# สาทิสสัณฐานของไฮเพอร์ริงบางชนิด

นางมณีนาถ แก้วเนียม

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# HOMOMORPHISMS OF SOME HYPERRINGS

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มณีนาถ แก้วเนียม : สาทิสสัณฐานของไฮเพอร์ริงบางชนิด (HOMOMORPHISMS OF SOME HYPERRINGS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์, 48 หน้า.

*สาทิสสัณฐาน*ของกึ่งไฮเพอร์กรุป $(H,\circ)$ คือ ฟังก์ชัน $f: H \to H$ ซึ่ง $f(x \circ y) \subseteq f(x) \circ f(y)$ สำหรับทุก $x, y \in H$ *สาทิสสัณฐาน*ของไฮเพอร์ริง $(A, \oplus, \circ)$ คือฟังก์ชัน $f: A \to A$ ซึ่ง $f$ เป็นสาทิสสัณฐานของทั้ง $(A, \oplus)$ และ $(A, \circ)$ เราให้สัญลักษณ์ Hom $(A, \oplus, \circ)$ แทนเซตของสาทิสสัณฐานทั้งหมดของ $(A, \oplus, \circ)$ ไปยังตัวเอง

ถ้า $(R, +, \cdot)$ เป็นริง และ $I$ เป็นไอดีลของ $R$ เราให้ $(R, +, \circ_I)$ แทนไฮเพอร์ริงการคูณ โดยที่ $x \circ_I y = xy + I$ สำหรับทุก $x, y \in R$ วัตถุประสงค์แรกคือการให้ลักษณะเฉพาะว่าเมื่อใดที่ Hom $(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ = Hom $(\mathbb{Z}, +)$ และ Hom $(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$ = Hom $(\mathbb{Z}_n, +)$ เป็นจริง เราแสดงด้วยว่า Hom $(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ เป็นเซตอนันต์ เมื่อ $m > 0$, $|$Hom $(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| \geq \frac{2n}{(m,n)}$ เมื่อ $(m, n) > 1$ และการเท่ากันเป็นจริง ถ้า $(m, n)$ เป็นเลขยกกำลังที่มีฐานเป็นจำนวนเฉพาะ

เราพิจารณาคราสเนอร์ไฮเพอร์ริง $(G^0, \oplus_1, \cdot)$ และ $(G^0, \oplus_2, \cdot)$ ที่นิยามจากกรุป $(G, \cdot)$ โดย $G^0 = G \,\dot{\cup}\, \{0\}$, $0 \oplus_1 0 = \{0\}$, $x \oplus_1 0 = 0 \oplus_1 x = \{x\}$, $x \oplus_1 x = G^0 \setminus \{x\}$ สำหรับทุก $x \in G$, $x \oplus_1 y = \{x, y\}$ สำหรับทุก $x, y \in G$ ที่แตกต่างกัน $0 \oplus_2 0 = \{0\}$, $x \oplus_2 0 = 0 \oplus_2 x = \{x\}$, $x \oplus_2 x = \{x, 0\}$ สำหรับทุก $x \in G$, $x \oplus_2 y = G \setminus \{x, y\}$ สำหรับทุก $x, y \in G$ ที่แตกต่างกัน และ $0 \cdot x = x \cdot 0 = 0$ สำหรับทุก $x \in G^0$ เราต้องกำหนดว่า $|G| > 3$ สำหรับคราสเนอร์ไฮเพอร์ริง $(G^0, \oplus_2, \cdot)$ วัตถุประสงค์ที่สองคือการให้ลักษณะเฉพาะของสมาชิกของ Hom $(G^0, \oplus_1, \cdot)$ และ Hom $(G^0, \oplus_2, \cdot)$ เราพิจารณาคราสเนอร์ไฮเพอร์ริง $(R/\rho, \oplus, *)$ โดยที่ $(R, +, \cdot)$ เป็นริงสลับที่, $x \rho y$ ก็ต่อเมื่อ $x = y$ หรือ $x = -y$, $x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\}$ และ $x\rho * y\rho = (xy)\rho$ สำหรับทุก $x, y \in R$ เราให้ลักษณะเฉพาะของสมาชิกของ Hom $(\mathbb{Z}/\rho, \oplus, *)$ และของ $f \in$ Hom $(\mathbb{Z}_n/\rho, \oplus, *)$ ซึ่ง $f(\bar{0}\rho) = \bar{0}\rho$ และ $f(\bar{1}\rho) = \bar{1}\rho$ ยิ่งไปกว่านั้น เราให้ลักษณะเฉพาะของสมาชิกของ Hom $([0,\infty), \oplus, \cdot)$ โดยที่ $x \oplus x = [0, x]$ สำหรับทุก $x \in [0,\infty)$ และ $x \oplus y = \{$ค่าสูงสุดของ $x$ และ $y\}$ สำหรับทุก $x, y \in [0,\infty)$ ที่แตกต่างกัน

ให้ $(R, \oplus_{P_1}, \circ_{P_2})$ เป็น P-ไฮเพอร์ริง ของริง $(R, +, \cdot)$ ที่เกิดจากเซตย่อย $P_1$, $P_2$ ของ $R$ ที่ไม่เป็นเซตว่าง วัตถุประสงค์ที่สามคือ หา Hom $(\mathbb{Z}, +) \cap$ Hom $(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}})$ และบอกว่าเมื่อใด Hom $(\mathbb{Z}_n, +)$ จึงจะเป็นเซตย่อยของ Hom $(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ เราแสดงด้วยว่าเซต Hom $(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}}) \setminus$ Hom $(\mathbb{Z}, +)$ และ Hom $(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n}) \setminus$ Hom $(\mathbb{Z}_n, +)$ ไม่เป็นเซตว่างสำหรับบางค่าของ $l, m$

ภาควิชา...คณิตศาสตร์... ลายมือชื่อนิสิต ......................................................

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MANEENAT KAEWNEAM : HOMOMORPHISMS OF SOME HYPERRINGS. ADVISOR : PROF. YUPAPORN KEMPRASIT, Ph.D., 48 pp.



A *homomorphism* of a semihypergroup $(H,\circ)$ is a function $f: H \to H$ such that $f(x \circ y) \subseteq f(x) \circ f(y)$ for all $x, y \in H$. A *homomorphism* of a hyperring $(A, \oplus, \circ)$ is a function $f: A \to A$ such that $f$ is a homomorphism of both $(A, \oplus)$ and $(A, \circ)$. Denote by Hom $(A, \oplus, \circ)$ the set of all homomorphisms of $(A, \oplus, \circ)$ into itself.

If $(R, +, \cdot)$ is a ring and $I$ is an ideal of $R$, we write $(R, +, \circ_I)$ for the multiplicative hyperring where $x \circ_I y = xy + I$ for all $x, y \in R$. The first purpose is to characterize when Hom $(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ = Hom $(\mathbb{Z}, +)$ and Hom $(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$ = Hom $(\mathbb{Z}_n, +)$ hold. We also show that Hom $(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ is infinite when $m > 0$, $|$Hom $(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| \geq \frac{2n}{(m,n)}$ when $(m, n) > 1$ and the equality holds if $(m, n)$ is a prime power.

We consider the two Krasner hyperrings $(G^0, \oplus_1, \cdot)$ and $(G^0, \oplus_2, \cdot)$ defined from a group $(G, \cdot)$ by $G^0 = G \dot{\cup} \{0\}$, $0 \oplus_1 0 = \{0\}$, $x \oplus_1 0 = 0 \oplus_1 x = \{x\}$, $x \oplus_1 x = G^0 \setminus \{x\}$ for all $x \in G$, $x \oplus_1 y = \{x, y\}$ for all distinct $x, y \in G$, $0 \oplus_2 0 = \{0\}$, $x \oplus_2 0 = 0 \oplus_2 x = \{x\}$, $x \oplus_2 x = \{x, 0\}$ for all $x \in G$, $x \oplus_2 y = G \setminus \{x, y\}$ for all distinct $x, y \in G$ and $0 \cdot x = x \cdot 0 = 0$ for all $x \in G^0$. For the Krasner hyperring $(G^0, \oplus_2, \cdot)$, the condition that $|G| > 3$ must be assumed. The second purpose is to characterize the elements of Hom $(G^0, \oplus_1, \cdot)$ and Hom $(G^0, \oplus_2, \cdot)$. The Krasner hyperring $(R/\rho, \oplus, *)$ is considered where $(R, +, \cdot)$ is a commutative ring, $x \rho y \Leftrightarrow x = y$ or $x = -y$, $x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\}$ and $x\rho * y\rho = (xy)\rho$ for all $x, y \in R$. We characterize the elements of Hom $(\mathbb{Z}/\rho, \oplus, *)$ and $f \in$ Hom $(\mathbb{Z}_n/\rho, \oplus, *)$ with $f(\overline{0}\rho) = \overline{0}\rho$ and $f(\overline{1}\rho) = \overline{1}\rho$. Moreover, the elements of Hom $([0,\infty), \oplus, \cdot)$ are characterized where $x \oplus x = [0, x]$ for all $x \in [0,\infty)$ and $x \oplus y = \{\max\{x, y\}\}$ for all distinct $x, y \in [0,\infty)$.

Let $(R, \oplus_{P_1}, \circ_{P_2})$ be the P-hyperring of a ring $(R, +, \cdot)$ induced by nonempty subsets $P_1, P_2$ of $R$. The third purpose is to find Hom $(\mathbb{Z}, +) \cap$ Hom $(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}})$ and determine when Hom $(\mathbb{Z}_n, +)$ is contained in Hom $(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$. The sets Hom $(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}}) \setminus$ Hom $(\mathbb{Z}, +)$ and Hom $(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n}) \setminus$ Hom $(\mathbb{Z}_n, +)$ are also shown to be nonempty for certain $l$, $m$.


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# INTRODUCTION

The concept of homomorphism has been introduced and studied in every algebraic structure. We know that the concept of ring plays a crucial role in algebra. There are many kinds of hyperrings defined in the area of algebraic hyperstructures. However, all of them are nice generalizations of rings. Hyperring homomorphisms are defined naturally and generalize ring homomorphisms. Hyperrings of our interest are multiplicative hyperrings ([1], p.177), Krasner hyperrings ([1], p.167) and P-hyperrings ([1], p.179). M. Krasner introduced Krasner hyperrings in 1966 at a conference. They may be called a simple hyperring. In 1982, R. Rota [9] initiated the study of multiplicative hyperrings. V-S-hyperrings were studied by T. Vougiouklis, L. Konguetsof and S. Spartalis ([1], p.179). By the definitions, V-S-hyperrings are generalizations of both multiplicative hyperrings and Krasner hyperrings. P-hyperrings are V-S-hyperrings of a special type. Note that the addition of a Krasner hyperring and the multiplication of a multiplicative hyperring are hyperoperations while both the addition and the multiplication of a V-S-hyperring are hyperoperations.

We denote by $\text{Hom}(A, \oplus, \circ)$ the set of all homomorphisms of a V-S-hyperring $(A, \oplus, \circ)$ into itself.

D.M. Olson and V.K. Ward [6] gave a nice result concerning when a strongly distributive multiplicative hyperring becomes a ring as follows: A strongly distributive multiplicative hyperring $(A, +, \circ)$ is a ring if and only if there exist $a, b$ in $A$ such that $a \circ b$ contains exactly one element. If $(R, +, \cdot)$ is a ring and $I$ is an ideal of $R$, let $(R, +, \circ_I)$ be the multiplicative hyperring where $x \circ_I y = xy + I$ ([1], p.177). A necessary and sufficient condition for the multiplicative hyperring $(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$ to be regular was given in [8]. In [5], the authors characterized the elements of $\text{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ and $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$ where $m$ is a prime number. The cardinalities of these two sets were also given. Some results on homomorphisms of some other multiplicative hyperrings were studied in [7]. In Chapter II, we characterize when $\text{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) = \text{Hom}(\mathbb{Z}, +)$ and $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) =$

$\text{Hom}(\mathbb{Z}_n, +)$ hold. In addition, we also show that $\text{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ is infinite when $m > 0$, $|\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| \geq \frac{2n}{(m,n)}$ when $(m, n) > 1$ and the equality holds if $(m, n)$ is a prime power.

Semigroups admitting ring structure have long been studied. Since the multiplicative structure of a Krasner hyperring is a semigroup, it is reasonable to study semigroups admitting Krasner hyperring structure. In [4], the author characterized multiplicative interval semigroups on $\mathbb{R}$ which admit a Krasner hyperring structure. We also know that every group admits a Krasner hyperring structure. Chapter III deals with homomorphisms of some Krasner hyperrings. We characterize the elements of $\text{Hom}(G^0, \oplus_1, \cdot)$ and $\text{Hom}(G^0, \oplus_2, \cdot)$ where $(G^0, \oplus_1, \cdot)$ and $(G^0, \oplus_2, \cdot)$ are the Krasner hyperrings defined from a group $(G, \cdot)$ by $G^0 = G \cup \{0\}$, $0 \oplus_1 0 = \{0\}$, $x \oplus_1 0 = 0 \oplus_1 x = \{x\}$, $x \oplus_1 x = G^0 \smallsetminus \{x\}$ for all $x \in G$, $x \oplus_1 y = \{x, y\}$ for all distinct $x, y \in G$, $0 \oplus_2 0 = \{0\}$, $x \oplus_2 0 = 0 \oplus_2 x = \{x\}$, $x \oplus_2 x = \{x, 0\}$ for all $x \in G$, $x \oplus_2 y = G \smallsetminus \{x, y\}$ for all distinct $x, y \in G$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in G^0$ ([1], p.170 and [3], p.76). For the Krasner hyperring $(G^0, \oplus_2, \cdot)$, the condition that $|G| > 3$ must be assumed. The Krasner hyperring $(R/\rho, \oplus, *)$ is defined from a commutative ring $(R, +, \cdot)$ as follows: $x\rho y \iff x = y$ or $x = -y$, $x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\}$ and $x\rho * y\rho = (xy)\rho$ for all $x, y \in R$ ([3], p.75). We characterize the elements of $\text{Hom}(\mathbb{Z}/\rho, \oplus, *)$ and $f \in \text{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ with $f(\bar{0}\rho) = \bar{0}\rho$ and $f(\bar{1}\rho) = \bar{1}\rho$. The Krasner hyperring $([0, \infty), \oplus, \cdot)$ defined in [4] is also considered in this chapter, i.e., $x \oplus x = [0, x]$ for all $x \in [0, \infty)$ and $x \oplus y = \{max\{x, y\}\}$ for all distinct $x, y \in [0, \infty)$. We give necessary and sufficient conditions for $f : [0, \infty) \to [0, \infty)$ to be an element of $\text{Hom}([0, \infty), \oplus, \cdot)$ and show that $\text{Hom}([0, \infty), \oplus, \cdot)$ must be an uncountable set.

In the last chapter, we study homomorphisms of some P-hyperrings. Let $(R, \oplus_{P_1}, \circ_{P_2})$ denote the P-hyperring defined from a ring $(R, +, \cdot)$ and nonempty subsets $P_1, P_2$ of $R$, i.e., $P_1P_2R \cup RP_2P_1 \subseteq P_1$, $x \oplus_{P_1} y = x + y + P_1$ and $x \circ_{P_2} y = xP_2y$ for all $x, y \in R$ ([1], p.179). For integers $l$ and $m$, the set $\text{Hom}(\mathbb{Z}, +) \cap \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}})$ is investigated. We also determine when $\text{Hom}(\mathbb{Z}_n, +)$ is a subset of $\text{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$. We also show that the sets $\text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}}) \smallsetminus \text{Hom}(\mathbb{Z}, +)$

and $\mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n}) \setminus \mathrm{Hom}(\mathbb{Z}_n, +)$ are nonempty for certain $l, m$.

The definitions and quoted results used in this research are provided in Chapter I.

# CHAPTER I

# PRELIMINARIES

The cardinality of a set $X$ is denoted by $|X|$.

The set of all integers, the set of all rational numbers and the set of all real numbers are denoted by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$, respectively. Let $\mathbb{Z}^+ = \{x \in \mathbb{Z} \mid x > 0\}$. For $x, y \in \mathbb{Z}$ and $x \neq 0$, $x \,|\, y$ stands for "$x$ divides $y$". Recall that a positive integer $n$ is said to be *square-free* if there is no integer $a > 1$ such that $a^2 \,|\, n$. Then $n$ is square-free if and only if either $n = 1$ or $n$ is a product of distinct primes. For a positive integer $n$, let $\mathbb{Z}_n$ be the set of integers modulo $n$. The equivalence class of $x \in \mathbb{Z}$ modulo $n$ is denoted by $\overline{x}$. Then

$$\mathbb{Z}_n = \{\overline{x} \mid x \in \mathbb{Z}\} = \{\overline{0}, \overline{1}, ..., \overline{n-1}\}, \;\; |\mathbb{Z}_n| = n$$

and $(\mathbb{Z}_n, +, \cdot)$ is a ring where $\overline{x} + \overline{y} = \overline{x+y}$ and $\overline{x} \cdot \overline{y} = \overline{xy}$ for all $x, y \in \mathbb{Z}$. For $a \in \mathbb{Z}$, define $g_a : \mathbb{Z} \to \mathbb{Z}$ and $h_{\overline{a}} : \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$g_a(x) = ax \;\; \text{and} \;\; h_{\overline{a}}(\overline{x}) = \overline{ax} \;\; \text{for all } x \in \mathbb{Z}.$$

If $G$ is a group, let $\text{Hom}(G)$ denote the set of all homomorphisms $f : G \to G$. Then

$$\text{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\} \text{ and } \text{Hom}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}.$$

Since $g_a \neq g_b$ if $a \neq b$ and $h_{\overline{a}} \neq h_{\overline{b}}$ if $\overline{a} \neq \overline{b}$, it follows that $|\text{Hom }(\mathbb{Z}, +)| = \aleph_0$ and $|\text{Hom}(\mathbb{Z}_n, +)| = n$. For $a, b \in \mathbb{Z}$, not both 0, let $(a, b)$ be the g.c.d. of $a$ and $b$. It is clearly seen that if $n$ is square-free, then $(a, n) = (a^k, n)$ for all $a, k \in \mathbb{Z}$ with $k > 0$.

