]$ (see Fig. 2.11). This hysteresis can be considered as a composition of the nonlinear function $\tanh$ and a backlash with unity slope (see Fig. 2.10). This

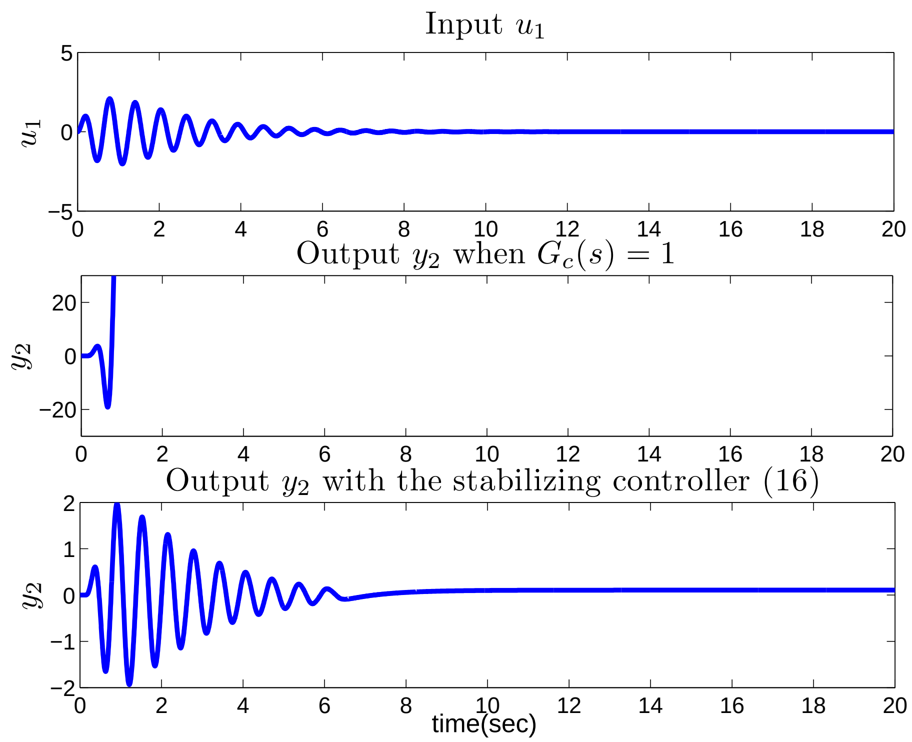

Figure 2.9: Simulation results of Example 2.6.2

idea can be found in [15]. Assume that the plant is a heat-conduction process whose transfer function is given by

$$G_p(s) = \frac{20}{\sqrt{s}\sinh(\sqrt{s})}. \tag{2.18}$$

It is known (see, for example, [10]) that the impulse response g_p is given by

$$g_p(t) = 20 + 40\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t}, \quad t \geq 0 \tag{2.19}$$

and the transfer function $G_p(s)$ has one pole at the origin while others on the negative real axis.

The structure of the controller is chosen as

$$G_c(s, \mathbf{p}) = \frac{p_3(s + p_1)}{s + p_2} \tag{2.20}$$

where $\mathbf{p} = [p_1,\ p_2,\ p_3]^T$ is the design parameter. From the starting point $\mathbf{p}_0 = [5,\ \ 20,\ \ 5]^T$, which is an unstable point, the stability point $\mathbf{p} = [4,\ \ 46,\ \ 4.75]^T$ is located. The simulation results are shown in Fig. 2.12. As can be seen, when $G_c(s) = 1$, the system has a sustained oscillation and when the stabilizing controller (2.20) is used, the output y_2 goes to a finite constant, which means that the sustained oscillation is eliminated.

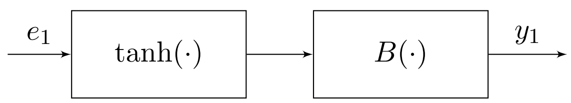

Figure 2.10: Composition of the hysteresis element used in Example 2.6.3

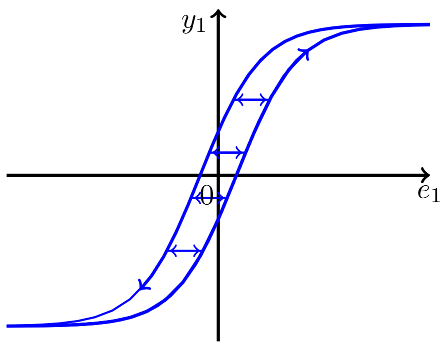

Figure 2.11: Hysteresis element in Example 2.6.3

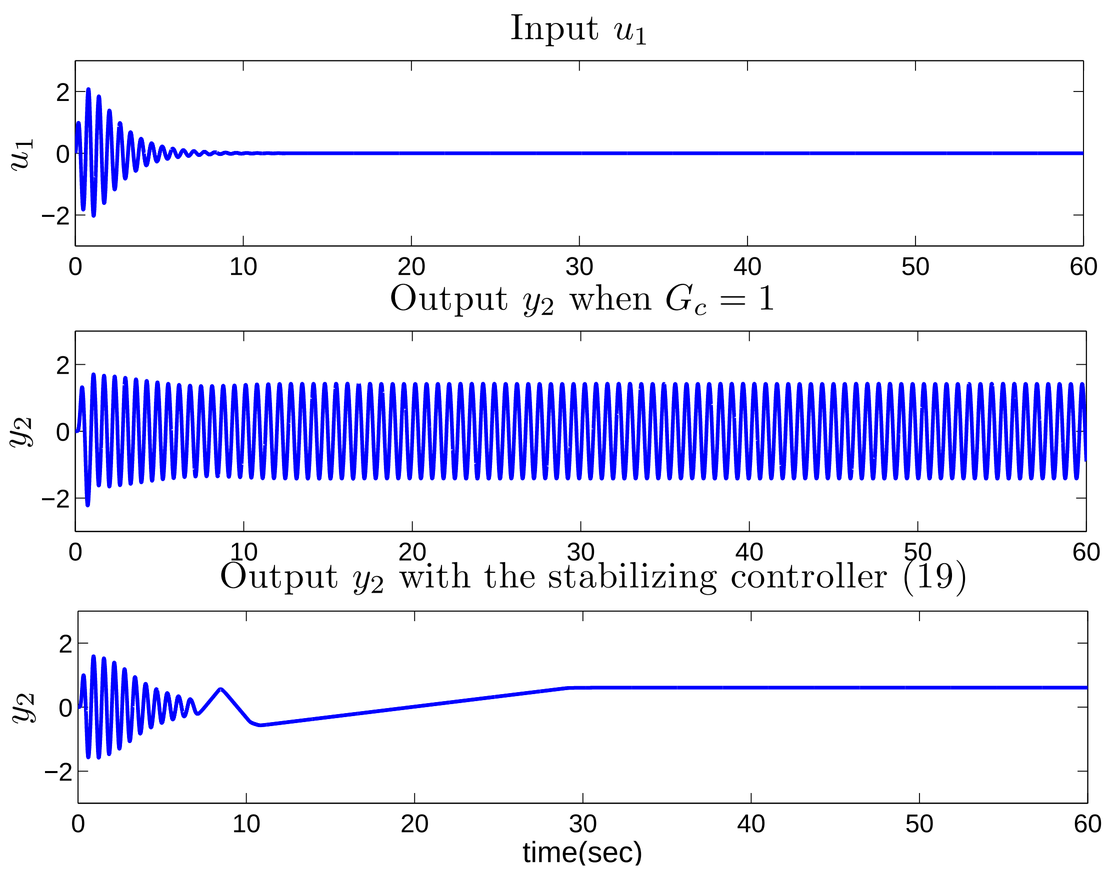

Figure 2.12: Simulation results of Example 2.6.3

2.7 Conclusions

This work has considered the input-output stability of feedback control systems with a hysteresis, in which the linear subsystem can be either a rational or a non-rational transfer function belonging to a subclass of $\mathcal{A}_-$. Based on the result that for all the inputs whose first derivatives belong to $\mathbb{N}_2$, if the linear subsystem satisfies the Popov-type condition, then the outputs are bounded and go to finite constants, we have developed a practical inequality to obtain stability points, which is in keeping with the method of inequalities and is always soluble by numerical methods. The merit of the contribution is clearly demonstrated by the given examples.

The input space considered in this work is appropriate to characterize some types of vibration signals such as seismic signals (see, for example, [5]) whose magnitudes and slopes are enveloped by decreasing exponential functions.

CHAPTER III

STABILITY CONDITIONS FOR FEEDBACK SYSTEM WITH BACKLASH SUBJECT TO BOUNDED INPUTS

3.1 Introduction

This chapter considers the BIBO stability of nonlinear feedback systems with backlash under inputs having bounded magnitude, which is a different set of inputs with inputs considered in Chapter 2. The reason we consider this input set is that, the input set considered in Chapter 2 is not suitable to characterize persistent inputs.

The key theorem (Theorem 3.1) stated in this chapter is from [6]. Based on this, we develop BIBO stability conditions (Theorem 3.2) for control feedback systems which includes a controller, a backlash in cascade connection with a linear time-invariant plant. Theorem 3.2 is used in the following chapter to guarantee the finiteness of the error and the controller output in the design process.

3.2 System Description

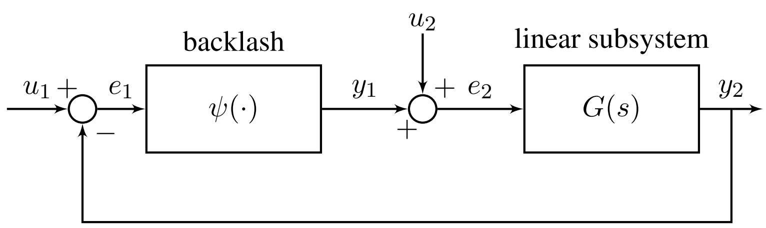

Figure 3.1: Feedback control system with backlash

The system in Figure 3.1 is described by the following equations.

$$\left.\begin{aligned} e_1(t) &= u_1(t) - y_2(t) \\ e_2(t) &= u_2(t) + y_1(t) \\ y_1(t) &= \psi(e_1) \\ y_2(t) &= (g * e_2)(t) \end{aligned}\right\}, \quad \forall t \geq 0. \tag{3.1}$$

The notation $\psi(\cdot)$ in equation (2.1) expresses the input-output characteristic of the backlash element in the system, g is the impulse response of the linear subsystem and has Laplace transform equal to $G(s)$. Suppose that the system is at rest for $t \leq 0$.

The inputs u_1, u_2 belong to the space $\mathbb{L}_\infty$.

