เศษส่วนต่อเนื่องอย่างชัดแจ้งที่เกี่ยวข้องกับลำดับฟีโบนักชี

นางสาวมนต์ฤดี ยอดโพธิ์ทอง

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2546

ISBN 974-17-5600-3

ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

EXPLICIT CONTINUED FRACTIONS RELATED TO FIBONACCI SEQUENCES

Miss Monrudee Yodphothong

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2003
ISBN 974-17-5600-3

Thesis Title	Explicit continued fractions related to Fibonacci sequences
By	Miss Monrudee Yodphothong
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Ajchara Harnchoowong, Ph.D.
Thesis Co-advisor	Associate Professor Vichian Laohakosol, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science
(Professor Piamsak Menasveta, Ph.D.)

Thesis Committee

.... Chairman
(Assistant Professor Amorn Wasanawichit, Ph.D.)

.... Thesis Advisor
(Assistant Professor Ajchara Harnchoowong, Ph.D.)

.... Thesis Co-advisor
(Associate Professor Vichian Laohakosol, Ph.D.)

.... Member
(Paisan Nakmahachalasint, Ph.D.)

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

มนต์ฤดี ยอดโพธิ์ทอง : เศษส่วนต่อเนื่องอย่างชัดแจ้งที่เกี่ยวข้องกับลำดับฟีโบนักชี (EXPLICIT CONTINUED FRACTIONS RELATED TO FIBONACCI SEQUENCES) อ.ที่ปรึกษา : ผศ. ดร. อัจฉรา หาญชูวงศ์, อ.ที่ปรึกษาร่วม : รศ. ดร. วิเชียร เลาหโกศล, 42 หน้า. ISBN 974-17-5600-3.

ในปี 1977 เดวิสันค้นพบเศษส่วนต่อเนื่องสำหรับจำนวนจริงกลุ่มหนึ่งที่แทนได้ด้วยผลบวกของส่วนกลับของกำลังของ 2 โดยที่กำลังมาจากลำดับของภาคจำนวนเต็มของพหุคูณของจำนวนทอง ซึ่งเป็นรากของพหุนามลักษณะเฉพาะของลำดับฟีโบนักชี หลังจากนั้นไม่นานอาดัมและเดวิสันได้ขยายผลนี้โดยแทน 2 ด้วยจำนวนเต็มที่มากกว่า 1 ในปี 1986 บาวแมนใช้วิธีแตกต่างออกไปโดยส่วนหนึ่งอาศัยการที่จำนวนเต็มบวกสามารถเขียนเป็นผลบวกของพจน์ในลำดับฟีโบนักชี ซึ่งรู้จักกันในนามของการกระจายเซคเคนดอร์ฟขยายผลของเดวิสันออกไปหารูปแบบเศษส่วนต่อเนื่องอย่างชัดแจ้งของจำนวนที่แทนได้ด้วยเศษส่วนของอนุกรมในรูปแบบข้างต้น

ในวิทยานิพนธ์นี้ เราจะศึกษาปัญหาของการขยายผลงานของบาวแมนโดยการขยายการกระจายเซคเคนดอร์ฟไปสู่การกระจายโดยใช้ลำดับ (h,k) ฟีโบนักชีของเดย์คิน โดยที่ลำดับ$(2,2)$ ฟีโบนักชีคือลำดับฟีโบนักชีเดิม เราสามารถหาเศษส่วนต่อเนื่องอย่างชัดแจ้งสำหรับจำนวนจริงซึ่งอยู่ในรูปเศษส่วนของอนุกรมของเทอมซึ่งกำลังสามารถกระจายได้ในรูปลำดับ(h,k)

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต……………………...........
สาขาวิชา **คณิตศาสตร์**	ลายมือชื่ออาจารย์ที่ปรึกษา………...........
ปีการศึกษา 2546	ลายมือชื่ออาจารย์ที่ปรึกษาร่วม...……...........

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

4572436423 : MAJOR MATHEMATICS
KEY WORDS : CONTINUED FRACTIONS / FIBONACCI SEQUENCES
MONRUDEE YODPHOTHONG : EXPLICIT CONTINUED FRACTIONS RELATED TO FIBONACCI SEQUENCES. THESIS ADVISOR : ASSIST.PROF. AJCHARA HARNCHOOWONG, Ph.D., THESIS CO-ADVISOR : ASSOC. PROF. VICHIAN LAOHAKOSOL, Ph.D., 42 pp. ISBN 974-17-5600-3

In 1977, Davison discovered a remarkable class of continued fractions for real numbers representable as sums of reciprocals of powers of 2; these powers being taken from the sequence of the integral parts of multiples of the golden number, a root of the characteristic polynomial of the Fibonacci recurrence sequence. Soon after, Adams and Davison generalized this result by replacing 2 with any integer greater than 1. In 1986, Bowman using a different approach based partly on unique representation of positive integers by sums of integers belonging to the Fibonacci sequence, known as Zeckendorff expansion, generalized Davison's result even further by obtaining explicit continued fractions for numbers representable as quotients of series of the above form.

In this thesis, we deal with the problem of generalizing Bowman's results by extending Zeckendorff expansion to expansion via the $(h, k)^{th}$ Fibonacci sequence introduced by Daykin; the (2,2)-class is that of Fibonacci sequence. Explicit continued fractions are derived for real numbers which are quotients of series of terms whose powers are representable via (h, k)-class expansions.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

Department **Mathematics**	Student's signature..................................
Field of study **Mathematics**	Advisor's signature..................................
Academic year **2003**	Co-advisor's signature..............................

ACKNOWLEDGEMENTS

I would like to express my profound gratitudes and deep appreciation to Assistant Professor Dr. Ajchara Harnchoowong and Associate Professor Dr. Vichian Laohakosol, my thesis advisor and co-advisor, respectively, for their advice and encouragement. Sincere thanks and deep appreciation are also extended to Assistant Professor Dr. Amorn Wasanawichit, the chairman, and Dr. Paisan Nakmahachalasint, committee member, for their comments and suggestions. Also, I thank Ms. Angkana Sripayap and Mr. Somjate Chaiya who help sending some of the references from abroad and thank all teachers who have taught me.

In particular, I would like to express my deep gratitude to my parents, brother and friends for their encouragement throughout my graduate study.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

CONTENTS

CHAPTER I

Introduction

In this chapter, we collect definitions and theorems to be used throughout the entire thesis.

The following symbols will be standard:

$\mathbb{R}$ the set of real numbers,

$\mathbb{Z}$ the set of all integers,

$\mathbb{N}$ the set of all positive integers,

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$[x]$ the integer part of x, i.e. the largest integer less than or equal to x,

$a <_m b$ means that $a + m \leq b$, where m is a positive integer.

The expansion

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}} = a_0 + \frac{b_1}{a_1+}\,\frac{b_1}{a_2+}\,\frac{b_3}{a_3 + ...}$$

is call a *continued fraction.*

The quantities a_i and b_i may be taken to be integers, reals or functions. It is more convenient to use the notation

$$[a_0; b_1, a_1; b_2, a_2; ...; b_n, a_n; ...]$$

for the above continued fraction. The elements $b_1, b_2, b_3, ...$ are called its *partial numerators*; $a_1, a_2, a_3, ...$ its *partial denominators.*

When all $b_i = 1$ we use $[a_0, a_1, a_2 ...]$ for $[a_0; 1, a_1; 1, a_2; ...; 1, a_n; ...]$. We assume that all partial denominators are not equal to zero.