We know that for $I \subseteq \mathbb{Z}, I$ is an ideal of the ring $(\mathbb{Z}, +, \cdot)$ if and only if $I = m\mathbb{Z}$ for some $m \in \mathbb{Z}$. Since $x \mapsto \overline{x}$ is an epimorphism from the ring $(\mathbb{Z}, +, \cdot)$ onto the ring $(\mathbb{Z}_n, +, \cdot)$, it follows that for $J \subseteq \mathbb{Z}_n, J$ is an ideal of the ring $(\mathbb{Z}_n, +, \cdot)$ if and only if $J = m\mathbb{Z}_n$ for some $m \in \mathbb{Z}$ where $m\mathbb{Z}_n = \{m\overline{x} \mid x \in \mathbb{Z}\} = \{\overline{mx} \mid x \in \mathbb{Z}\}$. Notice that $m\mathbb{Z} = (-m)\mathbb{Z}$ and $m\mathbb{Z}_n = (-m)\mathbb{Z}_n = \overline{m}\mathbb{Z}_n = \mathbb{Z}_n\overline{m} = \mathbb{Z}\overline{m}$. We have that

$$m\mathbb{Z}_n = \overline{m}\mathbb{Z}_n = \mathbb{Z}\overline{m},$$

$$m\mathbb{Z}_n = (m,n)\mathbb{Z}_n = \left\{ \overline{0}, \overline{(m,n)}, ..., \left(\tfrac{n}{(m,n)} - 1\right)\overline{(m,n)} \right\}, \;\; |m\mathbb{Z}_n| = \tfrac{n}{(m,n)},$$

$$\mathbb{Z} = \bigcup_{i=0}^{m-1} (i + m\mathbb{Z}) \;\text{ if } m > 0 \;\text{ and }\; \mathbb{Z}_n = \bigcup_{i=0}^{(m,n)-1} (\overline{i} + (m,n)\mathbb{Z}_n)$$

which are disjoint unions. We shall verify that the last statement holds. Since $(m,n)\mathbb{Z}_n$ is a subgroup of the group $(\mathbb{Z}_n, +)$ and $\frac{|\mathbb{Z}_n|}{|(m,n)\mathbb{Z}_n|} = \frac{n}{\frac{n}{(m,n)}} = (m,n)$, it follows that the index of $(m,n)\mathbb{Z}_n$ in the group $(\mathbb{Z}_n, +)$ is $(m,n)$. Next, let $i, j \in \{0, 1, 2, ..., (m,n) - 1\}$ be such that $\overline{i} + (m,n)\mathbb{Z}_n = \overline{j} + (m,n)\mathbb{Z}_n$. Then $\overline{i} - \overline{j} = (m,n)\overline{s}$ for some $s \in \mathbb{Z}$. Thus $i - j - (m,n)s = nt$ for some $t \in \mathbb{Z}$, so $i - j = (m,n)s + nt$. Since $(m,n) \mid ((m,n)s + nt)$, we have that $(m,n) \mid (i - j)$. It follows that $i - j = 0$, so $i = j$. Hence the desired result follows.

A *hyperoperation* on a nonempty set $H$ is a function $\circ : H \times H \to \mathcal{P}(H) \smallsetminus \{\varnothing\}$ where $\mathcal{P}(H)$ is the power set of $H$. The value of $(x, y) \in H \times H$ under the hyperoperation $\circ$ is denoted by $x \circ y$. The system $(H, \circ)$ is called a *hypergroupoid*. For nonempty subsets $A, B$ of $H$ and an element $x$ of $H$, let

$$A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b, \; A \circ x = A \circ \{x\} \text{ and } x \circ A = \{x\} \circ A.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$x \circ (y \circ z) = (x \circ y) \circ z \quad \text{for all} \;\; x, y, z \in H.$$

A semihypergroup $(H, \circ)$ is called a *hypergroup* if

$$H \circ x = x \circ H = H \quad \text{for all} \;\; x \in H.$$

Then semihypergroups and hypergroups generalize semigroups and groups, respectively.

A *multiplicative hyperring* is a system $(A, +, \circ)$ such that

(1) $(A, +)$ is an abelian group,

(2) $(A, \circ)$ is a semihypergroup,

(3) for all $x, y, z \in A, x \circ (y + z) \subseteq x \circ y + x \circ z$ and $(y + z) \circ x \subseteq y \circ x + z \circ x$,

(4) for all $x, y \in A,\ x \circ (-y) = (-x) \circ y = -(x \circ y)$.

If in the condition (3), the equalities are valid, then the multiplicative hyperring $(A, +, \circ)$ is called *strongly distributive*.

**Example 1.1.** ([1], p.177) Let $(R, +, \cdot)$ be a ring, $I$ an ideal of $R$ and $\circ_I$ the hyperoperation defined on $R$ by

$$x \circ_I y = xy + I \quad \text{for all } x, y \in R.$$

Then $(R, +, \circ_I)$ is a strongly distributive multiplicative hyperring.

**Example 1.2.** ([7]) Let $(R, +, \cdot)$ be a ring and $\varnothing \neq P \subseteq R$. Define

$$x *_P y = xPy \quad \text{for all } x, y \in R.$$

Then $(R, +, *_P)$ is a multiplicative hyperring which is not necessarily strongly distributive.

A *Krasner hyperring* is a system $(A, \oplus, \cdot)$ where

(1) $(A, \oplus)$ is a hypergroup such that

(1.1) $x \oplus y = y \oplus x$ for all $x, y \in A$,

(1.2) there is an element $0 \in A$ such that $x \oplus 0 = \{x\}$ for all $x \in A$,

(1.3) for every element $x \in A$, there exists a unique element $-x \in A$ such that $0 \in x \oplus (-x)$,

(1.4) for $x, y, z \in A,\ x \in y \oplus z \Rightarrow y \in x \oplus (-z)$,

(2) $(A, \cdot)$ is a semigroup having 0 in (1.2) as its zero,

(3) for all $x, y, z \in A,\quad x \cdot (y \oplus z) = x \cdot y \oplus x \cdot z$ and $(y \oplus z) \cdot x = y \cdot x \oplus z \cdot x$.

The element 0 of $A$ may be called the *zero* of the Krasner hyperring $(A, \oplus, \cdot)$.

**Example 1.3.** ([4]) Define the hyperoperation $\oplus$ on $[0, \infty)$ by

$$x \oplus y = \begin{cases} [0, x] & \text{if } x = y, \\ \{\max\{x, y\}\} & \text{if } x \neq y. \end{cases}$$

Then $([0,\infty),\oplus,\cdot)$ is Krasner hyperring.

From Example 1.3, we have that the multiplicative semigroup $[0,\infty)$ admits a Krasner hyperring structure.

**Example 1.4.** ([1], p.170 and [3], p.76) Let $(G,\cdot)$ be a group and $G^0 = G \cup \{0\}$ where 0 is a symbol not representing any element of $G$.

(1) Let the hyperoperation $\oplus_1$ be defined on $G^0$ by

$$x \oplus_1 0 = 0 \oplus_1 x \ = \ \{x\} \qquad \text{for all } \ x \in G^0,$$

$$x \oplus_1 y = \begin{cases} \{x,y\} & \text{if } \ x,y \in G \ \text{ and } \ x \neq y, \\ G^0 \smallsetminus \{x\} & \text{if } \ x,y \in G \ \text{ and } \ x = y. \end{cases}$$

Define

$$x \cdot 0 = 0 \cdot x = 0 \quad \text{for all } x \in G^0.$$

Then $(G^0,\oplus_1,\cdot)$ is a Krasner hyperring.

(2) Assume that $|G| > 3$. Define the hyperoperation $\oplus_2$ on $G^0$ as follows:

$$x \oplus_2 0 = 0 \oplus_2 x = \{x\} \ \text{ for all } x \in G^0,$$

$$x \oplus_2 x = \{x, 0\} \quad \text{for all } x \in G,$$

$$x \oplus_2 y = G \smallsetminus \{x,y\} \ \text{ for all distinct } x,y \in G.$$

Define

$$x \cdot 0 = 0 \cdot x = 0 \quad \text{for all } x \in G^0.$$

Then $(G^0,\oplus_2,\cdot)$ is a Krasner hyperring.

We can see from Example 1.4 that every group admits a Krasner hyperring structure.

**Example 1.5.** ([3], p.75) Let $R$ be a commutative ring and $\rho$ the equivalence relation on $R$ defined by

$$x\rho y \iff x = y \text{ or } x = -y.$$

Then $x\rho = \{x, -x\}$ for all $x \in R$. Define the hyperoperation $\oplus$ and the operation $*$ on $R/\rho$ by

$$x\rho \oplus y\rho = \{(x+y)\rho, (x-y)\rho\},$$
$$x\rho * y\rho = (xy)\rho \text{ for all } x, y \in R.$$

It follows that $(R/\rho, \oplus, *)$ is a Krasner hyperring.

A *V-S-hyperring* is a triple $(A, \oplus, \circ)$ where

(1) $(A, \oplus)$ is a hypergroup,

(2) $(A, \circ)$ is a semihypergroup,

(3) for all $x, y, z \in A$, $x \circ (y \oplus z) \subseteq x \circ y \oplus x \circ z$ and $(y \oplus z) \circ x \subseteq y \circ x \oplus z \circ x$.

Notice that multiplicative hyperrings and Krasner hyperrings are also V-S-hyperrings.

**Example 1.6.** ([1], p.179) Let $P_1$ and $P_2$ be nonempty subsets of a ring $R$ such that $RP_2P_1 \subseteq P_1$ and $P_1P_2R \subseteq P_1$. Define the hyperoperations $\oplus_{P_1}$ and $\circ_{P_2}$ on R by

$$x \oplus_{P_1} y = x + y + P_1 \text{ and } x \circ_{P_2} y = xP_2y \text{ for all } x, y \in R.$$

Then $(R, \oplus_{P_1}, \circ_{P_2})$ is a V-S-hyperring.

Notice that Example 1.2 is a special case of Example 1.6 with $P_1 = \{0\}$ and $P_2 = P$.

The V-S-hyperring defined in Example 1.6 is called a *P-hyperring*. Hence if $l, m \in \mathbb{Z}$, then $(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}})$ and $(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ are P-hyperrings defined from the rings $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_n, +, \cdot)$, respectively.

A *homomorphism* of a semihypergroup $(H, \circ)$ is a function $f : H \to H$ such that

$$f(x \circ y) \subseteq f(x) \circ f(y) \text{ for all } x, y \in H.$$

([1], p.12). Denote by $\mathrm{Hom}(H, \circ)$ the set of all homomorphisms of $(H, \circ)$. By a *homomorphism* of a V-S-hyperring $(A, \oplus, \circ)$ we mean a function $f : A \to A$ such that $f$ is a homomorphism of both the hypergroup $(A, \oplus)$ and the semihypergroup

$(A, \circ)$. The set of all homomorphisms of the hyperring $(A, \oplus, \circ)$ is denoted by $\text{Hom}(A, \oplus, \circ)$. Notice that $\text{Hom}(A, \oplus, \circ) = \text{Hom}(A, \oplus) \cap \text{Hom}(A, \circ)$, i.e.,

$$\text{Hom}(A, \oplus, \circ) = \{f : A \to A \mid f(x \oplus y) \subseteq f(x) \oplus f(y) \text{ and } f(x \circ y) \subseteq f(x) \circ f(y) \text{ for all } x, y \in A\}.$$

In particular, for a multiplicative hyperring $(A, +, \circ)$,

$$\text{Hom}(A, +, \circ) = \{f : A \to A \mid f(x + y) = f(x) + f(y) \text{ and } f(x \circ y) \subseteq f(x) \circ f(y) \text{ for all } x, y \in A\}$$

and for a Krasner hyperring $(A, \oplus, \cdot)$,

$$\text{Hom}(A, \oplus, \cdot) = \{f : A \to A \mid f(x \oplus y) \subseteq f(x) \oplus f(y) \text{ and } f(x \cdot y) = f(x) \cdot f(y) \text{ for all } x, y \in A\}.$$

If $G$ is a group, then $\text{Hom}(G)$ is clearly a semigroup under composition. In fact, $\text{Hom}(A, \oplus, \circ)$ is also a semigroup under composition. The identity mapping on $A$ is clearly an element of $\text{Hom}(A, \oplus, \circ)$. For $f, g \in \text{Hom}(A, \oplus, \circ)$ and $x, y \in A$, we have that

$$\begin{aligned}(gf)(x \oplus y) = g(f(x \oplus y)) &\subseteq g(f(x) \oplus f(y)) \\ &\subseteq g(f(x)) \oplus g(f(y)) \\ &= (gf)(x) \oplus (gf)(y)\end{aligned}$$

and

$$\begin{aligned}(gf)(x \circ y) = g(f(x \circ y)) &\subseteq g(f(x) \circ f(y)) \\ &\subseteq g(f(x)) \circ g(f(y)) \\ &= (gf)(x) \circ (gf)(y),\end{aligned}$$

so $gf \in \text{Hom}(A, \oplus, \circ)$. Since $F(A)$ is a semigroup under composition where $F(A)$ is the set of all functions from $A$ into itself, it follows that $\text{Hom}(A, \oplus, \circ)$ is a subsemigroup of $F(A)$.

Recall that a monomorphism of a group $G$ is a $1-1$ homomorphism of $G$. We let $\text{Mono}(G)$ denote the set of all monomorphisms of $G$. Then $\text{Mono}(G)$ is clearly a subsemigroup of the semigroup $\text{Hom}(G)$ under composition.

Let $\phi$ denote the Euler-phi function, i.e., for a positive integer $n, \phi(n)$ is the number of $x \in \{1, 2, 3, ..., n\}$ relatively prime to $n$. Then

$$\phi(n) = |\{x \mid x \in \{1, 2, 3, ..., n\} \text{ and } (x, n) = 1\}|.$$

It is well-known that for $a \in \mathbb{Z}, \bar{a}$ is a generator of the group$(\mathbb{Z}_n, +)$ if and only if $(a, n) = 1$, i.e., for $a \in \mathbb{Z}, \mathbb{Z}\bar{a} = \mathbb{Z}_n$ if and only if $(a, n) = 1$. Then the number of all generators of $(\mathbb{Z}_n, +)$ is $\phi(n)$.

An element $a$ of a semigroup $S$ is called an *idempotent* if $a^2 = a$. If $f : S \to S'$ is a semigroup homomorphism and $a$ is an idempotent of $S$, then $f(a)$ is clearly an idempotent of $S'$.

Note that in the Krasner hyperring $(R/\rho, \oplus, *)$ in Example 1.5, $0\rho$ is an idempotent of the semigroup $(R/\rho, *)$. If $R$ has an identity 1, then $1\rho$ is also an idempotent of $(R/\rho, *)$. From the definition of $\rho$, we can check that in the Krasner hyperring $(\mathbb{Z}_6/\rho, \oplus, *)$, every element of $\mathbb{Z}_6/\rho$ is an idempotent of the semigroup $(\mathbb{Z}_6/\rho, *)$.

# CHAPTER II
# HOMOMORPHISMS OF MULTIPLICATIVE HYPERRINGS

This chapter is concerned with the strongly distributive multiplicative hyperrings $(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ and $(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$ defined as in Example 1.1. By the definitions, $\text{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) \subseteq \text{Hom}(\mathbb{Z}, +)$ and $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) \subseteq \text{Hom}(\mathbb{Z}_n, +)$. Our purpose is to give characterizations determining when $\text{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) = \text{Hom}(\mathbb{Z}, +)$ and $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) = \text{Hom}(\mathbb{Z}_n, +)$ hold. We show that $\text{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ is an infinite set if $m > 0$, $|\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| \geq \frac{2n}{(m,n)}$ when $(m, n) > 1$ and the equality holds if $(m, n)$ is a prime power.