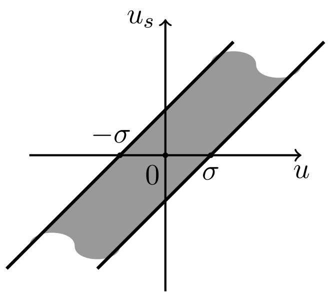

Figure 3.2: The uncertainty model of backlash

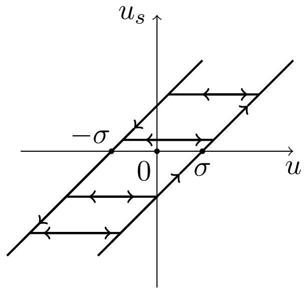

Figure 3.3: The friction-driven hysteresis model of backlash

Assumption 3.1. *Let the backlash $\psi(\cdot)$ be represented by the uncertain band model ([6])*

$$\left.\begin{aligned} \psi(x) &= Kx + n(x) \\ n(x) &= [-h, h] \quad \forall x \end{aligned}\right\}, \tag{3.2}$$

where K is a constant gain and $n(\cdot)$ denotes the interval valued function mapping $\mathbb{R}$ to $2^{\mathbb{R}}$.

The uncertain band model (3.2) is useful in the sense that the backlash width does not need to be known exactly (see Fig. 3.2). Hence, it can be used to find a robust controller to compensate the backlash effect which may not be known accurately. In addition, the uncertain model can be used to represent some other backlash models such as a friction-driven hysteresis model (see Fig. 3.3).

Assumption 3.2. *The transfer function $G(s)$ is strictly proper.*

3.3 Stability Conditions

Theorem 3.1 ([6])**.** *Consider the system 3.1 and let Assumptions 3.1 and 3.2 hold. Define $G_b(s) \triangleq G(s)(1+KG(s))^{-1}$ and let g_b be its inverse Laplace transform. If $g_b, \dot{g}_b \in \mathcal{A}$ then the responses e_1 and e_2 are bounded for any bounded inputs u_1 and u_2.*

For the sake of completeness, the proof is given as follows.

Proof. For a function $x : \mathbb{R}_+ \to \mathbb{R}$ and for a fixed $T > 0$, define

$$x_T(t) \triangleq \begin{cases} x(t), & 0 \le t \le T \\ 0, & t > T \end{cases}.$$

Also, for a given $X \subset \mathbb{L}_\infty$, define $X_T \triangleq \{x_T | x \in X\}$.

Now consider the system (3.1), and let Assumptions 3.1 and 3.2 hold, we have

$$e_1 = g_a * u_1 - g_b * [u_2 + n(e_1)] \tag{3.3}$$

where g_a and g_b are respectively the impulse responses of $G_a(s) \triangleq (1 + KG(s))^{-1}$ and $G_b(s) \triangleq G(s)[1 + KG(s)]^{-1}$. Define a mapping

$$\Upsilon(e_1) = g_a * u_1 - g_b * [u_2 + n(e_1)] \tag{3.4}$$

It is easy to see that

$$||\Upsilon||_\infty \leq ||g_a||_\mathcal{A} ||u_1||_\infty + ||g_b||_\mathcal{A}(||u_2||_\infty + h) \tag{3.5}$$

Since $g_b \in \mathcal{A}$, the right hand side of inequality 3.5 is finite. Choose a constant M such that

$$||g_a||_\mathcal{A} ||u_1||_\infty + ||g_b||_\mathcal{A}(||u_2||_\infty + h) \leq M. \tag{3.6}$$

For a chosen M, define the set S as follows.

$$\mathcal{S} \triangleq \{x : \mathbb{R}_+ \rightarrow \mathbb{R} | \; ||x||_\infty \leq M\}.$$

Following (3.3), define a truncated mapping $\Upsilon_T : \mathcal{S}_T \rightarrow 2^{\mathcal{S}_T}$ for a given $T > 0$ and for any $u_1, u_2 \in \mathbb{L}_{\infty T}$. It is easy to see that: (i) $\mathcal{S}_T$ is a nonempty, closed, convex set in Banach space and Υ_T is upper semi-continuous; (ii) for all $e_1 \in \mathcal{S}_T$, the set $\Upsilon_T(e_1)$ is nonempty, closed and convex; (iii) by the compactness of G_b (see [18]), the set $\Upsilon_T(\mathcal{S}_T)$ is relatively compact. As a result, using Kakutani's fixed point theorem (see, for example, [40]), Υ_T has a fixed point that guarantees the existence of a $\tilde{e_1} \in \mathcal{S}_T$ such that $\tilde{e_1} \in \Upsilon_T(\tilde{e_1})$. Since the constant M can be chosen arbitrarily large, it means that there exists at least a solution $\tilde{e_1} \in \mathbb{L}_\infty$ such that $\tilde{e_1} \in \Upsilon_T(\tilde{e_1})$, which is also the error e_1 of the feedback system (3.1). Then the boundedness of e_2 readily follows. □

3.4 Determing of Stabilizing Controller

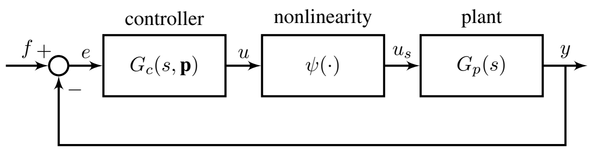

Figure 3.4: Feedback system with nonlinearity

Consider the feedback control system displayed in Fig. 3.4, which is described by

$$\left.\begin{aligned} u &= g_c * e \\ e &= f - u_s * g_p = f - \psi(u) * g_p \end{aligned}\right\} \tag{3.7}$$

where ψ is a backlash, g_p is the impulse response of the plant and has Laplace transform equal to $G_p(s)$, g_c is the impulse response of the controller and has Laplace transform equal to $G_c(s, \mathbf{p})$, and $\mathbf{p} \in \mathbb{R}^n$ denotes a design parameter vector. As usual, the asterisk denotes the convolution; that is

$$(g_c * e)(t) = \int_0^t g_c(t - \tau) e(\tau) d\tau, \quad t \geq 0. \tag{3.8}$$

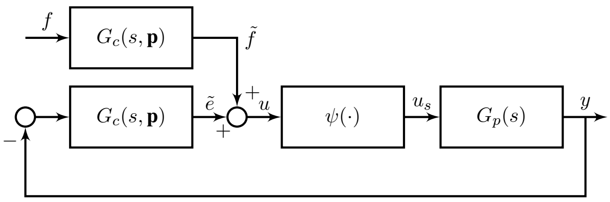

Figure 3.5: Equivalent closed-loop system

In this section, we will develop a theorem to determine a controller G_c stabilizing the system (3.7).

Assumption 3.3. *For every input f, there exists at least a solution (e, u) that satisfy (4.1), where $e : \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$. All initial conditions are assumed to be zero for $t \leq 0$.*

Assumption 3.4. *Let the backlash $\psi(\cdot)$ be represented by the uncertain band model ([6])*

Assumption 3.5. *For $G(s)$ defined by $G(s) \triangleq G_p(s)G_c(s)[1 + KG_p(s)G_c(s)]^{-1}$, the impulse response g satisfies conditions that $g \in \mathcal{A}$ and $\dot{g} \in \mathcal{A}$.*

By using Theorem 3.1, the following result is obtained.

Theorem 3.2. *Consider the system* (3.7) *and let Assumptions 3.3, 3.4 and 3.5 hold. Define the composite transfer function $G_2(s) \triangleq G_c(s, \mathbf{p})G_p(s)$. Let $G_2(s)$ be strictly proper and suppose that $g_c \in \mathcal{A}$ where g_c is the impulse response of $G_c(s, \mathbf{p})$. Then it follows that the responses u and e are bounded for any $f \in \mathcal{P}$.*

Proof. By rearranging the block diagram in Fig. 3.4 to the one shown in Fig. 3.5, one can easily see that

$$u = \tilde{f} + \tilde{e} \tag{3.9}$$

where $\tilde{f} = g_c * f$. It is easy to show (see, for example, [11]) that $\tilde{f} \in \mathbb{L}_\infty$ if $f \in \mathcal{P}$ and $g_c \in \mathcal{A}$. Assumption 3.5 implies that the transfer function $G(s) = G_p(s)G_c(s)[1+KG_p(s)G_c(s)]^{-1}$ is BIBO stable; moreover, $g, \dot{g} \in \mathcal{A}$. Thus, by Theorem 3.1, it follows that $\tilde{e} \in \mathbb{L}_\infty$. Hence, for the system (4.1), u and e are bounded for any $f \in \mathcal{P}$. □

CHAPTER IV

DESIGN OF FEEDBACK SYSTEMS WITH BACKLASH FOR INPUTS RESTRICTED IN MAGNITUDE AND SLOPE

This chapter develops a design method for unity feedback systems comprising a backlash and linear time-invariant convolution subsystems, where the main design objective is to ensure that the error and the controller output stay within prescribed bounds for all time and for all possible inputs having bounded magnitude and bounded slope. The design formulation is based on the principle of matching, thereby explicitly considering the peak error and the peak controller output for such a possible set. The original design inequalities are replaced with the surrogate design criteria that are in keeping with the method of inequalities. Essentially, the backlash is replaced with a gain and an equivalent disturbance; thus, the nominal system used during the design process becomes linear and the associated performance measures are readily obtainable by known methods. To illustrate the usefulness of the method, a design example is given where the plant has a time-delay.

4.1 Introduction

Consider the feedback control system displayed in Fig. 4.1, which is described by

$$\left.\begin{aligned} u &= g_c * e \\ e &= f - u_s * g_p = f - \psi(u) * g_p \end{aligned}\right\} \tag{4.1}$$

where ψ is a backlash, g_p is the impulse response of the plant and has Laplace transform equal to $G_p(s)$, g_c is the impulse response of the controller and has Laplace transform equal to $G_c(s, \mathbf{p})$, and $\mathbf{p} \in \mathbb{R}^n$ denotes a design parameter vector. As usual, the asterisk denotes the convolution; that is

$$(g_c * e)(t) = \int_0^t g_c(t-\tau)e(\tau)d\tau, \quad t \geq 0. \tag{4.2}$$

Suppose that the input f is known only to the extent that it belongs to a *possible set* $\mathcal{P}$ (that is, the set of possible inputs) described by

$$\mathcal{P} \triangleq \{f : \mathbb{R}_+ \to \mathbb{R} \mid ||f||_\infty \leq M, ||\dot{f}||_\infty \leq D\} \tag{4.3}$$

where $\mathbb{R}_+$ denotes the half line $[0, \infty)$ and the bounds M and D are given. Notice that $\mathcal{P} \subset \mathbb{L}_\infty$, where $\mathbb{L}_\infty \triangleq \{x : \mathbb{R}_+ \to \mathbb{R} \mid |x(t)| < \infty \ \forall t \geq 0\}$.

It may be noted that $\mathcal{P}$ is suitable for characterizing *persistent signals* (that is, signals that vary persistently for all time). Since $\mathbb{L}_\infty$ includes some inputs that have stepwise discontinuities, using $\mathcal{P}$ as the possible set can make the design formulation be more realistic and appropriate. For detailed discussion, see [31, 38].