The *finite continued fraction* $[a_0; b_1, a_1; b_2, a_2; ...; b_n, a_n]$ is called the n^{th} *convergent* of the continued fraction $[a_0; b_1, a_1; b_2, a_2; ...; b_n, a_n; ...]$.

Define $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$,

$p_n = a_n p_{n-1} + b_n p_{n-2}$ and $q_n = a_n q_{n-1} + b_n q_{n-2}$ for all $n \geq 0$.

Then $\dfrac{p_n}{q_n}$ is the n^{th} convergent of $[a_0; b_1, a_1; b_2, a_2; ...; b_n, a_n; ...]$.

In $\mathbb{R}$, it is known that any real number can be represented as a continued fraction of the form

$$[a_0, a_1, a_2, ...]$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$, for all $i \geq 1$. This is called a *simple* continued fraction and the a_i are called its *partial quotients*.

Definition 1.1. The *Fibonacci sequence* is the sequence (f_n) where

$$f_0 = 1,\ f_1 = 1 \qquad \text{and}$$
$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2.$$

In 1977 J.L. Davison[4] proved that:

Theorem 1.2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{[n\alpha]}} = [t_0, t_1, t_2, t_3, ..., t_n, ...]$ where $\alpha = \frac{1+\sqrt{5}}{2}$, $t_0 = 0$, $t_1 = 1$, $t_n = 2^{f_{n-2}}$ $(n \geq 2)$ and (f_n) is the Fibonacci sequence.

In the same year W.W. Adams and J.L. Davison[1] generalized the above theorem to:

Theorem 1.3. Let $x > 0$ be any irrational real number and $C > 1$ an integer. Let $[a_0, a_1, a_2, a_3, ...]$ be the continued fraction for x^{-1} and $\dfrac{p_n}{q_n}$ its n^{th} convergent.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

Then $$(C-1)\sum_{n=1}^{\infty}\frac{1}{C^{[nx]}} \;=\; [t_0, t_1, t_2, t_3, ...]$$

where (t_n) is the sequence defined by $t_0 = a_0C,\ t_n \;=\; \dfrac{C^{q_n} - C^{q_{n-2}}}{C^{q_{n-1}} - 1}$ for all $n \geq 1$.

Theorem 1.2 is a special case where $C = 2$ and $x = \frac{1+\sqrt{5}}{2}$ of Theorem 1.3, as we now show.

Let $C = 2$ and $x = \frac{1+\sqrt{5}}{2}$.
Then $x^{-1} = [0, 1, 1, 1, 1, 1, ...]$, $a_0 = 0$ and $a_i = 1$ for all $i \geq 1$.
Since $q_0 = 1, q_1 = 1$ and $q_n = q_{n-1} + q_{n-2}$ for $n \geq 2$, $(q_n) = (f_n)$, $(n \geq 0)$.
By Theorem 1.3,

$$t_0 = 0,\ t_1 \;=\; \frac{2^{q_1} - 2^{q_{-1}}}{2^{q_0} - 1} \;=\; \frac{2-1}{2-1} \;=\; 1$$

and for all $n \geq 2$,
$$\begin{aligned} t_n &= \frac{2^{q_n} - 2^{q_{n-2}}}{2^{q_{n-1}} - 1} \\ &= \frac{2^{f_n} - 2^{f_{n-2}}}{2^{f_{n-1}} - 1} \\ &= \frac{2^{f_{n-1}+f_{n-2}} - 2^{f_{n-2}}}{2^{f_{n-1}} - 1} \\ &= \frac{2^{f_{n-2}}(2^{f_{n-1}} - 1)}{2^{f_{n-1}} - 1} \\ &= 2^{f_{n-2}}. \end{aligned}$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

In 1986 D. Bowman[2] generalized W.W. Adams and J.L. Davison's Theorem by using Euler-Minding Theorem and Lemma 2.5(in Chapter II) about Zeckendorff representation of natural numbers.

We now show that Adams-Davison's result is a special case of that of Bowman.

Define Y_n for any integer n as follows: let Y_0 and Y_1 be any real numbers such that $Y_0 + Y_1\alpha > 0$ and $Y_{n+2} = Y_{n+1} + Y_n$ for $n \in \mathbb{Z}$.

D. Bowman proved the following theorem:

Theorem 1.4. $\dfrac{\sum_{n=1}^{\infty}(\frac{1}{C})^{Y_0 n + Y_{-1}[n\alpha^{-1}]}}{\sum_{n=1}^{\infty}(\frac{1}{C})^{Y_1 n + Y_0[n\alpha^{-1}]}} = C^{Y_{-1}} + \dfrac{1}{C^{Y_0}+}\,\dfrac{1}{C^{Y_1}+}\,\dfrac{1}{C^{Y_2}+\ldots}$ where $C > 1$ is a real number and $\alpha = \frac{1+\sqrt{5}}{2}$.

Let $Y_0 = 0$, $Y_1 = 1$. Then $Y_0 + Y_1\alpha \geq 0$ and $Y_{-1} = 1$.By Bowman's theorem, we have

$$C^{Y_{-1}} + \frac{1}{C^{Y_0}} + \frac{1}{C^{Y_1}} + \frac{1}{C^{Y_2}+\ldots} = \frac{\sum_{n=1}^{\infty}(\frac{1}{C})^{[n\alpha^{-1}]}}{\sum_{n=1}^{\infty}(\frac{1}{C})^{n}}$$

$$= (C-1)\sum_{n=1}^{\infty}\frac{1}{C^{[n\alpha^{-1}]}}$$

The continued fraction for $\alpha = \frac{1+\sqrt{5}}{2}$ is $[1, 1, 1, 1, 1, \ldots]$.
Since $q_{-1} = 0 = Y_0, q_0 = 1 = Y_1$ and $q_n = q_{n-1} + q_{n-2}$ for $n \geq 1$, $q_n = Y_{n+1}$.
Now the result of Theorem 1.3 is

$$(C-1)\sum_{n=1}^{\infty}\frac{1}{C^{[n\alpha^{-1}]}} = t_0 + \frac{1}{t_1+}\,\frac{1}{t_2+}\,\frac{1}{t_3+\ldots}$$

where $t_0 = C$ and for all $n \geq 1$,

$$t_n = \frac{C^{q_n} - C^{q_{n-2}}}{C^{q_{n-1}} - 1}$$

$$= \frac{C^{Y_{n+1}} - C^{Y_{n-1}}}{C^{Y_n} - 1}$$

$$= \frac{C^{Y_n + Y_{n-1}} - C^{Y_{n-1}}}{C^{Y_n} - 1}$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$= \frac{C^{Y_{n-1}}(C^{Y_n} - 1)}{C^{Y_n} - 1}$$

$$= C^{Y_{n-1}}.$$

In [5] D.E. Daykin generalized of the so-called Zeckendorff's theorem about unique representation of natural numbers by sums of integer form the Fibonacci sequence.

We briefly recall his results.

