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GENERAL FORM OF SLIGHTLY COMPRESSIBLE MODULES

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ในวิทยานิพนธ์นี้ เรากำหนดรูปทั่วไปของสไลต์ลีคอมเพรสซิเบิลมอดูล กำหนดให้ R เป็นริงเปลี่ยนหมู่ที่มีเอกลักษณ์ และ M เป็น R-มอดูลทางขวา จะเรียก R-มอดูลทางขวา N ว่าเป็น *M-สไลต์ลีคอมเพรสซิเบิลมอดูล* ถ้าทุกๆ สับมอดูล A ที่ไม่ใช่ศูนย์ของ N มี R-มอดูลโฮโมมอร์ฟิซึมที่ไม่ใช่ศูนย์จาก M ไปยัง A ในกรณีที่ $M = N$ เราได้ว่า N เป็นสไลต์ลีคอมเพรสซิเบิลมอดูล นอกจากนี้เราให้เงื่อนไขสำหรับการที่ R-มอดูลทางขวา จะเป็น M-สไลต์ลีคอมเพรสซิเบิลมอดูล และศึกษาสมบัติต่างๆของ M-สไลต์ลีคอมเพรสซิเบิลมอดูล ต่อจากนั้นเราแนะนำแนวคิดของ M-สไลต์ลีคอมเพรสซิเบิลอินเจคทีฟมอดูล โดยจะเรียก R-มอดูลทางขวา N ว่าเป็น *M-สไลต์ลีคอมเพรสซิเบิลอินเจคทีฟมอดูล* ถ้า ทุก ๆ R-มอดูลโฮโมมอร์ฟิซึมจาก M-สไลต์ลีคอมเพรสซิเบิลสับมอดูลของ M ไปยัง N สามารถขยายไปบน M นอกจากนี้เราศึกษาสมบัติต่างๆของ M-สไลต์ลีคอมเพรสซิเบิลอินเจคทีฟมอดูล และหาตัวอย่างที่สอดคล้อง ในส่วนสุดท้าย เราแนะนำแนวคิดของ สับ-M-พรินซิเพิลลีอินเจคทีฟมอดูล โดยจะเรียก R-มอดูลทางขวา N ว่าเป็น *สับ-M-พรินซิเพิลลีอินเจคทีฟมอดูล* ถ้าทุกๆ สับมอดูล A ที่ไม่ใช่ศูนย์ของ M และทุก ๆ R-มอดูลโฮโมมอร์ฟิซึมจาก A-ไซคลิกสับมอดูลของ A ไปยัง N สามารถขยายไปบน M นอกจากนี้เราศึกษาสมบัติต่างๆของสับ -M-พรินซิเพิลลีอินเจคทีฟมอดูล และหาตัวอย่างที่สอดคล้อง

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In this thesis, we determine a general form of slightly compressible modules. Let R be an associative ring with identity and M a right R-module. A right R-module N is called an *M-slightly compressible module* if, for every nonzero submodule A of N, there exists a nonzero R-module homomorphism from M to A. In the case that $M = N$, N is, in fact, a slightly compressible module. Moreover, we provide conditions for any right R-module to be an M-slightly compressible module and study some properties of M-slightly compressible modules. Next, we introduce the concept of M-slightly compressible injective modules. A right R-module N is called an *M-slightly compressible injective module* if every R-module homomorphism from an M-slightly compressible submodule of M to N can be extended to M. Moreover, we study some properties of M-slightly compressible injective modules and also provide examples of them. Finally, we introduce the concept of sub-M-principally injective modules. A right R-module N is called a *sub-M-principally injective module* if for any nonzero submodule A of M, any R-module homomorphism from A-cyclic submodule of A to N can be extended to M. Moreover, we study some properties of sub-M-principally injective modules and also provide examples of them.

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CHAPTER I

INTRODUCTION

Ring theorists began to concentrate more on special areas of subject such as representation theory of finite dimensional algebras, Noetherian rings and group rings since fifties to seventies of the last century. Afterward, questions in general module theory continue to be interested by people worldwide. Here the emphasis has been on the structure of modules themselves, independent of the structure of underlying rings.

In 1976, Zelmanowitz[23] introduced the notion of compressible modules. According to Zelmanowitz, let R be an associative ring with identity, a right R-module M is called ***compressible*** provided for each nonzero submodule N of M there exists an R-module monomorphism from M to N. For example, if R is a domain, i.e., a ring which has no zero divisors, then the right R-module R is compressible. Generalizations of compressible modules have been studied in many papers [8], [13], [27]. Recently, Smith[20] introduced the concept of a slightly compressible module which is a generalization of compressible module. According to Smith, let R be an associative ring with identity, a right R-module M is called ***slightly compressible*** if for a nonzero submodule N of M, there exists a nonzero R-module homomorphism from M to N and he also studied the properties of slightly compressible modules. For example [[20], Example 1.2], let S be any nonzero ring and let R denote the ring of 2×2 upper triangular matrices over S. Then the right R-module R is slightly compressible.

Moreover, we are interested in the injectivity of modules. Injective modules became familiar to any module theoretics from the work of Baer[2] in 1940 and had many applications in characterization some classes of rings.

In 1940, Bear[2] established a very useful test for injectivity. This test called the ***Baer's Criterion***, said that let R be an associative ring with identity and Q a right R-module, any R-module homomorphism of a right ideal I of R into Q can be extended to an R-module homomorphism of R into Q if and only if Q is injective.

The Baer Criterion has been generalized by many authors. For example, in 1989, Camillo[4] introduced the notion of principally injective modules for commutative rings. Let R be an associative ring with identity. A right R-module M is called ***principally injective*** (or ***p-injective***) if every R-module homomorphism from a principal right ideal of R to M can be extended to an R-module homomorphism from R to M. Next in 1999, Sanh, Shum, Dhompongsa and Wongwai[19] extended the notion of principally injective modules for commutative rings to M-principal injectivity for a given right R-module M. Let R be an associative ring with identity and M a right R-module. A right R-module N is called M***-principally injective*** if every R-module homomorphism from an M-cyclic submodule of M to N can be extended to an R-module homomorphism from M to N.

The first chapter of this thesis, we determine a general form of slightly compressible modules. Let R be an associative ring with identity and M a right R-module. A right R-module N is called M***-slightly compressible*** if, for every nonzero submodule A of N, there exists a nonzero R-module homomorphism from M to A. In the case that $M = N$, N is, in fact, a slightly compressible module. Moreover, we provide conditions for any right R-module to be an M-slightly compressible module and examples of M-slightly compressible modules.

In the second chapter of this thesis, we introduce the concept of M-slightly compressible injective modules, which extended from the Baer Criterion. Let R be an associative ring with identity and M a right R-module. A right R-module N is called M***-slightly compressible injective*** if every R-module homomorphism from an M-slightly compressible submodule of M to N can be extended to an R-module homomorphism from M to N. Moreover, we study some properties of M-slightly compressible injective modules and relationship between M-principally injective modules and M-slightly compressible injective modules and we provide examples of them.

In the third chapter of this thesis, we introduce the concept of sub-M-principally injective modules. Let R be an associative ring with identity and M a right R-module. A right R-module N is called ***sub-M-principally injective*** if for any nonzero submodule A of M, any R-module homomorphism from A-cyclic submodule of A to N can be extended to an R-module homomorphism from M to N. Moreover, we study some properties of sub-M-principally injective modules and relationship between M-principally injective modules, M-slightly compressible injective modules and sub-M-principally injective modules and we provide examples of them.

CHAPTER II

PRELIMINARIES

In this chapter, we present basic definitions, notations and theorems on rings and modules which will be used for this thesis.

2.1 Modules and Submodules

Throughout this thesis, unless otherwise stated, let R and S be associative rings with identities 1_R and 1_S, respectively.

Definition 2.1.1. [7] Let M be a nonempty set. A ***unital right R-module*** M is

(i) an additive abelian group M together with

(ii) a mapping

$$M \times R \to M \text{ with } (m, r) \mapsto mr,$$

called the *module multiplication*, for which we have

(a) Associative law: $(mr_1)r_2 = m(r_1r_2)$,

(b) Distributive laws: $(m_1 + m_2)r = m_1r + m_2r$, $m(r_1 + r_2) = mr_1 + mr_2$,

(c) Unitary law: $m1_R = m$

for all $m, m_1, m_2 \in M$ and $r, r_1, r_2 \in R$.

An analogous definition holds for left R-modules. Moreover, by a right R-module we mean a unital right R-module. We write M_R for a right R-module M. We denote 0_M the identity under addition of a right R-module M and 0_R the identity under addition of a ring R. Then $0_M r = 0_M = m0_R$ for all $r \in R$ and $m \in M$.

Example 2.1.2. [1]

(i) For every abelian group M, there is a unique right $\mathbb{Z}$-module structure on M. This is simply the structure given by the usual **multiple function**

$$(x, n) \mapsto xn \quad \text{for all } x \in M, n \in \mathbb{Z}$$

$$\text{where } xn = \begin{cases} \underbrace{x + \cdots + x}_{n \ terms} & \text{for} \quad n \in \mathbb{Z}^+ \\ -\left(\underbrace{x + \cdots + x}_{|n| \ terms}\right) & \text{for} \quad n \in \mathbb{Z}^- \\ 0_M & \text{for} \quad n = 0. \end{cases}$$

(ii) Let $\phi : R \to S$ be a ring homomorphism. Then ϕ induces a left and a right R-module structure on the additive group of S. Indeed, the module multiplication, for the left R-module S, is given by

$$(r, s) \mapsto \phi(r)s \quad \text{for all } r \in R, s \in S$$

where the product $\phi(r)s$ is computed in the ring S. The right R-module structure on S is defined similarly.

(iii) Each ring R induces a left R-module L structure on its additive group and a right R-module M structure on its additive group via the module multiplications

$$(a, x) \mapsto ax \text{ for all } a \in R, x \in L \quad \text{and} \quad (x, a) \mapsto xa \quad \text{for all } x \in M, a \in R$$

where ax and xa denote the products in the ring R. These modules induced on the additive group of a ring R are called the **regular left** and **regular right modules** of R, respectively. Then every left ideal of R is a regular left module of R and every right ideal of R is a regular right module of R. The ${}_RR$ is a left R-module and R_R is a right R-module by product in R.

Definition 2.1.3. [21] Let M be a right R-module. A subgroup N of $(M,+)$ is called a ***submodule*** of M if N is closed under multiplication with elements in R, i.e., $nr \in N$ for all $r \in R, n \in N$. We write $N \hookrightarrow M$ for a submodule N of M.

Then $N \hookrightarrow M$ is also a right R-module by the operations induced from M:

$$N \times R \to N,\ (n,r) \mapsto nr \text{ for all } r \in R, n \in N.$$

The subset $\{0_M\}$ of a right R-module M is clearly a submodule of M. We call it the ***zero submodule*** and usually denote it by 0 alone.

Remark. *Every submodule of ${}_RR$ is a left ideal of R and every submodule of R_R is a right ideal of R.*

Definition 2.1.4. [21] A right R-module M is called ***simple*** if $M \neq 0$ and it has no submodules except 0 and M.

For nonempty subsets N, N_1, N_2 of a right R-module M and a nonempty subset A of a ring R we define:

$$N_1 + N_2 = \{n_1 + n_2 \,|\, n_1 \in N_1, n_2 \in N_2\},$$
$$NA = \left\{\sum_{i=1}^{k} n_i a_i \mid n_i \in N, a_i \in A, k \in \mathbb{N}\right\}.$$

If N_1 and N_2 are submodules of a right R-module M, then $N_1 + N_2$ is also a submodule of M. For a right ideal A of R, the product NA is always a submodule of M.

For any finite family $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of M_R, the sum $\sum\limits_{\lambda\in\Lambda} N_\lambda$ is defined by

$$\sum_{\lambda\in\Lambda} N_\lambda = \left\{\sum_{\lambda\in\Lambda} n_\lambda \,|\, n_\lambda \in N_\lambda \text{ for all } \lambda \in \Lambda\right\}.$$

This is a submodule of M.

For any infinite family $\{N_\lambda\}_{\lambda\in\Lambda}$ of submodules of M_R, the sum $\sum\limits_{\lambda\in\Lambda} N_\lambda$ is defined by

$$\sum_{\lambda\in\Lambda} N_\lambda = \left\{ \sum_{k=1}^{r} n_{\lambda_k} \,|\, r \in \mathbb{N}, \lambda_k \in \Lambda, n_{\lambda_k} \in N_{\lambda_k} \right\}.$$

This is a submodule of M. Also the intersection $\bigcap\limits_{\lambda\in\Lambda} N_\lambda$ is a submodule of M. $\sum\limits_{\lambda\in\Lambda} N_\lambda$ is the smallest submodule of M which contains all N_λ and $\bigcap\limits_{\lambda\in\Lambda} N_\lambda$ is the largest submodule of M which is contained in all N_λ.

Proposition 2.1.5. [1] *Let M be a right R-module and let X be a nonempty subset of M. Then XR is a submodule of M.*

Proposition 2.1.6. [1] *Let M be a right R-module and N a nonempty subset of M. Then the followings are equivalent:*

(i) N is a submodule of M.

(ii) $NR = N$.

(iii) For all $a, b \in R$ and all $x, y \in N$,

$$xa + yb \in N.$$

Definition 2.1.7. [21] Let M be a right R-module and $\{B_i \,|\, i \in I\}$ a nonempty family of submodules of M. If

$$(i)\ M = \sum_{i\in I} B_i \quad \text{and} \quad (ii)\ \forall j \in J \left[B_j \cap \sum_{i\in I, i\neq j} B_i = 0 \right],$$

then M is called the ***(internal) direct sum*** of the family of submodules $\{B_i \,|\, i \in I\}$. This is written as $M = \bigoplus\limits_{i\in I} B_i$ and the B_i are called ***direct summands*** of M.