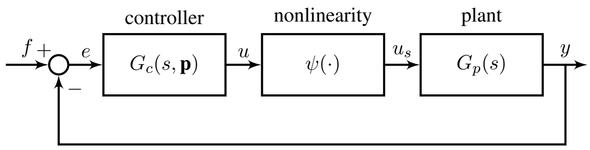

Figure 4.1: Feedback system with nonlinearity

Following [20–22], the problem considered here is to determine a design parameter **p**, and hence the controller transfer function $G_c(s,\mathbf{p})$, such that the following design criteria are satisfied.

$$\hat{e} \le E_{max} \text{ and } \hat{u} \le U_{max} \tag{4.4}$$

where the bounds E_{max} and U_{max} are given and $\hat{e}$ and $\hat{u}$ are defined by

$$\hat{e} \triangleq \sup_{f\in\mathcal{P}} \|e\|_\infty \text{ and } \hat{u} \triangleq \sup_{f\in\mathcal{P}} \|u\|_\infty. \tag{4.5}$$

The performance measures $\hat{e}$ and $\hat{u}$ are sometimes called the peak error and the peak controller output, respectively, for the possible set $\mathcal{P}$. Obviously, $\hat{e}$ and $\hat{u}$ depend upon **p**.

It is worth noting that the design criteria (4.4) are used to ensure that the error e and the controller output u stay within their respective bounds E_{max} and U_{max} for all time and for all $f \in \mathcal{P}$. It is evident that (4.4) are equivalent to

$$\left.\begin{aligned} |e(f,t)| &\le E_{max}, \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \\ |u(f,t)| &\le U_{max}, \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \end{aligned}\right\}. \tag{4.6}$$

The main objective of the chapter is to extend the results developed previously by [20, 21] so as to enable one to determine, by numerical methods, the controller $G_c(s,\mathbf{p})$ for the feedback system (4.1) so that the criteria (4.4) are satisfied, in which ψ is a backlash and the plant $G_p(s)$ can be a rational or non-rational transfer function.

Note that the main results stated in [20, 21] were proved by applying Schauder's fixed point theorem (see, for example, [8]), while that of this work is proved by using a multivalued version of the fixed point theorem known as Kakutani's theorem (see, for example, [6, 40]), which was used by [6]. In addition, the stability conditions used in [20, 21] and that in the present paper are different.

The organization of the chapter is as follows. Section 4.2 presents the main theoretical result. Section 4.3 derives the surrogate design criteria, thereby providing practical inequalities for designing the system (4.1) so that the original criteria (4.4) are satisfied. Section 4.4 provides the stability conditions for ensuring that the peaks $\hat{e}$ and $\hat{u}$ are finite, so as to enable a numerical algorithm to search for a design solution (if exists) in the design parameter space. To illustrate the usefulness and the potential of the developed method, a design example for a time-delay plant is carried out in Section 4.5. Finally, the conclusions are given.

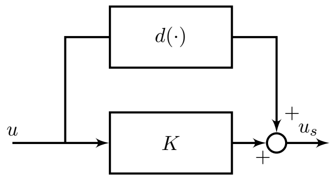

Figure 4.2: Decomposition of the backlash

4.2 Main Theoretical Result

In this section, the main theoretical result is derived by: (i) the decomposition technique used by [25] and [18, 20, 21], in which the backlash is replaced by a constant gain K and an equivalent bounded disturbance, and (ii) Kakutani's fixed point theorem. The result is stated in Theorem 4.1, providing sufficient conditions for the satisfaction of the design criteria (4.4). The conditions will be used in Section 3 to derive practical design inequalities that can be used for determining $G_c(s, \mathbf{p})$ by numerical methods so that the crieria (4.4) are satisfied.

Assumption 4.1. *For every input $f \in \mathcal{P}$, there exists at least a solution (e, u) that satisfy (4.1), where $e : \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$. All initial conditions are assumed to be zero for $t \leq 0$.*

Assumption 4.2. *Let the backlash $\psi(\cdot)$ be represented by the uncertain band model ([6])*

From Assumption 4.2, it is easy to see that for a backlash with bandwidth σ (see Fig. 3.2),

$$||n(\cdot)||_\infty \leq h \text{ where } h = K\sigma. \tag{4.7}$$

In connection with the technique used in [25] and [18,20,21], if u is bounded then $\psi(\cdot)$ can be decomposed as

$$\psi[u(t)] = Ku(t) + d[u(t)] \text{ for } t \geq 0 \tag{4.8}$$

where K is a constant gain and $d(\cdot)$ is the equivalent disturbance (see Fig. 4.2). From (3.2) and the decomposition (4.8), one can easily see that

$$d[u(t)] = n[u(t)] \text{ for } t \geq 0. \tag{4.9}$$

Since $\psi(\cdot)$ is a multivalued function, so is $d(\cdot)$. From (4.7), it is obvious that

$$||d[u(t)]||_\infty \leq h \quad \forall f \in \mathcal{P}. \tag{4.10}$$

Instead of considering the system (4.1), now we consider the system shown in Fig. 4.3, called the *auxiliary* linear system, where the backlash is replaced by a fixed gain K and an equivalent disturbance $d(u)$. The auxiliary system is described by

$$\left.\begin{aligned} u' &= g_c * e' \\ e' &= f - g_p * [Ku' + d(u)] \end{aligned}\right\} \tag{4.11}$$

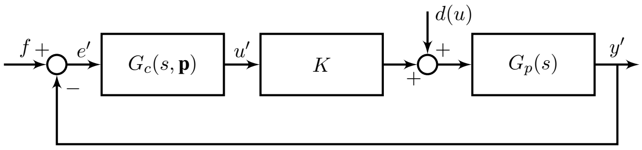

Figure 4.3: Auxiliary linear system

where $f \in \mathcal{P}$, $d(u) \in \mathcal{D}_u$ and the set $\mathcal{D}_u$ is defined by

$$\mathcal{D}_u \triangleq \{d(u) = \psi(u) - Ku \mid ||u||_\infty \leq U_{max}\}. \tag{4.12}$$

Note that $\mathcal{D}_u \subset \mathbb{L}_\infty$ and every element of $\mathcal{D}_u$ depends on u for $||u||_\infty \leq U_{max}$.

In connection with the auxiliary system (4.11), let g be the impulse response of the transfer function from $d(u)$ to u', that is,

$$G(s) \triangleq G_p(s)G_c(s)[1 + KG_p(s)G_c(s)]^{-1}. \tag{4.13}$$

Assumption 4.3. *For $G(s)$ defined by* (4.13)*, the impulse response g satisfies conditions that $g \in \mathcal{A}$ and $\dot{g} \in \mathcal{A}$.*

In the following, the relation between the systems (4.1) and (4.11) is described and can be proved by using the technique used in [6] and [18], which is basically the application of Kakutani's theorem (see, for example, [40]).

Theorem 4.1. *Consider the system* (4.1) *where $\hat{u}$ is finite, and let Assumptions 4.1, 4.2 and 4.3 hold. The original design criteria* (4.4) *are satisfied if, for the auxiliary system* (4.11)*, the following conditions hold:*

$$\left.\begin{array}{ll} \hat{e}' \leq E_{max}, & \hat{e}' \triangleq \displaystyle\sup_{f\in\mathcal{P}, d\in\mathcal{D}_u} \|e'\|_\infty \\ \hat{u}' \leq U_{max}, & \hat{u}' \triangleq \displaystyle\sup_{f\in\mathcal{P}, d\in\mathcal{D}_u} \|u'\|_\infty \end{array}\right\}. \tag{4.14}$$

Proof. For a function $x : \mathbb{R}_+ \to \mathbb{R}$ and for a fixed $T > 0$, define

$$x_T(t) \triangleq \begin{cases} x(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}.$$

Also, for a given $X \subset \mathbb{L}_\infty$, define $X_T \triangleq \{x_T | x \in X\}$. Define the sets of acceptable u's and acceptable e's as follows.

$$\mathcal{U} \triangleq \{x : \mathbb{R}_+ \to \mathbb{R} |\ ||x||_\infty \leq U_{max}\}.$$

$$\mathcal{E} \triangleq \{x : \mathbb{R}_+ \to \mathbb{R} |\ ||x||_\infty \leq E_{max}\}.$$

Now consider the system (4.11), and let (4.14) hold. Consequently, it follows that $e' \in \mathcal{E}$ and $u' \in \mathcal{U}$ for all $f \in \mathcal{P}$ and all $u \in \mathcal{U}$. From (4.11), we have

$$u' = -g * d(u) + g_1 * f \triangleq \Phi(u) \tag{4.15}$$

where g and g_1 are respectively the impulse responses of $G(s)$ and $G_1(s) \triangleq G_c(s)[1+KG_p(s)G_c(s)]^{-1}$. Following (4.15), define a truncated mapping $\Phi_T : \mathcal{U}_T \to 2^{\mathcal{U}_T}$ for a given $T > 0$ and for any $f \in \mathcal{P}_T$. It is easy to see that: (i) $\mathcal{U}_T$ is a nonempty, closed, convex set in Banach space and Φ_T is upper semi-continuous; (ii) for all $u \in \mathcal{U}_T$, the set $\Phi_T(u)$ is nonempty, closed and convex; (iii) by the compactness of G (see [18]), the set $\Phi_T(\mathcal{U}_T)$ is relatively compact. As a result, using Kakutani's fixed point theorem (see, for example, [40]), Φ_T has a fixed point that guarantees the existence of a $\tilde{u} \in \mathcal{U}_T$ such that $\tilde{u} \in \Phi_T(\tilde{u})$. Let $\tilde{e} : [0, T] \to \mathbb{R}$ denote the associated error. Then it follows that

$$\left.\begin{aligned} \tilde{u} &= g_c * \tilde{e} \\ \tilde{e} &= f - g_p * [K\tilde{u} + d(\tilde{u})] \end{aligned}\right\}. \tag{4.16}$$

By the finiteness of $\hat{u}$, (4.16) are equivalent to

$$\left.\begin{aligned} \tilde{u} &= g_c * \tilde{e} \\ \tilde{e} &= f - g_p * \psi(\tilde{u}) \end{aligned}\right\}. \tag{4.17}$$

It readily follows that $\tilde{u}$ and $\tilde{e}$ are also the error and the controller output of the system (4.1) for any $T > 0$. The conditions $\tilde{u} \in \mathcal{U}$ and $\tilde{e} \in \mathcal{E}$ imply that (4.4) are satisfied, and therefore the proof is completed. $\square$

It should be noted that by virtue of Assumption 4.3, Theorem 4.1 is applicable to both lumped- and distributed-parameter plants, that is to say, rational and non-rational plant transfer functions.

4.3 Surrogate Design Criteria

Based on the results in Section 4.2, this section develops the design criteria in the form of inequalities that can be solved in practice by numerical methods.