Definition 1.5. Let h, k be natural numbers such that $h \leq k \leq h+1$. The $(h,k)^{th}$ *Fibonacci sequence* (v_n) is defined by

$$\begin{aligned} v_n &= n && \text{for } 1 \leq n \leq k, \\ v_n &= v_{n-1} + v_{n-h} && \text{for } k < n < h+k, \\ v_n &= v_{n-1} + v_{n-k} + (k-h) && \text{for } n \geq h+k. \end{aligned}$$

The Fibonacci sequence (f_n) is the $(2,2)^{\text{th}}$ Fibonacci sequence.

Well-known is the following theorem of Zeckendorff.

Theorem 1.6. For each natural number N there is one and only one system of natural numbers $i_1, i_2, ..., i_d$ such that

$$N = f_{i_1} + f_{i_2} + ... + f_{i_d}$$

where $1 \leq i_1 <_2 i_2 <_2 ... <_2 i_d$ and (f_n) is the Fibonacci sequence.

D.E. Daykin's generalization of Theorem 1.6 is the following:

Theorem 1.7. If (v_n) is an $(h,k)^{th}$ Fibonacci sequence, $h \leq k \leq h+1$, then for each natural number N there is one, and only one system of natural numbers $i_1, i_2, ..., i_d$ such that

$$N = v_{i_1} + v_{i_2} + ... + v_{i_d} \tag{1.1}$$

where $1 \leq i_1 <_h i_2 <_k ... <_k i_d$.

Our derivation in Chapter II makes use of continued fractions whose partial denominators are all equal to 1. Such continued fractions converge as confirmed by the following theorem:

Theorem 1.8(Worpitzky's Theorem).[8] Let for all $n \geq 1$, $|b_n| \leq \frac{1}{4}$. Then $[0; b_1, 1; b_2, 1; ...; b_n, 1, ...]$ converges.

In this thesis we apply Bowman's method to find explicit continued fractions using the generalized Fibonacci sequence.

In Chapter II, we describe the P-and Q-systems which will be used in Lemma 2.5 and Bowman's method.

In Chapter III,we derive main theorem.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

CHAPTER II

Bowman's generalization

In this chapter we discuss Bowman's method and the representation systems of integers called the P-and Q-systems.

2.1 P-and Q-systems

Let $1 = v_1 < v_2 < v_3 < ...$ be a finite or an infinite sequence of integers. Every $N \in \mathbb{N}$ can be represented uniquely in the form([6])

$$N = \sum_{i=1}^{n} d_i v_i \quad , \; d_i \geq 0, \; d_i v_i + ... + d_1 v_1 < v_{i+1}, \; i \geq 1,$$

where v_n is the largest sequence member $\leq N$.(If the sequence is finite and v_j is its last term, we assume here and throughout that $v_{j+1} = \infty$.) A special case is the following :

Let $a_0, a_1, a_2, ...$ be a finite or an infinite sequence of positive integers, and let

$$p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2} \;\; (n \geq 1),$$

$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \;\; (n \geq 1).$$

Then every $N \in \mathbb{N}$ can be expressed uniquely in the form

$$N \ = \ \sum_{i=\epsilon_0}^{n} s_i p_i, \qquad 0 \le s_{-1} \le a_0 - 1, \ 0 \le s_i \le a_{i+1} \quad (i \ge 0),$$

$$s_i = a_{i+1} \ \Rightarrow \ s_{i-1} \ = \ 0 \qquad\qquad (i \ge 1 + \epsilon_0),$$

and also in the form

$$N \ = \ \sum_{i=\epsilon_1}^{n} t_i q_i, \qquad 0 \le t_0 \le a_1 - 1, \ 0 \le t_i \le a_{i+1} \quad (i \ge 1),$$

$$t_i = a_{i+1} \ \Rightarrow \ t_{i-1} \ = \ 0 \qquad\qquad (i \ge 1 + \epsilon_1),$$

where

$$\epsilon_0 = \begin{cases} -1, & \text{if } \ a_0 > 1; \\ 0, & \text{if } \ a_0 = 1. \end{cases} \qquad \epsilon_1 = \begin{cases} 0, & \text{if } \ a_1 > 1; \\ 1, & \text{if } \ a_1 = 1. \end{cases}$$

Note that the $p_i, q_i (i \ge 0)$ are the numerators and the denominators of the convergents of the simple continued fraction of $\zeta = [a_0, a_1, a_2, ...]$, respectively. The counting system based on the p_i and q_i are called the $P^{(\zeta)} - system$ and $Q^{(\zeta)} - system$ respectively, or simply the $P - system$ and $Q - system$.

The representation of N in terms of q_i also includes the case of $N = 0$.

Lemma 2.1. Let $n \ge 0$. Then every $N \ge 0$ has a unique representation in the form $N = \sum_{i=0}^{n} t_i q_i$, where

(1) $0 \le t_0 \le a_1 - 1$,

(2) $0 \le t_i \le a_{i+1}, \quad (i \ge 1)$

(3) $t_i = a_{i+1} \Rightarrow t_{i-1} = 0, \quad (i \ge 1)$.

The representation of each $N \in \mathbb{N}_0$ as in Lemma 2.1 is called the Zeckendorff representation of N.

We will show how to find examples of P-and Q-system for the positive integer 7 in the following example.

Example 2.2. Let $\alpha = \dfrac{1+\sqrt{5}}{2} = [1,1,1,1,...]$. The numerators and denominators of the convergents of α are $p_0 = 1,\ p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 8, ...$ and $q_0 = 1,\ q_1 = 1,\ q_2 = 2,\ q_3 = 3,\ q_4 = 5,\ q_5 = 8, ...$. Then

$7 = s_3p_3 + s_2p_2 + s_1p_1 + s_0p_0$ where $s_3 = 1,\ s_1 = 1$ and $s_2 = s_0 = 0$ in $P^{(\alpha)}$-system and $7 = t_4q_4 + t_3q_3 + t_2q_2 + t_1q_1 + t_0q_0$ where $t_4 = 1,\ t_2 = 1$ and $t_3 = t_1 = t_0 = 0$ in $Q^{(\alpha)}$-system.

2.2 Bowman's method

D. Bowman proved Theorem 1.4 in [2] which is a generalization of W.W. Adams and J.L. Davison's Theorem. In this section we discuss his method, to be applied in Chapter III.

D. Bowman used Euler-Minding Theorem. We give a proof of this theorem using the following lemma.

Lemma 2.3. Let $\dfrac{A_p}{B_p},\ p \geq 1,$ denote the p^{th} convergents of the continued fraction $[a_0; b_1, a_1; b_2, a_2; ...]$ and $A_{-1} = 1,\ A_0 = a_0,\ B_{-1} = 0$ and $B_0 = 1$.