Notice that $(-m)\mathbb{Z} = m\mathbb{Z}$, $(-m)\mathbb{Z}_n = m\mathbb{Z}_n$, $(\mathbb{Z}, +, \circ_{0\mathbb{Z}}) = (\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}_n, +, \circ_{0\mathbb{Z}_n}) = (\mathbb{Z}_n, +, \cdot)$. We know that $\text{Hom}(\mathbb{Z}, +, \cdot) = \{g_0, g_1\}$. Hence Hom $(\mathbb{Z}, +, \circ_{0\mathbb{Z}}) \neq \text{Hom}(\mathbb{Z}, +)$. We have that, $\text{Hom}(\mathbb{Z}_n, +, \cdot) = \{h_{\bar{a}} \mid a \in \mathbb{Z}$ and $\bar{a} = \bar{a}^2\}$. To see this, let $f \in \text{Hom}(\mathbb{Z}_n, +, \cdot)$. Then $f \in \text{Hom}(\mathbb{Z}_n, +)$, so $f = h_{\bar{a}}$ for some $a \in \{0, 1, 2, ..., n-1\}$. Thus

$$\bar{a} = h_{\bar{a}}(\bar{1}) = h_{\bar{a}}(\bar{1} \cdot \bar{1}) = h_{\bar{a}}(\bar{1})h_{\bar{a}}(\bar{1}) = \bar{a}\bar{a} = \bar{a}^2.$$

If $a \in \{0, 1, 2, ..., n-1\}$ such that $\bar{a} = \bar{a}^2$, then for all $x, y \in \mathbb{Z}$,

$$h_{\bar{a}}(\bar{x}\bar{y}) = \bar{a}(\bar{x}\bar{y}) = \bar{a}^2(\bar{x}\bar{y}) = (\bar{a}\bar{x})(\bar{a}\bar{y}) = h_{\bar{a}}(\bar{x})h_{\bar{a}}(\bar{y}).$$

Thus $h_{\bar{a}} \in \text{Hom}(\mathbb{Z}_n, +, \cdot)$. Hence we have

$$\text{Hom}(\mathbb{Z}_n, +, \cdot) = \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} = \bar{a}^2\}.$$

We can see that $\bar{2} \neq \bar{2}^2$ in $\mathbb{Z}_n$ for all $n \geq 3$. It is evident that if $n = 1$ or $n = 2$, then $\text{Hom}(\mathbb{Z}_n, +, \circ_{0\mathbb{Z}_n}) = \text{Hom}(\mathbb{Z}_n, +)$. Consequently,

$$\text{Hom}(\mathbb{Z}_n, +, \circ_{0\mathbb{Z}_n}) = \text{Hom}(\mathbb{Z}_n, +) \Longleftrightarrow n = 1 \text{ or } n = 2.$$

Throughout this chapter, let $m$ be a positive integer. In fact, the results obtained in Section 2.2 are valid when $m = 0$ since $(0, n) = n > 0$.

## 2.1 Multiplicative Hyperrings Defined from the Ring $(\mathbb{Z},+,\cdot)$ and Its Ideals

In this section, we deal with the homomorphisms of the multiplicative hyperring $(\mathbb{Z},+,\circ_{m\mathbb{Z}})$. Recall that $x \circ_{m\mathbb{Z}} y = xy + m\mathbb{Z}$ for all $x, y \in \mathbb{Z}$.

The following three lemmas are needed.

**Lemma 2.1.1.** *For $a \in \mathbb{Z}$, $g_a \in \mathrm{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}})$ if and only if $m \,|(a^2 - a)$.*

**Proof.** Assume that $g_a \in \mathrm{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}})$. Then $g_a(1 \circ_{m\mathbb{Z}} 1) \subseteq g_a(1) \circ_{m\mathbb{Z}} g_a(1)$, so

$$\begin{aligned} a + am\mathbb{Z} &= a(1+m\mathbb{Z}) \\ &= a(1\cdot 1 + m\mathbb{Z}) \\ &= g_a(1 \circ_{m\mathbb{Z}} 1) \\ &\subseteq g_a(1) \circ_{m\mathbb{Z}} g_a(1) \\ &= a \circ_{m\mathbb{Z}} a \\ &= a^2 + m\mathbb{Z}. \end{aligned}$$

This implies that $a = a^2 + mt$ for some $t \in \mathbb{Z}$. Thus $m \,|(a^2 - a)$.

Conversely, assume that $m \,|(a^2 - a)$. Then $a^2 - a = mt$ for some $t \in \mathbb{Z}$, so $a = a^2 - mt$. Thus for all $x, y \in \mathbb{Z}$,

$$\begin{aligned} g_a(x \circ_{m\mathbb{Z}} y) &= g_a(xy + m\mathbb{Z}) \\ &= a(xy + m\mathbb{Z}) \\ &= axy + am\mathbb{Z} \\ &= (a^2 - mt)xy + am\mathbb{Z} \\ &\subseteq a^2xy + m\mathbb{Z} + am\mathbb{Z} \\ &= a^2xy + m\mathbb{Z} \\ &= (ax)(ay) + m\mathbb{Z} \\ &= g_a(x)g_a(y) + m\mathbb{Z} \\ &= g_a(x) \circ_{m\mathbb{Z}} g_a(y). \end{aligned}$$

Hence $g_a \in \mathrm{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}})$, as desired. □

**Lemma 2.1.2.** $\{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\} \subseteq \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$.

**Proof.** If $a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)$, then $m \mid a$ or $m \mid (a-1)$, so $m \mid (a^2 - a)$. By Lemma 2.1.1, the result follows. □

**Lemma 2.1.3.** *If* $m > 2$, *then* $\{ g_a \mid a \in m\mathbb{Z}+2\} \subseteq \mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$.

**Proof.** Assume that $m > 2$ and let $a \in m\mathbb{Z} + 2$. Then $a = mk + 2$ for some $k \in \mathbb{Z}$. But

$$a^2 - a = m^2k^2 + 3mk + 2,$$

so $m \nmid (a^2 - a)$. By Lemma 2.1.1, $g_a \notin \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$. Hence the desired result follows. □

**Theorem 2.1.4.** *The following statements hold.*

(i) $\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ *is infinite.*

(ii) $\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Z}, +)$ *if and only if* $m \leq 2$.

(iii) *If* $m > 2$, *then* $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$ *is infinite.*

(iv) *If* $m$ *is a prime power, then*

$$\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) = \{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}.$$

**Proof.** (i) Since $g_a \neq g_b$ if $a \neq b$ in $\mathbb{Z}$, (i) follows from Lemma 2.1.2.

(ii) If $m > 2$, then by Lemma 2.1.3, $\mathrm{Hom}(\mathbb{Z}, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) \neq \varnothing$, so $\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) \neq \mathrm{Hom}(\mathbb{Z}, +)$. This shows that if $\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Z}, +)$, then $m \leq 2$.

Assume that $m \leq 2$. Then $m\mathbb{Z} \cup (m\mathbb{Z}+1) = \mathbb{Z}$. It follows that $\{ g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\} = \mathrm{Hom}(\mathbb{Z}, +)$. Hence by Lemma 2.1.2, $\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$. But $\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) \subseteq \mathrm{Hom}(\mathbb{Z}, +)$, so $\mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Z}, +)$.

(iii) follows directly from Lemma 2.1.3.

(iv) Assume that $m$ is a prime power. Let $a \in \mathbb{Z}$ be such that $g_a \in \mathrm{Hom}(\mathbb{Z}, +, \circ_{m\mathbb{Z}})$. By Lemma 2.1.1, $m \mid (a^2 - a)$. Since $a^2 - a = a(a-1)$, and $a$ and $a - 1$ are relatively prime, we have that $m \mid a$ or $m \mid (a-1)$. Therefore $a \in m\mathbb{Z}$ or

$a-1 \in m\mathbb{Z}$. Hence $a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)$. This shows that $\text{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}}) \subseteq \{\, g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$. This implies by Lemma 2.1.2 that $\text{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}}) = \{\, g_a \mid a \in m\mathbb{Z} \cup (m\mathbb{Z}+1)\}$. □

**Remark 2.1.5.** Since $\text{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}}) \subseteq \text{Hom}(\mathbb{Z},+)$, $\text{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}})$ is infinite by Theorem 2.1.4(i) and $|\text{Hom}(\mathbb{Z},+)| = \aleph_0$, it follows that $|\text{Hom}(\mathbb{Z},+,\circ_{m\mathbb{Z}})| = \aleph_0$.

**Example 2.1.6.** By Theorem 2.1.4(iv),

$$\text{Hom}(\mathbb{Z},+,\circ_{4\mathbb{Z}}) = \{\, g_a \mid a \in 4\mathbb{Z} \cup (4\mathbb{Z}+1)\}$$

and hence

$$\text{Hom}(\mathbb{Z},+) \smallsetminus \text{Hom}(\mathbb{Z},+,\circ_{4\mathbb{Z}}) = \{\, g_a \mid a \in (4\mathbb{Z}+2) \cup (4\mathbb{Z}+3)\}.$$

## 2.2 Multiplicative Hyperrings Defined from the Ring $(\mathbb{Z}_n,+,\cdot)$ and Its Ideals

In this section, the homomorphisms of the multiplicative hyperring $(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n})$ are considered. Let us recall that $\bar{x} \circ_{m\mathbb{Z}_n} \bar{y} = \bar{x}\bar{y} + m\mathbb{Z}_n$ for all $x, y \in \mathbb{Z}$.

First, the following three lemmas are provided.

**Lemma 2.2.1.** *For $a \in \mathbb{Z}$, $h_{\bar{a}} \in \text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$ if and only if $(m,n) \mid (a^2-a)$.*

**Proof.** Assume that $h_{\bar{a}} \in \text{Hom}(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n})$. Then

$$\begin{aligned}
\bar{a} + am\mathbb{Z}_n &= \bar{a}(\bar{1}\cdot\bar{1} + m\mathbb{Z}_n)\\
&= \bar{a}(\bar{1} \circ_{m\mathbb{Z}_n} \bar{1})\\
&= h_{\bar{a}}(\bar{1} \circ_{m\mathbb{Z}_n} \bar{1})\\
&\subseteq h_{\bar{a}}(\bar{1}) \circ_{m\mathbb{Z}_n} h_{\bar{a}}(\bar{1})\\
&= \bar{a} \circ_{m\mathbb{Z}_n} \bar{a}\\
&= \bar{a}^2 + m\mathbb{Z}_n\\
&= \bar{a}^2 + (m,n)\mathbb{Z}_n,
\end{aligned}$$

so $\bar{a} - \bar{a}^2 = (m,n)\bar{s}$ for some $s \in \mathbb{Z}$. Hence $a - a^2 - (m,n)s = nt$ for some $t \in \mathbb{Z}$. Thus $a - a^2 = (m,n)s + nt$. But $(m,n) \mid ((m,n)s + nt)$, so $(m,n) \mid (a^2 - a)$.

For the converse, assume that $(m,n) \mid (a^2 - a)$. Then $a^2 - a = (m,n)s$ for some $s \in \mathbb{Z}$, so $a = a^2 - (m,n)s$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned}
h_{\bar{a}}(\bar{x} \circ_{m\mathbb{Z}_n} \bar{y}) &= h_{\bar{a}}(\overline{xy} + m\mathbb{Z}_n) \\
&= \bar{a}(\overline{xy} + m\mathbb{Z}_n) \\
&= \overline{axy} + am\mathbb{Z}_n \\
&= \overline{(a^2 - (m,n)s)xy} + am\mathbb{Z}_n \\
&= \overline{a^2xy} - \overline{(m,n)sxy} + am\mathbb{Z}_n \\
&\subseteq \overline{a^2xy} + (m,n)\mathbb{Z}_n + am\mathbb{Z}_n \\
&= \overline{a^2xy} + m\mathbb{Z}_n + am\mathbb{Z}_n \\
&= \overline{a^2xy} + m\mathbb{Z}_n \\
&= \overline{ax}\ \overline{ay} + m\mathbb{Z}_n \\
&= h_{\bar{a}}(\bar{x})h_{\bar{a}}(\bar{y}) + m\mathbb{Z}_n \\
&= h_{\bar{a}}(\bar{x}) \circ_{m\mathbb{Z}_n} h_{\bar{a}}(\bar{y}).
\end{aligned}$$

Hence $h_{\bar{a}} \in \mathrm{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$. □

**Lemma 2.2.2.** $\{ h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z} + 1)\} \subseteq \mathrm{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$.

**Proof.** If $a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z} + 1)$, then $(m,n) \mid a$ or $(m,n) \mid (a-1)$, thus $(m,n) \mid (a^2 - a)$. Hence by Lemma 2.2.1, the result follows. □

**Lemma 2.2.3.** *If* $(m,n) > 2$, *then* $\{ h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} + 2\} \subseteq \mathrm{Hom}(\mathbb{Z}_n, +) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$.

**Proof.** If $(m,n) > 2$ and $a \in (m,n)\mathbb{Z} + 2$, then $a = (m,n)k + 2$ for some $k \in \mathbb{Z}$, so

$$a^2 - a = (m,n)^2k^2 + 3(m,n)k + 2$$

which is not divisible by $(m,n)$, so by Lemma 2.2.1, $h_{\bar{a}} \notin \text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$, i.e., $h_{\bar{a}} \in \text{Hom}(\mathbb{Z}_n, +) \smallsetminus \text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})$, so the result follows. □

**Theorem 2.2.4.** *The following statements hold.*

(i) *If* $(m,n) > 1$, *then* $|\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| \geq \frac{2n}{(m,n)}$.

(ii) $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) = \text{Hom}(\mathbb{Z}_n, +)$ *if and only if* $(m,n) \leq 2$.

(iii) *If* $(m,n) > 2$, *then* $|\text{Hom}(\mathbb{Z}_n, +) \smallsetminus \text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| \geq \frac{n}{(m,n)}$.

(iv) *If* $(m,n)$ *is a prime power, then*

$$\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) = \{\, h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z} + 1)\}$$

*and hence* $|\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| = \frac{2n}{(m,n)}$.

**Proof.** (i) Assume that $(m,n) > 1$. Then $|(m,n)\mathbb{Z}_n| = \frac{n}{(m,n)} < n$. This implies that $(m,n)\mathbb{Z}_n \cap ((m,n)\mathbb{Z}_n + 1) = \varnothing$. Since $h_{\bar{a}} \neq h_{\bar{b}}$ for all distinct $\bar{a}, \bar{b} \in \mathbb{Z}_n$, it follows that

$$\begin{aligned}
|\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n})| &\geq |\{\, h_{\bar{a}} \mid a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z} + 1)\}| \\
&= |\{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in (m,n)\mathbb{Z}_n \cup ((m,n)\mathbb{Z}_n + \bar{1})\}| \\
&= |(m,n)\mathbb{Z}_n| + |(m,n)\mathbb{Z}_n + \bar{1}| \\
&= \frac{n}{(m,n)} + \frac{n}{(m,n)} = \frac{2n}{(m,n)}.
\end{aligned}$$

(ii) If $(m,n) > 2$, then by Lemma 2.2.3, $\text{Hom}(\mathbb{Z}_n, +) \smallsetminus \text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) \neq \varnothing$, so $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) \neq \text{Hom}(\mathbb{Z}_n, +)$. Hence if $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) = \text{Hom}(\mathbb{Z}_n, +)$, then $(m,n) \leq 2$.

Assume that $(m,n) \leq 2$. Then $(m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z} + 1) = \mathbb{Z}$. This implies that $\{h_{\bar{a}} | \, a \in (m,n)\mathbb{Z} \cup ((m,n)\mathbb{Z} + 1)\} = \text{Hom}(\mathbb{Z}_n, +)$. Therefore by Lemma 2.2.2, we have that $\text{Hom}(\mathbb{Z}_n, +, \circ_{m\mathbb{Z}_n}) = \text{Hom}(\mathbb{Z}_n, +)$.

(iii) Assume that $(m,n) > 2$. Then

$$
\begin{aligned}
&|\mathrm{Hom}(\mathbb{Z}_n,+)\smallsetminus \mathrm{Hom}(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n})|\\
&\qquad\qquad \geq |\{h_{\bar{a}}|\, a\in (m,n)\mathbb{Z}+2\}| \quad \text{by Lemma 2.2.3}\\
&\qquad\qquad = |\{h_{\bar{a}}|\, a\in \mathbb{Z} \text{ and } \bar{a}\in (m,n)\mathbb{Z}_n+\bar{2}\}|\\
&\qquad\qquad = |(m,n)\mathbb{Z}_n+\bar{2}|\\
&\qquad\qquad = |(m,n)\mathbb{Z}_n| = \frac{n}{(m,n)}.
\end{aligned}
$$

(iv) Let $(m,n)$ be a prime power and $a\in\mathbb{Z}$ such that $h_{\bar{a}}\in \mathrm{Hom}(\mathbb{Z}_n,\ +,\ \circ_{m\mathbb{Z}_n})$. By Lemma 2.2.1, $(m,n)\,|\,(a^2-a)$. But $a^2-a=a(a-1)$ and $(a,a-1)=1$, so $(m,n)\,|\,a$ or $(m,n)\,|\,(a-1)$. Thus $a\in(m,n)\mathbb{Z}\cup((m,n)\mathbb{Z}+1)$. This shows that $\mathrm{Hom}(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n})\subseteq\{\,h_{\bar{a}}\mid a\in(m,n)\mathbb{Z}\cup((m,n)\mathbb{Z}+1)\,\}$. Hence by Lemma 2.2.2, we have that $\mathrm{Hom}(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n}) = \{\,h_{\bar{a}}\mid a\in(m,n)\mathbb{Z}\cup((m,n)\mathbb{Z}+1)\}$. □

**Remark 2.2.5.** If $(m,n)=1$, then by Theorem 2.2.4(ii), $|\mathrm{Hom}(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n})| = n < \frac{2n}{(m,n)}$. Therefore the condition that $(m,n)>1$ in Theorem 2.2.4(i) can not be omitted.