If only (ii) is satisfied, then $\{B_i \,|\, i \in I\}$ is called an ***independent family*** of submodules.

In the case of finite index set, say $I = \{1, 2, \ldots, n\}$, M is also written as

$$M = B_1 \oplus B_2 \oplus \cdots \oplus B_n.$$

Lemma 2.1.8. [7] *Let* M *be a right* R*-module with* $M = \sum_{i \in I} B_i$ *where* $B_i \hookrightarrow M$ *for all* $i \in I$*. Then (ii) of the previous definition is equivalent to :*

For $x \in M$*, the representation* $x = \sum_{i \in I'} b_i$ *with* $b_i \in B_i$*,* $I' \subset I$*, where* I' *is finite, is unique in the following sense :*

If

$$x = \sum_{i \in I'} b_i = \sum_{i \in I'} c_i \text{ with } b_i, c_i \in B_i,$$

then it follows that

$$\forall i \in I' \left[b_i = c_i \right].$$

Definition 2.1.9. [7]

(i) A submodule B of a right R-module M is called a ***direct summand*** of M if there exists $C \hookrightarrow M$ such that $M = B \oplus C$.

(ii) A nonzero right R-module M is called ***directly indecomposable*** if 0 and M are the only direct summand of M.

We write $B \overset{\oplus}{\hookrightarrow} M$ for a direct summand B of M.

Example 2.1.10. [21]

(i) Let K be a field, V a vector space over K and let $\{x_i \mid i \in I\}$ be a basis of V. Then clearly we have

$$V = \bigoplus_{i \in I} x_i K.$$

Further every submodule of V is a direct summand.

(ii) Let $\mathbb{Z}$ be the set of all integers. Then $\mathbb{Z}$ is a right $\mathbb{Z}$-module. Hence the ideal $n\mathbb{Z}$ is not a direct summand of $\mathbb{Z}_{\mathbb{Z}}$ for all $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$.

Proof. Suppose there exists $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ such that $\mathbb{Z} = n\mathbb{Z} \oplus V$ for some submodule V of $\mathbb{Z}_{\mathbb{Z}}$. Thus $V = m\mathbb{Z}$ for some $m \in \mathbb{Z}$ and $\mathbb{Z} = n\mathbb{Z} \oplus m\mathbb{Z}$. Then $nm \in n\mathbb{Z} \cap m\mathbb{Z} = \{0\}$. Since $\mathbb{Z}$ does not have zero divisors, $m = 0$. Then $\mathbb{Z} = n\mathbb{Z}$, i.e., $n = -1$ or $n = 1$ which is a contradiction. It follows that $\mathbb{Z}_{\mathbb{Z}}$ is directly indecomposable. $\square$

(iii) Every simple module M is directly indecomposable because it has only 0 and M as submodules.

Let M be a right R-module and let K be a submodule of M. Then it is easy to see that the set of cosets

$$M/K = \{x + K \mid x \in M\}$$

is a right R-module relative to the addition and the scalar multiplication defined via

$$(m_1 + K) + (m_2 + K) = (m_1 + m_2) + K, \qquad (m + K)r = mr + K$$

where $m, m_1, m_2 \in M, r \in R$. Of course, the additive identity and inverse are given by

$$K = 0 + K \quad \text{and} \quad -(x + K) = -x + K.$$

In order to show that M/K is a right R-module, it is sufficient to show that

$$M/K \times R \to M/K \text{ with } (m + K, r) \mapsto mr + K$$

is a mapping since the other module properties follow directly from those of M.

Let $m_1 + K, m_2 + K \in M/K$ with $m_1 + K = m_2 + K$. Then $m_1 - m_2 \in K$. Since $K \hookrightarrow M$, $(m_1 - m_2)r \in K$. Hence $m_1 r - m_2 r \in K$, so we have $m_1 r + K = m_2 r + K$. The resulting module M/K is called ***the right R-factor module of M modulo K***.

Definition 2.1.11. [21] A submodule K of a right R-module M is called ***essential*** or ***large in*** M if, for every nonzero submodule L of M, we have $K \cap L \neq 0$.

Example 2.1.12. [21]

(i) Every right R-module M is an essential submodule in M.

(ii) In $\mathbb{Z}_{\mathbb{Z}}$, every nonzero submodule is essential.

Definition 2.1.13. [21] A right R-module M is called a ***uniform module*** if, every nonzero submodule is essential in M, i.e., the intersection of any two nonzero submodules is nonzero.

Example 2.1.14. In $\mathbb{Z}_{\mathbb{Z}}$, since every nonzero submodule is essential, $\mathbb{Z}$ is a uniform $\mathbb{Z}$-module.

It is easy to check that every nonzero submodule of a uniform right R-module is uniform.

Definition 2.1.15. [21] A subset L of a right R-module M is called a ***generating set*** of M if $LR = M$. We also say L ***generates*** M or M **is *generated by*** L.

If there is a finite generating set of M, then M is called ***finitely generated***.

If M is generated by one element, then it is called ***cyclic***.

Example 2.1.16. [21]

(i) Every ring is generated by its unit.

(ii) Every principal right ideal of a ring R is just the cyclic submodule of R_R.

Definition 2.1.17. [21] A right R-module M is called ***divisible*** if, for every $s \in R$ which is not a zero divisor and every $n \in M$, there exists $m \in M$ with $ms = n$.

Example 2.1.18. [21] Let $\mathbb{Q}$ be the set of all rational numbers and $\mathbb{R}$ the set of all real numbers. Then $\mathbb{Q}$ and $\mathbb{R}$ are divisible $\mathbb{Z}$-modules.

2.2 Homomorphisms of Modules

Definition 2.2.1. [7] Let M and N be right R-modules. A map $f : M \to N$ is an ***R-module homomorphism*** provided

(i) $f : M \to N$ is a homomorphism of abelian groups and

(ii) if $r \in R$ and $m \in M$, then $f(mr) = f(m)r$.

In this thesis, we write R-homomorphism instead of R-module homomorphism. We denote $Hom_R(M, N)$, the abelian group of the R-homomorphism from M to N and $End_R(M)$ is used to denote the endomorphism ring of M.

For $f \in Hom_R(M, N)$, we define the ***kernel*** and ***image*** by

$$Ker\,(f) = \{m \in M \mid f\,(m) = 0_N\} \quad \text{and} \quad Im\,(f) = \{f\,(m) \in N \mid m \in M\}.$$

Theorem 2.2.2. [21] *For $f \in Hom_R(M, N)$, $Ker\,(f)$ is a submodule of M and $Im\,(f)$ is a submodule of N.*

The ***coimage*** of f and the ***cokernel*** of f are defined, respectively, by

$$Coim\,(f) = M/Ker\,(f) \quad \text{and} \quad Coker\,(f) = N/Im\,(f)\,.$$

Definition 2.2.3. [1] Let M and N be right R-modules and $f : M \to N$ an R-homomorphism.

(i) $f : M \to N$ is called an ***R-epimorphism*** in case it is surjective.

(ii) $f : M \to N$ is called an ***R-monomorphism*** in case it is injective.

(iii) $f : M \to N$ is called an ***R-isomorphism*** in case it is injective and surjective.

Definition 2.2.4. [1] Let M and N be right R-modules. Then M and N are said to be ***isomorphic*** if there is an R-isomorphism between M and N. We write $M \cong N$ to represent that M is isomorphic to N.

Remark. [1]

(i) If M is a right R-module, then every submodule of M is actually the image of some monomorphism. Let K be a submodule of M, then the ***inclusion***

map *$i_K : K \to M$, defined by*

$$i_K(k) = k$$

for all $k \in K$, is an R-monomorphism, also called the ***natural embedding of*** *K* ***in*** *M, with image K.*

(ii) *Every submodule of a right R-module M is also the kernel of an epimorphism. Let K be a submodule of M. Then the mapping $\pi_K : M \to M/K$ from M onto the factor module M/K defined by*

$$\pi_K(x) = x + K$$

for all $x \in M$ is seen to be an R-epimorphism with kernel K. We call π_K the ***natural epimorphism of*** *M* ***onto*** *M/K or* ***canonical homomorphism (projection) of*** *M* ***onto*** *M/K.*

Theorem 2.2.5. [1] *Let M, M', N and N' be right R-modules and $f : M \to N$ an R-homomorphism.*

(i) *If $g : M \to M'$ is an R-epimorphism with $Ker\,(g) \subseteq Ker\,(f)$, then there exists a unique R-homomorphism $h : M' \to N$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & N \\ M' \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

commutes, i.e., $f = hg$. Moreover, $Ker\,(h) = g(Ker\,(f))$ and $Im\,(h) = Im\,(f)$, so

(a) *h is an R-monomorphism if and only if $Ker\,(g) = Ker\,(f)$ and*

(b) *h is an R-epimorphism if and only if f is an R-epimorphism.*

(ii) If $g : N' \to N$ *is an* R*-monomorphism with* $Im\,(f) \subseteq Im\,(g)$*, then there exists a unique* R*-homomorphism* $h : M \to N'$ *such that*

$$\begin{tikzcd}
N' \arrow[r, "g"] & N \\
 & M \arrow[u, "f"'] \arrow[ul, dotted, "h"]
\end{tikzcd}$$

commutes, i.e., $f = gh$*. Moreover,* $Ker\,(h) = Ker\,(f)$ *and* $Im\,(h) = g^{-1}(Im\,(f))$*, so*

(a) h *is an* R*-monomorphism if and only if* f *is an* R*-monomorphism and*

(b) h *is an* R*-epimorphism if and only if* $Im\,(g) = Im\,(f)$*.*

Example 2.2.6. [7]

(i) Let A and B be right R-modules. The zero R-homomorphism of A into B is defined by

$$\begin{aligned} 0 : A &\to B \\ a &\mapsto 0 \text{ for all } a \in A. \end{aligned}$$

(ii) Let M be a right R-module. The **identity map I_M on M** defined by

$$\begin{aligned} I_M : M &\to M \\ m &\mapsto m \text{ for all } m \in M. \end{aligned}$$

(iii) Let B be a right R-module and A a submodule of B. The **inclusion map** i_A of A is defined by

$$\begin{aligned} i_A : A &\to B \\ a &\mapsto a \text{ for all } a \in A. \end{aligned}$$

(iv) Let A be a right R-module and B a submodule of A. The natural(canonical) R-homomorphism of A onto the factor module A/B is defined by

$$\begin{aligned}\pi_B : A &\to A/B \\ a &\mapsto a + B \text{ for all } a \in A.\end{aligned}$$

Theorem 2.2.7. [7] *If* $\alpha : A \to B$ *is an* R*-homomorphism, then* $\hat{\alpha} : A/Ker\,(\alpha) \to Im\,(\alpha)$*, defined by*

$$\hat{\alpha}\,(a + Ker\,(\alpha)) = \alpha\,(a)$$

for all $a + Ker\,(\alpha) \in A/Ker\,(\alpha)$*, is an* R*-isomorphism, thus we have*

$$A/Ker\,(\alpha) \cong Im\,(\alpha)\,.$$

Definition 2.2.8. [1] Let M, M' and M'' be right R-modules. A pair of R-homomorphisms $M' \xrightarrow{f} M \xrightarrow{g} M''$ is said to be ***exact at*** M if $Im\,(f) = Ker\,(g)$.

Definition 2.2.9. [1] Let M_j be a right R-module and f_j an R-homomorphism from M_{j-1} to M_j for all $j \in \{n \pm i \,|\, i \in \mathbb{N} \cup \{0\}\}$ where $n \in \mathbb{Z}$. Let

$$\boldsymbol{A} \;= \ldots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M_{n+1} \to \ldots$$

be a sequence(finite or infinite) of R-homomorphisms f_j where $j \in \{n \pm i \,|\, i \in \mathbb{N} \cup \{0\}\}$ and $n \in \mathbb{Z}$.

(i) $\boldsymbol{A}$ is called an ***exact sequence*** if each pair of R-homomorphisms

$$M_{j-1} \xrightarrow{f_j} M_j \xrightarrow{f_{j+1}} M_{j+1}$$

is exact at M_j, i.e., $Im\,(f_j) = Ker\,(f_{j+1})$ for all $j \in \{n \pm i \,|\, i \in \mathbb{N} \cup \{0\}\}$.

(ii) An exact sequence $\boldsymbol{A}$ is called a ***split exact sequence*** if $Im\,(f_j) = Ker\,(f_{j+1})$ is a direct summand of M_j for all $j \in \{n \pm i \,|\, i \in \mathbb{N} \cup \{0\}\}$.

Definition 2.2.10. [1] Let M, M' and M'' be right R-modules. An exact sequence of the form

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is called a ***short exact sequence***. This means that f is an R-monomorphism, g is an R-epimorphism and $Ker\,(g) = Im\,(f)$.

Lemma 2.2.11. [7] *Let N, M and W be right R-modules and $\boldsymbol{A} = 0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} W \rightarrow 0$ a short exact sequence.*

(i) The followings are equivalent :

  (a) $\boldsymbol{A}$ splits.

  (b) There exists an R-homomorphism $f' : M \rightarrow N$ with $f'f = I_N$.

  (c) There exists an R-homomorphism $g' : W \rightarrow M$ with $gg' = I_W$.

(ii) If $\boldsymbol{A}$ splits, then f' and g' exist as in above and the sequence

$$0 \leftarrow N \xleftarrow{f'} M \xleftarrow{g'} W \leftarrow 0$$

is exact and splits.

Lemma 2.2.12. [7] *Let M and N be right R-modules. For an R-homomorphism $\alpha : M \rightarrow N$ the followings are equivalent:*

(i) $Ker(\alpha)$ is a direct summand of M and $Im(\alpha)$ is a direct summand of N.