Then for all $p \geq 1$,

$$A_p = a_pA_{p-1} + b_pA_{p-2} \quad and \quad B_p = a_pB_{p-1} + b_pB_{p-2}.$$

Theorem 2.4(Euler-Minding Theorem). If $\dfrac{A_p}{B_p} = 1 + \dfrac{C_1}{1+}\ \dfrac{C_2}{1+}\ \dfrac{C_3}{1+...}\ \dfrac{C_p}{1}$

where (C_k) is sequence of nonzero real numbers, then

$$A_p = 1 + \sum_{n\geq 1, 1\leq V_1 <_2 ... <_2 V_n \leq p} C_{V_1}C_{V_2}...C_{V_n} \qquad \text{and}$$

$$B_p = 1 + \sum_{n\geq 1, 2\leq V_1 <_2 ... <_2 V_n \leq p} C_{V_1}C_{V_2}...C_{V_n}.$$

Proof. For $p = 1$: $\dfrac{A_1}{B_1} = 1 + \dfrac{C_1}{1} = 1 + C_1.$

In this case, the sum A_1 consists of only one term, that is $n = 1$ and $V_1 = 1$ and the sum of B_1 is the empty sum. Thus

$$A_1 = 1 + C_1 = 1 + \sum_{n\geq 1, 1\leq V_1 <_2 ... <_2 V_n \leq 1} C_{V_1}C_{V_2}...C_{V_n}$$

and

$$B_1 = 1 \qquad = 1 + \sum_{n\geq 1, 2\leq V_1 <_2 ... <_2 V_n \leq 1} C_{V_1}C_{V_2}...C_{V_n}\ .$$

Now, assume that the statement is true for $1 \leq m < p$.

By Lemma 2.3, $A_p = A_{p-1} + C_p A_{p-2}$ and

$$B_p = B_{p-1} + C_p B_{p-2}.$$

Thus

$$A_p = \left(1 + \sum_{n\geq 1, 1\leq V_1 <_2 ... <_2 V_n \leq p-1} C_{V_1}C_{V_2}...C_{V_n}\right) + C_p\left(1 + \sum_{n\geq 1, 1\leq V_1 <_2 ... <_2 V_n \leq p-2} C_{V_1}C_{V_2}...C_{V_n}\right)$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$= \quad 1 + \sum_{n\geq 1, 1\leq V_1 <_2 \ldots <_2 V_n \leq p-1} C_{V_1}C_{V_2}\ldots C_{V_n}$$

$$+ \; C_p \; + \sum_{n\geq 1, 1\leq V_1 <_2 \ldots <_2 V_n \leq p-2} C_{V_1}C_{V_2}\ldots C_{V_n}C_p.$$

Since $\displaystyle\sum_{n\geq 1, 1\leq V_1 <_2 \ldots <_2 V_n \leq p-1} C_{V_1}C_{V_2}\ldots C_{V_n} + C_p + \sum_{n\geq 1, 1\leq V_1 <_2 \ldots <_2 V_n \leq p-2} C_{V_1}C_{V_2}\ldots C_{V_n}C_p =$

$\displaystyle\sum_{n\geq 1, 1\leq V_1 <_2 \ldots <_2 V_n \leq p} C_{V_1}C_{V_2}\ldots C_{V_n},$

$$A_p \; = \; 1 \; + \sum_{n\geq 1, 1\leq V_1 <_2 \ldots <_2 V_n \leq p} C_{V_1}C_{V_2}\ldots C_{V_n}.$$

Similarly,

$$\begin{aligned} B_p \; &= \; \left(1 + \sum_{n\geq 1, 2\leq V_1 <_2 \ldots <_2 V_n \leq p-1} C_{V_1}C_{V_2}\ldots C_{V_n}\right) \\ &\quad + C_p\left(1 + \sum_{n\geq 1, 2\leq V_1 <_2 \ldots <_2 V_n \leq p-2} C_{V_1}C_{V_2}\ldots C_{V_n}\right) \\ &= \; 1 + \sum_{n\geq 1, 2\leq V_1 <_2 \ldots <_2 V_n \leq p-1} C_{V_1}C_{V_2}\ldots C_{V_n} \\ &\quad + \; C_p \; + \sum_{n\geq 1, 2\leq V_1 <_2 \ldots <_2 V_n \leq p-2} C_{V_1}C_{V_2}\ldots C_{V_n}C_p. \\ &= \; 1 \; + \sum_{n\geq 1, 2\leq V_1 <_2 \ldots <_2 V_n \leq p} C_{V_1}C_{V_2}\ldots C_{V_n}. \end{aligned}$$

□

Let $\dfrac{A_m(C_n, C_{n+1}, \ldots, C_{n+m-1})}{B_m(C_n, C_{n+1}, \ldots, C_{n+m-1})} = [1; C_n, 1; C_{n+1}, 1; \ldots; C_{n+m-1}, 1]$ where n and m are positive integers.

Since, for $p \geq 1$,

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$1+\frac{C_1}{1+}\,\frac{C_2}{1+}\,\frac{C_3}{1+...}\,\frac{C_p}{1} \;=\; 1 \;+\; \cfrac{C_1}{1+\cfrac{C_2}{1+...\cfrac{C_p}{1}}}$$

$$= \; 1 \;+\; \cfrac{C_1}{\cfrac{A_{p-1}(C_2,C_3,...,C_p)}{B_{p-1}(C_2,C_3,...,C_p)}}$$

$$= \frac{A_{p-1}(C_2,C_3,...,C_p) \;+\; C_1B_{p-1}(C_2,C_3,...,C_p)}{A_{p-1}(C_2,C_3,...,C_p)},$$

$$B_p(C_1,C_2,...,C_p) \;=\; A_{p-1}(C_2,C_3,...,C_p)\;.$$

Now, let $p \to \infty$ and we have

$$1+\frac{C_1}{1+}\,\frac{C_2}{1+}\,\frac{C_3}{1+...} \;=\; \frac{A_\infty(C_1,C_2,...)}{A_\infty(C_2,C_3,...)}. \tag{2.1}$$

The continued fraction (2.1) converges by Worpitzky's theorem provided e.g. that $|C_i| \leq \frac{1}{4}$ for all i.

In Bowman's proof, he concluded wihtout a proof the following result from P-and Q-systems of A.S. Fraenkel, J. Levitt and M. Shimshoni[6].

We will prove it using T.C. Brown's theorem from [3].

Lemma 2.5. Let the Fibonacci representation of an integer $N \geq 1$ be written as

$$N \;=\; f_{i_1} \;+\; f_{i_2} \;+...+\; f_{i_n}, \tag{2.2}$$

where $1 \leq i_1 <_2 i_2 <_2 ... <_2 i_n$.

Define $e(N) = 0$ if $N = 0$,

$e(N) = f_{i_1 - 1} + f_{i_2 - 1} + ... + f_{i_n - 1}$, where N has the representation (2.2).

Then $e(N) = [(N+1)\alpha^{-1}]$ $(N \geq 0)$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Let β be an irrational number such that $0 < \beta < 1$. The *characteristic sequence* associated with β is the sequence

$$g(\beta) = g_1 g_2 g_3 ...,$$

where $g_n = [(n+1)\beta] - [n\beta]$ $(n \geq 1)$.

Remark 2.6. $g(\beta)$ is a sequence of 0's and 1's, and $[k\beta] = g_1 + g_2 + ... + g_{k-1}$, $k \geq 2$.

Now, let $[0, a_1, a_2, a_3, ...]$ be the simple continued fraction for β, let $\dfrac{p_n}{q_n} = [0, a_1, a_2, ..., a_n]$, $n \geq 1$, and $X_n = g_1 g_2 ... g_{q_n}$, $n \geq 1$.

Then X_n is the initial segment of $g(\beta)$ of length q_n.

Define $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$ and $q_0 = 1$.

By Lemma 2.3, $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $(n \geq 1)$.