If $(m,n)$ is a prime power, then by Theorem 2.2.4(iv), $|\mathrm{Hom}(\mathbb{Z}_n,+,\circ_{m\mathbb{Z}_n})| = \frac{2n}{(m,n)}$. This shows that $\frac{2n}{(m,n)}$ is the most suitable number for the inequality in Theorem 2.2.4(i).

**Example 2.2.6.** By Theorem 2.2.4(iv), $|\mathrm{Hom}(\mathbb{Z}_{20},+,\circ_{4\mathbb{Z}_{20}})| = \dfrac{2\times 20}{(4,20)} = 10$ and

$$
\begin{aligned}
\mathrm{Hom}(\mathbb{Z}_{20},+,\circ_{4\mathbb{Z}_{20}}) &= \{\,h_{\bar{a}}\mid a\in 4\mathbb{Z}\cup(4\mathbb{Z}+1)\}\\
&= \{\,h_{\bar{a}}\mid a\in\mathbb{Z} \text{ and } \bar{a}\in 4\mathbb{Z}_{20}\cup(4\mathbb{Z}_{20}+\bar{1})\}\\
&= \{\,h_{\bar{0}},h_{\bar{4}},h_{\bar{8}},h_{\overline{12}},h_{\overline{16}},h_{\bar{1}},h_{\bar{5}},h_{\bar{9}},h_{\overline{13}},h_{\overline{17}}\}.
\end{aligned}
$$

Thus

$$
\mathrm{Hom}(\mathbb{Z}_{20},+)\smallsetminus\mathrm{Hom}(\mathbb{Z}_{20},+,\circ_{4\mathbb{Z}_{20}}) = \{h_{\bar{2}},h_{\bar{3}},h_{\bar{6}},h_{\bar{7}},h_{\overline{10}},h_{\overline{11}},h_{\overline{14}},h_{\overline{15}},h_{\overline{18}},h_{\overline{19}}\}.
$$

It follows from Theorem 2.2.4(i) and (iii) that

$$
|\mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}})| \geq \frac{2\times 18}{(6,18)} = 6
$$

and

$$|\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}})| \geq \frac{18}{(6,18)} = 3.$$

From Lemma 2.2.2 and Lemma 2.2.3, we have respectively that

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}}) &\supseteq \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z} \cup (6\mathbb{Z}+1)\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in 6\mathbb{Z}_{18} \cup (6\mathbb{Z}_{18}+\bar{1})\} \\ &= \{\, h_{\bar{0}}, h_{\bar{6}}, h_{\overline{12}}, h_{\bar{1}}, h_{\bar{7}}, h_{\overline{13}}\,\},\end{aligned}$$

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}}) &\supseteq \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z}+2\} \\ &= \{\, h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a} \in 6\mathbb{Z}_{18}+\bar{2}\} \\ &= \{\, h_{\bar{2}}, h_{\bar{8}}, h_{\overline{14}}\,\}.\end{aligned}$$

Let us consider $h_{\bar{a}}$ where $a \in (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4) \cup (6\mathbb{Z}+5)$. If $k \in \mathbb{Z}$, then

$$6 \mid (6k+3)^2 - (6k+3),\, 6 \mid (6k+4)^2 - (6k+4) \text{ and } 6 \nmid (6k+5)^2 - (6k+5),$$

so by Lemma 2.2.1,

$$\{\, h_{\bar{a}} \mid a \in (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4)\} \subseteq \mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}})$$

and

$$\{\, h_{\bar{a}} \mid a \in 6\mathbb{Z}+5\} \subseteq \mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}}).$$

Consequently,

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}}) &= \{\, h_{\bar{a}} \mid a \in 6\mathbb{Z} \cup (6\mathbb{Z}+1) \cup (6\mathbb{Z}+3) \cup (6\mathbb{Z}+4)\} \\ &= \{h_{\bar{0}}, h_{\bar{6}}, h_{\overline{12}}, h_{\bar{1}}, h_{\bar{7}}, h_{\overline{13}}, h_{\bar{3}}, h_{\bar{9}}, h_{\overline{15}}, h_{\bar{4}}, h_{\overline{10}}, h_{\overline{16}}\},\end{aligned}$$

$$|\mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}})| = 12,$$

$$\begin{aligned}\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}}) &= \{\, h_{\bar{a}} \mid a \in (6\mathbb{Z}+2) \cup (6\mathbb{Z}+5)\} \\ &= \{h_{\bar{2}}, h_{\bar{8}}, h_{\overline{14}}, h_{\bar{5}}, h_{\overline{11}}, h_{\overline{17}}\},\end{aligned}$$

$$|\mathrm{Hom}(\mathbb{Z}_{18},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{18},+,\circ_{6\mathbb{Z}_{18}})| = 6.$$

# CHAPTER III

# HOMOMORPHISMS OF KRASNER HYPERRINGS

In this chapter we deal with homomorphisms of the Krasner hyperrings defined in Example 1.3, Example 1.4 and in Example 1.5 when $R = (\mathbb{Z}, +, \cdot)$ and $R = (\mathbb{Z}_n, +, \cdot)$. Notice that the zero mapping on a Krasner hyperring $(A, \oplus, \cdot)$ is clearly a homomorphism of $(A, \oplus, \cdot)$. We characterize the elements of $\mathrm{Hom}(G^0, \oplus_1, \cdot)$ and $\mathrm{Hom}(G^0, \oplus_2, \cdot)$ where the Krasner hyperrings $(G^0, \oplus_1, \cdot)$ and $(G^0, \oplus_2, \cdot)$ defined from a group $(G, \cdot)$ in Example 1.4(1) and Example 1.4(2), respectively. The elements of $\mathrm{Hom}(\mathbb{Z}/\rho, \oplus, *)$ and the elements of $\mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ fixing $\bar{0}\rho$ and $\bar{1}\rho$ are characterized where $(\mathbb{Z}/\rho, \oplus, *)$ and $(\mathbb{Z}_n/\rho, \oplus, *)$ are the Krasner hyperrings defined as in Example 1.5. Finally we give necessary and sufficient conditions for $f : [0, \infty) \to [0, \infty)$ to be an element of $\mathrm{Hom}([0, \infty), \oplus, \cdot)$ and show that the set $\mathrm{Hom}([0, \infty), \oplus, \cdot)$ is uncountable where $([0, \infty), \oplus, \cdot)$ is the Krasner hyperring defined in Example 1.3.

## 3.1 Krasner Hyperrings Defined from Groups

Let $G$ be a group and recall the definitions of the Krasner hyperrings $(G^0, \oplus_1, \cdot)$ and $(G^0, \oplus_2, \cdot)$ as follows:

$$x \oplus_1 0 = 0 \oplus_1 x = \{x\} \qquad \text{for all } x \in G^0,$$

$$x \oplus_1 y = \begin{cases} \{x, y\} & \text{if } x, y \in G \text{ and } x \neq y, \\ G^0 \smallsetminus \{x\} & \text{if } x, y \in G \text{ and } x = y, \end{cases}$$

$$x \oplus_2 0 = 0 \oplus_2 x = \{x\} \qquad \text{for all } x \in G^0,$$

$$x \oplus_2 x = \{x, 0\} \qquad \text{for all } x \in G,$$

$$x \oplus_2 y = G \smallsetminus \{x, y\} \qquad \text{for all distinct } x, y \in G$$

and

$$x \cdot 0 = 0 \cdot x = 0 \quad \text{for all } x \in G^0.$$

Let $e$ be the identity of the group $G$.

To characterize the elements of $\text{Hom}(G^0, \oplus_1, \cdot)$, we first show that every element of $\text{Hom}(G^0, \oplus_1, \cdot)$ fixes the element 0.

**Lemma 3.1.1.** *If* $f \in \text{Hom}(G^0, \oplus_1, \cdot)$, *then* $f(0) = 0$.

*Proof.* Assume that $f \in \text{Hom}(G^0, \oplus_1, \cdot)$. Suppose that $f(0) \neq 0$. Then $f(0) \in G$, so

$$\{f(0)\} = f(\{0\}) = f(0 \oplus_1 0) \subseteq f(0) \oplus_1 f(0) = G^0 \smallsetminus \{f(0)\}$$

which is a contradiction. Thus $f(0) = 0$. □

**Theorem 3.1.2.** *For* $f : G^0 \to G^0,\ f \in \text{Hom}(G^0, \oplus_1, \cdot)$ *if and only if either*

(i) $f$ *is the zero mapping on* $(G^0, \oplus_1, \cdot)$ *or*

(ii) $f|_G \in \text{Mono}(G)$ *and* $f(0) = 0$.

*Proof.* Assume that $f \in \text{Hom}(G^0, \oplus_1, \cdot)$. From Lemma 3.1.1, $f(0) = 0$.

**Case 1:** $f(a) = 0$ for some $a \in G$. Then

$$f(G^0 \smallsetminus \{a\}) = f(a \oplus_1 a) \subseteq f(a) \oplus_1 f(a) = 0 \oplus_1 0 = \{0\}.$$

Thus $f(x) = 0$ for all $x \in G^0$, i.e., $f$ satisfies (i).

**Case 2:** $f(a) \neq 0$ for all $a \in G$. Then $f(G) \subseteq G$. Since $f \in \text{Hom}(G^0, \cdot)$, it follows that $f|_G \in \text{Hom}(G)$. Next, to show that $f$ is 1-1, let $x, y \in G$ be such that $x \neq y$. Suppose that $f(x) = f(y)$. Then

$$\{f(x)\} = \{f(x), f(y)\} = f(\{x, y\}) = f(x \oplus_1 y) \subseteq f(x) \oplus_1 f(y) = G^0 \smallsetminus \{f(x)\},$$

a contradiction. Thus $f(x) \neq f(y)$. Hence $f \in \text{Mono}(G)$, so $f$ satisfies (ii).

Conversely, assume that $f$ satisfies (i) or (ii). If $f$ satisfies (i), then $f \in \text{Hom}(G^0, \oplus_1, \cdot)$. Next, assume that $f$ satisfies (ii). Since $f(0) = 0$ and $f|_G \in \text{Hom}(G)$, it is clear that $f \in \text{Hom}(G^0, \cdot)$. Next we show that $f \in \text{Hom}(G^0, \oplus_1)$. We have that

$$f(0 \oplus_1 0) = f(\{0\}) = \{0\} = 0 \oplus_1 0 = f(0) \oplus_1 f(0)$$

and for every $x \in G$,

$$f(x \oplus_1 0) = f(\{x\}) = \{f(x)\} = f(x) \oplus_1 0 = f(x) \oplus_1 f(0).$$

Since $f(G) \subseteq G, f(0) = 0$ and $f$ is $1-1$, it follows that for $x \in G$,

$$\begin{aligned} f(x \oplus_1 x) = f(G^0 \smallsetminus \{x\}) &= \{f(t) \mid t \in G^0 \smallsetminus \{x\}\} \\ &\subseteq G^0 \smallsetminus \{f(x)\} \\ &= f(x) \oplus_1 f(x). \end{aligned}$$

It remains to show that for distinct $x, y \in G, f(x \oplus_1 y) \subseteq f(x) \oplus_1 f(y)$. If $x, y \in G$ are distinct, then $f(x) \neq f(y)$ since $f$ is $1-1$, so

$$f(x \oplus_1 y) = f(\{x, y\}) = \{f(x), f(y)\} = f(x) \oplus_1 f(y).$$

This shows that $f \in \text{Hom}(G^0, \oplus_1)$, so $f \in \text{Hom}(G^0, \oplus_1, \cdot)$, as desired.

The proof is thereby complete. □

For each $f \in \text{Hom}(G)$, let $\bar{f} : G^0 \to G^0$ be defined by

$$\bar{f}(0) = 0 \text{ and } \bar{f}(x) = f(x) \text{ for all } x \in G.$$

The following corollary is a direct consequence of Theorem 3.1.2.

**Corollary 3.1.3.** $\text{Hom}(G^0, \oplus_1, \cdot) = \{\bar{f} \mid f \in \text{Mono}(G)\} \cup \{$*the zero mapping on* $(G^0, \oplus_1, \cdot)\}$

**Example 3.1.4.** (1) For each odd positive integer $n$, let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = x^n \text{ for all } x \in \mathbb{R}.$$

Then $f_n|_{\mathbb{R} \smallsetminus \{0\}} \in \text{Mono}(\mathbb{R} \smallsetminus \{0\}, \cdot)$ and $f_n(0) = 0$ for every odd positive integer $n$. It follows that

$$\{f_n \mid n \text{ is an odd positive integer }\} \subseteq \text{Hom}(\mathbb{R}, \oplus_1, \cdot).$$

Here we let $(\mathbb{R} \setminus \{0\})^0 = \mathbb{R}$. This implies that $\mathrm{Hom}(\mathbb{R}, \oplus_1, \cdot)$ is an infinite set. We obtain similarly that $\mathrm{Hom}(F, \oplus_1, \cdot)$ is an infinite set for every subfield $F$ of $\mathbb{R}$.

(2) Let $\theta$ be a symbol not representing any element of $\mathbb{Z}$ and define

$$x + \theta = \theta + x = \theta \;\text{ for all } x \in \mathbb{Z} \cup \{\theta\}.$$

We can see that

$$\mathrm{Mono}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z} \setminus \{0\}\}.$$

Let $\mu : \mathbb{Z} \cup \{\theta\} \to \mathbb{Z} \cup \{\theta\}$ be defined by $\mu(x) = \theta$ for all $x \in \mathbb{Z} \cup \{\theta\}$. By Corollary 3.1.3, we have that

$$\mathrm{Hom}(\mathbb{Z} \cup \{\theta\}, \oplus_1, +) = \{\bar{g}_a \mid a \in \mathbb{Z} \setminus \{0\}\} \cup \{\mu\},$$

and hence $|\mathrm{Hom}(\mathbb{Z} \cup \{\theta\}, \oplus_1, +)| = \aleph_0$.

(3) Define

$$\bar{x} + \theta = \theta + \bar{x} = \theta \;\text{ for all } \bar{x} \in \mathbb{Z}_n \cup \{\theta\}$$

where $\theta$ is a symbol not representing any element of $\mathbb{Z}_n$ and define $\lambda : \mathbb{Z}_n \cup \{\theta\} \to \mathbb{Z}_n \cup \{\theta\}$ by $\lambda(\bar{x}) = \theta$ for all $\bar{x} \in \mathbb{Z}_n \cup \{\theta\}$. We have that

$$\begin{aligned}
\mathrm{Mono}(\mathbb{Z}_n, +) &= \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } h_{\bar{a}} \text{ is } 1-1\} \\
&= \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } h_{\bar{a}}(\mathbb{Z}_n) = \mathbb{Z}_n\} \\
&= \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \bar{a}\mathbb{Z}_n = \mathbb{Z}_n\} \\
&= \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } \mathbb{Z}\bar{a} = \mathbb{Z}_n\} \\
&= \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } (a, n) = 1\}.
\end{aligned}$$

It follows from Corollary 3.1.3 that

$$\mathrm{Hom}(\mathbb{Z}_n \cup \{\theta\}, \oplus_1, +) = \{h_{\bar{a}} \mid a \in \mathbb{Z} \text{ and } (a, n) = 1\} \cup \{\lambda\}.$$

Hence

$$|\mathrm{Hom}(\mathbb{Z}_n \cup \{\theta\}, \oplus_1, +)| = \phi(n) + 1.$$

For example,

$$\mathrm{Hom}(\mathbb{Z}_{12} \cup \{\theta\}, \oplus_1, +) = \{h_{\bar{1}}, h_{\bar{5}}, h_{\bar{7}}, h_{\overline{11}}, \lambda\}.$$

The next theorem characterizes the homomorphisms of the Krasner hyperring $(G^0, \oplus_2, \cdot)$.

**Theorem 3.1.5.** *For $f : G^0 \to G^0,\ f \in \operatorname{Hom}(G^0, \oplus_2, \cdot)$ if and only if one of the following statements hold.*

(i) *$f$ is the zero mapping on $(G^0, \oplus_2, \cdot)$.*

(ii) *$f(x) = e \quad$ for all $x \in G^0$.*

(iii) *$f(0) = 0 \ $ and $f(x) = e$ for all $x \in G$.*

(iv) *$f|_G \in \operatorname{Mono}(G)$ and $f(0) = 0$.*

*Proof.* Assume that $f \in \operatorname{Hom}(G^0, \oplus_2, \cdot)$.