From (4.12), it can be seen that the set $\mathcal{D}_u$ cannot be readily employed in the design since every $d(u) \in \mathcal{D}_u$ depends on u. Thus, to eliminate the dependence of u, we replace $d(u)$ with d and $\mathcal{D}_u$ with $\mathcal{D}$ where

$$\mathcal{D} \triangleq \{d \in \mathbb{L}_\infty |\ ||d||_\infty \leq h\}. \tag{4.18}$$

Note that although both $\mathcal{D}$ and $\mathcal{D}_u$ contain bounded signals whose magnitudes do not exceed h, all members of $\mathcal{D}$ do not depend on u.

By the virtue of the linearity of (4.11), we have

$$\left.\begin{aligned} \hat{e}' &\leq \sup_{f \in \mathcal{P}, d \in \mathcal{D}} \|e'\|_\infty \triangleq \phi_{ef} + \phi_{ed} \\ \hat{u}' &\leq \sup_{f \in \mathcal{P}, d \in \mathcal{D}} \|u'\|_\infty \triangleq \phi_{uf} + \phi_{ud} \end{aligned}\right\} \tag{4.19}$$

where

$$\left.\begin{aligned} \phi_{ef} &\triangleq \sup\{||e'||_\infty \mid f \in \mathcal{P}, d = 0\} \\ \phi_{ed} &\triangleq \sup\{||e'||_\infty \mid f = 0, d \in \mathcal{D}\} \\ \phi_{uf} &\triangleq \sup\{||u'||_\infty \mid f \in \mathcal{P}, d = 0\} \\ \phi_{ud} &\triangleq \sup\{||u'||_\infty \mid f = 0, d \in \mathcal{D}\} \end{aligned}\right\}. \tag{4.20}$$

The numbers ϕ_{ef}, ϕ_{uf} can be computed by employing the method proposed in [31], in which the impulse responses need to be calculated first. Further, from a well-known result in linear systems theory (see, for example, [11]), it follows that

$$\left.\begin{array}{l}\phi_{ed} = h\int_0^\infty |e'_d(\delta,t)|dt \\ \phi_{ud} = h\int_0^\infty |u'_d(\delta,t)|dt\end{array}\right\}, \tag{4.21}$$

where $e'_d(\delta,t)$ and $u'_d(\delta,t)$ are the values of e' and u', respectively, at time t when f equals 0 and d is the Dirac delta function. Hence, ϕ_{ed} and ϕ_{ud} can readily be computed by standard numerical algorithms, provided that the responses $e'_d(\delta,\cdot)$ and $u'_d(\delta,\cdot)$ are obtained.

Next, the main result providing a useful computational tool is stated as follows.

Theorem 4.2. *Consider the system* (4.1) *where* $\hat{u}$ *is finite, and let Assumptions 4.1, 4.2 and 4.3 hold. If, for the auxiliary system* (4.11)*, the following hold:*

$$\left.\begin{array}{l}\phi_1(\mathbf{p}) \le E_{max} \;\; \textit{where} \;\; \phi_1(\mathbf{p}) \triangleq \phi_{ef} + \phi_{ed} \\ \phi_2(\mathbf{p}) \le U_{max} \;\; \textit{where} \;\; \phi_2(\mathbf{p}) \triangleq \phi_{uf} + \phi_{ud}\end{array}\right\} \tag{4.22}$$

then the original design criteria (4.4) *are satisfied.*

Proof. First note that $\phi_{ef}, \phi_{ed}, \phi_{uf}, \phi_{ud}$ are functions of the design parameter **p**. By Theorem 4.1, the proof follows readily from the above discussion. □

From the above, it is easy to see that $\phi_1(\mathbf{p})$ and $\phi_2(\mathbf{p})$ are readily computable. Hence, it follows from Theorem 4.2 that a design parameter **p** (that is, $G_c(s,\mathbf{p})$) satisfying the inequalities (4.22) is also a solution of the original design problem defined by (4.4). For this reason, (4.22) are called the *surrogate design criteria*.

4.4 Stability Condition

Following previous work ([31, 32, 38, 39] and also the references therein), it is readily appreciated that in solving the design inequalities (4.22) by numerical methods, a search algorithm needs to start from a *stability point* (that is, a point **p** for which all associated performance measures $\phi_{ef}, \phi_{ed}, \phi_{uf}, \phi_{ud}$ are all finite). Moreover, it is important to note that Theorems 4.1 and 4.2 require the assumption that $\hat{u}$ is finite. Hence, a stability condition that ensures the finiteness of $\hat{e}$ and $\hat{u}$ is needed and will be established in the following.

From Theorem 3.2, it readily follows that to ensure the finiteness of $\hat{e}$ and $\hat{u}$ of the system (4.1), one needs to determine the controller $G_c(s)$ that makes the transfer function $G(s) = G_p(s)G_c(s)[1 + KG_p(s)G_c(s)]^{-1}$ BIBO stable.

For retarded delay differential systems (which of course includes rational systems), it is well known (see, for example, [3] and the references therein) that the system (or alternatively its transfer function) is BIBO stable if and only if all the characteristic roots (or alternatively all the poles of the transfer function) have negative real parts.

Let $f(s)$ be the characteristic function of a retarded delay differential system. Let ϕ_0 denote the abscissa of stability of $f(s)$ defined by

$$\phi_0 \triangleq \sup\{\text{Re}(s) : \ f(s) = 0\}.$$

Then it follows that the system is BIBO stable if

$$\phi_0(\mathbf{p}) \leq -\varepsilon \ \text{ where } \ 0 < \varepsilon \ll 1. \tag{4.23}$$

It should be noted that the inequality (4.23) can be used in practice to determine a value of **p** for which the system is BIBO stable, by numerical methods. For further details, see [2, 3, 38, 39].

In this work, the abscissa of stability for retarded delay differential systems is computed by using the method developed in [2, 3].

4.5 Numerical Example

In this section, the usefulness of the developed method is illustrated by a numerical example which is an application arising in process control. The plant is a linear system with time delay whose transfer function is

$$G_p(s) = \frac{2e^{-0.2s}}{(s+1)(s+2)}. \tag{4.24}$$

As advocated in [14, 26], the backlash phenomenon appears in the linkage mechanism in the positioner and actuator of valves as the amount of friction increases. According to [14], a backlash of 10% increases the peak error at load disturbance with 50%. When the backlash becomes too large, the valve needs to be replaced. However, replacing the valve cannot be done without interrupting the process. For this and for economical reasons, it may be better to take into account the backlash phenomenon in the controller design process.

Assume that the backlash ψ has a unity slope and its bandwidth σ is known to the extent that it belongs to the interval $(0, 0.1]$.

Assume that the control objective is to keep the error e and the controller output u stay within the ranges ± 0.1 and ± 10, respectively, for all time and for $f \in \mathcal{P}$ where

$$M = 1 \quad \text{and} \quad D = 0.1. \tag{4.25}$$

Consequently, the design criteria are expressed as

$$\hat{e} \leq 0.1 \quad \text{and} \quad \hat{u} \leq 10. \tag{4.26}$$

By Theorems 4.2 and 3.2, a design solution **p** of the problem (4.26) is obtained by solving the following inequalities:

$$\phi_0(\mathbf{p}) \leq 10^{-4}, \ \ \phi_1(\mathbf{p}) \leq 0.1, \ \ \phi_2(\mathbf{p}) \leq 10, \tag{4.27}$$

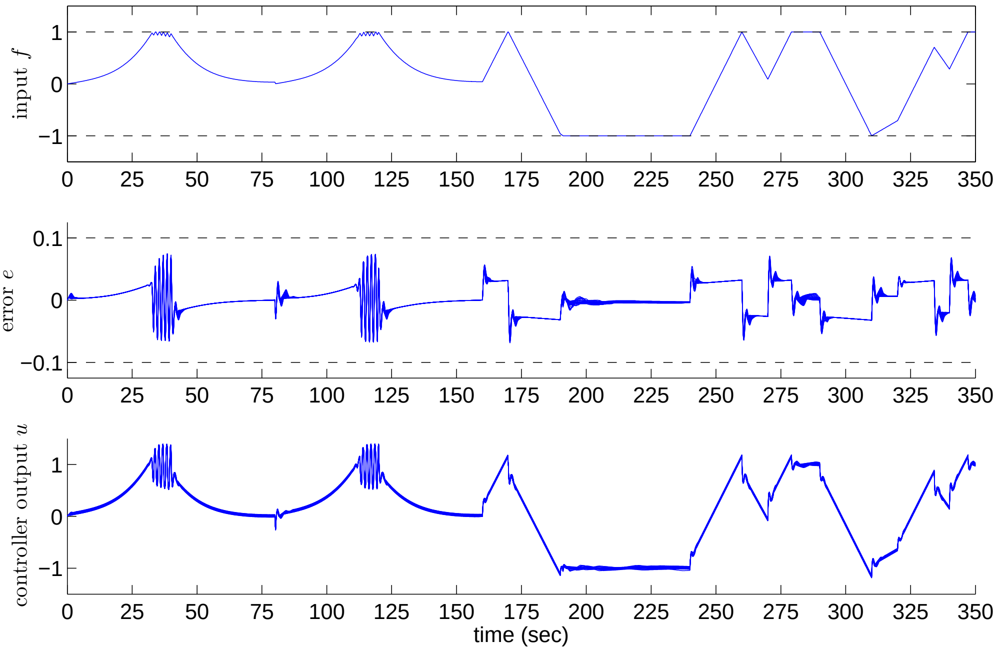

Figure 4.4: Simulation results with the controller (4.28) and the backlash with different values of bandwidth .

which are solved by using the numerical search algorithm called the moving boundaries process (MBP) (see [39] and also [38]) in this work. Alternatively, other algorithms for solving a set of inequalities may be used (see [38] and the references therein for details).

According to Assumption 4.3 and Theorem 3.2, it is required that the transfer function $G(s, \mathbf{p})$ be strictly proper and BIBO stable and that the composite transfer function $G_2(s, \mathbf{p})$ be strictly proper. The controller transfer function $G_c(s, \mathbf{p})$ should be chosen so that all these requirements are fulfilled.

In computing ϕ_1 and ϕ_2, the impulse responses of the system need to be obtained (see Section 3). In this connection, such responses are evaluated by efficient and reliable algorithms described in [2, 4], which are based on Zakian I_{MN} approximations.

It should be noted that, among all the possible controllers satisfying the requirements, the one with simple structure is usually preferred. Therefore, designers should begin with a simple one first.

After exhaustive searches with first- and second-order controllers, it was found that a solution could not be located. Thus, a third-order controller of the form

$$G_c(s, \mathbf{p}) = \frac{p_1(s^2 + p_2 s + p_3)}{(s + p_6)(s^2 + p_4 s + p_5)} \tag{4.28}$$

is to be used where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5, p_6]^T \in \mathbb{R}^6$ is the design parameter to be determined.