From the definition of X_n we have that for each $n \geq 2$,

$$X_n = X_{n-1}^{a_n} X_{n-2},$$

where $X_0 = 0$ and $X_1 = 0^{a_1 - 1}1$. (Here $X_{n-1}^{a_n}$ denotes $X_{n-1} X_{n-1} ... X_{n-1}$, with a_n repetitions. If $a_1 = 1$, then $X_1 = 1$.)

Theorem 2.7.[3] Let N be a positive integer whose Zeckendorff representation is

$N = t_n q_n + ... + t_1 q_1 + t_0 q_0$. Then the initial segment of $g(\beta)$ of length N is

$$g_1 g_2 g_3 ... g_N = X_n^{t_n} ... X_1^{t_1} X_0^{t_0}.$$

Lemma 2.8.[3] For each $n \geq 0$, the number of 1's in X_n is p_n.

We use Theorem 2.7 and Lemma 2.8 to prove Theorem 2.9.

Theorem 2.9.[3] Let β be an irrational number such that $0 < \beta < 1$ and $N \geq 0$ have Zeckendroff representation as $N = t_n q_n + ... + t_1 q_1 + t_0 q_0$.

Then $[(N+1)\beta] = t_n p_n + ... + t_1 p_1 + t_0 p_0$.

Proof. By Remark 2.6, $[(N+1)\beta] = g_1 + g_2 + ... + g_N$.

Let W_N be the initial segment of $g(\beta)$ of length N, i.e. $W_N = g_1 g_2 ... g_N$.

Note that $g_1 + g_2 + ... + g_N$ is equal the number of 1's in W_N.

By Theorem 2.7, $W_N = X_n^{t_n} ... X_1^{t_1} X_0^{t_0}$.

By Lemma 2.8, the number of 1's in each X_i is p_i, the number of 1's in W_N is $t_n p_n + ... + t_1 p_1 + t_0 p_0$.

Hence $[(N+1)\beta] = t_n p_n + ... + t_1 p_1 + t_0 p_0$. □

We show that Lemma 2.5 is the special case of Theorem 2.9.

Let $\beta = \alpha^{-1} = \left(\dfrac{1+\sqrt{5}}{2}\right)^{-1}$. Then $\alpha^{-1} = [0, 1, 1, 1, ...](= [a_0, a_1, a_2, ...])$. Define $p_{-1} = 1,\ p_0 = 0,\ q_{-1} = 0$ and $q_0 = 1$.

By Theorem 2.3, $p_n = p_{n-1} + p_{n-2}$ and $q_n = q_{n-1} + q_{n-2} \quad (n \geq 1)$. Note that (q_n) is Fibonacci sequence (f_n) and $(p_n) = (f_{n-1})$.

Write N in Q-system form: $N = t_m q_m + ... + t_1 q_1 + t_0 q_0$ where each t_i satisfies the conditions in Lemma 2.1.

Thus $N = f_{i_1} + f_{i_2} + ... + f_{i_m}$ where $1 \leq i_1 <_2 i_2 <_2 ... <_2 i_m$.

By Theorem 2.20, $[(N+1)\alpha^{-1}] = f_{i_1-1} + f_{i_2-1} + .. + f_{i_m-1}$,

which is the result of Lemma 2.5.

Next, we illustrate the method of Bowman by using it to prove Theorem 1.4.

Proof of Theorem 1.4 Let (f_n) be Fibonacci sequence and f_n defined for negative n by $f_{n+2} = f_{n+1} + f_n$.

Define Y_n for any integer n as follows: let Y_0 and Y_1 be any real numbers such that $Y_0 + Y_1\alpha > 0$ and $Y_{n+2} = Y_{n+1} + Y_n$ for $n \in \mathbb{Z}$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

Let the Fibonacci representation of an integer $N \geq 1$ be

$$N \;=\; f_{i_1} \;+\; f_{i_2} \;+\; ... \;+\; f_{i_n},$$

where $1 \leq i_1 <_2 i_2 <_2 ... <_2 i_n$.

Define $\quad e(N) \;=\; 0 \quad$ if $\;N = 0$,

$\qquad e(N) \;=\; f_{i_1-1} + f_{i_2-1} + ... + f_{i_n-1}$, where N has the representation above.

By Lemma 2.5, $\quad e(N) \;=\; [(N+1)\alpha^{-1}] \qquad (N \geq 0)$.

Set $C_n = a^{f_{n-2}} b^{f_{n-1}}$ in (2.1), with $0 < a, b \leq 1$, not both 1, to get

$$1 \;+\; \frac{a^{f_{-1}}a^{f_0}}{1+} \;+\; \frac{a^{f_0}a^{f_1}}{1+...} \;=\; \frac{A_\infty(a^{f_{-1}}a^{f_0}, a^{f_0}a^{f_1}, ...)}{A_\infty(a^{f_0}a^{f_1}, a^{f_1}a^{f_2}, ...)}.$$

By Theorem 2.4, we have

$$1 \;+\; \frac{a^{f_{-1}}a^{f_0}}{1+} \;+\; \frac{a^{f_0}a^{f_1}}{1+...} \;=\; \frac{1 + \displaystyle\sum_{n\geq 1,\; 1\leq i_1 <_2 ... <_2 i_n} a^{f_{i_1-2}+...+f_{i_n-2}} b^{f_{i_1-1}+...+f_{i_n-1}}}{1 + \displaystyle\sum_{n\geq 1,\; 1\leq i_1 <_2 ... <_2 i_n} a^{f_{i_1-1}+...+f_{i_n-1}} b^{f_{i_1}+...+f_{i_n}}}.$$

Denote the numerator by $F(a,b)$ and the denominator by $G(a,b)$.

Then

$$F(b,ab) \;=\; 1 \;+\; \sum_{n\geq 1,\ 1\leq i_1<_2\ldots<_2 i_n} a^{f_{i_1-1}+\ldots+f_{i_n-1}} b^{f_{i_1}+\ldots+f_{i_n}} \;=\; G(a,b) \tag{2.3}$$

and so

$$\frac{F(a,b)}{F(b,ab)} \;=\; 1 \;+\; \frac{a^{f_{-1}}a^{f_0}}{1+} \;+\; \frac{a^{f_0}a^{f_1}}{1+\ldots} \tag{2.4}$$

From this, it follows that

$$\frac{F(b,ab)}{F(ab,ab^2)} \;=\; 1 \;+\; \frac{a^{f_0}a^{f_1}}{1+} \;+\; \frac{a^{f_1}a^{f_2}}{1+\ldots},$$

so we find that

$$F(a,b) \;=\; F(b,ab) + bF(ab,ab^2), \tag{2.5}$$

with $0 < a, b \leq 1$, and not both 1.

Notice that if in(2.3) the exponent of b is k, then the exponent of a will be e(k) , k ranging over the integers > 0.