**Case 1:** $0 \in f(G)$. Then $f(a) = 0\ $ for some $a \in G$. Since

$$\{0\} = f(\{a\}) = f(a \oplus_2 0) \subseteq f(a) \oplus_2 f(0) = 0 \oplus_2 f(0) = \{f(0)\},$$

we have that $f(0) = 0$. To show that $f$ is the zero mapping on $(G^0, \oplus_2, \cdot)$, let $x \in G \smallsetminus \{a\}$. Since

$$f(a \oplus_2 x) = f(G \smallsetminus \{a, x\}) = \{f(t) \mid t \in G \smallsetminus \{a, x\}\},$$

$$f(a) \oplus_2 f(x) = 0 \oplus_2 f(x) = \{f(x)\}$$

and

$$f(a \oplus_2 x) \subseteq f(a) \oplus_2 f(x),$$

it follows that $f(G \smallsetminus \{a\}) = \{f(x)\}$. If $y \in G \smallsetminus \{e\}$, then $ay \in G \smallsetminus \{a\}$, so

$$f(x) = f(ay) = f(a)f(y) = 0 \cdot f(y) = 0.$$

This shows that $f(t) = 0$ for all $t \in G^0$, so $f$ satisfies (i).

**Case 2:** $0 \notin f(G^0)$. Then $f(0) \in G$. If $x \in G^0$, then

$$e = f(0)^{-1} f(0) = f(0)^{-1} f(0 \cdot x) = f(0)^{-1} f(0) f(x) = f(x).$$

This shows that $f$ satisfies (ii).

**Case 3:** $0 \notin f(G)$ and $0 \in f(G^0)$. Then $f(0) = 0\ $ and $f(G) \subseteq G$. Thus $f|_G \in \operatorname{Hom}(G)$, so $\ f(e) = e$. If $f|_G$ is $1-1$, then $f$ satisfies (iv). Next, assume that $f|_G$ is not $1-1$. Then there exists an element $a \in G \smallsetminus \{e\}$ such that $f(a) = e$. Since

$$f(e \oplus_2 a) = f(G \smallsetminus \{e, a\}) = \{f(t) \mid t \in G \smallsetminus \{e, a\}\},$$

$$f(e) \oplus_2 f(a) = e \oplus_2 e = \{e, 0\}$$

and

$$f(e \oplus_2 a) \subseteq f(e) \oplus_2 f(a),$$

it follows that $f(G \smallsetminus \{a\}) \subseteq \{e, 0\}$. But $f(G) \subseteq G$, so $f(G \smallsetminus \{a\}) = \{e\} = \{f(a)\}$. Hence $f(x) = e$ for all $x \in G$, so $f$ satisfies (iii).

Conversely, assume that $f$ satisfies (i), (ii), (iii) or (iv). If $f$ satisfies (i), then $f \in \text{Hom}(G^0, \oplus_2, \cdot)$. Assume that $f$ satisfies (ii). Then for $x, y \in G^0$,

$$f(x \oplus_2 y) = \{e\} \subseteq \{e, 0\} = e \oplus_2 e = f(x) \oplus_2 f(y),$$
$$f(xy) = e = ee = f(x)f(y).$$

Thus $f \in \text{Hom}(G^0, \oplus_2, \cdot)$.

Next, assume that $f$ satisfies (iii), i.e., $f(0) = 0$ and $f(G) = \{e\}$. We have that

$$f(0 \oplus_2 0) = f(\{0\}) = \{0\} = 0 \oplus_2 0 = f(0) \oplus_2 f(0)$$

and for all $x \in G$,

$$f(x \oplus_2 0) = f(\{x\}) = \{f(x)\} = f(x) \oplus_2 0 = f(x) \oplus_2 f(0)$$

and

$$f(x \oplus_2 x) = f(\{x, 0\}) = \{f(x), f(0)\} = \{f(x), 0\} = f(x) \oplus_2 f(x).$$

If $x, y \in G$ are distinct, then

$$f(x \oplus_2 y) = f(G \smallsetminus \{x, y\}) = \{e\} \subseteq \{e, 0\} = e \oplus_2 e = f(x) \oplus_2 f(y).$$

This proves that $f \in \text{Hom}(G^0, \oplus_2)$. It is clear that $f \in \text{Hom}(G^0, \cdot)$. Hence $f \in \text{Hom}(G^0, \oplus_2, \cdot)$.

Finally, assume that $f$ satisfies (iv), i.e., $f|_G \in \text{Mono}(G)$ and $f(0) = 0$. Since $f|_G \in \text{Hom}(G)$ and $f(0) = 0$, it follows that $f \in \text{Hom}(G^0, \cdot)$. We have that

$$f(0 \oplus_2 0) = f(\{0\}) = \{0\} = 0 \oplus_2 0 = f(0) \oplus_2 f(0)$$

and for all $x \in G$,

$$f(x \oplus_2 0) = f(\{x\}) = \{f(x)\} = f(x) \oplus_2 0 = f(x) \oplus_2 f(0)$$

and

$$f(x \oplus_2 x) = f(\{x, 0\}) = \{f(x), f(0)\} = \{f(x), 0\} = f(x) \oplus_2 f(x).$$

Let $x, y \in G$ be distinct. Since $f|_G$ is 1-1, we have that

$$\begin{aligned} f(x \oplus_2 y) = f(G \smallsetminus \{x, y\}) &= \{f(t) \mid t \in G \smallsetminus \{x, y\}\} \\ &= f(G) \smallsetminus \{f(x), f(y)\} \\ &\subseteq G \smallsetminus \{f(x), f(y)\} \\ &= f(x) \oplus_2 f(y). \end{aligned}$$

Hence $f \in \mathrm{Hom}(G^0, \oplus_2, \cdot)$.

Therefore the proof is complete. $\square$

## 3.2 The Krasner Hyperring Defined from the Ring $(\mathbb{Z}, +, \cdot)$ and an Equivalence Relation

The elements of $\mathrm{Hom}(\mathbb{Z}/\rho, \oplus, *)$ are investigated in this section. Recall that

$$\begin{aligned} x\rho y &\iff x = y \text{ or } x = -y, \\ x\rho \oplus y\rho &= \{(x+y)\rho, (x-y)\rho\}, \\ x\rho * y\rho &= (xy)\rho \text{ for all } x, y \in \mathbb{Z}. \end{aligned}$$

First, we give the following series of lemmas.

**Lemma 3.2.1.** *If* $f \in \mathrm{Hom}(\mathbb{Z}/\rho, \oplus)$, *then* $f(0\rho) = 0\rho$.

*Proof.* Assume that $f \in \mathrm{Hom}(\mathbb{Z}/\rho, \oplus)$. Let $f(0\rho) = a\rho$ for some $a \in \mathbb{Z}$. Since

$$\begin{aligned} f(0\rho \oplus 0\rho) &= f(\{(0+0)\rho, (0-0)\rho\}) \\ &= f(\{0\rho\}) \\ &= \{a\rho\}, \end{aligned}$$

we have that

$$\begin{aligned} a\rho \in f(0\rho \oplus 0\rho) &\subseteq f(0\rho) \oplus f\{0\rho) \\ &= a\rho \oplus a\rho \\ &= \{(a+a)\rho, (a-a)\rho\} \\ &= \{(2a)\rho, 0\rho\}. \end{aligned}$$

Then $a\rho = (2a)\rho$ or $a\rho = 0\rho$. If $a\rho = (2a)\rho$, then $a = 2a$ or $a = -2a$ which implies that $a = 0$, so the desired result follows. □

**Lemma 3.2.2.** *If $f \in \mathrm{Hom}(\mathbb{Z}/\rho, *)$, then $f(1\rho) = 0\rho$ or $f(1\rho) = 1\rho$.*

*Proof.* Let $f(1\rho) = b\rho$ for some $b \in \mathbb{Z}$. Then

$$b\rho = f(1\rho) = f(1\rho * 1\rho) = f(1\rho) * f(1\rho) = b\rho * b\rho = b^2\rho.$$

Thus $b = b^2$ or $b = -(b^2)$, so $b = 0$ or $1$ or $-1$. Since $1\rho = -1\rho$, we have that $f(1\rho) = 0\rho$ or $f(1\rho) = 1\rho$. □

**Lemma 3.2.3.** *If $f \in \mathrm{Hom}(\mathbb{Z}/\rho, *)$ and $f(1\rho) = 0\rho$, then $f$ is the zero mapping on $(\mathbb{Z}/\rho, \oplus, *)$.*

*Proof.* If $x \in \mathbb{Z}$, then

$$f(x\rho) = f((x \cdot 1)\rho) = f(x\rho * 1\rho) = f(x\rho) * f(1\rho) = f(x\rho) * 0\rho = 0\rho.$$

Hence $f(x\rho) = 0\rho$ for all $x \in \mathbb{Z}$, i.e., $f$ is the zero mapping on $(\mathbb{Z}/\rho, \oplus, *)$. □

**Lemma 3.2.4.** *If $f \in \mathrm{Hom}(\mathbb{Z}/\rho, \oplus, *)$ and $f(1\rho) = 1\rho$, then either*

(i) *$f$ is the identity mapping on $\mathbb{Z}/\rho$ or*

(ii) $f(x\rho) = \begin{cases} 0\rho & \text{if } x \text{ is even,} \\ 1\rho & \text{if } x \text{ is odd.} \end{cases}$

*Proof.* It follows from the assumption that

$$f(2\rho) \in f(1\rho \oplus 1\rho) \subseteq f(1\rho) \oplus f(1\rho) = 1\rho \oplus 1\rho = \{2\rho, 0\rho\}.$$

**Case 1:** $f(2\rho) = 0\rho$. Then for $x \in \mathbb{Z}$,

$$f(2x\rho) = f(2\rho * x\rho) = f(2\rho) * f(x\rho) = 0\rho * f(x\rho) = 0\rho$$

and

$$f((2x+1)\rho) \in f(2x\rho \oplus 1\rho) \subseteq f(2x\rho) \oplus f(1\rho) = 0\rho \oplus 1\rho = \{1\rho\}.$$

Hence $f(x\rho) = \begin{cases} 0\rho & \text{if } x \text{ is even,} \\ 1\rho & \text{if } x \text{ is odd,} \end{cases}$

so $f$ satisfies (ii).

**Case 2:** $f(2\rho) = 2\rho$. Since

$$f(3\rho) \in f(2\rho \oplus 1\rho) \subseteq f(2\rho) \oplus f(1\rho) = 2\rho \oplus 1\rho = \{3\rho, 1\rho\},$$

we have that either $f(3\rho) = 3\rho$ or $f(3\rho) = 1\rho$. We also have that

$$f(4\rho) = f(2\rho * 2\rho) = f(2\rho) * f(2\rho) = 2\rho * 2\rho = 4\rho.$$

To show that $f(3\rho) = 3\rho$, suppose that $f(3\rho) = 1\rho$. Then we have

$$4\rho = f(4\rho) \in f(3\rho \oplus 1\rho) \subseteq f(3\rho) \oplus f(1\rho) = 1\rho \oplus 1\rho = \{2\rho, 0\rho\}$$

which is a contradiction since $f(4\rho) = 4\rho$. Thus $f(3\rho) = 3\rho$.

Assume that $k \geq 4$ and $f(x\rho) = x\rho$ for all $x \in \{0, 1, 2, ..., k\}$.

**Subcase 2.1:** $k+1$ is even. Then $k+1 = 2a$ for some $a \in \mathbb{Z}^+$. Thus $a < k$. Then $f(a\rho) = a\rho$. Hence

$$f((k+1)\rho) = f((2a)\rho) = f(2\rho * a\rho) = f(2\rho) * f(a\rho) = 2\rho * a\rho = (2a)\rho = (k+1)\rho.$$

**Subcase 2.2:** $k+1$ is odd. Then $(k+1)+1$ is even, so $(k+1)+1 = 2b$ for some $b \in \mathbb{Z}^+$. Thus $b < k$, so $f(b\rho) = b\rho$. It follows that

$$\begin{aligned} f(((k+1)+1)\rho) = f((2b)\rho) &= f(2\rho * b\rho) \\ &= f(2\rho) * f(b\rho) \\ &= 2\rho * b\rho \\ &= 2b\rho \\ &= ((k+1)+1)\rho. \end{aligned}$$

Since

$$\begin{aligned} f((k+1)\rho) \in f(k\rho \oplus 1\rho) \subseteq f(k\rho) \oplus f(1\rho) \\ &= k\rho \oplus 1\rho \\ &= \{(k+1)\rho, (k-1)\rho\}, \end{aligned}$$

we have that $f((k+1)\rho) = (k+1)\rho$ or $f((k+1)\rho) = (k-1)\rho$. Suppose that $f((k+1)\rho) = (k-1)\rho$. Then

$$\begin{aligned} ((k+1)+1)\rho = f(((k+1)+1)\rho) &\in f((k+1)\rho \oplus 1\rho) \\ &\subseteq f((k+1)\rho) \oplus f(1\rho) \\ &= (k-1)\rho \oplus 1\rho \\ &= \{k\rho, (k-2)\rho\} \end{aligned}$$

which is a contradiction. Thus $f((k+1)\rho) = (k+1)\rho$.

Hence $f(x\rho) = x\rho$ for all $x \in \mathbb{Z}^+ \cup \{0\}$. Since $x\rho = -x\rho$ for all $x \in \mathbb{Z}$, $f(x\rho) = x\rho$ for all $x \in \mathbb{Z}$, so $f$ satisfies (i). □

**Lemma 3.2.5.** *Let $f : \mathbb{Z}/\rho \to \mathbb{Z}/\rho$ be defined by*

$$f(x\rho) = \begin{cases} 0\rho & \text{if } x \text{ is even,} \\ 1\rho & \text{if } x \text{ is odd.} \end{cases}$$

*Then $f \in \mathrm{Hom}(\mathbb{Z}/\rho, \oplus, *)$.*

*Proof.* By the definition of $\rho, f$ is well-defined. Let $x, y \in \mathbb{Z}$. Then

$$f(x\rho \oplus y\rho) = f(\{(x+y)\rho, (x-y)\rho\}) = \{f((x+y)\rho), f((x-y)\rho)\}$$

and

$$f(x\rho * y\rho) = f((xy)\rho).$$

**Case 1:** $x$ and $y$ are even. Then $x+y, x-y$ and $xy$ are even, so

$$f(x\rho \oplus y\rho) = \{0\rho\} = 0\rho \oplus 0\rho = f(x\rho) \oplus f(y\rho)$$

and

$$f(x\rho * y\rho) = 0\rho = 0\rho * 0\rho = f(x\rho) * f(y\rho).$$

**Case 2:** $x$ and $y$ are odd. Then $x+y$ and $x-y$ are even, and $xy$ is odd. Thus

$$f(x\rho \oplus y\rho) = \{0\rho\} \subseteq \{2\rho, 0\rho\} = 1\rho \oplus 1\rho = f(x\rho) \oplus f(y\rho)$$

and

$$f(x\rho * y\rho) = 1\rho = 1\rho * 1\rho = f(x\rho) * f(y\rho).$$

**Case 3:** $x$ is even and $y$ is odd. Then $x+y$ and $x-y$ are odd, and $xy$ is even. Thus

$$f(x\rho \oplus y\rho) = \{1\rho\} = 0\rho \oplus 1\rho = f(x\rho) \oplus f(y\rho)$$

and

$$f(x\rho * y\rho) = 0\rho = 0\rho * 1\rho = f(x\rho) * f(y\rho).$$

**Case 4:** $x$ is odd and $y$ is even. We can show similary to Case 3 that $f(x\rho \oplus y\rho) \subseteq f(x\rho) \oplus f(y\rho)$ and $f(x\rho * y\rho) = f(x\rho) * f(y\rho)$.

Hence we have that $f \in \mathrm{Hom}(\mathbb{Z}/\rho, \oplus, *)$, as desired. $\square$

It is evident that the zero mapping and the identity mapping on $(\mathbb{Z}/\rho, \oplus, *)$ are homomorphisms. The following theorem is directly obtained from Lemma 3.2.2, Lemma 3.2.3, Lemma 3.2.4 and Lemma 3.2.5.

**Theorem 3.2.6.** *Assume that $f : \mathbb{Z}/\rho \to \mathbb{Z}/\rho$. Then $f \in \mathrm{Hom}(\mathbb{Z}/\rho, \oplus, *)$ if and only if one of the following statements holds.*

(i) *$f$ is the zero mapping on $(\mathbb{Z}/\rho, \oplus, *)$.*

(ii) *$f$ is the identity mapping on $\mathbb{Z}/\rho$.*

(iii) $f(x\rho) = \begin{cases} 0\rho & \text{if } x \text{ is even,} \\ 1\rho & \text{if } x \text{ is odd.} \end{cases}$

## 3.3 The Krasner Hyperring Defined from the Ring $(\mathbb{Z}_n, +, \cdot)$ and an Equivalence Relation

This section deals with the Krasner hyperring $(\mathbb{Z}_n/\rho, \oplus, *)$ defined from the ring $(\mathbb{Z}_n, +, \cdot)$ and the equivalence relation $\rho$ (see Example 1.5), i.e.,

$$\bar{x}\rho\bar{y} \iff \bar{x} = \bar{y} \text{ or } \bar{x} = -\bar{y},$$

$$\bar{x}\rho \oplus \bar{y}\rho = \{\overline{(x+y)}\rho,\ \overline{(x-y)}\rho\},$$

$$\bar{x}\rho * \bar{y}\rho = \overline{(xy)}\rho \text{ for all } x, y \in \mathbb{Z}.$$

Notice that for $x \in \mathbb{Z}, \bar{x}\rho = \{\bar{x}, -\bar{x}\} = \{\bar{x}, \overline{n-x}\}$. If $n$ is even, then $\mathbb{Z}_n/\rho = \{\bar{0}\rho, \bar{1}\rho, ..., \overline{(\frac{n}{2})}\rho\}$ and $|\mathbb{Z}_n/\rho| = \frac{n}{2}+1$. If $n$ is odd, then $\mathbb{Z}_n/\rho = \{\bar{0}\rho, \bar{1}\rho, ..., \overline{(\frac{n-1}{2})}\rho\}$ and $|\mathbb{Z}_n/\rho| = \frac{n+1}{2}$.