(ii) There exists an R-homomorphism $\beta : N \rightarrow M$ with $\alpha = \alpha\beta\alpha$.

Proposition 2.2.13. [1] *Let M and N be right R-modules. If $f : M \rightarrow N$ is an R-homomorphism, then*

$$0 \rightarrow Ker\,(f) \xrightarrow{i} M \xrightarrow{f} N \xrightarrow{\pi} Coker\,(f) \rightarrow 0$$

is exact where i is the inclusion map and π is the natural epimorphism from N to $N/Im\,(f)$.

Definition 2.2.14. [21] Let M be a right R-module. An R-module N is called ***M-cyclic*** if it is isomorphic to M/L for some submodule L of M.

Example 2.2.15. [21] Factor modules of M are M-cyclic modules.

Remark. [19] *Any M-cyclic submodule X of M can be considered as the image of an R-endomorphism of M.*

2.3 Injective Modules

In this section, we present the definition and the basic properties of injective modules.

Definition 2.3.1. [9] Let A and B be right R-modules.

A right R-module I is ***injective*** if, for any R-monomorphism $g : A \to B$ and any R-homomorphism $h : A \to I$, there exists an R-homomorphism $h' : B \to I$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\overset{g}{1-1}"] \arrow[d, "h"'] & B \arrow[dl, dotted, "h'"] \\ I & \end{tikzcd} \tag{*}$$

commutes, i.e., $h = h'g$.

We refer to this property informally by saying that any $h : A \to I$ can be ***extended*** to B, or to an R-homomorphism $h' : B \to I$.

Example 2.3.2.

(i) Trivially, the zero module is injective.

(ii) $\mathbb{Q}_{\mathbb{Z}}$ and $\mathbb{R}_{\mathbb{Z}}$ are injective because $\mathbb{Q}_{\mathbb{Z}}$ and $\mathbb{R}_{\mathbb{Z}}$ are divisible.

The following remarkable criterion for injectivity, due to R. Baer, says that it is sufficient to test the extendibility condition in $(*)$ with B chosen to be the right regular module, R_R.

Theorem 2.3.3. ***Baer's Criterion or Baer's Test***[2] *A right R-module I is injective if and only if, for any right ideal $\mathfrak{U}$ of R, any R-homomorphism $f : \mathfrak{U} \to I$ can be extended to an R-homomorphism $f' : R \to I$.*

Remark. *An R-homomorphism $f' : R \to I$ is uniquely determined by specifying the image $f'(1_R) \in I$. If we can find an element $i \in I$ such that $f(r) = ir$ for every $r \in \mathfrak{U}$, then f can be extended to $f' : R \to I$ where $f'(1_R) = i$.*

For most rings R, R_R is simply not injective. But there exists a ring R for which R_R is injective; we say that such rings are ***right self-injective***. Some examples are given below.

Example 2.3.4.

(i) Let F be a field and $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}$. Then R is a right self-injective ring because R has no proper right ideal.

(ii) Let R be the set of all $n \times n$ upper triangular matrices over a ring $K \neq 0$, where $n \geq 2$. Then R is not right self-injective. To simplify the notations, we work in the case $n = 2$. Consider the ideal $\mathfrak{U} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in K \right\}$ and define $f : \mathfrak{U} \to R$ by $f\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ for all $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in \mathfrak{U}$. This is easily checked to be an R-homomorphism. If f can be extended to R, there exists a matrix $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$ such that

$$f\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix} \qquad (a \in K),$$

which is clearly impossible. This shows that R_R is not injective.

Proposition 2.3.5. [9] *For any right R-module Q, the followings are equivalent :*

(i) Q is a divisible module.

(ii) For any $a \in R$, any R-homomorphism $f : aR \to Q$ extends to an R-homomorphism from R_R to Q.

In [4], a module Q_R satisfying the condition *(ii)* in Proposition 2.3.5 is said to be ***principally injective***.

Theorem 2.3.6. [7] *The following properties of a right R-module Q are equivalent:*

(i) *Q is injective.*

(ii) *Any short exact sequence $0 \to Q \to M \to N \to 0$ splits.*

(iii) *Q is a direct summand of every right R-module containing it as a submodule.*

Theorem 2.3.7. [7] *Let A and Q be right R-modules. If Q is injective and $Q \cong A$, then A is injective.*

Remark. *Every vector space over a field F is injective.*

Proof. Let Q be a vector space over a field F. By Proposition 18.6[1], Q can be embedded in an injective left F-module, say V. Then Q is isomorphic to a subspace V' of V. Since every vector space has a basis, there exists a basis of V' and extend it to a basis of V. Then V is the internal direct sum of V' and K for some subspace K of V. By Theorem 5.3.4[7], V' is injective. Since $Q \cong V'$, by Theorem 2.3.7, Q is injective. $\square$

CHAPTER III

M-SLIGHTLY COMPRESSIBLE MODULES

In this chapter, we determine a general form of slightly compressible modules which subsequently are called ***M-slightly compressible modules*** for a right R-module M. Moreover, we provide conditions for any right R-module to be an M-slightly compressible module and also provide examples of M-slightly compressible modules.

3.1 Definitions and Examples

First, we begin with the concept of compressible modules which was introduced by Zelmanowitz in 1976.

Definition 3.1.1. [23] A right R-module M is called ***compressible*** if, for every nonzero submodule N of M there exists an R-monomorphism from M to N.

Example 3.1.2. Every simple right R-module is compressible. Since any simple right R-module M has only one nonzero submodule that is M, so an R-monomorphism from M to M is the identity map of M.

Next in 2005, Smith[20] introduced the concept of a slightly compressible module, which is a generalization of compressible modules.

Definition 3.1.3. [20] A right R-module M is called ***slightly compressible*** if, for every nonzero submodule N of M, there exists a nonzero R-homomorphism from M to N.

Example 3.1.4.

(i) Let I be any proper ideal of R. Then the right R-module R/I is slightly compressible.

Proof. Claim that a right R-module R/I is slightly compressible. Note that any nonzero submodule of R/I has the form E/I for some nonzero right ideal E of R properly containing I. Let $a \in E \setminus I$. Then the mapping $f : R/I \to E/I$ defined by

$$f(r+I) = ar + I \text{ for all } r \in R$$

is a nonzero R-homomorphism. Hence the right R-module R/I is slightly compressible. □

(ii) Let $\mathbb{Z}_3$ be the set of all integers modulo 3 and $R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$. Then R_R is a slightly compressible module.

Proof. Note that all nonzero submodules of R are

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}, \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\} \text{ and } R.$$

Define $f_1 : R \to \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}$ by

$$f_1\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$$

and define $f_2 : R \to \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}$ by

$$f_2\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R.$$

It is easy to check that f_1 and f_2 are R-homomorphisms. Next, we claim that R_R is not a compressible module by showing that every R-homomorphism from R_R to $E := \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}$ is not one to one. Suppose there exists an R-monomorphism α from R_R to E. Then $Ker\,(\alpha) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $\alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ for some $a \in \mathbb{Z}_3 \setminus \{0\}$. Then $\alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) = \alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \in Ker\,(\alpha)$, which is a contradiction. Hence every R-homomorphism from R_R to E is not one to one. Therefore R_R is a slightly compressible module but not a compressible module. □

Next, we determine a general form of slightly compressible modules called an ***M-slightly compressible modules*** for a right R-module M.

Definition 3.1.5. Let M be a right R-module. A right R-module N is called an ***M-slightly compressible module*** if, for every nonzero submodule A of N, there exists a nonzero R-homomorphism from M to A.

In the case that $N = M$, N is, in fact, a slightly compressible module.

Example 3.1.6.

(i) Let M be a right R-module. The zero right R-module is an M-slightly compressible module.

(ii) From [3], for right R-modules M and N, N is called a ***fully-M-cyclic module*** if, every submodule A of N, there exists $s \in Hom_R(M, N)$ such that $A = s(M)$. It is clear that every fully-M-cyclic module is an M-slightly compressible module but an M-slightly compressible module may not be a fully-M-cyclic module, for example, $\mathbb{R}_{\mathbb{Z}}$ is $\mathbb{Z}$-slightly compressible but not fully-$\mathbb{Z}$-cyclic module because $\mathbb{R}_{\mathbb{Z}}$ is not cyclic $\mathbb{Z}$-module.

(iii) Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}$$
$$\text{and } N_R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

Then N is an M-slightly compressible module.

Proof. Note that all nonzero submodules of N are

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\},\ E_k := \left\{ \begin{pmatrix} ak & 0 \\ a & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\} \text{ where } k \in \mathbb{Z}_p \text{ and } N.$$

Define $g : M \to \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$ by

$$g\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in M,$$

and for each $k \in \mathbb{Z}_p$, define $f_k : M \to E_k$ by

$$f_k\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} ka & 0 \\ a & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in M.$$

It is easy to check that g and f_k are nonzero R-homomorphisms for all $k \in \mathbb{Z}_p$ and g, f_k are also R-homomorphisms from M to N. Then N is an M-slightly compressible module. □

(iv) Let $\mathbb{Z}_m$ and $\mathbb{Z}_n$ be the set of all integers modulo m and n, respectively, where $m, n \in \mathbb{Z}^+$. Then a right $\mathbb{Z}$-module $\mathbb{Z}_n$ is a $\mathbb{Z}_m$-slightly compressible module for all $n \mid m$.

Proof. Let $m, n \in \mathbb{Z}^+$ be such that $n \mid m$ and $\phi : \mathbb{Z}_m \to \mathbb{Z}_n$ a $\mathbb{Z}$-homomorphism. Then we must have $m\phi([1]_m) = [0]_n$. Since $n \mid m$, all elements $[y]_n \in \mathbb{Z}_n$ satisfy $m[y]_n = [my]_n = [0]_n$. There are $n-1$ nonzero $\mathbb{Z}$-homomorphisms, given by $[1]_m \mapsto [1]_n$, $[1]_m \mapsto [2]_n$, …, $[1]_m \mapsto [n-1]_n$. Hence every nonzero submodule E of $\mathbb{Z}_n$, there exists a nonzero $\mathbb{Z}$-homomorphism from $\mathbb{Z}_m$ to E. □

3.2 Some Properties of M-Slightly Compressible Modules

In general, the class of slightly compressible R-modules is not closed under taking submodules.

Example 3.2.1. [20] Let F be a field,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\} \text{ and } A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}.$$

Then A is a cyclic right R-module which is not slightly compressible.

Proof. First, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$. Thus A is cyclic. Let $B = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in F \right\}$. Then $B \hookrightarrow A$. Next, we show that every R-homomorphism from A to B is zero. Let $f : A \to B$ be an R-homomorphism. Then

$$f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \quad \text{for some } x \in F.$$

Since f is an R-homomorphism,

$$\begin{aligned} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) &= f\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then $x = 0$ so that $f = 0$. Hence every R-homomorphism from A to B is zero. Therefore A is not slightly compressible. But from Example 1.2[20], R_R is slightly compressible. □

On the other hand, let M be a right R-module, every submodule of M-slightly compressible module is also an M-slightly compressible module.

Theorem 3.2.2. *Let M and N be right R-modules. Then N is M-slightly compressible if and only if every nonzero submodule of N is M-slightly compressible.*

Proof. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Assume that N is M-slightly compressible. Let A be a nonzero submodule of N and B a nonzero submodule of A. Then B is also a nonzero submodule of N. There exists a nonzero R-homomorphism from M to B. Hence A is an M-slightly compressible module. $\square$

Example 3.2.3.

(i) Let F be a field, $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$ and $A_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}$. By Example 3.2.1, R_R is a slightly compressible module, i.e., R_R is an R_R-slightly compressible module. Since $A \hookrightarrow R_R$, by Theorem 3.2.2, A is R_R-slightly compressible and every nonzero submodule of A is also an R_R-slightly compressible module.

(ii) Let $\mathbb{Z}_3$ be the set of all integers modulo 3,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\},\ M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$$
$$\text{and } N_R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}.$$

From Example 3.1.6(iii), we have N is an M-slightly compressible module. By Theorem 3.2.2, every nonzero submodule of N is an M-slightly compressible module.

Corollary 3.2.4. *Let M be a right R-module. Then M is slightly compressible if and only if every submodule of M is M-slightly compressible.*

We can change from submodules to essential submodules which is shown in the following result.

Proposition 3.2.5. *Let M and N be right R-modules. Then N is M-slightly compressible if and only if every essential submodule of N is M-slightly compressible.*

Proof. (⇒) From Theorem 3.2.2, we are done.

(⇐) Assume that every essential submodule of N is M-slightly compressible. Since N is an essential submodule of N, N is an M-slightly compressible module. □

Example 3.2.6. Let $\mathbb{Z}_3$ be the set of all integers modulo 3,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\},\ M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$$
$$\text{and } A_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}.$$

Clearly, only A and M are essential submodules of M. Since R_R is an R_R-slightly compressible module and M is a submodule of R_R, by Theorem 3.2.2, M is an R_R-slightly compressible module. By Proposition 3.2.5, A is an R_R-slightly compressible module.

Proposition 3.2.7. *Let M, P and Q be right R-modules with $P \cong Q$. If P is an M-slightly compressible module, then Q is an M-slightly compressible module.*

Proof. Assume that P is an M-slightly compressible module. Let L be a nonzero submodule of Q. Since $P \cong Q$, there exists an R-isomorphism $f : Q \to P$ and $f|_L : L \to P$ is an R-monomorphism. Then $f|_L(L) \hookrightarrow P$. Since P is an M-slightly compressible module, there exists a nonzero R-homomorphism $g : M \to f|_L(L)$. Since $f|_L$ is an R-momonorphism, the R-homomorphism $f|_L^{-1}$ from $f|_L(L)$ to L exists and $f|_L^{-1}g$ is an R-homomorphism from M to L. Hence Q is an M-slightly compressible module. □

Example 3.2.8. Let $\mathbb{Z}_3$ be the set of all integers modulo 3,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\}, M_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$$
$$\text{and } A_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}.$$

From Example 3.2.6, A is an R_R-slightly compressible module. Define $f : A \to M$ by

$$f\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in A.$$

It is easy to check that f is an R-isomorphism so $A \cong M$. By Proposition 3.2.7, M is an R_R-slightly compressible module.