Hence,

$$F(b,ab) \;=\; 1 \;+\; \sum_{n\geq 1} a^{e(n)}b^n \;=\; \sum_{n\geq 0} a^{e(n)}b^n.$$

Thus,

$$F(a,b) \;=\; \sum_{n\geq 0} a^{n-e(n)}b^{e(n)}.$$

Using Lemma 2.5, we have

$$F(a,b) \;=\; \sum_{n\geq 0} a^{n-[(n+1)\alpha^{-1}]}b^{[(n+1)\alpha^{-1}]}\ , \tag{2.6}$$

and

$$F(b,ab) \;=\; \sum_{n\geq 0} a^{[(n+1)\alpha^{-1}]}b^n\ . \tag{2.7}$$

Let $a \;=\; C^A$ and $b \;=\; C^B$ in (2.6) and (2.7) to get

$$F(C^A,C^B) \;=\; \sum_{n\geq 1} C^{A(n-1)+(B-A)[n\alpha^{-1}]}, \tag{2.8}$$

and

$$F(C^B, C^{A+B}) = \sum_{n\geq 1} C^{B(n-1)+A[n\alpha^{-1}]}. \qquad (2.9)$$

Set $A = Y_0 - Y_1$ and $B = -Y_0$ in (2.9) to get

$$F(C^{-Y_0}, C^{-Y_1}) = \sum_{n\geq 1} \left(\frac{1}{C}\right)^{Y_0(n+1)+(Y_1-Y_0)[n\alpha^{-1}]}, \quad |C| > 1, \qquad (2.10)$$ or

set $A = -Y_0$ and $B = -Y_1$ in (2.9) to get

$$F(C^{-Y_1}, C^{-Y_0-Y_1}) = \sum_{n\geq 1} \left(\frac{1}{C}\right)^{Y_1(n+1)+Y_0[n\alpha^{-1}]}, \quad |C| > 1. \qquad (2.11)$$

From (2.4), we see that

$$\frac{F(C^{Y_0}, C^{Y_1})}{F(C^{Y_1}, C^{Y_0+Y_1})} = 1 + \frac{C^{Y_0 f_{-1}+Y_1 f_0}}{1+} \frac{C^{Y_0 f_0+Y_1 f_1}}{1+} \frac{C^{Y_0 f_1+Y_1 f_2}}{1+...}, \quad 0 < C < 1.$$

It is easy to show by induction that $Y_n = Y_0 f_{n-2} + Y_1 f_{n-1}$, for integer n; hence,

$$\frac{F(C^{Y_0}, C^{Y_1})}{F(C^{Y_1}, C^{Y_0+Y_1})} = 1 + \frac{C^{Y_1}}{1+} \frac{C^{Y_2}}{1+} \frac{C^{Y_3}}{1+...}, \quad 0 < C < 1.$$

Replacing C with its reciprocal variable,

$$\frac{F(C^{-Y_0}, C^{-Y_1})}{F(C^{-Y_1}, C^{-Y_0-Y_1})} = 1 + \frac{C^{-Y_1}}{1+} \frac{C^{-Y_2}}{1+} \frac{C^{-Y_3}}{1+...}, \quad C > 1,$$

$$= 1 + \frac{C^{Y_0}C^{-Y_1}}{C^{Y_0}+} \frac{C^{Y_0}C^{Y_1}C^{-Y_2}}{C^{Y_1}+} \frac{C^{Y_1}C^{Y_2}C^{-Y_3}}{C^{Y_2}+...}, \quad C > 1,$$

$$= 1 + \frac{C^{Y_0-Y_1}}{C^{Y_0}+} \frac{1}{C^{Y_1}+} \frac{1}{C^{Y_2}+...}, \quad C > 1.$$

Hence,

$$\frac{F(C^{-Y_0}, C^{-Y_1})C^{-Y_0}}{F(C^{-Y_1}, C^{-Y_0-Y_1})C^{-Y_1}} = C^{Y_{-1}} + \frac{1}{C^{Y_0}+} \frac{1}{C^{Y_1}+} \frac{1}{C^{Y_2}+...}, \quad C > 1.$$

Substituting in (2.10) and (2.11) and simplifying yields the theorem. □

CHAPTER III

Explicit continued fractions

In this chapter, we apply the generalized Fibonacci representation in the Bowman's method to derive corresponding explicit continued fractions.

3.1 Explicit continued fractions whose partial numerators are 1

Let v_n be $(1,1)^{\text{th}}$ Fibonacci sequence:

$$v_1 = 1 \quad \text{and} \quad v_n = v_{n-1} + v_{n-1} = 2v_{n-1} \qquad \text{where} \qquad n \geq 2.$$

Set $v_0 \ = \ 2^{-1}$. Then $v_1 \ = \ 2v_0$.

Hence $\qquad v_n = 2^{n-1} \qquad$ for all $\qquad n \geq 0.$

By Theorem 1.7, the $(1,1)^{\text{th}}$ Fibonacci representation of an integer $N \geq 1$ is

$$N = v_{i_1} \ + \ v_{i_2} \ + \ ... \ + \ v_{i_d}$$

where $1 \leq i_1 < i_2 < i_3 < ... < i_d$.

Theorem 3.1. Let a be a real number such that $0 < a < 1$. Then

$$a^{-\frac{1}{2}} + \frac{1}{a^{-2^0}+} \ \frac{1}{a^{-2^1}+} \ \frac{1}{a^{-2^2}+...}$$

$$= a^{-\frac{1}{2}} \left(\frac{1 + \frac{a^{\frac{3}{2}}}{1 - a^{\frac{3}{2}}} - \sum_{n=1}^{\infty} \left(\frac{a^{9 \cdot 2^{n-2}}}{1 - a^{3 \cdot 2^n}} \left(1 + \sum_{m \geq 1, 1 \leq j_1 <_2 j_2 <_2 \ldots <_2 j_m \leq n-2} a^{3(2^{j_1 - 2} + \ldots + 2^{j_m - 2})} \right) \right)}{1 + \frac{a^3}{1 - a^3} - \sum_{n=1}^{\infty} \left(\frac{a^{9 \cdot 2^{n-1}}}{1 - a^{3 \cdot 2^{n+1}}} \left(1 + \sum_{m \geq 1, 1 \leq j_1 <_2 j_2 <_2 \ldots <_2 j_m \leq n-2} a^{3(2^{j_1 - 1} + \ldots + 2^{j_m - 1})} \right) \right)} \right).$$

Proof. Let $C_n = a^{v_{n-1}} a^{v_n} = a^{3v_{n-1}} = a^{3 \cdot 2^{n-2}}$ where a is a real number such that $0 < a < 1$.

Substitute C_n into (2.1) to get

$$1 + \frac{a^{3v_0}}{1+} \frac{a^{3v_1}}{1+} \frac{a^{3v_2}}{1 + \ldots} = 1 + \frac{C_1}{1+} \frac{C_2}{1+} \frac{C_3}{1 + \ldots}$$

$$= \frac{A_\infty(C_1, C_2, \ldots)}{A_\infty(C_2, C_3, \ldots)}$$

$$= \frac{1 + \sum_{n \geq 1,\ 1 \leq i_1 <_2 i_2 <_2 \ldots <_2 i_n} a^{3 \cdot 2^{i_1 - 2} + \ldots + 3 \cdot 2^{i_n - 2}}}{1 + \sum_{n \geq 1,\ 2 \leq i_1 <_2 i_2 <_2 \ldots <_2 i_n} a^{3 \cdot 2^{i_1 - 2} + \ldots + 3 \cdot 2^{i_n - 2}}}$$

$$= \frac{1 + \sum_{n \geq 1,\ 1 \leq i_1 <_2 i_2 <_2 \ldots <_2 i_n} a^{3 \cdot 2^{i_1 - 2} + \ldots + 3 \cdot 2^{i_n - 2}}}{1 + \sum_{n \geq 1,\ 1 \leq i_1 <_2 i_2 <_2 \ldots <_2 i_n} a^{3 \cdot 2^{i_1 - 1} + \ldots + 3 \cdot 2^{i_n - 1}}}.$$

$$:= \frac{F(a)}{G(a)} \tag{3.1}$$

Let $A = \sum_{d \geq 1,\ 1 \leq i_1 < i_2 < \ldots < i_d} a^{3(2^{i_1 - 2} + \ldots + 2^{i_d - 2})}$.