The following remark shows the possibilities of $f(\bar{0}\rho)$ where $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus)$.

**Remark 3.3.1.** *The following statements hold.*

(i) *If* $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus)$, *then*

$$f(\bar{0}\rho) = \begin{cases} \bar{0}\rho & \text{if } 3 \nmid n, \\ \bar{0}\rho \text{ or } \overline{(\frac{n}{3})}\rho & \text{if } 3 \mid n. \end{cases}$$

(ii) *If* $3 \mid n$ *and* $f : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ *is defined by* $f(\bar{x}\rho) = \overline{(\frac{n}{3})}\rho$ *for all* $x \in \mathbb{Z}$, *then* $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus)$.

*Proof.* (i) Let $f(\bar{0}\rho) = \bar{a}\rho$ for some $a \in \{0, 1, 2, ..., n-1\}$. Then

$$\bar{a}\rho = f(\bar{0}\rho) \in f(\bar{0}\rho \oplus \bar{0}\rho) \subseteq f(\bar{0}\rho) \oplus f(\bar{0}\rho) = \bar{a}\rho \oplus \bar{a}\rho = \{(\overline{2a})\rho, \bar{0}\rho\},$$

so $\bar{a}\rho = (\overline{2a})\rho$ or $\bar{a}\rho = \bar{0}\rho$. Assume that $\bar{a}\rho = (\overline{2a})\rho$. Then $\bar{a} = \overline{2a}$ or $\bar{a} = -(\overline{2a})$ which implies that $n \mid a$ or $n \mid 3a$. If $n \mid a$, then $\bar{a} = \bar{0}$, so $\bar{a}\rho = \bar{0}\rho$. Next, assume that $n \mid 3a$.

**Case 1:** $3 \nmid n$. Since $n \mid 3a$, it follows that $n \mid a$, so $\bar{a} = \bar{0}$, i.e., $\bar{a}\rho = \bar{0}\rho$.

**Case 2:** $3 \mid n$. Since $n \mid 3a$, we have that $\frac{n}{3} \mid a$. But $a \in \{0, 1, 2, ..., n-1\}$, so $a = 0, \frac{n}{3}$ or $\frac{2n}{3}$. Since $\overline{(\frac{n}{3})} = -\overline{(\frac{2n}{3})}$, it follows that $\overline{(\frac{n}{3})}\rho = \overline{(\frac{2n}{3})}\rho$. Hence $\bar{a}\rho = \bar{0}\rho$ or

$\overline{(\frac{n}{3})}\rho$.

(ii) Since $-\frac{n}{3} = \frac{2n}{3}$, it follows that $\overline{(\frac{n}{3})}\rho = \overline{(\frac{2n}{3})}\rho$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} f(\bar{x}\rho \oplus \bar{y}\rho) = \{\overline{(\frac{n}{3})}\rho\} \subseteq \{\overline{(\frac{n}{3})}\rho, \bar{0}\rho\} &= \{\overline{(\frac{2n}{3})}\rho, \bar{0}\rho\} \\ &= \overline{(\frac{n}{3})}\rho \oplus \overline{(\frac{n}{3})}\rho \\ &= f(\bar{x}\rho) \oplus f(\bar{y}\rho), \end{aligned}$$

so we have that $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus)$, as desired. □

To obtain the main theorems of this section, the following lemmas are needed.

**Lemma 3.3.2.** *Let $n$ be even and $f : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ defined by*

$$f(\bar{x}\rho) = \begin{cases} \bar{0}\rho & \text{if } x \text{ is even,} \\ \bar{1}\rho & \text{if } x \text{ is odd.} \end{cases}$$

*Then $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$.*

*Proof.* To show that $f$ is well-defined, let $x, y \in \mathbb{Z}$ be such that $\bar{x}\rho = \bar{y}\rho$. Then $\bar{x} = \bar{y}$ or $\bar{x} = -\bar{y}$, so $n \mid (x - y)$ or $n \mid (x + y)$. Since $n$ is even, it follows that $x - y$ or $x + y$ is even which implies that either $x$ and $y$ are even or $x$ and $y$ are odd. The remainder of the proof is given similarly to that of Lemma 3.2.5 □

**Lemma 3.3.3.** *Assume that $n$ is even. If $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ is such that $f(\bar{0}\rho) = \bar{0}\rho$ and $f(\bar{1}\rho) = \bar{1}\rho$, then either*

(i) *$f$ is the identity mapping on $\mathbb{Z}_n/\rho$ or*

(ii) $f(\bar{x}\rho) = \begin{cases} \bar{0}\rho & \text{if } x \text{ is even,} \\ \bar{1}\rho & \text{if } x \text{ is odd.} \end{cases}$

*Proof.* Recall that $\mathbb{Z}_n/\rho = \{\bar{0}\rho, \bar{1}\rho, \bar{2}\rho, ..., \overline{(\frac{n}{2})}\rho\}$ and $|\mathbb{Z}_n/\rho| = \frac{n}{2} + 1$. Let $A = \{0, 1, 2, ..., \frac{n}{2}\}$. Then $\mathbb{Z}_n/\rho = \{\bar{x}\rho \mid x \in A\}$. If $n = 2$, by assumption, we are done. Assume that $n \geq 4$. Since

$$f(\bar{2}\rho) \in f(\bar{1}\rho \oplus \bar{1}\rho) \subseteq f(\bar{1}\rho) \oplus f(\bar{1}\rho) = \bar{1}\rho \oplus \bar{1}\rho = \{\bar{2}\rho, \bar{0}\rho\},$$

we have that $f(\bar{2}\rho) = \bar{2}\rho$ or $f(\bar{2}\rho) = \bar{0}\rho$.

**Case 1:** $f(\bar{2}\rho) = \bar{2}\rho$. Claim that $f$ is the identity mapping on $\mathbb{Z}_n/\rho$, i.e., claim that $f(\bar{x}\rho) = \bar{x}\rho$ for all $x \in A$. Note that $f(\bar{0}\rho) = \bar{0}\rho, f(\bar{1}\rho) = \bar{1}\rho$ and $f(\bar{2}\rho) = \bar{2}\rho$.

Assume that $k \in A$, $k \geq 2$, $k+1 \in A$ and $f(\bar{x}\rho) = \bar{x}\rho$ for all $x \in \{0, 1, 2, ..., k\}$. Then

$$f((\overline{k+1})\rho) \in f(\bar{k}\rho \oplus \bar{1}\rho) \subseteq f(\bar{k}\rho) \oplus f(\bar{1}\rho) = \bar{k}\rho \oplus \bar{1}\rho = \{(\overline{k+1})\rho, (\overline{k-1})\rho\}.$$

**Subcase 1.1 :** $k+1$ is even. Then $k+1 = 2a$ for some $a \in A$, so $a < k$. Thus

$$\begin{aligned} f((\overline{k+1})\rho) = f((\overline{2a})\rho) = f(\bar{2}\rho * \bar{a}\rho) &= f(\bar{2}\rho) * f(\bar{a}\rho) \\ &= \bar{2}\rho * \bar{a}\rho \\ &= (\overline{2a})\rho \\ &= (\overline{k+1})\rho. \end{aligned}$$

**Subcase 1.2 :** $k+1$ is odd and $k+1 < \frac{n}{2}$. Then $k+2$ is even and $k+2 \in A$. Let $k+2 = 2b$ for some $b \in A$. Then $b \leq k$, and hence

$$\begin{aligned} f((\overline{k+2})\rho) = f((\overline{2b})\rho) = f(\bar{2}\rho * \bar{b}\rho) &= f(\bar{2}\rho) * f(\bar{b}\rho) \\ &= \bar{2}\rho * \bar{b}\rho \\ &= (\overline{2b})\rho \\ &= (\overline{k+2})\rho. \end{aligned}$$

To show that $f((\overline{k+1})\rho) = (\overline{k+1})\rho$, suppose not. Since $f((\overline{k+1})\rho) \in \{(\overline{k+1})\rho, (\overline{k-1})\rho\}$, we have that $f((\overline{k+1})\rho) = (\overline{k-1})\rho$. It follows that

$$\begin{aligned} (\overline{k+2})\rho = f((\overline{k+2})\rho) &\in f((\overline{k+1})\rho \oplus \bar{1}\rho) \\ &\subseteq f((\overline{k+1})\rho) \oplus f(\bar{1}\rho) \\ &= (\overline{k-1})\rho \oplus \bar{1}\rho \\ &= \{\bar{k}\rho, (\overline{k-2})\rho\} \end{aligned}$$

which is a contradiction since $k-2, k, k+2 \in A$. Thus $f((\overline{k+1})\rho) = (\overline{k+1})\rho$.

**Subcase 1.3 :** $k+1$ is odd and $k+1=\frac{n}{2}$. Then $n\geq 6$. Since $f((\overline{k+1})\rho)\in\{(\overline{k+1})\rho,(\overline{k-1})\rho\}$, we have that $f((\overline{k+1})\rho)\in\{(\overline{k+1})\rho,(\overline{\frac{n}{2}-2})\rho\}$. Suppose that $f((\overline{k+1})\rho)=(\overline{\frac{n}{2}-2})\rho$. Then

$$\begin{aligned}\bar{0}\rho=f(\bar{0}\rho)=f(\bar{n}\rho)&=f(\bar{2}\rho*(\overline{\frac{n}{2}})\rho)\\&=f(\bar{2}\rho)*f((\overline{\frac{n}{2}})\rho)\\&=f(\bar{2}\rho)*f((\overline{k+1})\rho)\\&=\bar{2}\rho*(\overline{\frac{n}{2}-2})\rho\\&=(\overline{n-4})\rho\end{aligned}$$

which is a contradiction since $n\geq 6$. Therefore $f((\overline{k+1})\rho)=(\overline{k+1})\rho$.

Hence we have the claim, i.e., $f$ satisfies (i).

**Case 2:** $f(\bar{2}\rho)=\bar{0}\rho$. Then for $k\in\mathbb{Z}$,

$$f((\overline{2k})\rho)=f(\bar{2}\rho*\bar{k}\rho)=f(\bar{2}\rho)*f(\bar{k}\rho)=\bar{0}\rho*f(\bar{k}\rho)=\bar{0}\rho$$

and so

$$f((\overline{2k+1})\rho)\in f((\overline{2k})\rho\oplus\bar{1}\rho)\subseteq f((\overline{2k})\rho)\oplus f(\bar{1}\rho)=\bar{0}\rho\oplus\bar{1}\rho=\{\bar{1}\rho\}.$$

Hence $f$ satisfies (ii).

Therefore the proof of the lemma is complete. □

**Lemma 3.3.4.** *Assume that $n$ is odd. If $f\in\mathrm{Hom}(\mathbb{Z}_n/\rho,\oplus,*)$ is such that $f(\bar{0}\rho)=\bar{0}\rho$ and $f(\bar{1}\rho)=\bar{1}\rho$, then $f$ is the identity mapping on $\mathbb{Z}_n/\rho$.*

*Proof.* Recall that $\mathbb{Z}_n/\rho=\{\bar{0}\rho,\ \bar{1}\rho,\ \bar{2}\rho,\ ...,\ (\overline{\frac{n-1}{2}})\rho\}$ and $|\mathbb{Z}_n/\rho|=\frac{n+1}{2}$. Let $A=\{0,\ 1,\ 2,...,\ \frac{n-1}{2}\}$. If $n=1$ or $3$, then we are done. Assume that $n\geq 5$. We can see from the proof of Lemma 3.3.3 that $f(\bar{2}\rho)\in\{\bar{2}\rho,\bar{0}\rho\}$. Suppose that $f(\bar{2}\rho)=\bar{0}\rho$. From the proof of Lemma 3.3.3 for Case 2, we have that for $k\in\mathbb{Z}$,

$$f((\overline{2k})\rho)=\bar{0}\rho \text{ and } f((\overline{2k+1})\rho)=\bar{1}\rho.$$

It follows that

$$f((\overline{\frac{n-1}{2}})\rho)\oplus f((\overline{\frac{n-1}{2}})\rho)=\begin{cases}\{\bar{0}\rho\} & \text{if } \frac{n-1}{2} \text{ is even,}\\ \{\bar{2}\rho,\ \bar{0}\rho\} & \text{if } \frac{n-1}{2} \text{ is odd}\end{cases}$$

and

$$f((\overline{\frac{n-1}{2}})\rho \oplus (\overline{\frac{n-1}{2}})\rho) = f(\{(\overline{n-1})\rho, \bar{0}\rho\}) = f(\{\bar{1}\rho, \bar{0}\rho\})$$
$$= \{f(\bar{1}\rho), f(\bar{0}\rho)\}$$
$$= \{\bar{1}\rho, \bar{0}\rho\}.$$

Since $n \geq 5$, we deduce that $f((\overline{\frac{n-1}{2}})\rho \oplus (\overline{\frac{n-1}{2}})\rho) \nsubseteq f((\overline{\frac{n-1}{2}})\rho) \oplus f((\overline{\frac{n-1}{2}})\rho)$, a contradiction. Hence $f(\bar{2}\rho) = \bar{2}\rho$.

Assume that $k \geq 2, k+1 \in A$ and $f(\bar{x}\rho) = \bar{x}\rho$ for all $x \in \{0, 1, 2, ..., k\}$.

**Case 1:** $k+1$ is even. We can see from the proof of Lemma 3.3.3 for Subcase 1.1 that $f((\overline{k+1})\rho) = (\overline{k+1})\rho$.

**Case 2:** $k+1$ is odd and $k+1 < \frac{n-1}{2}$. We can see from the proof of Lemma 3.3.3 for Subcase 1.2 that $f((\overline{k+1})\rho) = (\overline{k+1})\rho$.

**Case 3:** $k+1$ is odd and $k+1 = \frac{n-1}{2}$. Since $-(\overline{\frac{n-1}{2}}) = \overline{\frac{n-1}{2}} + \bar{1}$, it follows that $(\overline{k+1})\rho = (\overline{k+2})\rho$. Since $k+2$ is even, it can be seen from the proof of Lemma 3.3.3 for Subcase 1.2 that $f((\overline{k+2})\rho) = (\overline{k+2})\rho$. Hence

$$f((\overline{k+1})\rho) = f((\overline{k+2})\rho) = (\overline{k+2})\rho = (\overline{k+1})\rho.$$

Therefore we have that $f(\overline{x}\rho) = \overline{x}\rho$ for all $x \in A$, i.e., $f$ is the identity mapping on $\mathbb{Z}_n/\rho$. □

The following theorem is directly obtained from Lemma 3.3.2 and Lemma 3.3.3, and the next theorem is obtained from Lemma 3.3.4.

**Theorem 3.3.5.** *Assume that $n$ is even and $f : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ is such that $f(\bar{0}\rho) = \bar{0}\rho$ and $f(\bar{1}\rho) = \bar{1}\rho$. Then $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ if and only if either*

(i) *$f$ is the identity mapping on $\mathbb{Z}_n/\rho$ or*

(ii) $f(\bar{x}\rho) = \begin{cases} \bar{0}\rho & \text{if } x \text{ is even,} \\ \bar{1}\rho & \text{if } x \text{ is odd.} \end{cases}$

**Theorem 3.3.6.** *Assume that $n$ is odd and $f : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ is such that $f(\bar{0}\rho) = \bar{0}\rho$ and $f(\bar{1}\rho) = \bar{1}\rho$. Then $f \in \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ if and only if $f$ is the identity mapping on $\mathbb{Z}_n/\rho$.*

**Remark 3.3.7.** Theorem 3.3.5 and Theorem 3.3.6 characterize the elements of $\mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ which fix the elements $\bar{0}\rho$ and $\bar{1}\rho$ of $\mathbb{Z}_n/\rho$. In fact, the elements of $\mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$ need not have this property as shown by the following examples.