Theorem 3.2.9. *Let M, M' and N be right R-modules which N is an M-slightly compressible module.*

(i) *If M is an R-epimorphic image of M', then N is an M'-slightly compressible module.*

(ii) *If M is an M'-slightly compressible module, then N is also an M'-slightly compressible module.*

Proof. (i) Assume that M is an R-epimorphic image of M'. There exists an R-epimorphism α from M' to M, so $\alpha(M') = M$. Let A be a nonzero submodule of N. Since N is M-slightly compressible, there exists a nonzero R-homomorphism s from M to A. Then $s\alpha$ is a nonzero R-homomorphism from M' to A. Therefore N is an M'-slightly compressible module.

(ii) Assume that M is an M'-slightly compressible module. Let A be a nonzero submodule of N. Since N is an M-slightly compressible module, there exists a nonzero R-homomorphism g from M to A. Since M is an M'-slightly compressible module, there exists a nonzero R-homomorphism g' from M' to M. Then gg' is a nonzero R-homomorphism from M' to A. Hence N is an M'-slightly compressible module.

□

Example 3.2.10. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}, \; M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}$$

$$\text{and } N_R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

From Example 3.1.6(iii), N is M-slightly compressible. Define $f : R \to M$ by

$$f\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R.$$

It is easy to check that f is an R-epimorphism and $f(R) = M$. By Theorem 3.2.9(i), M is an R_R-slightly compressible module. By Theorem 3.2.9(ii), N is an R_R-slightly compressible module.

The following theorem indicates that every right R-module is an R_R-slightly compressible module.

Theorem 3.2.11. *Every right R-module is an R_R-slightly compressible module.*

Proof. Let M be a right R-module and A a nonzero submodule of M. There exists $a \in A \setminus \{0_A\}$. Then $aR \hookrightarrow A$. Define $f : R \to A$ by

$$f(r) = ar \text{ for all } r \in R.$$

Since M is a unital right R-module, $f(1_R) = a1_R = a \neq 0_A$, f is a nonzero R-homomorphism from R to A. Hence M is an R_R-slightly compressible module. □

Example 3.2.12.

(i) Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\} \text{ and } N_R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

By Theorem 3.2.11, N is an R_R-slightly compressible module.

(ii) Every right ideal of R is an R_R-slightly compressible submodule of R_R because every right ideal of R is a right R-module.

The following results show the characteristics of essential submodules and uniform submodules of M-slightly compressible modules where M is a right R-module.

Theorem 3.2.13. *Let M and N be right R-modules which N is M-slightly compressible and A is a submodule of N.*

(i) *A is essential in N if and only if for each $t \in Hom_R(M,N)\smallsetminus\{0\}$, $t(M) \cap A \neq 0$.*

(ii) *A is uniform if and only if for each $t \in Hom_R(M,A)\smallsetminus\{0\}$, $t(M)$ is essential in A.*

Proof. (i) ($\Rightarrow$) It is obvious.

($\Leftarrow$) Assume that for each $t \in Hom_R(M,N)\smallsetminus\{0\}$, $t(M) \cap A \neq 0$. Let B be a nonzero submodule of N. Since N is an M-slightly compressible module, there exists a nonzero R-homomorphism s from M to B. Thus s is also a nonzero R-homomorphism from M to N. By assumption, $s(M) \cap A \neq 0$. Since $s(M) \hookrightarrow B$, $B \cap A \neq 0$. Therefore A is essential in N.

(ii) ($\Rightarrow$) It is obvious.

($\Leftarrow$) Assume that for each $t \in Hom_R(M,A)\smallsetminus\{0\}$, $t(M)$ is essential in A. Let B and C be nonzero submodules of A. Since N is an M-slightly compressible module, there exists a nonzero R-homomorphism u from M to B and a nonzero R-homomorphism v from M to C. Thus u, v are also nonzero R-homomorphisms from M to A. By assumption, we have $u(M)$ and $v(M)$ are essential in A. Then $u(M) \cap v(M) \neq 0$. Since $u(M) \hookrightarrow B$ and $v(M) \hookrightarrow C$, $B \cap C \neq 0$. Therefore A is uniform.

□

Example 3.2.14. Let $\mathbb{Z}_3$ be the set of all integers modulo 3,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\},\ M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}$$

$$\text{and } A_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\}.$$

(i) Since M is a right R-module by Theorem 3.2.11, M is an R_R-slightly compressible module. Clearly, all nonzero submodules of M are only A and M, so A is essential in M. Since A is simple, for each $t \in Hom_R(R, M) \smallsetminus \{0\}$, $t(R) = M$ or $t(R) = A$. Then $t(R) \cap A \neq 0$ for all $t \in Hom_R(R, M) \smallsetminus \{0\}$.

(ii) Since A is simple, A is an uniform submodule of M. Then for each $t \in Hom_R(R, A) \smallsetminus \{0\}$, $t(R) = A$ and $t(R)$ is essential in A.

Proposition 3.2.15. *Let M and N be right R-modules with $Hom_R(M, N) \neq \{0\}$. Then N is a simple module if and only if N is an M-slightly compressible module with every nonzero R-homomorphism from M to N is an R-epimorphism.*

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) Assume that N is a M-slightly compressible module with every nonzero R-homomorphism from M to N is an R-epimorphism. Let A be a nonzero submodule of N. There exists a nonzero R-homomorphism s from M to A so s is also a nonzero R-homomorphism from M to N. By assumption, we have $N = s(M)$ and hence $N = A$. Therefore N is a simple module. □

Example 3.2.16. Let F be a field,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\} \text{ and } N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in F \right\}.$$

Clearly, N_R is a simple module. Define $f : R \to N$ by

$$f\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R.$$

It is easy to check that f is a nonzero R-homomorphism so $Hom_R(R, N) \neq \{0\}$. By Proposition 3.2.15, N is an R_R-slightly compressible module with every nonzero R-homomorphism from R to N is an epimorphism.

Following result is a neccesary and sufficient condition for any right R-modules to be M-slightly compressible modules where M is a right R-module.

Theorem 3.2.17. *Let M and N be right R-modules. Every nonzero submodule of N contains a nonzero M-cyclic module if and only if N is M-slightly compressible.*

Proof. ($\Rightarrow$) Assume that every nonzero submodule of N contains a nonzero M-cyclic module. Let A be a nonzero submodule of N. By assumption, there exists a nonzero submodule B of A such that $B \cong M/C$ for some submodule C of M, so there exists an R-isomorphism α from M/C to B. Let π_C be the natural epimorphism of M onto M/C. Thus $\alpha\pi_C : M \to B$ is an R-epimorphism and $\alpha\pi_C$ is also a nonzero R-homomorphism from M to A. Hence N is M-slightly compressible.

($\Leftarrow$) Assume that N is M-slightly compressible. Let A be a nonzero submodule of N. There exists a nonzero R-homomorphism s from M to A. Then $s(M)$ is a nonzero submodule of A. By Theorem 2.2.7, $s(M) \cong M/Ker(s)$, so $s(M)$ is a nonzero M-cyclic module. Hence every nonzero submodule of N contains a nonzero M-cyclic module. $\square$

Example 3.2.18. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}$$
$$\text{and } N_R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

From Example 3.1.6(iii), N is an M-slightly compressible module. By Theorem 3.2.17, every nonzero submodule of N contains a nonzero M-cyclic module.

Corollary 3.2.19. *Let M be a right R-module. Every nonzero submodule of M contains a nonzero M-cyclic submodule of M if and only if M is slightly compressible.*

CHAPTER IV

M-SLIGHTLY COMPRESSIBLE INJECTIVE MODULES

In 1940, Bear[2] established a very useful test for injectivity. This test called the ***Baer's Criterion*** said that for any right R-module Q,

any R-homomorphism of a right ideal $\mathfrak{U}$ of R into Q
can be extended to an R-homomorphism of R into Q
if and only if
Q is injective.

If R_R satisfies the Baer Criterion, that is, any R-homomorphism of a right ideal I of R into R can be extended to an R-homomorphism of R into R, then R is called a ***right self-injective ring***.

Since every right ideal of R is a right R-module and by Theorem 3.2.11, we see that every right ideal of R is an R_R-slightly compressible submodule of R_R and every R_R-slightly compressible submodule of R_R is a right ideal of R because every submodule of R_R is a right ideal of R. We use this fact to generalize the notion of injectivity to M-slightly compressible injective module for a given right R-module M.

Moreover, we investigate some properties of M-slightly compressible injective modules and also provide examples of them.

4.1 Definition and Examples

Definition 4.1.1. Let M be a right R-module. A right R-module N is called an ***M-slightly compressible injective module*** if every R-homomorphism from an M-slightly compressible submodule of M to N can be extended to an R-homomorphism from M to N.

In other words, given any diagram

$$\begin{tikzcd}
M\text{-slightly compressible submodule of } M \arrow[r, "i"] \arrow[d, "g"'] & M \\
N &
\end{tikzcd}$$

where i is the inclusion map of an M-slightly compressible submodule of M and g is an R-homomorphism from that M-slightly compressible submodule of M to N, there exists an R-homomorphism $h : M \to N$ such that the diagram

$$\begin{tikzcd}
M\text{-slightly submodule of } M \arrow[r, "i"] \arrow[d, "g"'] & M \arrow[dl, dotted, "h"] \\
N &
\end{tikzcd}$$

commutes, i.e., $hi = g$.

Example 4.1.2. Let $\mathbb{Z}_p$ be the set of all integers modulo p, where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$$
$$\text{and } M_R = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

Then

(i) N is an R_R-slightly compressible injective module,

(ii) M is an M-slightly compressible injective module.

Proof. (i) From previous chapter, R_R is a slightly compressible module and by Corollary 3.2.4, every submodule of R_R is an R_R-slightly compressible module. All nonzero proper submodules of R are

$$\left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p\right\} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R \text{ and } \left\{\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p\right\} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R.$$

Case I : $A_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$

We claim that every R-homomorphism from A_1 to N is zero. Let $\alpha : A_1 \to N$ be an R-homomorphism. Then

$$\alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \qquad \text{for some } x \in \mathbb{Z}_p.$$

Since α is an R-homomorphism,

$$\begin{aligned} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} &= \alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Then $x = 0$ so that $\alpha = 0$. Hence every R-homomorphism from A_1 to N is zero. Then every R-homomorphism from A_1 to N can be extended to the zero R-homomorphism from R to N.

Case II : $A_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R$

Let $\alpha : A_2 \to N$ be an R-homomorphism. Then

$$\alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \qquad \text{for some } x \in \mathbb{Z}_p.$$

Define $\bar{\alpha} : R \to N$ by

$$\bar{\alpha}\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & xb \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R.$$

It is easy to check that $\bar{\alpha}$ is an R-homomorphism from R to N and

$$\begin{aligned} \alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) &= \alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) = \alpha\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & xa \\ 0 & 0 \end{pmatrix} = \bar{\alpha}\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) \end{aligned}$$

for all $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \in A_2$. Hence $\bar{\alpha} i_{A_2} = \alpha$.

Therefore N is an R_R-slightly compressible injective module.

(ii) All nonzero M-slightly compressible submodules of M are

$$\begin{aligned} A_1 &:= \left\{\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p\right\} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R, \\ A_2 &:= \left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p\right\} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} R \\ &\text{and } M = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} R \end{aligned}$$

where the R-homomorphism $f_1 : M \to A_1$ defined by

$$f_1\left(\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \in M$$

and the R-homomorphism $f_2 : M \to A_2$ defined by

$$f_2\left(\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \in M.$$

Let $\alpha_1 : A_1 \to M$ and $\alpha_2 : A_2 \to M$ be R-homomorphisms. Then

$$\alpha_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \qquad \text{for some } a, b \in \mathbb{Z}_p,$$

$$\alpha_2\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} \qquad \text{for some } c, d \in \mathbb{Z}_p.$$

Since α_1, α_2 are R-homomorphisms,

$$\begin{aligned}
\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} &= \alpha_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \alpha_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{and } \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix} &= \alpha_2\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) \\
&= \alpha_2\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.
\end{aligned}$$

Then $a = 0, d = 0$ so that

$$\alpha_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \text{ and } \alpha_2\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

We define $\bar{\alpha_1} : M \to M$ by

$$\bar{\alpha_1}\left(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & by \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \in M$$

and define $\bar{\alpha_2} : M \to M$ by

$$\bar{\alpha_2}\left(\begin{pmatrix}0 & 0\\ x & y\end{pmatrix}\right) = \begin{pmatrix}0 & 0\\ cx & 0\end{pmatrix} \text{ for all } \begin{pmatrix}0 & 0\\ x & y\end{pmatrix} \in M.$$

Then $\bar{\alpha_1}$ and $\bar{\alpha_2}$ are R-homomorphisms from M to M.

$$\begin{aligned}\text{Thus } \alpha_1\left(\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}\right) &= \alpha_1\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}\right) = \alpha_1\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right)\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & bx\end{pmatrix} = \bar{\alpha_1}\left(\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}\text{and } \alpha_2\left(\begin{pmatrix}0 & 0\\ x & 0\end{pmatrix}\right) &= \alpha_2\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\right) = \alpha_2\left(\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix}0 & 0\\ c & 0\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ cx & 0\end{pmatrix} = \bar{\alpha_2}\left(\begin{pmatrix}0 & 0\\ x & 0\end{pmatrix}\right)\end{aligned}$$

for all $\begin{pmatrix}0 & 0\\ 0 & x\end{pmatrix} \in A_1$ and $\begin{pmatrix}0 & 0\\ x & 0\end{pmatrix} \in A_2$. Hence $\bar{\alpha_1} i_{A_1} = \alpha_1$ and $\bar{\alpha_2} i_{A_2} = \alpha_2$. Therefore M is an M-slightly compressible injective module.