By Theorem 1.7,

$$A = \sum_{d \geq 1,\ 1 \leq i_1 < i_2 < \ldots < i_d} a^{\frac{3}{2}(2^{i_1 - 1} + \ldots + 2^{i_d - 1})} = \sum_{k \in \mathbb{N}} a^{\frac{3}{2}k} = \frac{a^{\frac{3}{2}}}{1 - a^{\frac{3}{2}}}.$$

Now,

$$F(a) = 1 + \sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2 \ldots <_2 i_n} a^{3\cdot 2^{i_1-2}+3\cdot 2^{i_2-2}+\ldots+3\cdot 2^{i_n-2}}$$

$$= 1 + A - B$$

where B is a sum of the same form as in $F(a)$ but extends over $n \geq 1,\ 1 \leq i_1 < i_2 < \ldots < i_n$ such that there exists a pair $i_{k+1} = i_k + 1$ for some k. To evaluate B, we split the sum into subsums corresponding to all pairs of consecutive values of each possible consecutive indices, namely, $B = B_{1,2} + B_{2,3} + B_{3,4} + \ldots$, where $B_{m,m+1}$ denoted the subsum taken over all possible consecutive indices $i_k,\ i_{k+1}$ with value pair $(m, m+1)$.

For value-pair (1,2), possible indices are $i_1 = 1 < i_2 = 2$ and the subsum is

$$
\begin{aligned}
B_{1,2} &= \sum_{d\geq 2, 1=i_1<2=i_2<i_3<\ldots<i_d} a^{3(2^{i_1-2}+2^{i_2-2}+\ldots+2^{i_d-2})} \\
&= \sum_{d\geq 2, 2=i_2<i_3<\ldots<i_d} a^{3(\frac{1}{2}+2^{i_2-2}+2^{i_3-2}+\ldots+2^{i_d-2})} \\
&= \sum_{d\geq 2, 1=j_2\leq j_3<\ldots<j_d} a^{3(\frac{1}{2}+2^{j_2-1}+2^{j_3}+\ldots+2^{j_d})} \\
&= \sum_{d\geq 2, 1=j_2\leq j_3<\ldots<j_d} a^{3(\frac{1}{2}+2^{j_2-1}+2^1(2^{j_3-1}+\ldots+2^{j_d-1}))} \\
&= \sum_{k=\frac{1}{2}+2^0+2^1\mathbb{N}_0} a^{3k}. \\
&= \sum_{l=0}^{\infty} a^{3(\frac{1}{2}+2^0+2^1 l)} \\
&= \frac{a^{3(2^{-1}+2^0)}}{1-a^{3\cdot 2^1}}.
\end{aligned}
$$

For value-pair (2,3), possible indices are $i_2 = 2 < i_3 = 3$ and the subsum is

$$
\begin{aligned}
B_{2,3} &= \sum_{d \geq 3, 2 = i_2 < 3 = i_3 < i_4 < \ldots < i_d} a^{3(2^{i_2 - 2} + 2^{i_3 - 2} + \ldots + 2^{i_d - 2})} \\
&= \sum_{d \geq 3, 3 = i_3 < i_4 < \ldots < i_d} a^{3(2^0 + 2^{i_3 - 2} + 2^{i_4 - 2} + \ldots + 2^{i_d - 2})} \\
&= \sum_{d \geq 3, 1 = j_3 \leq j_4 < \ldots < j_d} a^{3(2^0 + 2^{j_3} + 2^{j_4 + 1} + \ldots + 2^{j_d + 1})} \\
&= \sum_{d \geq 3, 1 = j_3 \leq j_4 < \ldots < j_d} a^{3(2^0 + 2^{j_3} + 2^2(2^{j_4 - 1} + \ldots + 2^{j_d - 1}))} \\
&= \sum_{k = 2^0 + 2^1 + 2^2 \mathbb{N}_0} a^{3k} \\
&= \sum_{l=0}^{\infty} a^{3(2^0 + 2^1 + 2^2 l)} \\
&= \frac{a^{3(2^0 + 2^1)}}{1 - a^{3 \cdot 2^2}}.
\end{aligned}
$$

For value-pair (3,4), possible indices are $i_3 = 3 < i_4 = 4$ or $i_1 = 1 < i_3 = 3 < i_4 = 4$ and the subsum is

$$
\begin{aligned}
B_{3,4} = & \sum_{d \geq 4, 3 = i_3 < 4 = i_4 < i_5 < \ldots < i_d} a^{3(2^{i_3 - 2} + 2^{i_4 - 2} + \ldots + 2^{i_d - 2})} \\
& + \sum_{d \geq 4, 1 = i_1 < 3 = i_3 < 4 = i_4 < i_5 < \ldots < i_d} a^{3(2^{i_1 - 2} + 2^{i_3 - 2} + 2^{i_4 - 2} + \ldots + 2^{i_d - 2})} \\
= & \left(1 + \sum_{m \geq 1, 1 \leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1 - 2} + \ldots + 2^{j_m - 2})}\right) \left(\sum_{d \geq 4, 3 = i_3 < 4 = i_4 < i_5 < \ldots < i_d} a^{3(2^{i_3 - 2} + 2^{i_4 - 2} + \ldots + 2^{i_d - 2})}\right) \\
= & \left(1 + \sum_{m \geq 1, 1 \leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1 - 2} + \ldots + 2^{j_m - 2})}\right) \left(\sum_{d \geq 4, 4 = i_4 < i_5 < \ldots < i_d} a^{3(2^1 + 2^{i_4 - 2} + 2^{i_5 - 2} + \ldots + 2^{i_d - 2})}\right) \\
= & \left(1 + \sum_{m \geq 1, 1 \leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1 - 2} + \ldots + 2^{j_m - 2})}\right) \left(\sum_{d \geq 4, 1 = j_4 \leq j_5 < \ldots < j_d} a^{3(2^1 + 2^{j_4 + 1} + 2^{j_5 + 2} + \ldots + 2^{j_d + 2})}\right)
\end{aligned}
$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{d\geq 4, 1=j_4\leq j_5<\ldots<j_d} a^{3(2^1+2^{j_4+1}+2^3(2^{j_5-1}+\ldots+2^{j_d-1})}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{k=2^1+2^2+2^3\mathbb{N}_0} a^{3k}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{l=0}^{\infty} a^{3(2^1+2^2+2^3 l)}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\frac{a^{3(2^1+2^2)}}{1-a^{3\cdot 2^3}}\right).$$