Let $\bar{a}\rho$ be an idempotent of the semigroup $(\mathbb{Z}_n/\rho, *)$ and define $k_{\bar{a}\rho} : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ by

$$k_{\bar{a}\rho}(\bar{x}\rho) = (\overline{xa})\rho \quad \text{for all } x \in \mathbb{Z}.$$

If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} k_{\bar{a}\rho}(\bar{x}\rho \oplus \bar{y}\rho) &= k_{\bar{a}\rho}(\{\overline{(x+y)}\rho,\ \overline{(x-y)}\rho\}) \\ &= \{k_{\bar{a}\rho}(\overline{(x+y)}\rho),\ k_{\bar{a}\rho}(\overline{(x-y)}\rho)\} \\ &= \{(\overline{(x+y)a})\rho,\ (\overline{(x-y)a})\rho\} \\ &= \{\overline{(xa+ya)}\rho,\ \overline{(xa-ya)}\rho\} \\ &= \{(\overline{xa}+\overline{ya})\rho,\ (\overline{xa}-\overline{ya})\rho\} \\ &= (\overline{xa})\rho \oplus (\overline{ya})\rho \\ &= k_{\bar{a}\rho}(\bar{x}\rho) \oplus k_{\bar{a}\rho}(\bar{y}\rho) \end{aligned}$$

and

$$\begin{aligned} k_{\bar{a}\rho}(\bar{x}\rho * \bar{y}\rho) = k_{\bar{a}\rho}(\overline{xy})\rho &= (\overline{xya})\rho \\ &= (\overline{xy})\rho * \bar{a}\rho \\ &= (\overline{xy})\rho * \bar{a}\rho * \bar{a}\rho \\ &= (\overline{xa})\rho * (\overline{ya})\rho \\ &= k_{\bar{a}\rho}(\bar{x}\rho) * k_{\bar{a}\rho}(\bar{y}\rho). \end{aligned}$$

This proves that

$$\{k_{\bar{a}\rho} \mid \bar{a}\rho \text{ is an idempotent of } (\mathbb{Z}_n/\rho, *)\} \subseteq \mathrm{Hom}(\mathbb{Z}_n/\rho, \oplus, *).$$

We can see that for distinct idempotents $\bar{a}\rho, \bar{b}\rho$ of $(\mathbb{Z}_n/\rho, *)$, $k_{\bar{a}\rho} \neq k_{\bar{b}\rho}$.

Next, assume that $n$ is even. For an idempotent $\bar{a}\rho$ of $(\mathbb{Z}_n/\rho, *)$, define $l_{\bar{a}\rho} : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ by

$$l_{\bar{a}\rho}(\bar{x}\rho) = \begin{cases} \bar{0}\rho & \text{if } x \text{ is even,} \\ \bar{a}\rho & \text{if } x \text{ is odd.} \end{cases}$$

It can be seen from the proof of Lemma 3.3.2 that $l_{\bar{a}\rho}$ is well-defined for every idempotent $\bar{a}\rho$ of $(\mathbb{Z}_n/\rho, *)$. From the proof of Lemma 3.2.5 and the fact that $\bar{a}\rho * \bar{a}\rho = \bar{a}\rho$, we can see that $l_{\bar{a}\rho} \in \text{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$. Hence

$$\{l_{\bar{a}\rho} \mid \bar{a}\rho \text{ is an idempotent of } (\mathbb{Z}_n/\rho, *)\} \subseteq \text{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$$

and we can see that $l_{\bar{a}\rho} \neq l_{\bar{b}\rho}$ if $\bar{a}\rho \neq \bar{b}\rho$.

Moreover, if $3 \mid n$ and $\overline{(\frac{n}{3})}\rho$ is an idempotent of $(\mathbb{Z}_n/\rho, *)$. The mapping $q : \mathbb{Z}_n/\rho \to \mathbb{Z}_n/\rho$ defined by

$$q(\bar{x}\rho) = \overline{(\tfrac{n}{3})}\rho \text{ for all } x \in \mathbb{Z}$$

belongs to $\text{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$. By Remark 3.3.1(ii), $q \in \text{Hom}(\mathbb{Z}_n/\rho, \oplus)$. If $x, y \in \mathbb{Z}$, then

$$q(\bar{x}\rho * \bar{y}\rho) = \overline{(\frac{n}{3})}\rho = \overline{(\frac{n}{3})}\rho * \overline{(\frac{n}{3})}\rho = q(\bar{x}\rho) * q(\bar{y}\rho).$$

Thus $q \in \text{Hom}(\mathbb{Z}_n/\rho, \oplus, *)$, as desired. In particular, $q : \mathbb{Z}_6/\rho \to \mathbb{Z}_6/\rho$ defined by $q(\bar{x}\rho) = \bar{2}\rho$ for all $x \in \mathbb{Z}$ is an element of $\text{Hom}(\mathbb{Z}_6/\rho, \oplus, *)$. Hence $k_{\bar{2}\rho}$, $k_{\bar{3}\rho}$, $l_{\bar{2}\rho}$ and $l_{\bar{3}\rho}$ are elements of $\text{Hom}(\mathbb{Z}_6/\rho, \oplus, *)$ which do not fix $\bar{1}\rho$ and $q$ is an element of $\text{Hom}(\mathbb{Z}_6/\rho, \oplus, *)$ not fixing $\bar{0}\rho$ and $\bar{1}\rho$.

## 3.4 A Krasner Hyperring Defined from the Interval $[0, \infty)$ with the Usual Multiplication

In this section, we characterize the homomorphisms of the Krasner hyperring $([0, \infty), \oplus, \cdot)$ defined in Example 1.3, i.e.,

$$x \oplus y = \begin{cases} [0, x] & \text{if } x = y, \\ \{\max\{x, y\}\} & \text{if } x \neq y, \end{cases}$$

and also show that the set $\text{Hom}([0, \infty), \oplus, \cdot)$ is uncountable.

We first provide the following lemmas.

**Lemma 3.4.1.** *For $f : [0, \infty) \to [0, \infty)$, $f \in \text{Hom}([0, \infty), \oplus)$ if and only if $f$ is increasing.*

*Proof.* Let $f \in \operatorname{Hom}([0,\infty),\oplus)$ and let $x, y \in [0,\infty)$ be such that $x < y$. Suppose that $f(x) > f(y)$ Then

$$f(\{y\}) = f(x \oplus y) \subseteq f(x) \oplus f(y) = \{f(x)\}$$

which is a contradiction. Thus $f(x) \leq f(y)$. This shows that $f$ is increasing.

Conversely, assume that $f$ is increasing. If $x \in [0,\infty)$, then $f(0) \leq f(t) \leq f(x)$, for all $t \in [0, x]$, so

$$f(x \oplus x) = f([0,x]) \subseteq [f(0), f(x)] \subseteq [0, f(x)] = f(x) \oplus f(x).$$

If $x, y \in [0,\infty)$ are such that $x < y$, then $f(x) \leq f(y)$, so

$$\begin{aligned} f(x \oplus y) &= f(\{y\}) = \{f(y)\} \\ &\subseteq \begin{cases} [0, f(y)] = f(x) \oplus f(y) & \text{if } f(x) = f(y), \\ \{f(y)\} \ = f(x) \oplus f(y) & \text{if } f(x) < f(y). \end{cases} \end{aligned}$$

Therefore $f \in \operatorname{Hom}([0,\infty),\oplus)$, as desired. □

**Lemma 3.4.2.** *If $f \in \operatorname{Hom}([0,\infty),\cdot)$, then one of the following statements holds.*

(i) *$f$ is the zero mapping on the semigroup* $([0,\infty),\cdot)$.

(ii) $f(x) = 1$ *for all* $x \in [0,\infty)$.

(iii) $f(0) = 0$ *and* $f|_{(0,\infty)} \in \operatorname{Hom}((0,\infty),\cdot)$.

*Proof.* Assume that $f \in \operatorname{Hom}([0,\infty),\cdot)$.

**Case 1:** $f(0) \neq 0$. If $x \in [0,\infty)$, then $f(0) = f(0 \cdot x) = f(0) \cdot f(x)$ which implies that $f(x) = 1$. Therefore $f$ satisfies (ii).

**Case 2:** $f(0) = 0$ and $f(a) = 0$ for some $a \in (0,\infty)$. Then

$$f(1) = f(a \cdot a^{-1}) = f(a) \cdot f(a^{-1}) = 0 \cdot f(a^{-1}) = 0,$$

so for $x \in (0,\infty)$,

$$f(x) = f(x \cdot 1) = f(x) \cdot f(1) = f(x) \cdot 0 = 0.$$

Hence $f$ satisfies (i).

**Case 3:** $f(0) = 0$ and $f(a) \neq 0$ for all $a \in (0, \infty)$. Then $f((0, \infty)) \subseteq (0, \infty)$. This implies that $f|_{(0,\infty)} \in \text{Hom}((0, \infty), \cdot)$. Therefore $f$ satisfies (iii). □

We remark from Lemma 3.4.1 that every constant function from $[0, \infty)$ into itself is an element of $\text{Hom}([0, \infty), \oplus)$.

The following theorem is directly obtained from the above remark, Lemma 3.4.1 and Lemma 3.4.2.

**Theorem 3.4.3.** *For $f : [0, \infty) \to [0, \infty)$, $f \in \text{Hom}([0, \infty), \oplus, \cdot)$ if and only if one of the following statements holds.*

(i) *$f$ is the zero mapping on the semigroup* $([0, \infty), \cdot)$.

(ii) $f(x) = 1$ *for all* $x \in [0, \infty)$.

(iii) $f(0) = 0$, $f|_{(0,\infty)} \in \text{Hom}((0, \infty), \cdot)$ *and $f$ is increasing.*

**Theorem 3.4.4.** $\text{Hom}([0, \infty), \oplus, \cdot)$ *is an uncountable set.*

*Proof.* For $a \in [0, \infty)$, define $k_a : [0, \infty) \to [0, \infty)$ by

$$k_a(x) = x^a \text{ for all } x \in [0, \infty).$$

Then it is clear that $k_a \in \text{Hom}([0, \infty), \cdot)$ and $k_a$ is increasing on $[0, \infty)$ for all $a \in [0, \infty)$. By Lemma 3.4.1, $k_a \in \text{Hom}([0, \infty), \oplus)$ for all $a \in [0, \infty)$. If $a, b \in [0, \infty)$ are distinct, then $k_a(2) = 2^a \neq 2^b = k_b(2)$, so $k_a \neq k_b$. Thus $| \{k_a \mid a \in [0, \infty)\} | = | [0, \infty) |$ and $\{k_a \mid a \in [0, \infty)\} \subseteq \text{Hom}([0, \infty), \oplus, \cdot)$. But $[0, \infty)$ is an uncountable set, so the set $\text{Hom}([0, \infty), \oplus, \cdot)$ is uncountable. □

# CHAPTER IV

# HOMOMORPHISMS OF P-HYPERRINGS

In this chapter, we are concerned with the homomorphisms of the P-hyperings $(\mathbb{Z},\ \oplus_{l\mathbb{Z}},\ \circ_{m\mathbb{Z}})$ and $(\mathbb{Z}_n,\ \oplus_{l\mathbb{Z}_n},\ \circ_{m\mathbb{Z}_n})$ defined as in Example 1.6. First, we determine the set $\text{Hom}(\mathbb{Z},\ +) \cap \text{Hom}(\mathbb{Z},\ \oplus_{l\mathbb{Z}},\ \circ_{m\mathbb{Z}})$ and construct an element of $\text{Hom}(\mathbb{Z},\ \oplus_{l\mathbb{Z}},\ \circ_{m\mathbb{Z}}) \smallsetminus \text{Hom}(\mathbb{Z},\ +)$ for certain $l, m$. It is shown that $\text{Hom}(\mathbb{Z}_n,\ +) \subseteq \text{Hom}(\mathbb{Z}_n,\ \oplus_{l\mathbb{Z}_n},\ \circ_{m\mathbb{Z}_n})$ if and only if $\frac{n}{(m,n)}$ is square-free. We also construct $f \in \text{Hom}(\mathbb{Z}_n,\ \oplus_{l\mathbb{Z}_n},\ \circ_{m\mathbb{Z}_n}) \smallsetminus \text{Hom}(\mathbb{Z}_n,\ +)$ for certain $l, m$.

## 4.1 P-hyperrings Defined from the Ring $(\mathbb{Z}, +, \cdot)$

In this section, we determine $\text{Hom}(\mathbb{Z}, +) \cap \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}})$. Recall that

$$x \oplus_{l\mathbb{Z}} y = x + y + l\mathbb{Z} \quad \text{and} \quad x \circ_{m\mathbb{Z}} y = x(m\mathbb{Z})y \quad \text{for all } x, y \in \mathbb{Z}.$$

The following two lemmas are needed.

**Lemma 4.1.1.** $\text{Hom}(\mathbb{Z}, +) \subseteq \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}})$.

*Proof.* If $a, x, y \in \mathbb{Z}$, then

$$\begin{aligned} g_a(x \oplus_{l\mathbb{Z}} y) &= g_a(x + y + l\mathbb{Z}) \\ &= a(x + y + l\mathbb{Z}) \\ &= ax + ay + al\mathbb{Z} \\ &\subseteq ax + ay + l\mathbb{Z} \\ &= g_a(x) + g_a(y) + l\mathbb{Z} \\ &= g_a(x) \oplus_{l\mathbb{Z}} g_a(y) \end{aligned}$$

which implies that $g_a \in \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}})$. But $\text{Hom}(\mathbb{Z}, +) = \{g_a \mid a \in \mathbb{Z}\}$, so $\text{Hom}(\mathbb{Z}, +) \subseteq \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}})$. □

**Lemma 4.1.2.** *The following statements hold.*

(i) *If* $m \neq 0$, *then for* $a \in \mathbb{Z}, g_a \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$ *if and only if* $a \in \{0, 1, -1\}$.

(ii) *If* $m = 0$, *then* $\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$.

*Proof.* (i) Assume that $g_a \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. Then

$$\begin{aligned} am\mathbb{Z} = a(m\mathbb{Z}) &= a(1(m\mathbb{Z})1) \\ &= g_a(1 \circ_{m\mathbb{Z}} 1) \\ &= g_a(1) \circ_{m\mathbb{Z}} g_a(1) \\ &= a \circ_{m\mathbb{Z}} a \\ &= a(m\mathbb{Z})a \\ &= a^2 m\mathbb{Z} \end{aligned}$$

which implies that $\pm am = a^2 m$, so $\pm a = a^2$. Thus $a \in \{0, 1, -1\}$.

Conversely, assume that $a \in \{0, 1, -1\}$. If $x, y \in \mathbb{Z}$, then

$$g_a(x \circ_{m\mathbb{Z}} y) = g_a(x(m\mathbb{Z})y) = axm\mathbb{Z}y$$

and

$$g_a(x) \circ_{m\mathbb{Z}} g_a(y) = ax \circ_{m\mathbb{Z}} ay = ax(m\mathbb{Z})ay.$$

Since $\pm a = a^2, axm\mathbb{Z}y = axm\mathbb{Z}ay$. Thus $g_a(x \circ_{m\mathbb{Z}} y) = g_a(x) \circ_{m\mathbb{Z}} g_a(y)$. Hence $g_a \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$.

(ii) Assume that $m = 0$. Let $a, x, y \in \mathbb{Z}$. Then

$$g_a(x \circ_{m\mathbb{Z}} y) = g_a(x\{0\}y) = g_a(\{0\}) = a(\{0\}) = \{0\}$$

and

$$g_a(x) \circ_{m\mathbb{Z}} g_a(y) = ax \circ_{m\mathbb{Z}} ay = ax(\{0\})ay = \{0\}.$$

Thus $g_a(x \circ_{m\mathbb{Z}} y) = g_a(x) \circ_{m\mathbb{Z}} g_a(y)$. Hence $g_a \in \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. Therefore Hom $(\mathbb{Z}, +) \subseteq \mathrm{Hom}(\mathbb{Z}, \circ_{m\mathbb{Z}})$. □

From Lemma 4.1.1 and Lemma 4.1.2, we have the following theorem.

**Theorem 4.1.3.** $\mathrm{Hom}(\mathbb{Z}, +) \cap \mathrm{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}}) = \{g_0, g_1, g_{-1}\}$ *if* $m \neq 0$,

$$\mathrm{Hom}(\mathbb{Z}, +) \subseteq \mathrm{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{m\mathbb{Z}}) \text{ if } m = 0.$$

**Remark 4.1.4.** We shall construct an element $f \in \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{kl\mathbb{Z}}) \smallsetminus \text{Hom}(\mathbb{Z}, +)$ when $k > 0$ and $kl > 1$.