After a number of iterations, a design solution $\mathbf{p}$ is found where

$$\mathbf{p} = [915.694, 3.107, 2.382, 47.96, 632.51, 0.011]^T \tag{4.29}$$

and the corresponding performance measures are

$$\phi_0 = -1.5491, \ \phi_1 = 0.0999, \ \phi_2 = 1.5770. \tag{4.30}$$

To verify the performance of the obtained controller, a test input f is generated randomly so that its magnitude and its slope do not exceed the bounds given in (4.25). The backlash used in the simulation has a unity slope with different values of bandwidth $\sigma = 0.02$, $\sigma = 0.05$ and $\sigma = 0.1$. The waveform of f and the responses of the system are shown in Fig. 4.4. The simulation results clearly show that the error and the controller output responses stay within the specified bounds.

4.6 Conclusions

This chapter has developed a practical method for designing a controller $G_c(s)$ for the system (4.1) so as to ensure the error e and the controller output u stay within the ranges $\pm E_{max}$ and $\pm U_{max}$, respectively, for all time and for all possible inputs $f \in \mathcal{P}$. The useful decomposition (4.8) replaces the backlash with a constant gain and a bounded disturbance. Theorem 4.1 provides an essential basis for Theorem 4.2 to develop the design inequalities that are more computationally tractable than (4.4). Accordingly, a solution of the original design problem (4.4) is obtained by solving the surrogate design criteria (4.22), which is associated with the auxiliary linear system subject to the inputs $f \in \mathcal{P}$ and $d \in \mathcal{D}$ and whose associated performance measures are readily obtainable by known methods. The inequalities (4.22) are used in conjunction with (4.23) so as to enable a numerical algorithm to search for the solution in the space of design parameters. Since the linear subsystems are represented by using the convolution, the method developed in this work is applicable to both lumped- and distributed-parameter plants as long as Assumption 4.3 holds. The simulation results have illustrated the usefulness of the proposed method.

CHAPTER V

ROBUST CONTROLLER DESIGN FOR FEEDBACK SYSTEMS WITH UNCERTAIN BACKLASH AND PLANT UNCERTAINTIES SUBJECT TO INPUTS SATISFYING BOUNDING CONDITIONS

This chapter develops a method for designing unity feedback systems where an uncertain linear time-invariant plant is in cascade connection with an uncertain backlash and a controller. The design problem considered is to determine a robust controller so as to ensure that, despite plant uncertainties, the error and the controller output stay within prescribed bounds for all time and for all inputs satisfying given bounding conditions. In essence, the backlash is replaced with a constant gain and a bounded disturbance, thereby resulting in an auxiliary uncertain linear system. Then, by applying the multi-valued version of the fixed-point theorem and the extended version of the theory of majorants, we derive a practical condition in the form of inequalities that can be solved in practice. Further we show that if such inequalities are satisfied for a chosen nominal system, then the original design problem is solved. The usefulness of the method is illustrated by a design example where the plant has a time-delay.

5.1 Introduction

Backlash exists in many practical applications, and it has long been known that it can severely limit system performance. Moreover, the model of a backlash is often not known accurately, and the plant often has uncertainties in its parameters.

There are several ways to alleviate, or ideally eliminate, undesirable effects of backlash on the performance of the system. One among them is to design an adaptive controller for the system (see, for example, [29]). However, this approach often results in a complicated controller. An alternative way is to design a robust controller where the uncertainty of the backlash is taken into account (see, for example, Barreiro and Baños 2006). The advantage of this method is that although there may be a certain amount of conservatism, the controller obtained (if exists) is much simpler and easier to implement.

The purpose of this chapter is to develop a computational method for designing a robust controller for feedback systems in which an uncertain backlash model and the plant uncertainties are explicitly taken into account in the design formulation. Moreover, in the formulation, all inputs that can happen or are likely to happen in practice are explicitly taken into account as a set of functions that satisfy bounding conditions.

Specifically, the chapter considers the feedback control system displayed in Fig. 5.1, which is described by

$$\left.\begin{aligned} u &= g_c * e \\ e &= f - u_s * g_p = f - \psi(u) * g_p,\ G_p(s) \in \mathcal{G}_p \end{aligned}\right\} \tag{5.1}$$

where ψ is a backlash, g_p is the impulse response of the plant and has Laplace transform equal to $G_p(s)$, g_c is the impulse response of the controller and has Laplace transform equal to $G_c(s, \mathbf{p})$, and $\mathbf{p} \in \mathbb{R}^n$ denotes a design parameter vector. Assume that the plant has uncertainties such that $G_p(s)$ is known to belong to a set $\mathcal{G}_p$.

The backlash $\psi(\cdot)$ in the system (5.1) is described by the uncertain band model ([6]).

Now assume that the input f is known only to the extent that it belongs to a *possible set* $\mathcal{P}$ (defined as the set of inputs that can or are likely to happen in practice). For clarity, assume that the possible set $\mathcal{P}$ considered throughout the chapter is defined by

$$\mathcal{P} \triangleq \{f : \mathbb{R}_+ \to \mathbb{R} |\ ||f||_\infty \leq M, ||\dot{f}||_\infty \leq D\} \tag{5.2}$$

where the bounds M and D are given. However, it is important to note that the method to be developed is applicable to any possible set of bounded signals. For different ways of characterizing the possible set and the detailed discussion on this, see [31, 38] and the references therein.

The design problem considered in the chapter is to determine a design parameter **p** (or equivalently the controller transfer function $G_c(s, \mathbf{p})$) such that the following design criteria are satisfied:

$$\sup_{G_p \in \mathcal{G}_p} \hat{e} \leq E_{max} \text{ and } \sup_{G_p \in \mathcal{G}_p} \hat{u} \leq U_{max} \tag{5.3}$$

where the bounds E_{max} and U_{max} are given. The numbers $\hat{e}$ and $\hat{u}$ are sometimes called the peak error and the peak controller output, respectively, for the possible set $\mathcal{P}$ and defined by

$$\hat{e} \triangleq \sup_{f \in \mathcal{P}} \|e\|_\infty \text{ and } \hat{u} \triangleq \sup_{f \in \mathcal{P}} \|u\|_\infty. \tag{5.4}$$

Clearly, $\hat{e}$ and $\hat{u}$ depend upon **p** and the plant $G_p(s)$.

Note further that the criteria (5.3) are equivalent to

$$\left.\begin{aligned} |e(f,t)| \leq E_{max}, \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \quad \forall G_p(s) \in \mathcal{G}_p \\ |u(f,t)| \leq U_{max}, \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \quad \forall G_p(s) \in \mathcal{G}_p \end{aligned}\right\}. \tag{5.5}$$

It is evident that once (5.3) are satisfied, one can ensure that despite the plant uncertainties, the variables e and u lie within the respective bounds $\pm E_{max}$ and $\pm U_{max}$ for all time and for all inputs in the possible set $\mathcal{P}$.

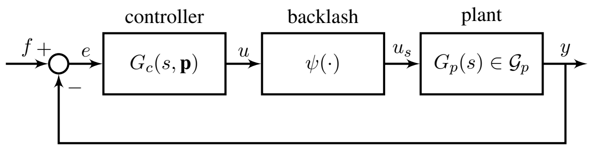

Figure 5.1: Uncertain feedback system with backlash

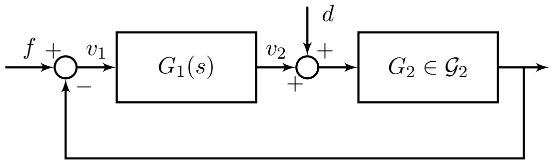

Figure 5.2: An uncertain linear system with two inputs.

The key tools used in the chapter are the multivalued version of the fixed point theorem which is known as Kakutani's theorem (see, for example, [6]) and the extended version of the theory of majorants ([18,22,34,38]). By decomposing the backlash as a constant gain and a bounded disturbance ([25]), we obtain an auxiliary linear system with uncertainties. Then, by applying the multi-valued version of the fixed-point theorem and the extended version of the theory of majorants, we derive a practical condition in the form of inequalities that can be solved in practice. Further we show that if such inequalities are satisfied for a chosen nominal system, then the original design problem is solved.

The organization of the chapter is as follows. Section 5.2 recapitulates the version of the theory of majorants that was extended by [22]. Section 5.3 derives the surrogate design criteria, thereby providing practical inequalities for designing the system (5.1) so that the original criteria (5.3) are satisfied; this is indeed the main contribution of the chapter. Section 5.4 provides the stability conditions for ensuring that the associated performance measures are finite, so as to enable a numerical algorithm to search for a design solution in the space of design parameters. To illustrate the usefulness and the potential of the developed method, a design example for a time-delay plant is carried out in Section 5.5. Finally, the discussion and conclusions are given.

5.2 Theory of Majorants for Uncertain Linear Systems

The theory of majorants ([33,34,37,38]) has been used for the design of robust control systems in [7,27,28]. The theory have been extended by [22] (see also [18]) in a straightforward manner to the case of uncertain linear feedback systems with inputs f and d (see below). The following summarizes the theory to used in Section 5.3.

Consider the uncertain linear system that is described by

$$\left.\begin{aligned} v_2 &= v_1 * g_1 \\ v_1 &= f - g_2 * (d + v_2), \;\; G_2(s) \in \mathcal{G}_2 \end{aligned}\right\}, \tag{5.6}$$

where $G_p(s)$ belongs to a set of plant transfer functions $\mathcal{G}_p$ (see Fig. 5.2). As before, the Laplace transforms of g_1 and g_2 are $G_1(s)$ and $G_2(s)$, respectively. The inputs f and d are assumed to belong to the sets $\mathcal{P}$ and $\mathcal{D}$, respectively, where

$$\mathcal{D} \triangleq \{d \in \mathbb{L}_\infty | ||d||_\infty \leq h\}. \tag{5.7}$$

Let the nominal system be obtained by replacing $G_2(s)$ in the the system (5.6) by a fixed

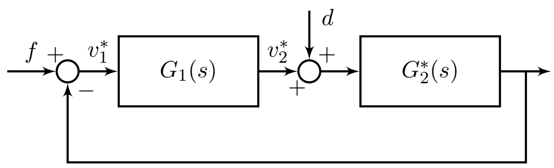

Figure 5.3: The nominal system for the system (5.6).

transfer function $G_2^*(s)$ (see Fig. 5.3). Therefore, the nominal system is given by

$$\left.\begin{aligned} v_2^* &= v_1^* * g_1 \\ v_1^* &= f - g_2^* * (d + v_2^*) \end{aligned}\right\}. \tag{5.8}$$

Define the peak outputs for the systems (5.6) and (5.8) as follows.

$$\left.\begin{aligned} \hat{v}_i &\triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|v_i\|_\infty \\ \hat{v}_i^* &\triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|v_i^*\|_\infty \end{aligned}\right\},\quad i = 1, 2, \tag{5.9}$$

In regard to the nominal system (5.8), the following theorem provides a useful sufficient condition for ensuring

$$\sup_{G_2\in\mathcal{G}_2} \hat{v}_i \le V_i \quad (i = 1, 2) \tag{5.10}$$

where the bounds V_1 and V_2 are given.