□

4.2 Some Properties of M-Slightly Compressible Injective Modules

This section is concerned with M-slightly compressible injective modules and the main properties of these modules are derived in this section.

Proposition 4.2.1. *Let M, N and K be right R-modules with $N \cong K$. If N is an M-slightly compressible injective module, then K is an M-slightly compressible injective module.*

Proof. Assume that N is an M-slightly compressible injective module. Let A be an M-slightly compressible submodule of M and α an R-homomorphism from A to K. Since $N \cong K$, there exists an R-isomorphism β from K to N. Then $\beta\alpha$ is an R-homomorphism from A to N. Since N is an M-slightly compressible injective module, there exists an R-homomorphism γ from M to N such that $\gamma i_A = \beta\alpha$ where i_A is the inclusion map. We choose $\bar{\alpha} = \beta^{-1}\gamma$, so $\bar{\alpha} i_A = \beta^{-1}\gamma i_A = \beta^{-1}\beta\alpha = \alpha$. Hence K is an M-slightly compressible injective module. $\square$

Example 4.2.2. Let $\mathbb{Z}_p$ be the set of all integers modulo p, where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$$
$$\text{and } M_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

From Example 4.1.2, N is an R_R-slightly compressible injective module. Define $\alpha : N \to M$ by

$$\alpha\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in N.$$

It is clear that α is an R-isomorphism, so $M \cong N$. By Proposition 4.2.1, M is also an R_R-slightly compressible injective module.

Proposition 4.2.3. *Let M be a right R-module. If M is a simple module, then every right R-module is M-slightly compressible injective.*

Proof. Suppose that M is a simple module. Then there is only one M-slightly compressible submodule of M, i.e., M. Hence every right R-module is M-slightly compressible injective. $\square$

Example 4.2.4. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_p$ the set of all integers modulo p, where p is a prime number. Since $\mathbb{Z}_p$ is a simple $\mathbb{Z}$-module, every right $\mathbb{Z}$-module is $\mathbb{Z}_p$-slightly compressible injective.

Next result is concerned with the necessary condition for an M-slightly compressible submodule of M is an M-slightly compressible injective module.

Proposition 4.2.5. *Let M be a right R-module and N an M-slightly compressible submodule of M. If N is an M-slightly compressible injective module, then N is a direct summand of M.*

Proof. Assume that N is an M-slightly compressible injective module. There exists $\alpha : M \to N$ such that $\alpha i_N = I_N$ where I_N is the identity map. By Lemma 2.2.11 (i), the short exact sequence

$$0 \to N \xrightarrow{i_N} M \xrightarrow{\pi_N} M/N \to 0$$

splits where π_N is the canonical projection of M onto M/N and i_N is the inclusion map. Therefore N is a direct summand of M. $\square$

Example 4.2.6. Let $\mathbb{Z}_p$ be the set of all integers modulo p, where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\} \text{ and } M_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

From Theorem 3.2.11, R_R is an R_R-slightly compressible module. Since $M \hookrightarrow R_R$, by Theorem 3.2.2, M is an R_R-slightly compressible submodule of R_R. From Example 4.2.2, M is an R_R-slightly compressible injective module. By Proposition 4.2.5, $M \overset{\oplus}{\hookrightarrow} R_R$.

On the other hand, the converse of Proposition 4.2.5 is not true in general, for example, in the $\mathbb{Z}$-module $\mathbb{Z}$, we know that $\mathbb{Z}_\mathbb{Z}$ is indecomposable so only 0 and $\mathbb{Z}_\mathbb{Z}$ are direct summands of $\mathbb{Z}_\mathbb{Z}$ and $\mathbb{Z}_\mathbb{Z}$ is a $\mathbb{Z}_\mathbb{Z}$-slightly compressible submodule of $\mathbb{Z}_\mathbb{Z}$ but $\mathbb{Z}_\mathbb{Z}$ is not $\mathbb{Z}_\mathbb{Z}$-slightly compressible injective.

Indeed, $m\mathbb{Z}_\mathbb{Z}$ is a $\mathbb{Z}_\mathbb{Z}$-slightly compressible submodule of $\mathbb{Z}_\mathbb{Z}$ where $m \in \mathbb{Z} \setminus \{0\}$, let $f : m\mathbb{Z} \to \mathbb{Z}$ be the $\mathbb{Z}$-homomorphism defined by $f(ma) = a$ for all $ma \in m\mathbb{Z}$.

Suppose there is a $\mathbb{Z}$-homomorphism $\delta : \mathbb{Z} \to \mathbb{Z}$ which extends f. Then

$$1 = f(m) = \delta(i(m)) = \delta(m) = m\delta(1),$$

which cannot hold. Therefore $\mathbb{Z}_\mathbb{Z}$ is not $\mathbb{Z}_\mathbb{Z}$-slightly compressible injective.

Theorem 4.2.7. *Let M and N be right R-modules and $A \overset{\oplus}{\hookrightarrow} N$. If N is an M-slightly compressible injective module, then A and N/A are M-slightly compressible injective modules.*

Proof. Assume that N is an M-slightly compressible injective module.

(i) Claim that A is an M-slightly compressible injective module. Let B be an M-slightly compressible submodule of M and α an R-homomorphism from B to A. Since $A \overset{\oplus}{\hookrightarrow} N$, the short exact sequence

$$0 \to A \xrightarrow{i_A} N \xrightarrow{\pi_A} N/A \to 0$$

splits where i_A is the inclusion map and π_A is the canonical projection of N onto N/A. By Lemma 2.2.11(i), there exists an R-homomorphism $f' : N \to A$ with $f'i_A = I_A$ where I_A is the identity map on A. Since N is an M-slightly compressible injective module, there exists $f : M \to N$ such that $fi_B = i_A\alpha$ where $i_B : B \to M$ is the inclusion map. Let $\bar{\alpha} = f'f$. Then $\bar{\alpha}i_B = f'fi_B = f'i_A\alpha = I_A\alpha = \alpha$. Hence A is an M-slightly compressible injective module.

(ii) Claim that N/A is an M-slightly compressible injective module. Since $A \overset{\oplus}{\hookrightarrow} N$, there exists $A' \hookrightarrow N$ such that $N = A \oplus A'$ so $A' \overset{\oplus}{\hookrightarrow} N$ and $A' \cong N/A$. From (i), A' is an M-slightly compressible injective module. By Proposition 4.2.1, N/A is an M-slightly compressible injective module.

□

Example 4.2.8. Let $\mathbb{Z}_p$ be the set of all integers modulo p, where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ M_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$$

$$\text{and } N_R = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

We want to show that M, N, R/M and R/N are R_R-slightly compressible injective modules.

Proof. First, we claim that R_R is an R_R-slightly compressible injective module. By Theorem 3.2.11, R_R is an R_R-slightly compressible module so all nonzero R_R-slightly compressible submodules of R_R are N, M and R_R. Let $\alpha_1 : N \to R$ and $\alpha_2 : M \to R$ be R-homomorphisms. Then

$$\alpha_1\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} \qquad \text{for some } a, b \in \mathbb{Z}_p,$$

$$\alpha_2\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) = \begin{pmatrix}c & 0\\ 0 & d\end{pmatrix} \qquad \text{for some } c, d \in \mathbb{Z}_p.$$

Since α_1, α_2 are R-homomorphisms,

$$\begin{aligned}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} &= \alpha_1\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)\\ &= \alpha_1\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)\\ &= \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{and } \begin{pmatrix}c & 0\\ 0 & d\end{pmatrix} &= \alpha_2\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right)\\ &= \alpha_2\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right)\\ &= \begin{pmatrix}c & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}.\end{aligned}$$

Then $b = 0, c = 0$ so that

$$\alpha_1\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right) = \begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix} \text{ and } \alpha_2\left(\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) = \begin{pmatrix}0 & 0\\ 0 & d\end{pmatrix}.$$

We define $\bar{\alpha}_1 : R \to R$ by

$$\bar{\alpha}_1\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \begin{pmatrix} ax & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R$$

and define $\bar{\alpha}_2 : R \to R$ by

$$\bar{\alpha}_2\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & dy \end{pmatrix} \text{ for all } \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R.$$

Then $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are R-homomorphisms from R to R and

$$\begin{aligned} \alpha_1\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) &= \alpha_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = \alpha_1\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & 0 \end{pmatrix} = \bar{\alpha}_1\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) \\ \text{and } \alpha_2\left(\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) &= \alpha_2\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) = \alpha_2\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & dx \end{pmatrix} = \bar{\alpha}_2\left(\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) \end{aligned}$$

for all $\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in N$ and $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \in M$. Hence $\bar{\alpha}_1 i_N = \alpha_1$ and $\bar{\alpha}_2 i_M = \alpha_2$. Therefore R_R is an R_R-slightly compressible injective module. By Example 4.2.6, $M \overset{\oplus}{\hookrightarrow} R_R$, so $N \overset{\oplus}{\hookrightarrow} R_R$. By Theorem 4.2.7, M, N, R/M and R/N are R_R-slightly compressible injective modules. □

Theorem 4.2.9. *Let M, N be right R-modules and A an M-slightly compressible submodule of M. If N is an M-slightly compressible injective module, then N is A-slightly compressible injective.*

Proof. Assume that N is an M-slightly compressible injective module. Let B be an A-slightly compressible submodule of A and α an R-homomorphism from B to

N. Since A is an M-slightly compressible submodule of M, by Theorem 3.2.2, B is an M-slightly compressible submodule of M. There exists $\bar{\alpha} : M \to N$ such that $\bar{\alpha} i_B = \alpha$. Then we choose $\bar{\alpha}|_A : A \to N$ which extends α. □

Example 4.2.10. Let $\mathbb{Z}_p$ be the set of all integers modulo p, where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\} \text{ and } N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

From Example 4.1.2(i), N is an R_R-slightly compressible injective module and $A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} R$ is an R_R-slightly compressible submodule of R_R. By Theorem 4.2.9, N is an A-slightly compressible injective module.

The converse of Theorem 4.2.9 is not true in general, for example, let $\mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$ be the set of all integers modulo p and p^2, respectively, where p is a prime number.

Let $R = \mathbb{Z}$, $N = \mathbb{Z}_p$ and $A = \{[0]_{p^2}, [p]_{p^2}, [2p]_{p^2}, \ldots, [(p-1)p]_{p^2}\}$. Thus A is a $\mathbb{Z}_{p^2}$-slightly compressible submodule of $\mathbb{Z}_{p^2}$ because there is a $\mathbb{Z}$-homomorphism $\gamma : \mathbb{Z}_{p^2} \to A$ given by

$$\gamma\left([n]_{p^2}\right) = [np]_{p^2}$$

for all $[n]_{p^2} \in \mathbb{Z}_{p^2}$. Clearly, A is simple by Proposition 4.2.3, $\mathbb{Z}_p$ is A-slightly compressible injective but $\mathbb{Z}_p$ is not $\mathbb{Z}_{p^2}$-slightly compressible injective because any $\mathbb{Z}$-homomorphism $\lambda : \mathbb{Z}_{p^2} \to \mathbb{Z}_p$ satisfies $\lambda(A) = 0$.

Theorem 4.2.11. *Let Q be a right R-module. Then Q is injective if and only if Q is R_R-slightly compressible injective.*

Proof. ($\Rightarrow$) It is obvious.

($\Leftarrow$) Assume that Q is an R_R-slightly compressible injective module. We claim that Q is injective by using the Baer's Criterion that is, we show that any R-homomorphism of a right ideal $\mathfrak{U}$ of R into Q can be extended to an R-homomorphism of R into Q. Let $\mathfrak{U}$ be a right ideal of R and α an R-homomorphism from

$\mathfrak{U}$ to Q. From Example 2.1.2(iii), $\mathfrak{U}$ is a right R-module. By Theorem 3.2.11, $\mathfrak{U}$ is an R_R-slightly compressible submodule of R_R. Then α can be extened to an R-homomorphism from R into Q. By Baer's Criterion, Q is injective. $\square$

Example 4.2.12. Let $\mathbb{Z}_p$ be the set of all integers modulo p, where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\} \text{ and } N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

From Example 4.1.2(i), N is an R_R-slightly compressible injective module. By Theorem 4.2.11, N is an injective right R-module.

Corollary 4.2.13. *R_R is an R_R-slightly compressible injective module if and only if R is a right self-injective ring.*

4.3 Relationship between M-Slightly Compressible and M-Principally Injective Modules

Recall that a right R-module M is called ***principally injective*** (or ***p-injective***) if, every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M.

If R_R is an injective module, then R_R is a principally injective module. By Theorem 4.2.11, an R_R-slightly compressible injective module implies a principally injective module.

In 1999, Sanh and his group[19] introduced the notion of M-principally injective module which extended from principally injective module.

In this section, we study relationship between M-slightly compressible injective modules and M-principally injective modules where M is a right R-module.

Recall that a right R-module N is called M***-cyclic*** if it is isomorphic to M/L for some submodule L of M.

Definition 4.3.1. [19] Let M be a right R-module. A right R-module N is called an M***-principally injective module*** if every R-homomorphism from an M-

cyclic submodule of M to N can be extended to an R-homomorphism from M to N.