Assume that $k > 0$ and $kl > 1$. We know that $\mathbb{Z} = \bigcup_{i=0}^{kl-1}(i + kl\mathbb{Z})$ which is a disjoint union. Since $\mathbb{Z} = \bigcup_{i=0}^{k-1}(i + k\mathbb{Z})$, it follows that $l\mathbb{Z} = \bigcup_{i=0}^{k-1}(il + kl\mathbb{Z})$. Define $f : \mathbb{Z} \to \mathbb{Z}$ by

$$f(i + kl\mathbb{Z}) = \{i\} \text{ for all } i \in \{0, 1, 2, \dots, kl - 1\}.$$

To show that $f \in \text{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{kl\mathbb{Z}})$, let $x, y \in \mathbb{Z}$. Then $x \in i + kl\mathbb{Z}$ and $y \in j + kl\mathbb{Z}$ for some $i, j \in \{0, 1, \dots, kl - 1\}$ and for $t \in \{0, 1, 2, \dots, k - 1\}$, $x + y + tl + kl\mathbb{Z} = i + j + tl + kl\mathbb{Z} = a_t + kl\mathbb{Z}$ for some $a_t \in \{0, 1, \dots, kl - 1\}$. Then

$$\begin{aligned}
f(x \oplus_{l\mathbb{Z}} y) &= f(x + y + l\mathbb{Z}) \\
&= f(x + y + (\bigcup_{t=0}^{k-1} tl + kl\mathbb{Z})) \\
&= f(\bigcup_{t=0}^{k-1} x + y + tl + kl\mathbb{Z}) \\
&= f(\bigcup_{t=0}^{k-1} a_t + kl\mathbb{Z}) \\
&= \bigcup_{t=0}^{k-1}\{a_t\},
\end{aligned}$$

$$\begin{aligned}
f(x) \oplus_{l\mathbb{Z}} f(y) &= i \oplus_{l\mathbb{Z}} j \\
&= i + j + l\mathbb{Z} \\
&= i + j + (\bigcup_{t=0}^{k-1} tl + kl\mathbb{Z}) \\
&= \bigcup_{t=0}^{k-1}(i + j + tl + kl\mathbb{Z}) \\
&= \bigcup_{t=0}^{k-1}(a_t + kl\mathbb{Z}),
\end{aligned}$$

$$\begin{aligned} f(x \circ_{kl\mathbb{Z}} y) &= f(x(kl\mathbb{Z})y) \\ &= f(xykl\mathbb{Z}) \\ &\subseteq f(kl\mathbb{Z}) \\ &= \{0\}, \end{aligned}$$

$$\begin{aligned} f(x) \circ_{kl\mathbb{Z}} f(y) &= i \circ_{kl\mathbb{Z}} j \\ &= i(kl\mathbb{Z})j \end{aligned}$$

which imply that $f(x \oplus_{l\mathbb{Z}} y) \subseteq f(x) \oplus_{l\mathbb{Z}} f(y)$ and $f(x \circ_{kl\mathbb{Z}} y) \subseteq f(x) \circ_{kl\mathbb{Z}} f(y)$. Thus $f \in \mathrm{Hom}(\mathbb{Z}, \oplus_{l\mathbb{Z}}, \circ_{kl\mathbb{Z}_n})$.

Next, to show that $f \notin \mathrm{Hom}(\mathbb{Z}, +)$, suppose on the contrary that $f \in \mathrm{Hom}(\mathbb{Z}, +)$. Then $f = \{g_0, g_1, g_{-1}\}$. Let $x = kl + 1$. Then $x > 2$, $x \in 1 + kl\mathbb{Z}$ and

$$1 = f(x) \in \{g_0(x),\ g_1(x),\ g_{-1}(x)\} = \{0,\ x, -x\}$$

which is a contradiction. Hence $f \notin \mathrm{Hom}(\mathbb{Z}, +)$.

## 4.2 P-hyperrings Defined from the Ring $(\mathbb{Z}_n, +, \cdot)$

In this section, we characterize when $\mathrm{Hom}(\mathbb{Z}_n, +) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ holds. Recall that

$$\bar{x} \oplus_{l\mathbb{Z}_n} \bar{y} = \overline{x} + \overline{y} + l\mathbb{Z}_n \quad \text{and} \quad \bar{x} \circ_{m\mathbb{Z}_n} \bar{y} = \bar{x}(m\mathbb{Z}_n)\bar{y} \quad \text{for all } x, y \in \mathbb{Z}.$$

The following series of lemmas is needed.

**Lemma 4.2.1.** $\mathrm{Hom}(\mathbb{Z}_n, +) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n})$.

*Proof.* If $a, x, y \in \mathbb{Z}$, then

$$\begin{aligned} h_{\overline{a}}(\overline{x} \oplus_{l\mathbb{Z}_n} \overline{y}) &= h_{\overline{a}}(\overline{x} + \overline{y} + l\mathbb{Z}_n) \\ &= \overline{a}(\overline{x} + \overline{y} + l\mathbb{Z}_n) \\ &= \overline{ax} + \overline{ay} + al\mathbb{Z}_n \\ &\subseteq \overline{ax} + \overline{ay} + l\mathbb{Z}_n \\ &= h_{\overline{a}}(\overline{x}) + h_{\overline{a}}(\overline{y}) + l\mathbb{Z}_n \\ &= h_{\overline{a}}(\overline{x}) \oplus_{l\mathbb{Z}_n} h_{\overline{a}}(\overline{y}) \end{aligned}$$

which implies that $h_{\overline{a}} \in \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n})$. But $\mathrm{Hom}(\mathbb{Z}_n, +) = \{h_{\overline{a}} \mid a \in \mathbb{Z}\}$, so $\mathrm{Hom}(\mathbb{Z}_n, +) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n})$. $\square$

**Lemma 4.2.2.** *For $a \in \mathbb{Z}$, $h_{\overline{a}} \in \mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$ if and only if $(am, n) = (a^2m, n)$.*

*Proof.* Assume that $h_{\overline{a}} \in \mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$. Then

$$\begin{aligned} am\mathbb{Z}_n &= \overline{a}(\overline{1}\ m\mathbb{Z}_n\overline{1}) \\ &= h_{\overline{a}}(\overline{1} \circ_{m\mathbb{Z}_n} \overline{1}) \\ &\subseteq h_{\overline{a}}(\overline{1}) \circ_{m\mathbb{Z}_n} h_{\overline{a}}(\overline{1}) \\ &= \overline{a} \circ_{m\mathbb{Z}_n} \overline{a} \\ &= \overline{a}(m\mathbb{Z}_n)\overline{a} \\ &= a^2m\mathbb{Z}_n \\ &= am(a\mathbb{Z}_n) \\ &\subseteq am\mathbb{Z}_n, \end{aligned}$$

so $am\mathbb{Z}_n = a^2m\mathbb{Z}_n$. Thus $(am, n)\mathbb{Z}_n = (a^2m, n)\mathbb{Z}_n$, and therefore $\frac{n}{(am,n)} = |(am, n)\mathbb{Z}_n| = |(a^2m, n)\mathbb{Z}_n| = \frac{n}{(a^2m,n)}$. This implies that $(am, n) = (a^2m, n)$.

Conversely, assume that $(am, n) = (a^2m, n)$. Then $am\mathbb{Z}_n = (am, n)\mathbb{Z}_n = (a^2m, n)\mathbb{Z}_n = a^2m\mathbb{Z}_n$. If $x, y \in \mathbb{Z}$, then

$$\begin{aligned} h_{\overline{a}}(\overline{x} \circ_{m\mathbb{Z}_n} \overline{y}) &= h_{\overline{a}}(\overline{x}(m\mathbb{Z}_n)\overline{y}) \\ &= h_{\overline{a}}(xym\mathbb{Z}_n) \\ &= axym\mathbb{Z}_n \\ &= xyam\mathbb{Z}_n \\ &= xya^2m\mathbb{Z}_n \\ &= \overline{ax}(m\mathbb{Z}_n)\overline{ay} \\ &= h_{\overline{a}}(\overline{x}) \circ_{m\mathbb{Z}_n} h_{\overline{a}}(\overline{y}) \end{aligned}$$

which implies that $h_{\overline{a}} \in \mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$. $\square$

Since $\mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n}) = \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}) \cap \mathrm{Hom}(\mathbb{Z}_n, \circ_{m\mathbb{Z}_n})$, from Lemma 4.2.1 and Lemma 4.2.2, we directly obtain the following lemma.

**Lemma 4.2.3.** $\mathrm{Hom}(\mathbb{Z}_n,+) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ *if and only if* $(am,n) = (a^2m,n)$ *for all* $a \in \mathbb{Z}$.

It is evident that for $a \in \mathbb{Z}, (am,n) = (a(m,n),n) = (m,n)(a, \frac{n}{(m,n)})$ and $(a^2m,n) = (a^2(m,n),n) = (m,n)(a^2, \frac{n}{(m,n)})$. Therefore from Lemma 4.2.2 and Lemma 4.2.3, we have respectively that

**Lemma 4.2.4.** *For* $a \in \mathbb{Z}, h_{\overline{a}} \in \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ *if and only if* $(a, \frac{n}{(m,n)}) = (a^2, \frac{n}{(m,n)})$.

**Lemma 4.2.5.** $\mathrm{Hom}\ (\mathbb{Z}_n,+) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ *if and only if* $(a, \frac{n}{(m,n)}) = (a^2, \frac{n}{(m,n)})$ *for all* $a \in \mathbb{Z}$.

**Theorem 4.2.6.** $\mathrm{Hom}(\mathbb{Z}_n,+) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ *if and only if* $\frac{n}{(m,n)}$ *is square-free.*

*Proof.* Assume that $\frac{n}{(m,n)}$ is not square-free. Then there is an integer $b > 1$ such that $b^2 | \frac{n}{(m,n)}$. Thus $(b, \frac{n}{(m,n)}) = b \neq b^2 = (b^2, \frac{n}{(m,n)})$. By Lemma 4.2.4, we have that $h_{\overline{b}} \notin \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$, so $\mathrm{Hom}(\mathbb{Z}_n,+) \nsubseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ This proves that $\mathrm{Hom}(\mathbb{Z}_n,+) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$ implies that $\frac{n}{(m,n)}$ is square-free.

If $\frac{n}{(m,n)}$ is square-free, then $(a, \frac{n}{(m,n)}) = (a^2, \frac{n}{(m,n)})$ for all $a \in \mathbb{Z}$, so by Lemma 4.2.5, $\mathrm{Hom}(\mathbb{Z}_n,+) \subseteq \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{m\mathbb{Z}_n})$. □

**Example 4.2.7.** Since $\frac{12}{(2,12)} = 6$, by Theorem 4.2.6, $\mathrm{Hom}(\mathbb{Z}_{12},+) \subseteq \mathrm{Hom}\ (\mathbb{Z}_{12}, \oplus_{l\mathbb{Z}_{12}}, \circ_{2\mathbb{Z}_{12}})$ for every $l \in \mathbb{Z}$. But since $\frac{12}{(3,12)} = 4$, by Theorem 4.2.6, $\mathrm{Hom}\ (\mathbb{Z}_{12},+) \nsubseteq \mathrm{Hom}(\mathbb{Z}_{12}, \oplus_{l\mathbb{Z}_{12}}, \circ_{3\mathbb{Z}_{12}})$ for every $l \in \mathbb{Z}$.

From Lemma 4.2.4, $\mathrm{Hom}(\mathbb{Z}_{12},+) \smallsetminus \mathrm{Hom}(\mathbb{Z}_{12}, \oplus_{l\mathbb{Z}_{12}}, \circ_{3\mathbb{Z}_{12}}) = \{h_{\overline{2}}, h_{\overline{6}}, h_{\overline{10}}\}$.

**Remark 4.2.8.** We shall construct an element $f \in \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{kl\mathbb{Z}_n}) \smallsetminus \mathrm{Hom}(\mathbb{Z}_n,+)$ when $1 < (kl,n) < n$. This implies from this fact and Example 4.2.7 that $\mathrm{Hom}(\mathbb{Z}_{12},+) \subsetneq \mathrm{Hom}(\mathbb{Z}_{12}, \oplus_{2\mathbb{Z}_{12}}, \circ_{2\mathbb{Z}_{12}})$.

Assume that $1 < (kl,n) < n$. Recall that $\mathbb{Z}_n = \bigcup\limits_{i=0}^{(kl,n)-1} (\overline{i} + (kl,n)\mathbb{Z}_n)$ which is a disjoint union. Let $r = \frac{(kl,n)}{(l,n)}$. Then $r \in \mathbb{Z}^+$ and $r|n$, so $\mathbb{Z}_n = \bigcup_{i=0}^{r-1}(\bar{i} + r\mathbb{Z}_n)$. This implies that

$$\begin{aligned} l\mathbb{Z}_n = (l,n)\mathbb{Z}_n &= \bigcup_{i=0}^{r-1}(\overline{il} + rl\mathbb{Z}_n) \\ &= \bigcup_{i=0}^{r-1}(\overline{il} + \frac{(kl,n)}{(l,n)}(l,n)\mathbb{Z}_n) \\ &= \bigcup_{i=0}^{r-1}(\overline{il} + (kl,n)\mathbb{Z}_n). \end{aligned}$$

Define $f : \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$f(\bar{i} + (kl,n)\mathbb{Z}_n) = \{\bar{i}\} \text{ for all } i \in \{0, 1, \ldots, (kl,n) - 1\}.$$

To show that $f \in \mathrm{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{kl\mathbb{Z}_n})$, let $\bar{x}, \bar{y} \in \mathbb{Z}_n$. Then $\bar{x} \in \bar{i} + (kl,n)\mathbb{Z}_n$ and $\bar{y} \in \bar{j} + (kl,n)\mathbb{Z}_n$ for some $i, j \in \{0, 1, \ldots, (kl,n) - 1\}$. Thus $\bar{x} + (kl,n)\mathbb{Z}_n = \bar{i} + (kl,n)\mathbb{Z}_n$, $\bar{y} + (kl,n)\mathbb{Z}_n = \bar{j} + (kl,n)\mathbb{Z}_n$ and for each $t \in \{0, 1, 2, ..., r-1\}$, $\bar{x} + \bar{y} + \overline{tl} + (kl,n)\mathbb{Z}_n = \bar{i} + \bar{j} + \overline{tl} + (kl,n)\mathbb{Z}_n = \bar{a}_t + (kl,n)\mathbb{Z}_n$ for some $a_t \in \{0, 1, \ldots, (kl,n) - 1\}$. Therefore

$$\begin{aligned} f(\bar{x} \oplus_{l\mathbb{Z}_n} \bar{y}) &= f(\bar{x} + \bar{y} + l\mathbb{Z}_n) \\ &= f(\bar{x} + \bar{y} + (l,n)\mathbb{Z}_n) \\ &= f(\bar{x} + \bar{y} + (\bigcup_{t=0}^{r-1} \overline{tl} + (kl,n)\mathbb{Z}_n)) \\ &= f(\bigcup_{t=0}^{r-1} \bar{x} + \bar{y} + \overline{tl} + (kl,n)\mathbb{Z}_n) \\ &= f(\bigcup_{t=0}^{r-1} \bar{a}_t + (kl,n)\mathbb{Z}_n) \\ &= \bigcup_{t=0}^{r-1} f(\bar{a}_t + (kl,n)\mathbb{Z}_n) \\ &= \bigcup_{t=0}^{r-1}\{\bar{a}_t\}, \end{aligned}$$

$$\begin{aligned}
f(\bar{x}) \oplus_{l\mathbb{Z}_n} f(\bar{y}) &= \bar{i} \oplus_{l\mathbb{Z}_n} \bar{j} \\
&= \bar{i} + \bar{j} + l\mathbb{Z}_n \\
&= \bar{i} + \bar{j} + \bigcup_{t=0}^{r-1} (\overline{tl} + (kl,n)\mathbb{Z}_n) \\
&= \bigcup_{t=0}^{r-1} (\bar{i} + \bar{j} + \overline{tl} + (kl,n)\mathbb{Z}_n) \\
&= \bigcup_{t=0}^{r-1} (\bar{a}_t + (kl,n)\mathbb{Z}_n), \\
f(\bar{x} \circ_{kl\mathbb{Z}_n} \bar{y}) &= f(\bar{x}(kl\mathbb{Z}_n)\bar{y}) \\
&= f(xykl\mathbb{Z}_n) \\
&\subseteq f(kl\mathbb{Z}_n) \\
&= \{\bar{0}\}, \\
f(\bar{x}) \circ_{kl\mathbb{Z}_n} f(\bar{y}) &= \bar{i} \circ_{kl\mathbb{Z}_n} \bar{j} \\
&= \bar{i}(kl\mathbb{Z}_n)\bar{j} \\
&= ijkl\mathbb{Z}_n
\end{aligned}$$

which imply that $f(\bar{x} \oplus_{l\mathbb{Z}_n} \bar{y}) \subseteq f(\bar{x}) \oplus_{l\mathbb{Z}_n} f(\bar{y})$ and $f(\bar{x} \circ_{kl\mathbb{Z}_n} \bar{y}) \subseteq f(\bar{x}) \circ_{kl\mathbb{Z}_n} f(\bar{y})$. Thus $f \in \text{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{kl\mathbb{Z}_n})$.

Next, to show that $f \notin \text{Hom}(\mathbb{Z}_n, +)$, suppose on the contrary that $f \in \text{Hom}(\mathbb{Z}_n, +)$. Then $f = h_{\bar{a}}$ for some $a \in \mathbb{Z}$. Since

$$\{\bar{0}\} = f((kl,n)\mathbb{Z}_n) = h_{\bar{a}}((kl,n)\mathbb{Z}_n) = \bar{a}(kl,n)\mathbb{Z}_n$$

and

$$\{\bar{1}\} = f(\bar{1} + (kl,n)\mathbb{Z}_n) = h_{\bar{a}}(\bar{1} + (kl,n)\mathbb{Z}_n) = \bar{a} + \bar{a}(kl,n)\mathbb{Z}_n,$$

it follows that $\bar{a} = \bar{1}$ and $(kl,n)\mathbb{Z}_n = \{\bar{0}\}$. This implies that $(kl,n) = n$ or $(kl,n) = 0$ which is a contradiction.

From Theorem 4.2.6 and this fact, we conclude that if $\frac{n}{(kl,n)}$ is square-free and $1 < (kl,n) < n$, then $\text{Hom}(\mathbb{Z}_n, +) \subsetneq \text{Hom}(\mathbb{Z}_n, \oplus_{l\mathbb{Z}_n}, \circ_{kl\mathbb{Z}_n})$.

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