Theorem 5.1 ([22]). *Let $v_i^*(t, \mathbf{1})$ is the value of v_i^* at time t when the input f is the unit step function $\mathbf{1}$ and d is zero. Define*

$$\tilde{\mu}_i \triangleq A|\sigma_i| + B\|v_i^*(\mathbf{1}) - \sigma_i\|_1, \quad \sigma_i = \lim_{t\to\infty} v_i^*(t, \mathbf{1}), \tag{5.11}$$

where

$$\left.\begin{aligned} A &= \sup\{\|z\|_1 : G_2 \in \mathcal{G}_2\}, \quad z = g_2 - g_2^* \\ B &= \sup\{|z(0)| + \|\dot{z}\|_1 : G_2 \in \mathcal{G}_2\} \end{aligned}\right\}. \tag{5.12}$$

Suppose that the nominal system (5.8) *is BIBO stable, and let $\tilde{\mu}_1 < \infty$ and $\tilde{\mu}_2 < 1$. The design criteria* (5.10) *are satisfied if*

$$\hat{\phi}_i \le V_{i\,\max}, \quad i = 1, 2, \tag{5.13}$$

where

$$\hat{\phi}_i \triangleq \frac{\hat{v}_i^* + \tilde{\mu}_i h}{1 - \tilde{\mu}_2}. \tag{5.14}$$

Now let $G_1(s) = G_1(s, \mathbf{p})$ be characterized by a design parameter vector $\mathbf{p}$. [34, 37, 38] advocates that in solving the inequalities (5.13) by numerical methods, the numbers v_i^*, σ_i and $\tilde{\mu}_i$ have to be computed repeatedly for different values of $\mathbf{p}$. However, it is clear from (5.12) that the numbers A and B do not depend on $G_1(s)$ and thus need to be computed only once.

From the above, one can see for a chosen nominal transfer function $G_2^*(s)$, the condition (5.13) provides useful inequalities for determining $G_1(s)$ by numerical methods so that the criteria (5.10) are satisfied.

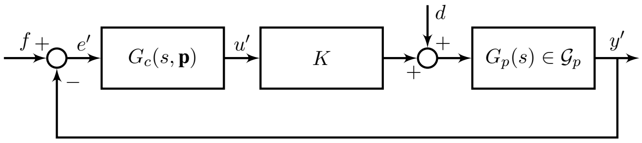

Figure 5.4: The auxiliary linear system for the system (5.1).

5.3 Design of Uncertain Nonlinear System

This section develops the design criteria in the form of inequalities that can be solved by numerical methods for the uncertain nonlinear system (5.1) to satisfy (5.3). Indeed, Theorems 5.2 and 5.3 are the main contribution of the chapter.

Assumption 5.1. *For every input $f \in \mathcal{P}$, there exists at least a solution (e, u) that satisfy (5.1), where $e : \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$. Assume that all initial conditions are zero for $t \leq 0$.*

Assumption 5.2. *Let the backlash $\psi(\cdot)$ be represented by the uncertain band model (3.2).*

By using the decomposition technique (4.8) (see [25]), the backlash is replaced by a constant gain and an equivalent disturbance. Thus, we obtain the auxiliary system displayed in Fig. 5.4 and described by

$$\left.\begin{aligned} u' &= g_c * e' \\ e' &= f - g_p * [Ku' + d],\ G_p(s) \in \mathcal{G}_p \end{aligned}\right\}, \tag{5.15}$$

where $f \in \mathcal{P}$, $d \in \mathcal{D}$ and the set $\mathcal{D}$ is defined by

$$\mathcal{D} \triangleq \{d \in \mathbb{L}_\infty |\ ||d||_\infty \leq h\}. \tag{5.16}$$

Note that the decomposition (4.8) is valid when $\hat{u}$ is finite.

Next, define the peak values of e' and u' for each $G_p(s) \in \mathcal{G}_p$ as follows.

$$\hat{e}' \triangleq \sup_{f\in\mathcal{P},\ d\in\mathcal{D}} ||e'||_\infty, \text{ and } \hat{u}' \triangleq \sup_{f\in\mathcal{P},\ d\in\mathcal{D}} ||u'||_\infty \tag{5.17}$$

In connection with the system (5.15), let g be the impulse response of the transfer function from d to u'; that is,

$$G(s) \triangleq G_p(s)G_c(s)[1 + KG_p(s)G_c(s)]^{-1}. \tag{5.18}$$

Assumption 5.3. *For $G(s)$ defined by* (5.18)*, the impulse response g satisfies the conditions that $g \in \mathcal{A}$ and $\dot{g} \in \mathcal{A}$ for every $G_p(s) \in \mathcal{G}_p$.*

The following theorem reveals that a design solution associated with the auxiliary system (5.15) is also a solution to the original design problem (5.3).

Theorem 5.2. *Consider the system* (5.1) *where $\hat{u} < \infty$. Let Assumptions 1, 2 and 3 hold. The original design criteria* (5.3) *are satisfied if, for the auxiliary system* (5.15)*, the following conditions hold:*

$$\sup_{G_p\in\mathcal{G}_p} \hat{e}' \leq E_{\max}, \quad \sup_{G_p\in\mathcal{G}_p} \hat{u}' \leq U_{\max}. \tag{5.19}$$

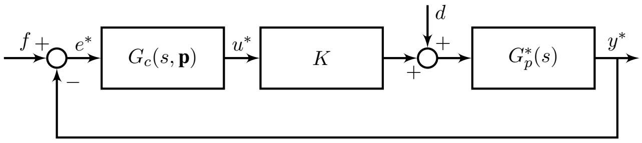

Figure 5.5: The nominal system for the uncertain system (5.15).

From Theorem 5.2, it readily follows that a solution to the inequalities (5.19), which is associated with the uncertain linear system (5.15), is also a solution to the inequalities (5.3). However, the inequalities (5.19) are not suitable for solution by numerical methods because the computation of $\sup_{G_p \in \mathcal{G}_p} \hat{e}'$ and $\sup_{G_p \in \mathcal{G}_p} \hat{u}'$ is intractable. Therefore, (5.19) will be replaced by readily computable inequalities to be derived by using Theorem 5.1.

Consider the nominal system shown in Fig. 5.5 where $f \in \mathcal{P}, d \in \mathcal{D}$ and $G_p^*(s)$ denotes the nominal transfer function for $G_p(s) \in \mathcal{G}_p$.

Assume that the nominal system is BIBO stable. Consequently, the following limits exist

$$\sigma_1 \triangleq \lim_{t \to \infty} e^*(t, \mathbf{1}), \quad \sigma_2 \triangleq \lim_{t \to \infty} u^*(t, \mathbf{1}), \tag{5.20}$$

where $e^*(t, \mathbf{1})$ and $u^*(t, \mathbf{1})$ are the values of e^* and u^* at the time t in response to the inputs $f = \mathbf{1}(t)$ and $d(t) = 0$. Define

$$\left.\begin{aligned} \hat{\mu}_1 &\triangleq A|\sigma_1| + B||e^*(\mathbf{1}) - \sigma_1||_1 \\ \hat{\mu}_2 &\triangleq A|\sigma_2| + B||u^*(\mathbf{1}) - \sigma_2||_1 \end{aligned}\right\}, \tag{5.21}$$

where

$$\left.\begin{aligned} A &= \sup\{||z||_1 : G_p \in \mathcal{G}_p, \;\; z = g_p - g_p^*\} \\ B &= \sup\{|z(0)| + ||\dot{z}||_1 : G_p \in \mathcal{G}_p\}. \end{aligned}\right\}. \tag{5.22}$$

Let $\hat{e}^*$ and $\hat{u}^*$ denote the peak values of e^* and u^* and be given by

$$\hat{e}^* \triangleq \sup_{f \in \mathcal{P},\, d \in \mathcal{D}} ||e^*||_\infty, \quad \hat{u}^* \triangleq \sup_{f \in \mathcal{P},\, d \in \mathcal{D}} ||u^*||_\infty. \tag{5.23}$$

The sufficient conditions for ensuring the satisfaction of the inequalities (5.19) is stated as follows and this is the main contribution of the chapter.

Theorem 5.3. *Consider the system* (5.1) *where* $\hat{u} < \infty$. *Let Assumptions 5.1, 5.2 and 5.3 hold. Assume that the nominal system in Fig. 5.5 is BIBO stable and that* $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$ *defined in* (5.22) *are finite. The inequalities* (5.19) *for the auxiliary system* (5.15)*, and hence the criteria* (5.3)*, are satisfied if* $\tilde{\mu}_2 < 1$ *and if*

$$\left.\begin{aligned} \hat{\phi}_e &\le E_{\max}, \;\; \textit{where } \hat{\phi}_e \triangleq \frac{\hat{e}^* + \tilde{\mu}_1 h}{1 - \tilde{\mu}_2} \\ \hat{\phi}_u &\le U_{\max}, \;\; \textit{where } \hat{\phi}_u \triangleq \frac{\hat{u}^* + \tilde{\mu}_2 h}{K(1 - \tilde{\mu}_2)} \end{aligned}\right\}. \tag{5.24}$$

For the possible set $\mathcal{P}$ in (5.2), the formulae for computing $\hat{\phi}_e$ and $\hat{\phi}_u$ are given in the following. Since $\hat{e}^*$ and $\hat{u}^*$ are the peak outputs of the nominal linear system, which has no uncertainty, one can easily deduce by the superposition ([22]) that

$$\hat{e}^* = \phi_{ef} + h||e_d^*(\delta)||_1, \;\; \hat{u}^* = \phi_{uf} + h||u_d^*(\delta)||_1, \tag{5.25}$$

where $e_d^*(\delta)$ and $u_d^*(\delta)$ are the responses of e^* and u^*, respectively, subject to the inputs $d(t) = \delta(t)$ and $f(t) = 0$, and

$$\phi_{ef} \triangleq \sup_{f\in\mathcal{P},\, d=0} ||e^*||_\infty, \;\; \phi_{uf} \triangleq \sup_{f\in\mathcal{P},\, d=0} ||u^*||_\infty. \tag{5.26}$$

It should be noted that ϕ_{ef} and ϕ_{uf} can be computed by using method developed in [31]. Therefore, the values of $\hat{e}^*$ and $\hat{u}^*$ can be readily obtained in practice.