In other words, given any diagram

$$\begin{tikzcd}
\text{M-cyclic submodule of M} \arrow[r, "i"] \arrow[d, "g"'] & M \\
N &
\end{tikzcd}$$

where i is the inclusion map of an M-cyclic submodule of M and g is an R-homomorphism from that M-cyclic submodule of M to N, there exists an R-homomorphism $g' : M \to N$ such that the diagram

$$\begin{tikzcd}
\text{M-cyclic submodule of M} \arrow[r, "i"] \arrow[d, "g"'] & M \arrow[dl, dotted, "g'"] \\
N &
\end{tikzcd}$$

commutes, i.e., $g'i = g$.

Note that every principally injective module is an R_R-principally injective module so an R_R-slightly compressible injective module implies an R_R-principally injective module. However, in case $M_R \neq R_R$, an M_R-slightly compressible injective module may not be an M_R-principally injective module.

Example 4.3.2. Let $\mathbb{Z}_2$ be the set of all integers modulo 2,

$$R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \end{pmatrix} := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{Z}_2 \right\},$$

$$M_R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix} := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c, d, e \in \mathbb{Z}_2 \right\},$$

$$\text{and } N_R = \begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} := \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\}.$$

We claim that

(i) N is an M-slightly compressible injective module, but

(ii) N is not an M-principally injective module.

Proof. (i) Note that all nonzero submodules of M are

$$\begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R,\ E_k := \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} R \text{ where } k \in \mathbb{Z}_2,$$

$$\begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left\{ \begin{pmatrix} 0 & a & b \\ 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\} \text{ and } M.$$

It is clear that $E' := \begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $E_k := \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} R$ are simple right R-modules for all $k \in \mathbb{Z}_2$. First, we claim that E' and E_k are not M-cyclic submodules of M for all $k \in \mathbb{Z}_2$, that is, every R-homomorphism from M to E' and every R-homomorphism from M to E_k are zero for all $k \in \mathbb{Z}_2$.

Step I : Claim that every R-homomorphism from M to E' is zero.

Let $f : M \to E'$ be an R-homomorphism. Then

$$f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&0&x\\0&0&0\\0&0&0\end{pmatrix}\text{ for some } x\in\mathbb{Z}_2.$$

Since f is an R-homomorphism,

$$\begin{aligned}\begin{pmatrix}0&0&x\\0&0&0\\0&0&0\end{pmatrix}&=f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)=f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)\\&=\begin{pmatrix}0&0&x\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.\end{aligned}$$

Then $x=0$, and $f:M\to E'$ is the zero R-homomorphism. Hence every R-homomorphism from M to E' is zero.

Step II : Claim that every R-homomorphism from M to E_k is zero for all $k\in\mathbb{Z}_2$. Let $k\in\mathbb{Z}_2$ and $f:M\to E_k$ be an R-homomorphism. Then

$$f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&0&kx\\0&0&x\\0&0&0\end{pmatrix}\text{ for some } x\in\mathbb{Z}_2.$$

Since f is an R-homomorphism,

$$\begin{aligned}\begin{pmatrix}0&0&kx\\0&0&x\\0&0&0\end{pmatrix}=f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)&=f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)\\&=f\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\\&=\begin{pmatrix}0&0&kx\\0&0&x\\0&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.\end{aligned}$$

Then $x = 0$ so that $f : M \to E_k$ is the zero R-homomorphism. Hence every R-homomorphism from M to E_k is zero for all $k \in \mathbb{Z}_2$. Since

$$E' \text{ is a submodule of } \begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{and } E_1 = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \mid b \in \mathbb{Z}_2 \right\} \hookrightarrow \left\{ \begin{pmatrix} 0 & a & b \\ 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in \mathbb{Z}_2 \right\},$$

every nonzero submodule of M is not an M-slightly compressible submodule of M. Hence N is an M-slightly compressible injective module.

(ii) First, we show that $N_R = \begin{pmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is an M-cyclic submodule of M.

Define $f : M \to N$ by

$$f\left(\begin{pmatrix} x & y & z \\ 0 & w & u \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & w & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} x & y & z \\ 0 & w & u \\ 0 & 0 & 0 \end{pmatrix} \in M.$$

It is clear that f is an R-homomorphism. Next, we show that f is onto.

Let $\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in N$ where $a, b \in \mathbb{Z}_2$. We choose $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & 0 & 0 \end{pmatrix} \in M$. Then

$$f\left(\begin{pmatrix}0&0&0\\0&a&b\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix}.$$

Then f is an R-epimorphism, i.e., $f(M)=N$ so N is an M-cyclic submodule of M. Next, we claim that there exists a nonzero R-homomorphism α from N to N such that $\bar{\alpha}i_N \neq \alpha$ for all $\bar{\alpha} \in Hom_R(M,N)$. We choose $\alpha = I_N$, the identity map on N, and we show that $\bar{\alpha}i_N \neq I_N$ for all $\bar{\alpha} \in Hom_R(M,N)$. Let $\bar{\alpha} \in Hom_R(M,N)$. Then

$$\bar{\alpha}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix} \text{ for some } a,b\in\mathbb{Z}_2.$$

$$\begin{aligned}\text{Then } \bar{\alpha}\left(\begin{pmatrix}x&y&z\\0&w&u\\0&0&0\end{pmatrix}\right) &= \bar{\alpha}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\begin{pmatrix}x&y&z\\0&w&u\\0&0&0\end{pmatrix}\right)\\ &= \bar{\alpha}\left(\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}\right)\begin{pmatrix}x&y&z\\0&w&u\\0&0&0\end{pmatrix}\\ &= \begin{pmatrix}0&a&b\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x&y&z\\0&w&u\\0&0&0\end{pmatrix}=\begin{pmatrix}0&aw&au\\0&0&0\\0&0&0\end{pmatrix}\end{aligned}$$

for all $\begin{pmatrix}x&y&z\\0&w&u\\0&0&0\end{pmatrix} \in M$ so $\bar{\alpha}\left(\begin{pmatrix}0&y&z\\0&0&0\\0&0&0\end{pmatrix}\right)=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

for all $\begin{pmatrix}0&y&z\\0&0&0\\0&0&0\end{pmatrix} \in N$. Hence $\bar{\alpha}i_N = 0 \neq I_N$ for all $\bar{\alpha} \in Hom_R(M,N)$.

Therefore, N is not an M-principally injective module.

□

In the following example, we can show that an M-principally injective module may not be an M-slightly compressible injective module.

Example 4.3.3 ([15], Example 6.6 **(Clark Example)).** Let D be a discrete valuation ring, that is a commutative integral domain with ideal lattice

$$0 \subset \cdots \subset p^n D \subset \cdots \subset p^2 D \subset pD \subset D.$$

[For example, $D = \mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} \,|\, p \nmid b\}$ is the ring of integers localized at the prime p where p is a prime number or $D = F[x]$ is the set of all polynomials over F where F is a field (we take $p = x$)]. Let U be the group of units of D. Then $p^{n+1}D - p^n D = p^n U$ and the field of quotients is $Q = \{up^k \,|\, k \in \mathbb{Z} \text{ and } u \in U\}$. Define $V_D = Q/D$ and $v_m = p^{-m} + D \in V, m \geq 0$ (so $v_0 = 1 + D = 0$). Then $pv_k = v_{k-1}$ for each $k \geq 1$. Let R be the trivial extension of D by V that is $R = D \oplus V$ where the multiplication is defined by $(d+v)(d'+v') = dd' + (dv' + d'v)$ for all $d + v, d' + v' \in R$. Then $Rv_m = R(0 + v_m) = 0 \oplus Dv_m$ for all $m \geq 0$ and $Rp^n = R(p^n + 0) = Dp^n \oplus V$ for all $n \geq 0$ because $V = p^n V$. Then R is a commutative ring with ideal lattice

$$0 = v_0 R \subset v_1 R \subset v_2 R \subset \cdots \subset V \subset \cdots \subset p^2 R \subset pR \subset R,$$

where p and $v_i, i \geq 0$ satisfy $pv_k = v_{k-1}$ for all $k \geq 1$ and V is the only nonprincipal ideal. But V is not finitely generated because $V = \sum_m v_m R = \cup_m v_m R$. However, R is p-injective; indeed every ideal is an annihilator. In fact one verifies that

$$v_m R = r(p^m R) \text{ and } p^m R = r(v_m R) \text{ for all } m \geq 0 \text{ and } r(V) = V.$$

However, R is not self-injective. Indeed $\gamma : V \to R$ is well-defined by

$$\gamma(0 + v_m d) = 0 + v_{m-1} d$$

because $v_m p = v_{m-1}$. Then γ is an R-homomorphism but γ cannot be extended to an R-homomorphism from R to R. Then R is not self-injective. Since every principal right ideal of R can be considered as a homomorphic image of R and vice versa, R_R is an R_R-principally injective module. By Corollary 4.2.13, R is right self-injective ring if and only if R_R is an R_R-slightly compressible injective module, then R_R is not R_R-slightly compressible injective but is R_R-principally injective.

In fact, M-slightly compressible submodules of M and M-cyclic submodules of M are different where M is a right R-module, that is, M-cyclic submodules of M may not be M-slightly compressible submodules of M, for example, let F be a field, $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}$, $M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}$. From Example 3.2.1, M is not an M-slightly compressible submodule of M but M is an M-cyclic submodule of M because $I_M(M) = M$ where I_M is the identity map on M.

On the other hand, M-slightly compressible submodules of M may not be M-cyclic submodules of M, for example, let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_p \right\} \text{ and } N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

By Theorem 3.2.11, N is an R_R-slightly compressible submodule of R_R.

Since any R_R-cyclic submodule of R_R can be considered as the image of an endomorphism of R_R, we will show that every R-homomorphism from R to N is not onto. Suppose there exists an R-epimorphism α from R to N. Then

$$\alpha\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \qquad \text{for some } x, y \in \mathbb{Z}_p \setminus \{0\}.$$

Let $\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \in N \setminus 0$.

Case I : $a \neq b$.

Since α is onto, there exists $\begin{pmatrix} m & n \\ 0 & q \end{pmatrix} \in R$ such that $\alpha\left(\begin{pmatrix} m & n \\ 0 & q \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$.

Since α is an R-homomorphism,

$$\begin{aligned}\begin{pmatrix}0 & a\\0 & b\end{pmatrix} &= \alpha\left(\begin{pmatrix}m & n\\0 & q\end{pmatrix}\right) = \alpha\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}m & n\\0 & q\end{pmatrix}\right)\\ &= \begin{pmatrix}0 & x\\0 & y\end{pmatrix}\begin{pmatrix}m & n\\0 & q\end{pmatrix} = \begin{pmatrix}0 & qx\\0 & qy\end{pmatrix}.\end{aligned}$$

Then $qx = a \neq b = qy$, $q \neq 0$ and $x \neq y$.

Case II : $a = b$.

Since α is onto, there exists $\begin{pmatrix}m & n\\0 & q\end{pmatrix} \in R$ such that $\alpha\left(\begin{pmatrix}m & n\\0 & q\end{pmatrix}\right) = \begin{pmatrix}0 & a\\0 & b\end{pmatrix}$.

Since α is an R-homomorphism,

$$\begin{aligned}\begin{pmatrix}0 & a\\0 & b\end{pmatrix} &= \alpha\left(\begin{pmatrix}m & n\\0 & q\end{pmatrix}\right) = \alpha\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}m & n\\0 & q\end{pmatrix}\right)\\ &= \begin{pmatrix}0 & x\\0 & y\end{pmatrix}\begin{pmatrix}m & n\\0 & q\end{pmatrix} = \begin{pmatrix}0 & qx\\0 & qy\end{pmatrix}.\end{aligned}$$

Then $qx = a = b = qy$ but $a, b, x, y \neq 0$ so $q \neq 0$ and $x = y$.

From two cases, α is not well-defined, which is a contradiction. Then N is not the image of any endomorphisms of R_R. Hence N is not an R_R-cyclic submodule of R_R.

Moreover, we find a right R-module M which makes M-slightly compressible injective modules and M-principally injective modules be the same.

In 2009, Ghorbani and Vedadi[5] introduced the concept of epi-retractable module. A right R-module M is called ***epi-retractable*** if every submodule of M_R is a homomorphic image of M.

Theorem 4.3.4. *Let M be an epi-retractable right R-module and N a right-R-module. Then N is an M-slightly compressible injective module if and only if N is an M-principally injective module.*

Proof. ($\Rightarrow$) Assume that N is an M-slightly compressible injective module. Let A

be an M-cyclic submodule of M and α an R-homomorphism from A to N. Since submodules of A are also submodules of M, we have that every submodule of A is a homomorphic image of M. By Theorem 3.2.17, A is an M-slightly compressible submodule of M. Thus α can be extended to an R-homomorphism from M to N. Therefore N is an M-principally injective module.

($\Leftarrow$) Assume that N is an M-principally injective module. Let A be an M-slightly compressible submodule of M and $\alpha : A \to N$ an R-homomorphism. By assumption, A is an M-cyclic submodule of M. Then α can be extended to an R-homomorphism from M to N. Therefore N is an M-slightly compressible injective module. $\square$

Example 4.3.5. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$$
$$\text{and } M_R = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

By Example 3.1.4(ii), R_R is an epi-retractable module and by Example 4.1.2(ii), M is an epi-retractable module, then

(i) from Example 4.1.2, N is an R_R-slightly compressible injective module. By Theorem 4.3.4, N is an R_R-principally injective module,

(ii) from Example 4.1.2, M is an M-slightly compressible injective module. By Theorem 4.3.4, M is an M-principally injective module.

Corollary 4.3.6. [9] *If Q_R is injective, then it is divisible, i.e., it is a p-injective module. The converse holds if R is a principal right ideal ring, that is, a right in which all right ideals are principal.*

Recall in [1], a right R-module M is called ***semisimple*** if every submodule of M is a direct summand of M.