From the above, it is easy to deduce that

$$\left.\begin{aligned} \hat{\phi}_e &= (h||e_d^*(\delta))||_1 + \phi_{ef} + h\tilde{\mu}_1)/(1-\tilde{\mu}_2) \\ \hat{\phi}_u &= (hK||u_d^*(\delta))||_1 + K\phi_{uf} + h\tilde{\mu}_2)/K(1-\tilde{\mu}_2) \end{aligned}\right\}. \tag{5.27}$$

Therefore, the design problem now becomes the problem of determining a design parameter **p** satisfying

$$\left.\begin{aligned} \hat{\phi}_e(\mathbf{p}) &\le E_{\max} \\ \hat{\phi}_u(\mathbf{p}) &\le U_{\max} \end{aligned}\right\}, \tag{5.28}$$

with the constraint $\tilde{\mu}_2(\mathbf{p}) < 1$. It should be noted that, if there is no pole-zero cancellation, $\tilde{\mu}_2(\mathbf{p}) < 1$ implies that $\tilde{\mu}_1(\mathbf{p})$ is finite.

From the above discussion, it is easy to see that $\hat{\phi}_e$ and $\hat{\phi}_u$ can be obtained numerically in practice. Thus, the inequalities (5.28) are called the *surrogate design criteria*.

5.4 Stability Conditions

In solving the design inequalities (5.28) by numerical methods, a search algorithm needs to start from a *stability point* (that is, a point **p** for which the associated performance measures $\hat{\phi}_e$ and $\hat{\phi}_u$ are finite).

It is important to note that Theorems 5.2 and 5.3 require the assumption that $\hat{u}$ is finite in order to guarantee the validation of the decomposition of ψ. Hence, a stability condition that ensures the finiteness of $\hat{e}$ and $\hat{u}$ is needed. In additon, Theorem 5.3 also requires the nominal system to be BIBO stable.

The following reveals that the BIBO stability of the auxiliary linear system (5.15) implies that of the original nonlinear system (5.1).

Theorem 5.4. *Consider the nonlinear system* (5.1)*. Let Assumptions 5.1, 5.2 and 5.3 hold. Define the composite transfer function $G_2(s) \triangleq G_c(s,\mathbf{p})G_p(s)$. For every $G_p(s) \in \mathcal{G}_p$, let $G_2(s)$ be strictly proper and suppose that $g_c \in \mathcal{A}$ where g_c is the impulse response of $G_c(s,\mathbf{p})$. Then it follows that the responses u and e are bounded for any $f \in \mathcal{P}$.*

Proof. Note that Assumption 5.3 implies that the transfer function

$$G(s) = G_p(s)G_c(s)[1 + KG_p(s)G_c(s)]^{-1}$$

is BIBO stable for every $G_p(s) \in \mathcal{G}_p$. Then by extending the Theorem 3.2 in Chapter 3 to the case of (5.15), theorem is obtained. □

From Theorem 5.4, it readily follows that to ensure the finiteness of $\sup_{G_p \in \mathcal{G}_p} \hat{e}$ and $\sup_{G_p \in \mathcal{G}_p} \hat{u}$ for the system (5.1), one needs to determine the controller $G_c(s)$ that makes the transfer function $G(s) = G_p(s)G_c(s)[1 + KG_p(s)G_c(s)]^{-1}$ BIBO stable for every $G_p(s) \in \mathcal{G}_p$.

For retarded delay differential systems (which of course includes rational systems), it is well known (see, for example, [3] and the references therein) that the system (or alternatively its transfer function) is BIBO stable if and only if all the characteristic roots (or alternatively all the poles of the transfer function) have negative real parts. Let $f(s)$ be the characteristic function of a retarded delay differential system. Let ϕ_0 denote the abscissa of stability of $f(s)$ defined by

$$\phi_0 \triangleq \sup\{\text{Re}(s) : \ f(s) = 0\}.$$

Then it follows that the system is BIBO stable if

$$\phi_0(\mathbf{p}) \leq -\varepsilon \ \text{ where } \ 0 < \varepsilon \ll 1. \tag{5.29}$$

It should be noted that the inequality (5.29) can be used in practice to determine a value of $\mathbf{p}$ for which the system is BIBO stable, by numerical methods. For further details, see [2, 3, 38, 39]. In this work, the abscissa of stability for retarded delay differential systems is computed by using the method developed in [2, 3].

5.5 Numerical Example

In this section, we consider a case study in process control system where the plant has time delay.

As shown in [14, 26], in process control systems, wear (or erosion) leads to the appearance of the backlash phenomenon in the linkage mechanism in the positioner and actuator of valves. According to [14], it is reported that a backlash of 10% increases the peak error at load disturbance with 50%. When the backlash becomes too large, the valve needs to be replaced. However, replacing the valve cannot be done without interrupting the process. For this and for economical reasons, it may be better to take into account the backlash phenomenon in the controller design process.

Consider an uncertain plant with time-delay whose transfer function is described by

$$G_p(s) = \frac{ae^{-0.2s}}{s^2 + bs + 2}, \ \ a \in [2.5, 3.5], \ \ b \in [2.9, 3.1]. \tag{5.30}$$

The backlash is assumed to have a unity slope and its bandwidth is only known to the extent that it is in the interval (0,0.1]. Assume that the control objective is to keep the error e and the controller output u staying within the bounds ± 0.2 and ± 10, respectively, for all time and for all inputs $f \in \mathcal{P}$ where

$$M = 1 \text{ and } D = 0.1. \tag{5.31}$$

Accordingly, the design criteria are expressed as

$$\hat{\phi}_0(\mathbf{p}) \leq -0.01,\ \tilde{\mu}_2(\mathbf{p}) \leq 0.5,\ \hat{\phi}_e(\mathbf{p}) \leq 0.2,\ \hat{\phi}_u(\mathbf{p}) \leq 10, \tag{5.32}$$

where

$$\hat{\phi}_0 = \sup\{\phi_0 : a \in [2.5, 3.5], b \in [2.9, 3.1]\}. \tag{5.33}$$

It is worth noting that from (5.27) one can see that if $\tilde{\mu}_2(\mathbf{p})$ is close to 1, then the values of $\hat{\phi}_e(\mathbf{p})$ and $\hat{\phi}_u(\mathbf{p})$ become very large. Therefore, any $\mathbf{p}$ that makes the value of $\tilde{\mu}_2(\mathbf{p})$ close to 1 will be automatically eliminated by the search algorithm.

The nominal plant transfer function $G_p^*(s)$ is chosen with

$$a = 3 \text{ and } b = 3. \tag{5.34}$$

According to Assumption 5.3 and Theorem 5.4, it is required for every $a \in [2.5, 3.5]$ and every $b \in [2.9, 3.1]$ that the transfer function $G(s, \mathbf{p})$ be strictly proper and BIBO stable and that the composite transfer function $G_2(s, \mathbf{p})$ be strictly proper. All these requirements can be fulfilled only if G_c is chosen to be a proper transfer function.

In order to compute $\hat{\phi}_e$ and $\hat{\phi}_u$, the impulse responses of the system need to be obtained (see Section 3). In this example, we use the efficient and reliable algorithms described in [2, 4], which are based on Zakian I_{MN} approximations, to evaluate such responses. Furthermore, the design inequalities are solved by using a numerical search algorithm called the moving boundaries process (MBP) ([37–39]).

After exhaustive searches with first order controllers, no solution was found. Thus, a second-order controller which has the form

$$G_c(s, \mathbf{p}) = \frac{p_1(s^2 + p_2 s + p_3)}{s^2 + p_4 s + p_5} \tag{5.35}$$

is to be tried where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5]^T \in \mathbb{R}^5$ is the design parameter to be determined.

After a number of iterations, a design solution $\mathbf{p}$ is found where

$$\mathbf{p} = [6.770, 2.860, 1.901, 12.127, 0.010]^T \tag{5.36}$$

and the corresponding performance measures are

$$\begin{aligned} \hat{\phi}_0(\mathbf{p}) &= -0.969, \tilde{\mu}_2(\mathbf{p}) = 0.273, \\ \hat{\phi}_e(\mathbf{p}) &= 0.199,\ \hat{\phi}_u(\mathbf{p}) = 1.191. \end{aligned} \tag{5.37}$$

To verify the performance of the obtained controller, a test input f is generated randomly so that its magnitude and its slope do not exceed the bounds given in (5.31). The waveform of f and the responses of the system are shown in Fig. 5.6. The simulation results clearly show that the error and the controller output responses stay within the specified bounds.

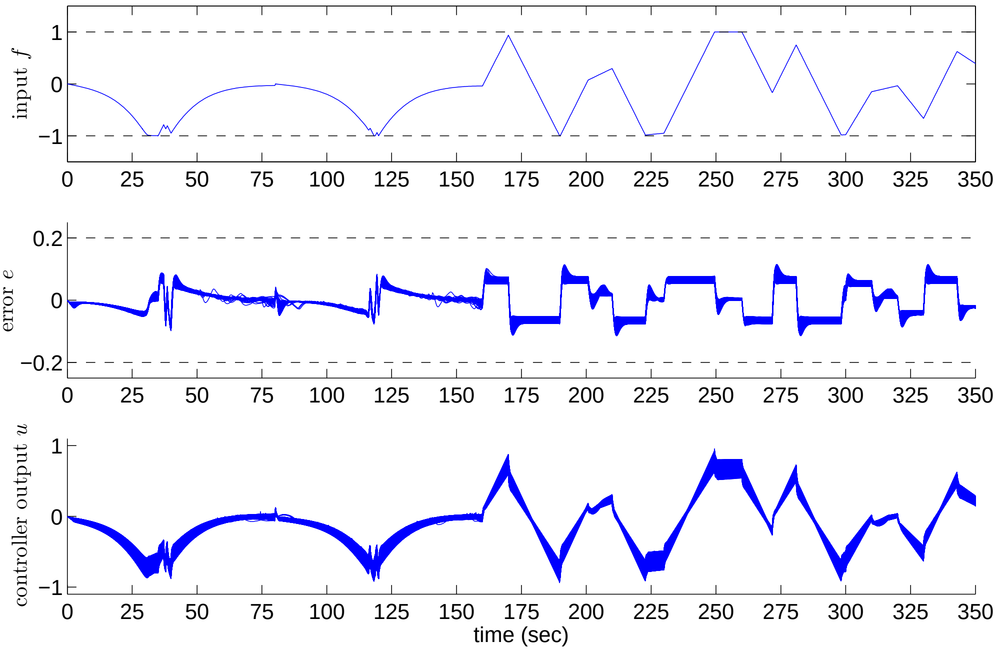

Figure 5.6: Simulation results with the controller (5.35) and the backlash with different values of bandwidth in the range $(0, 0.1]$ and the parameters $a = 2.5(0.1)3.5$, $b = 2.90(0.02)3.10$

5.6 Discussion and Conclusions

This chapter has developed a practical method for designing the feedback control system (5.1). The control objective is to ensure that the error e and the controller output u stay within the prescribed ranges $\pm E_{\max}$ and $\pm U_{\max}$, respectively, for all time and for all inputs $f \in \mathcal{P}$ in the presence of uncertainties appearing in both the backlash and the plant. The backlash is decomposed using the technique due to [25]. By using Kakutani's theorem (see, for example, [6]) in conjunction with the extension of the theory of majorants in [22], the chapter has developed computationally tractable design inequalities (5.28) associated with a chosen nominal system. Since the nominal system is linear and has no uncertainty, the performance measures are readily computed by known methods. Since the plant is represented by using the convolution, the method developed in this work is applicable to both lumped- and distributed-parameter plants as long as Assumption 5.3 holds. The simulation results have illustrated the usefulness of the proposed method.