Theorem 4.3.7. *Let M be a semisimple right R-module and N a right R-module. Then N is an M-slightly compressible injective module if and only if N is an M-principally injective module.*

Proof. ($\Rightarrow$) Assume that N is an M-slightly compressible injective module. Let A be an M-cyclic submodule of M and α an R-homomorphism from A to N. Since submodules of A are also submodules of M, we have that every submodule of A is a direct summand of M. Then every submodule of A is an M-cyclic submodule of M. By Thorem 3.2.17, A is an M-slightly compressible submodule of M. Thus α can be extended to an R-homomorphism from M to N. Therefore N is an M-principally injective module.
($\Leftarrow$) Assume that N is an M-principally injective module. Let A be an M-slightly compressible submodule of M and α an R-homomorphism from A to N. By assumption, A is a direct summand of M so A is an M-cyclic submodule of M. Then α can be extended to an R-homomorphism from M to N. Therefore N is an M-slightly compressible injective module. $\square$

Example 4.3.8. Let $\mathbb{Z}_3$ be the set of all integers modulo 3,

$$R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z}_3 \right\},$$

$$I_R = \begin{pmatrix} 0 & \mathbb{Z}_3 \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_3 \right\},$$

$$\text{and } M_R = \begin{pmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_3 \right\}.$$

Then R/I and M/I are right R-modules. We claim that R/I is a semisimple right R-module, that is, every nonzero submodule of R/I is a direct summand of R/I.

Proof. All nonzero submodules of R/I are

$$E'/I = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & 0 \end{pmatrix} /I, \ E_k/I = \left(\begin{pmatrix} 0 & k \\ 0 & 1 \end{pmatrix} R \right) /I \ \text{ where } k \in \mathbb{Z}_3,$$

$$M/I = \begin{pmatrix} 0 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix} /I \text{ and } R/I.$$

It is clear that E'/I and E_k/I are simple modules for all $k \in \mathbb{Z}_3$. Next, we claim that E'/I, E_k/I and M/I are direct summands of R/I for all $k \in \mathbb{Z}_3$.

Case I : E'/I.

Define $s' : R/I \to E'/I$ by

$$s'\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + I\right) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + I \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + I \in R/I.$$

It is easy to show that s' is an R-homomorphism. Next, we will show that $s' i_{E'/I} = I_{E'/I}$. Let $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + I \in E'/I$ where $a, b \in \mathbb{Z}_3$. Then

$$s'\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + I\right) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + I = I_{E'/I}\left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + I\right).$$

Thus s' is an R-epimorphism and $s' i_{E'/I} = I_{E'/I}$, by Lemma 2.2.12, E'/I is a direct summand of R/I.

Case II : E_k/I where $k \in \mathbb{Z}_3$.

For each $k \in \mathbb{Z}_3$, define $s_k : R/I \to E_k/I$ by

$$s_k\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + I\right) = \begin{pmatrix} 0 & kc \\ 0 & c \end{pmatrix} + I \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + I \in R/I.$$

It is easy to show that s_k is an R-homomorphism for all $k \in \mathbb{Z}_3$. Next, we will show that $s_k i_{E_k/I} = I_{E_k/I}$. Let $\begin{pmatrix} 0 & ka \\ 0 & a \end{pmatrix} + I \in E_k/I$ where $a \in \mathbb{Z}_3$. Then

$$s_k\left(\begin{pmatrix} 0 & ka \\ 0 & a \end{pmatrix} + I\right) = \begin{pmatrix} 0 & ka \\ 0 & a \end{pmatrix} + I = I_{E_k/I}\left(\begin{pmatrix} 0 & ka \\ 0 & a \end{pmatrix} + I\right).$$

Thus s_k is an R-epimorphism and $s_k i_{E_k/I} = I_{E_k/I}$ for all $k \in \mathbb{Z}_3$, by Lemma 2.2.12, E_k/I is a direct summand of R/I for all $k \in \mathbb{Z}_3$.

Case III : M/I.

Define $s'' : R/I \to M/I$ by

$$s''\left(\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}+I\right)=\begin{pmatrix}0 & a\\0 & 1\end{pmatrix}+I \text{ for some } a\in\mathbb{Z}_3.$$

Then $s''\left(\begin{pmatrix}x & y\\0 & z\end{pmatrix}+I\right)=\begin{pmatrix}0 & az\\0 & z\end{pmatrix}+I$ for all $\begin{pmatrix}x & y\\0 & z\end{pmatrix}+I\in R/I$. It is easy to show that s'' is an R-homomorphism. Next, we will show that $s''i_{M/I}=I_{M/I}$. Let $\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I\in M/I$. Then

$$s''\left(\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I\right)=\begin{pmatrix}0 & ay\\0 & y\end{pmatrix}+I=\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I=I_{M/I}\left(\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I\right)$$

because $\begin{pmatrix}0 & ay\\0 & y\end{pmatrix}-\begin{pmatrix}0 & x\\0 & y\end{pmatrix}\in I$. Thus s'' is an R-epimorphism and $s''i_{M/I}=I_{M/I}$, by Lemma 2.2.12, M/I is a direct summand of R/I. Hence R/I is a semisimple right R-module.

Finally, we claim that M/I is a R/I-principally injective module.

Case I : E'/I.

We claim that every R-homomorphism from E'/I to M/I is zero.

Let s' be an R-homomorphism from E'/I to M/I. Then

$$s'\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}+I\right)=\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I \text{ for some } x,y\in\mathbb{Z}_3.$$

Since s' is an R-homomorphism,

$$\begin{aligned}\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I=s'\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}+I\right)&=s'\left(\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\right)+I\right)\\&=s'\left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}+I\right)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}=\left(\begin{pmatrix}0 & x\\0 & y\end{pmatrix}+I\right)\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\\&=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}+I=I.\end{aligned}$$

Then $y = 0$ and then $s' = 0$. Then every R-homomorphism from E'/I to M/I is zero. Then we are done in this case.

Case II : E_k/I where $k \in \mathbb{Z}_3$.

We claim that every R-homomorphism from E_k/I to M/I can be extended to an R-homomorphism from R/I to M/I.

Let $k \in \mathbb{Z}_3$ and s_k be an R-homomorphism from E_k/I to M/I. Then

$$s_k\left(\begin{pmatrix} 0 & k \\ 0 & 1 \end{pmatrix} + I\right) = \begin{pmatrix} 0 & x_k \\ 0 & y_k \end{pmatrix} + I \text{ for some } x_k, y_k \in \mathbb{Z}_3.$$

Then for $\begin{pmatrix} 0 & bk \\ 0 & k \end{pmatrix} + I \in E_k/I$, we have

$$\begin{aligned} s_k\left(\begin{pmatrix} 0 & bk \\ 0 & k \end{pmatrix} + I\right) &= s_k\left(\left(\begin{pmatrix} 0 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}\right) + I\right) \\ &= s_k\left(\begin{pmatrix} 0 & x_k \\ 0 & y_k \end{pmatrix} + I\right)\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & x_k b \\ 0 & y_k b \end{pmatrix} + I. \end{aligned}$$

Then we choose $\bar{s_k} : R/I \to M/I$ defined by

$$\bar{s_k}\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + I\right) = \begin{pmatrix} 0 & ky_k c \\ 0 & y_k c \end{pmatrix} + I \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + I \in R/I.$$

It is easy to show that $\bar{s_k}$ is an R-homomorphism. Next, we claim that $\bar{s_k} i_{E_k/I} = s_k$.

Let $\begin{pmatrix} 0 & bk \\ 0 & b \end{pmatrix} + I \in E_k/I$ where $b \in \mathbb{Z}_3$. Then

$$\bar{s_k}\left(\begin{pmatrix} 0 & bk \\ 0 & b \end{pmatrix} + I\right) = \begin{pmatrix} 0 & ky_k b \\ 0 & y_k b \end{pmatrix} + I = \begin{pmatrix} 0 & x_k b \\ 0 & y_k b \end{pmatrix} + I = s_k\left(\begin{pmatrix} 0 & bk \\ 0 & b \end{pmatrix} + I\right)$$

because $\begin{pmatrix} 0 & ky_k b \\ 0 & y_k b \end{pmatrix} - \begin{pmatrix} 0 & x_k b \\ 0 & y_k b \end{pmatrix} \in I$. Then every R-homomorphism from E_k/I to M/I can be extended to an R-homomorphism from R/I to M/I.

Case III : M/I.

We claim that every R-homomorphism from M/I to M/I can be extended to an

R-homomorphism from R/I to M/I.

Let $\alpha : M/I \to M/I$ be an R-homomorphism. Define $s'' : R/I \to M/I$ by

$$s''\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + I\right) = \begin{pmatrix}0 & a\\ 0 & 1\end{pmatrix} + I \text{ for some } a \in \mathbb{Z}_3.$$

Then $s''\left(\begin{pmatrix}x & y\\ 0 & z\end{pmatrix} + I\right) = \begin{pmatrix}0 & az\\ 0 & z\end{pmatrix} + I$ for all $\begin{pmatrix}x & y\\ 0 & z\end{pmatrix} + I \in R/I$. It is easy to show that s'' is an R-homomorphism. We choose $\bar{\alpha} = \alpha s''$. Next, we claim that $\bar{\alpha} i_{M/I} = \alpha$. Let $\begin{pmatrix}0 & x\\ 0 & y\end{pmatrix} + I \in M/I$ where $x, y \in \mathbb{Z}_3$. Then

$$\begin{aligned}\bar{\alpha}\left(\begin{pmatrix}0 & x\\ 0 & y\end{pmatrix} + I\right) &= \alpha s''\left(\begin{pmatrix}0 & x\\ 0 & y\end{pmatrix} + I\right) = \alpha\left(s''\left(\begin{pmatrix}0 & x\\ 0 & y\end{pmatrix} + I\right)\right)\\ &= \alpha\left(\begin{pmatrix}0 & ay\\ 0 & y\end{pmatrix} + I\right) = \alpha\left(\begin{pmatrix}0 & x\\ 0 & y\end{pmatrix} + I\right)\end{aligned}$$

because $\begin{pmatrix}0 & ay\\ 0 & y\end{pmatrix} + I = \begin{pmatrix}0 & x\\ 0 & y\end{pmatrix} + I$ and α is an R-homomorphism. Then every R-homomorphism from M/I to M/I can be extended to an R-homomorphism from R/I to M/I. Thus M/I is an R/I-principally injective module and R/I is a semisimple module by Theorem 4.3.7, M/I is also an R/I-slightly compressible injective module. $\square$

CHAPTER V

SUB-M-PRINCIPALLY INJECTIVE MODULES

From Chapter IV, the notion of M-slightly compressible injective modules and M-principally injective modules are different, that is, there exists a right R-module M,

$$M\text{-principally injective module} \not\Rightarrow M\text{-slightly compressible injective module}$$
$$M\text{-principally injective module} \not\Leftarrow M\text{-slightly compressible injective module.}$$

In this chapter, we introduce the notion of sub-M-principally injective modules which implies M-slightly compressible injective modules and M-principally injective modules seen in Proposition 5.3.2 and Proposition 5.3.1, respectively.

Moreover, we study some properties of sub-M-principally injective modules and relationship between sub-M-principally injective modules, M-principally injective modules and M-slightly compressible injective modules and also provide examples of them.

5.1 Definition and Examples

Definition 5.1.1. Let M be a right R-module. A right R-module N is called a ***sub-M-principally injective module*** if for any nonzero submodule A of M, every R-homomorphism from A-cyclic submodule of A to N can be extended to an R-homomorphism from M to N.

Example 5.1.2. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$$

$$\text{and } M_R = \left\{ \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

Then

(i) N is a sub-R_R-principally injective module,

(ii) M is a sub-M-principally injective module.

Proof. (i) All nonzero submodules of R are

$$A := \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\},\ B := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\} \text{ and } R.$$

Then A and B are simple right R-modules so A is the only one A-cyclic submodule of A and B is the only one B-cyclic submodule of B. Next, we show that A and B are R_R-cyclic submodules of R. Define $f_1 : R \to A$ by

$$f_1\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R,$$

and define $f_2 : R \to B$ by

$$f_2\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R.$$

It is easy to check that f_1 and f_2 are R-epimorphisms. Then A and B are R_R-cyclic submodules of R. Similarly in the proof of Example 4.1.2(i), we can conclude that N is a sub-R_R-principally injective module.

(ii) Similarly in the proof of Example 4.1.2(ii), we can conclude that M is a sub-M-principally injective module.

□

5.2 Some Properties of Sub-M-Principally Injective Modules

This section is concerned with sub-M-principally injective modules and the main properties of these modules are derived in this section.

Proposition 5.2.1. *Let M, N and K be right R-modules with $N \cong K$. If N is a sub-M-principally injective module, then K is a sub-M-principally injective module.*

Proof. Assume that N is a sub-M-principally injective module. Let A be a submodule of M, B an A-cyclic submodule of A and α an R-homomorphism from B to K. Since $N \cong K$, there exists an R-isomorphism β from K to N. Then $\beta\alpha$ is an R-homomorphism from B to N. Since N is a sub-M-principally injective module, there exists an R-homomorphism γ from M to N such that $\gamma i_B = \beta\alpha$ where i_B is the inclusion map. We choose $\bar{\alpha} = \beta^{-1}\gamma$, so $\bar{\alpha} i_B = \beta^{-1}\gamma i_B = \beta^{-1}\beta\alpha = \alpha$. Hence K is a sub-M-principally injective module. □

Example 5.2.2. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}$$
$$\text{and } A_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

From Example 5.1.2(i), N is sub-R_R-principally injective. Define $\alpha : N \to A$ by

$$\alpha\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in N.$$

It is clear that α is an R-isomorphism so $N \cong A$. Hence A is also a sub-R_R-principally injective module.