The method developed in the chapter is applied to the design of a robust controller for process control systems where backlash characteristics appear in the linkage mechanism in the actuator of valves. Taking the uncertain backlash into account is an advantageous alternative solution since replacing the valve in which the backlash becomes too large is not feasible all the time, especially when the system is in operation.

It is interesting to note that, unlike design methods based on a direct cancellation that requires the backlash to be at the plant input ([6]), the approach used in developing the design method in the chapter can be applied to the cases where the backlash is at the output of the plant (see [21]).

CHAPTER VI

CONCLUSIONS

6.1 Contributions

The contributions of the thesis are as follows.

First, we have developed a practical method for determining stabilizing controllers by numerical methods based on the result that for any input such that the two norm of the product of its slope and an increasing exponential function is finite, if the linear subsystem satisfies a Popov inequality for a sector bound and if the slope of the hysteresis lies within the same sector bound, then the outputs of the system are ensured to be finite for all time. The usefulness of the method is illustrated by numerical methods.

Second, we also consider the BIBO stability of nonlinear feedback systems with backlash described by the uncertain band model (see, for example, [6]). It is shown that, the BIBO stability of the nonlinear system with backlash is guaranteed by guaranteeing the BIBO stability of the corresponding auxiliary linear system ([6]).

Third, we develop a design method for unity feedback systems comprising a backlash describing by the uncertain band model and linear time-invariant convolution subsystems, where the main design objective is to ensure that the error and the controller output stay within prescribed bounds for all time and for all possible inputs having bounded magnitude and bounded slope. The design formulation is based on the principle of matching, thereby explicitly considering the peak error and the peak controller output for such a possible set. The original design inequalities are replaced with the surrogate design criteria that are in keeping with the method of inequalities. Specifically, the design procedure is as follows.

1. Replacing the backlash ψ with a constant gain K and a bounded disturbance $d \in \mathcal{D}_u$ results in the corresponding auxiliary linear system.

2. Using Kakutani 's fixed point theorem to prove that the original design problem of the nonlinear system can be replaced by that of the auxiliary linear system.

3. Replacing the set $\mathcal{D}_u$ with a more tractable set D.

4. If the plant transfer function is known accurately, then the surrogate design criteria are obtained by the virtue of the linearity of the auxiliary system. If the plant is vague, we need to replace the auxiliary system with the nominal one by using the extension of Zakian's majorants. After this step, the surrogate design criteria are obtained by the virtue of the linearity of the nominal system.

5. The surrogate criteria are in keeping with the method of inequalities and suitable for solving by numerical methods.

The usefulness of the method has been illustrated by some numerical examples.

6.2 Future works

The work can be extended in some directions as follows.

First, the design method is also applicable to the case where the plant has a backlash at the output. It is similar to that work [20] (see also [18]).

Second, the design method proposed in Chapters 4 and 5 for the input set $\mathcal{P}$ is applicable to the input set considered in Chapter 2. This is because the method to compute the peak outputs of linear systems in [31] is applicable to many possible sets defined with many (two or more than two) bounding conditions on the two or/and infinity norms of the inputs and their slopes. See [31] for details. However, it should be noted that, in that case, the stability conditions to guarantee the finiteness of the error e and the controller u are different. To be more specific, we have to use the Popov-like conditions developed in Chapter 2 instead.

Third, the nonlinearity $\psi(\cdot)$ can be generalized to some classes of hysteresis that satisfy some conditions as discussed in the following. The condition for the function $n(\cdot)$ of the uncertain-band model of backlash (3.2) to guarantee that the Theorem 4.1 holds is that $n(\cdot)$ is upper semi-continuous. For the backlash, $n(\cdot)$ is defined as $n(\cdot) = [-h, h]$ which is trivially upper semi-continuous. If we can define $n(\cdot)$ in a more complicated way but still ensure that $n(\cdot)$ is upper semi-continuous, then we may describe the characteristics of some hysteresis with this model, and it follows that the design method developed here may be applicable to the systems having those classes of hysteresis.

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APPENDICES

APPENDIX A

Lemma 7.1 ([18]). *Let $X \triangleq \{x \in \mathbb{L}_\infty \mid ||x||_\infty \leq C\}$ where C is a finite number. For a given $T > 0$, define the convolution operator over X_T, denoted as H, given by*

$$Hx(t) = \int_0^T h(t-\tau)x(\tau)d\tau. \tag{7.1}$$

If $h, \dot{h} \in \mathcal{A}$, then H is compact.

For the sake of completeness, the proof is given as follows.

Proof. In order to prove Lemma 7.1, we need the following theorems.

Theorem 7.1. *Let X and Y denote normed spaces and $H : X \longrightarrow Y$ be a linear operator. Then H is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Hx_n\}$ in Y such that $\{Hx_n\}$ has a convergent subsequence.*

Theorem 7.2. *If $\mathcal{F}$ is a set of functions defined, equicontinuous and uniformly bounded on a bounded closed set, then from every sequence $f_n \in \mathcal{F}$ it is possible to select a uniformly convergent subsequence.*

Assume $h, \dot{h} \in \mathcal{A}$. Consider a sequence $\{x_n\} \in X_T$ and define the sequence $\{y_n\}$ as follows.

$$y_n(t) = Hx_n(t). \tag{7.2}$$

For any $x \in X_T$, it readily follows that

$$\left|\int_0^T h(t-\tau)x(\tau)d\tau\right| \leq \int_0^T |h(t-\tau)||x(\tau)|d\tau \leq C\int_0^\infty |h(\tau)|d\tau. \tag{7.3}$$

Since $h \in \mathcal{A}$, there exists a finite number M such that $\int_0^\infty |h(\tau)|d\tau < M$. Thus,

$$||y||_\infty = \sup_{t\geq 0}\left|\int_0^T h(t-\tau)x(\tau)d\tau\right| \leq MC. \tag{7.4}$$

Therefore, $\{y_n\}$ is uniformly bounded on $[0, T]$ for any $T > 0$. Next, we will prove that $\{y_n\}$ is equicontinuous. For any $t_1, t_2 \in [0, T]$ and any $k > 0$, let $\Delta t \triangleq t_1 - t_2$ and consider

$$\begin{aligned} &\lim_{\Delta t\to 0}|y_k(t_1) - y_k(t_2)| \leq \lim_{\Delta t\to 0}\int_0^T |h(t_1-\tau) - h(t_2-\tau)||x_k(\tau)|d\tau \\ &\leq \lim_{\Delta t\to 0} C|\Delta t|\int_0^\infty |\dot{h}(\tau)|d\tau \leq \lim_{\Delta t\to 0} C|\Delta t|M_1 = 0. \end{aligned} \tag{7.5}$$

Therefore, $\{y_n\}$ is equicontinuous by definition.

Since $[0, T]$ is a bounded and closed set, $\{y_n\}$ has a convergent subsequence according to Theorem 7.2. Then, by applying Theorem 7.1, it follows that the operator H is compact. □

APPENDIX B

Proposition 7.1. *Consider the system shown in Figure 2.6. If* $\dot{u}_1 \in \mathbb{N}_2$ *and if the impulse response* $g_c \in \mathcal{A}_{-}$*, then* $\dot{\tilde{u}}_1 \in \mathbb{N}_2$.

Proof. Since $\dot{u}_1 \in \mathbb{N}_2$, there exists $\sigma_1 > 0$ such that $e^{\sigma_1 t}\dot{u}_1 \in \mathbb{L}_2$. Since $g_c \in \mathcal{A}_{-}$, there exists $\sigma_2 > 0$ such that $e^{\sigma_2 t} g_c \in \mathbb{L}_1$. Choose $\sigma = \min(\sigma_1, \sigma_2)$ and consider

$$e^{\sigma t}\dot{\tilde{u}}_1 = e^{\sigma t}(g_c * \dot{u}_1) = e^{\sigma t}\int_0^t g_c(t-\tau)\dot{u}_1(\tau)\mathrm{d}\tau = \int_0^t g_c(t-\tau)e^{\sigma(t-\tau)}e^{\sigma\tau}\dot{u}_1(\tau)\mathrm{d}\tau = [e^{\sigma t}g_c] * [e^{\sigma t}\dot{u}_1] \tag{7.6}$$

Since $e^{\sigma t}g_c \in \mathbb{L}_1$ and $e^{\sigma t}\dot{u}_1 \in \mathbb{L}_2$, from a well-known fact (see, for example, [11]), it is easy to see that $e^{\sigma t}\dot{\tilde{u}}_1 \in \mathbb{L}_2$, which means that there exists a positive σ such that $e^{\sigma t}\dot{\tilde{u}}_1 \in \mathbb{L}_2$. Hence, $\dot{\tilde{u}}_1 \in \mathbb{N}_2$. □

Biography

Hoang Hai Nguyen was born in Haiphong, Vietnam, in 1987. He received his Bachelor's degree in Automatic Control from Faculty of Electrical Engineering, Hanoi University of Science and Technology, Vietnam, in 2010. He has been granted a scholarship by the AUN/SEED-Net to pursue his Master's degree in electrical engineering at Chulalongkorn University, Thailand since 2011. He conducted his graduate study with the Control Systems Research Laboratory, Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University. His research interests include nonlinear systems and control systems design by the Method of Inequalities and the Principle of Matching.

List of Publications

1. H. H. Nguyen and S. Arunsawatwong, "Robust Controller Design for Feedback Systems with Uncertain Backlash and Plant Uncertainties Subject to Inputs Satisfying Bounding Conditions," *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014. (Accepted)

2. H. H. Nguyen and S. Arunsawatwong, "Design of Feedback Systems with Backlash for Inputs Restricted in Magnitude and Slope," *Proc of SICE Annual Conference*, Nagoya, Japan, (2013): 229–234. (This paper appears in the finalist for SICE Annual Conference International Award 2013)

3. H. H. Nguyen and S. Arunsawatwong, "Input-output Stability of Feedback Systems with Hysteresis for Inputs Satisfying Bounding Conditions," *10th International Conference on Electrical Engineering/Electronics, Computer, Telecommunications and Information Technology*, Krabi, Thailand, (2013): 1–4.

4. H. H. Nguyen and S. Arunsawatwong, "Input-output Stability of Feedback Systems with Backlash for Inputs Satisfying Bounding Conditions," *Proc of the 5th AUN/SEED-Net Regional Conference in Electrical and Electronics Engineering*, Bangkok, Thailand, (2013): 1–4.