Proposition 5.2.3. *Let M be a right R-module and A a submodule of M. If A is a sub-M-principally injective module, then A is a direct summand of M.*

Proof. Assume that A is a sub-M-principally injective module. Then every R-homomorphism from A-cyclic submodule of A to A can be extended to an R-homomorphism from M to A. Since A is an A-cyclic submodule of A, the identity map I_A on A can be extended to an R-homomorphism $\alpha : M \to A$ such that $\alpha i_A = I_A$. By Lemma 2.2.11(i), the short exact sequence $0 \to A \xrightarrow{i_A} M \xrightarrow{\pi_A} M/A \to 0$ splits where π_A is the canonical projection of M onto M/A and i_A is the inclusion map. Therefore A is a direct summand of M. $\square$

Example 5.2.4. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\} \text{ and } A_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

From Example 5.2.2, A is a sub-R_R-principally injective module. Since A is a submodule of R_R and by Proposition 5.2.3, A is a direct summand of R_R.

On the other hand, the converse of Proposition 5.2.3 is not true in general, for example, in the $\mathbb{Z}$-module $\mathbb{Z}$, we know that $\mathbb{Z}_{\mathbb{Z}}$ is indecomposable so only 0 and $\mathbb{Z}_{\mathbb{Z}}$ are direct summands of $\mathbb{Z}_{\mathbb{Z}}$ but $\mathbb{Z}_{\mathbb{Z}}$ is not sub-$\mathbb{Z}_{\mathbb{Z}}$-principally injective.

Theorem 5.2.5. *Let M and N be right R-modules and $A \overset{\oplus}{\hookrightarrow} N$. If N is a sub-M-principally injective module, then A and N/A are sub-M-principally injective modules.*

Proof. Assume that N is a sub-M-principally injective module.

(i) Claim that A is a sub-M-principally injective module. Let B be a nonzero submodule of M, C a B-cyclic submodule of B and α an R-homomorphism from C to A. Since $A \overset{\oplus}{\hookrightarrow} N$, the short exact sequence

$$0 \to A \xrightarrow{i_A} N \xrightarrow{\pi_A} N/A \to 0$$

splits where i_A is the inclusion map and π_A is the canonical projection of N onto N/A. By Lemma 2.2.11(i), there exists an R-homomorphism $f' : N \to$

A with $f'i_A = I_A$ where I_A is the identity map on A. Since N is a sub-M-principally injective module, there exists an R-homomorphism f from M to N such that $fi_C = i_A\alpha$ where $i_C : C \to M$ is the inclusion map. Let $\bar{\alpha} = f'f$. Then $\bar{\alpha}i_C = f'fi_C = f'i_A\alpha = I_A\alpha = \alpha$. Therefore A is a sub-M-principally injective module.

(ii) Claim that N/A is a sub-M-principally injective module. Since $A \overset{\oplus}{\hookrightarrow} N$, there exists $A' \hookrightarrow N$ such that $N = A \oplus A'$ so $A' \overset{\oplus}{\hookrightarrow} N$ and $A' \cong N/A$. From (i), A' is a sub-M-principally injective module. By Proposition 5.2.1, N/A is a sub-M-principally injective module.

□

Example 5.2.6. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\} \text{ and } A_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

We want to show that A and R/A are sub-R_R-principally injective modules.

Proof. First, we claim that R_R is a sub-R_R-principally injective module. All nonzero submodules of R are

$$A,\ B := \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\} \text{ and } R.$$

Then A and B are simple right R-modules so A is the only one A-cyclic submodule of A and B is the only one B-cyclic submodule of B by the identity map I_B on B. By Example 5.1.2(i), A and B are R_R-cyclic submodules of R_R. Let $f_1 : A \to R$ and $f_2 : B \to R$ be R-homomorphisms. Then

$$f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} \text{ and } f_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}$$

for some $x_1, x_2, y_1, y_2, \in \mathbb{Z}_p$.

Since $f_1,\ f_2$ are R-homomorphisms,

$$\begin{aligned}\begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix} &= f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} x_1 & 0 \\ 0 & y_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y_1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\text{and } \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix} &= f_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = f_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} x_2 & 0 \\ 0 & y_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

Then $x_1 = 0, y_2 = 0$ so that

$$f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & y_1 \end{pmatrix} \text{ and } f_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Define $\bar{f}_1 : R \to R$ by

$$\bar{f}_1\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & y_1 b \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$$

and define $\bar{f}_2 : R \to R$ by

$$\bar{f}_2\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} x_2 a & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R.$$

It is easy to check that $\bar{f}_1$ and $\bar{f}_2$ are R-homomorphisms and

$$\begin{aligned}f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) &= f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right) = f_1\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & y_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y_1 a \end{pmatrix} = \bar{f}_1\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}\right)\end{aligned}$$

$$\begin{aligned}\text{and } f_2\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) &= f_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = f_2\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x_2a & 0 \\ 0 & 0 \end{pmatrix} = \bar{f}_2\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)\end{aligned}$$

for all $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \in A$ and $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in B$ so $\bar{f}_1 i_A = f_1$ and $\bar{f}_2 i_B = f_2$. Hence R_R is a sub-R_R-principally injective module.

From Example 5.2.4, $A \overset{\oplus}{\hookrightarrow} R$, so by Theorem 5.2.5, A and R/A are sub-R_R-principally injective modules. □

Proposition 5.2.7. *Let M and N be right R-modules with N a sub-M-principally injective module. Then*

(i) *N is sub-K-principally injective for all nonzero submodule K of M,*

(ii) *H is sub-K-principally injective for all direct summand H of N and nonzero submodule K of M.*

Proof. (i) Let K be a nonzero submodule of M and A a nonzero submodule of K. Then $A \hookrightarrow M$. Let B be an A-cyclic submodule of A and α an R-homomorphism from B to N. Since N is a sub-M-principally injective module, there exists an R-homomorphism $\bar{\alpha} : M \to N$ such that $\bar{\alpha} i_B = \alpha$ where $i_B : B \to M$ is the inclusion map. Since $B \hookrightarrow K$, $\bar{\alpha}|_K : K \to N$ is an R-homomorphism, $\bar{\alpha}|_K i_B = \alpha$. Therefore N is a sub-K-principally injective module.

(ii) Let K be a nonzero submodule of M and $H \overset{\oplus}{\hookrightarrow} N$. From (i), N is sub-K-principally injective. By Theorem 5.2.5, H is sub-K-principally injective. □

Example 5.2.8. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z}_p \right\},\ N_R = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\},$$

$$A_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{Z}_p \right\} \text{ and } B_R = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.$$

Then

(i) clearly, $A, B \hookrightarrow R_R$. From Example 5.1.2, N is a sub-R_R-principally injective module. By Proposition 5.2.7(i), N is a sub-A-principally injective module and N is a sub-B-principally injective module,

(ii) from Example 5.2.6, R_R is a sub-R_R-principally injective module and from Example 5.2.4, $B \overset{\oplus}{\hookrightarrow} R_R$ so by Proposition 5.2.7(ii), B is a sub-A-principally injective module and B is a sub-B-principally injective module.

The following result is a sufficient condition for a right R-module N is a sub-M-principally injective module.

Theorem 5.2.9. *Let M and N be right R-modules. If N is a sub-M-principally injective module, then for each nonzero submodule A of M and $s \in End_R(A)$, $Hom_R(A, N)s = \{f \in Hom_R(A, N) : f(Ker(s)) = 0\}$.*

Proof. Assume that N is a sub-M-principally injective module. Let A be a nonzero submodule of M and $s \in End_R(A)$. We claim that $Hom_R(A, N)s \subseteq \{f \in Hom_R(A, N) : f(Ker(s)) = 0\}$. Let $fs \in Hom_R(A, N)s$. Then $fs \in Hom_R(A, N)$ and $fs(Ker(s)) = 0$. Hence $fs \in \{f \in Hom_R(A, N) : f(Ker(s)) = 0\}$. Then $Hom_R(A, N)s \subseteq \{f \in Hom_R(A, N) : f(Ker(s)) = 0\}$. Next, we claim that $Hom_R(A, N)s \supseteq \{f \in Hom_R(A, N) : f(Ker(s)) = 0\}$. Let $f \in Hom_R(A, N)$ be such that $f(Ker(s)) = 0$. Then $Ker(s) \subset Ker(f)$. If $f = 0$, we are done so suppose $f \neq 0$. By Theorem 2.2.5(i), there exists a unique nonzero R-homomorphism $h : s(A) \to N$ such that $f = hs$. Since N is a sub-M-principally injective and $s(A)$ is an A-cyclic submodule of A, there exists a nonzero R-homomorphism $\bar{h} : M \to N$ such that $h = \bar{h} i_{s(A)}$ where $i_{s(A)}$ is the inclusion map from $s(A)$ to M. Since $s(A) \hookrightarrow A$, $\bar{h}|_A$ is a nonzero R-homomorphism from A to N and $\bar{h}|_A i_{s(A)} = h$. Hence $f = hs = \bar{h}|_A s$ so $f \in Hom_R(A, N)s$. $\square$

5.3 Relationship between M-Principally, M-Slightly Compressible and Sub-M-Principally Injective Modules

In this section, we study relationship between sub-M-principally injective modules, M-principally injective modules and M-slightly compressible injective modules.

Proposition 5.3.1. *Let M be a right R-module. Every sub-M-principally injective module is an M-principally injective module.*

Proof. Let N be a sub-M-principally injective module. Since M is a nonzero submodule of M and by definition of sub-M-principally injective module, every R-homomorphism from an M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. Hence N is an M-principally injective module. □

Proposition 5.3.2. *Let M be a right R-module. Every sub-M-principally injective module is an M-slightly compressible injective module.*

Proof. Let N be a sub-M-principally injective module. Let A be an M-slightly compressible submodule of M. Then A is an A-cyclic submodule of A, so every R-homomorphism from A to N can be extended to an R-homomorphism from M to N. Hence N is an M-slightly compressible injective module. □

But the converse of Propositions 5.3.1 and 5.3.2 are not true in general shown in the following example.

Example 5.3.3. Let F be a field,

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F \right\}, M_R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in F \right\}, N_R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in F \right\}.$$

Then

(i) N is an M-principally injective module and

(ii) N is an M-slightly compressible injective module, but

(iii) N is not a sub-M-principally injective module.

Proof. (i) $E := \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in F\right\}$ is a simple right R-module and only one nonzero proper submodule of M but from Example 3.2.1, E is not an M-cyclic submodule of M. Then only 0 and M are M-cyclic submodules of M. Hence N is an M-principally injective module.

(ii) Since M has only two nonzero submodules, i.e., E, M and from Example 3.2.1, E, M are not M-slightly compressible submodules of M so only 0 is an M-slightly compressible submodule of M. Then N is an M-slightly compressible injective module.

(iii) We claim there exists an R-homomorphism α from E to N which cannot be extended to any R-homomorphisms from M to N, that is, $\varphi i_E \neq \alpha$ for all $\varphi \in Hom_R(M, N)$. Define $\alpha : E \to N$ by

$$\alpha\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \text{ for all } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in E.$$

It is easy to show that α is a nonzero R-isomorphism. Let φ be an R-homomorphism from M to N. Then

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \text{ for some } x \in F, \text{ so}$$

$$\varphi\left(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in E$. Hence $\varphi i_E = 0 \neq \alpha$ for all $\varphi \in Hom_R(M, N)$. Therefore N is not a sub-M-principally injective module.

□

Next, we characterize relationship between sub-M-principally injective modules, M-principally injective modules and M-slightly compressible injective modules.

Clearly, every X-cyclic submodule of X is an M-cyclic submodule of M for every M-cyclic submodule X of M. Thus we have the following result.

Proposition 5.3.4. *Let M be an epi-retractable right R-module and N a right R-module. Then N is an M-principally injective module if and only if N is a sub-M-principally injective module.*

Proof. ($\Leftarrow$) By Proposition 5.3.1.
($\Rightarrow$) Assume that N is an M-principally injective module. Let A be a nonzero submodule of M, B an A-cyclic submodule of A and α an R-homomorphism from B to N. By assumption, A is an M-cyclic submodule of M so B is an M-cyclic submodule of M. Then α can be extended to an R-homomorphism from M to N. Therefore N is a sub-M-principally injective module. □

Proposition 5.3.5. *Let M and N be right R-modules. If N is a sub-M-principally injective module, then N is an A-principally injective module for all nonzero submodule A of M.*

Proof. Assume that N is a sub-M-principally injective module. Let A be a nonzero submodule of M. Claim that N is an A-principally injective module. Let B be an A-cyclic submodule of A and α an R-homomorphism from B to N. Since N is a sub-M-principally injective module, there exists an R-homomorphism $\bar{\alpha}$ from M to N such that $\bar{\alpha} i_B = \alpha$ where i_B is the inclusion map. Since $B \hookrightarrow A$, $\bar{\alpha}|_A : A \to N$ is an R-homomorphism and $\bar{\alpha}|_A i_B = \alpha$. Hence N is an A-principally injective module. Therefore N is an A-principally injective module for all nonzero submodule A of M. □

Corollary 5.3.6. *Let M be an epi-retractable right R-module and N a right R-module. Then N is an M-slightly compressible injective module if and only if N is a sub-M-principally injective module.*

Corollary 5.3.7. *Let N be a right R-module. Then N is an R_R-slightly compressible injective module if and only if N is a sub-R_R-principally injective module.*

Proof. ($\Leftarrow$) By Proposition 5.3.2.

($\Rightarrow$) Assume that N is an R_R-slightly compressible injective module. By Theorem 4.2.11, N is an injective right R-module. Hence N is a sub-R_R-principally injective module. $\square$

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