For value-pair (4,5), possible indices are $i_4 = 4 < i_5 = 5$ or $i_1 = 1 < i_4 = 4 < i_5 = 5$ or $i_1 = 2 < i_4 = 4 < i_5 = 5$ and the subsum is

$$B_{4,5} = \sum_{d\geq 5, 4=i_4<5=i_5<i_6<\ldots<i_d} a^{3(2^{i_4-2}+2^{i_5-2}+2^{i_6-2}+\ldots+2^{i_d-2})}$$

$$+ \sum_{d\geq 5, 1=i_1<4=i_4<5=i_5<i_6<\ldots<i_d} a^{3(2^{i_1-2}+2^{i_4-2}+2^{i_5-2}+\ldots+2^{i_d-2})}$$

$$+ \sum_{d\geq 5, 2=i_1<4=i_4<5=i_5<i_6<\ldots<i_d} a^{3(2^{i_1-2}+2^{i_4-2}+2^{i_5-2}+\ldots+2^{i_d-2})}$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{d\geq 5, 4=i_4<5=i_5<i_6<\ldots<i_d} a^{3(2^{i_4-2}+2^{i_5-2}+\ldots+2^{i_d-2})}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{d\geq 5, 5=i_5<i_6<\ldots<i_d} a^{3(2^2+2^{i_5-2}+2^{i_6-2}+\ldots+2^{i_d-2})}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{d\geq 5, 1=j_5\leq j_6<\ldots<j_d} a^{3(2^2+2^{j_5+2}+2^{j_6+3}+\ldots+2^{j_d+3})}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{d\geq 5, 1=j_5\leq j_6<\ldots<j_d} a^{3(2^2+2^{j_5+2}+2^4(2^{j_6-1}+\ldots+2^{j_d-1})}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{k=2^2+2^3+2^4\mathbb{N}_0} a^{3k}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\sum_{l=0}^{\infty} a^{3(2^2+2^3+2^4 l)}\right)$$

$$= \left(1 + \sum_{m\geq 1, 1\leq j_1<_2\ldots<_2 j_m<_2 4} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\left(\frac{a^{3(2^2+2^3)}}{1-a^{3\cdot 2^4}}\right).$$

The computations of other $B_{m,m+1}$ are carried out in a similar manner and we arrive at

$$F(a) = 1 + \frac{a^{\frac{3}{2}}}{1-a^{\frac{3}{2}}} - \sum_{n=1}^{\infty}\left(\frac{a^{3(2^{n-2}+2^{n-1})}}{1-a^{3\cdot 2^n}}\left(1 + \sum_{m\geq 1, 1\leq j_1<_2\ldots<_2 j_m<_2 n} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\right)$$

$$= 1 + \frac{a^{\frac{3}{2}}}{1-a^{\frac{3}{2}}} - \sum_{n=1}^{\infty}\left(\frac{a^{9\cdot 2^{n-2}}}{1-a^{3\cdot 2^n}}\left(1 + \sum_{m\geq 1, 1\leq j_1<_2\ldots<_2 j_m\leq n-2} a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\right).$$

Let $A' = \sum_{d\geq 1,\ 1\leq i_1<i_2<\ldots<i_d} a^{3(2^{i_1-1}+\ldots+2^{i_d-1})}$.

By Theorem 1.7, $A' = \sum_{k\in\mathbb{N}} a^{3k} = \dfrac{a^3}{1-a^3}$.

As for the denominator, we have

$$G(a) = 1 + \sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{3\cdot 2^{i_1-1}+3\cdot 2^{i_2-1}+\ldots+3\cdot 2^{i_n-1}} = 1 + A' - B',$$

where B' is a sum of the same form as in $G(a)$ but extends over $n \geq 1,\ 1 \leq i_1 < i_2 < \ldots < i_n$ such that there exists a pair $i_{k+1} = i_k + 1$ for some k. To evaluate B', we split the sum into subsums corresponding to all pairs of consecutive values of each possible consecutive indices, namely, $B' = B'_{1,2} + B'_{2,3} + B'_{3,4} + \ldots$, where $B'_{m,m+1}$ denotes the subsum taken over all possible consecutive indices $i_k,\ i_{k+1}$ with value pair $(m, m+1)$.

Using the same computation as for $F(a)$, we get

$$G(a) = 1 + \frac{a^3}{1-a^3} - \left(a^{3(2^0+2^1)}\left(\frac{1}{1-a^{3\cdot 2^2}}\right) + a^{3(2^1+2^2)}\left(\frac{1}{1-a^{3\cdot 2^3}}\right)\right.$$

$$+ \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 3} a^{3(2^{j_1-1}+\ldots+2^{j_m-1})}\right) a^{3(2^2+2^3)} \left(\frac{1}{1-a^{3\cdot 2^4}}\right)$$

$$+ \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 4} a^{3(2^{j_1-1}+\ldots+2^{j_m-1})}\right) a^{3(2^3+2^4)} \left(\frac{1}{1-a^{3\cdot 2^5}}\right)$$

$$+ \ \ldots \Bigg)$$

$$= 1 + \frac{a^3}{1-a^3} - \sum_{n=1}^{\infty} \left(\frac{a^{3(2^{n-1}+2^n)}}{1-a^{3\cdot 2^{n+1}}} \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m <_2 n} a^{3(2^{j_1-1}+\ldots+2^{j_m-1})}\right)\right)$$

$$= 1 + \frac{a^3}{1-a^3} - \sum_{n=1}^{\infty} \left(\frac{a^{9\cdot 2^{n-1}}}{1-a^{3\cdot 2^{n+1}}} \left(1 + \sum_{m\geq 1, 1\leq j_1 <_2 \ldots <_2 j_m \leq n-2} a^{3(2^{j_1-1}+\ldots+2^{j_m-1})}\right)\right).$$

Thus

$$1 + \frac{C_1}{1+} \frac{C_2}{1+} \frac{C_3}{1+\ldots} = 1 + \frac{a^{3v_0}}{1+} \frac{a^{3v_1}}{1+} \frac{a^{3v_2}}{1+\ldots}$$

$$= 1 + \frac{a^{-v_1}a^{3v_0}}{a^{-v_1}+} \frac{a^{-v_1}a^{3v_1}a^{-v_2}}{a^{-v_2}+} \frac{a^{-v_2}a^{3v_2}a^{-v_3}}{a^{-v_3}+\ldots} .$$

$$1 + \frac{C_1}{1+} \frac{C_2}{1+} \frac{C_3}{1+\ldots} = 1 + \frac{a^{v_0}}{a^{-v_1}+} \frac{1}{a^{-v_2}+} \frac{1}{a^{-v_3}+\ldots} .$$

By (3.1), $$\frac{F(a)}{G(a)} = 1 + \frac{a^{v_0}}{a^{-v_1}+} \frac{1}{a^{-v_2}+} \frac{1}{a^{-v_3}+\ldots} .$$

$$a^{-\frac{1}{2}} \frac{F(a)}{G(a)} = a^{-v_0} + \frac{1}{a^{-v_1}+} \frac{1}{a^{-v_2}+} \frac{1}{a^{-v_3}+\ldots}$$

$$= a^{-\frac{1}{2}} + \frac{1}{a^{-2^0}+} \frac{1}{a^{-2^1}+} \frac{1}{a^{-2^2}+\ldots} .$$

Hence

$$a^{-\frac{1}{2}}\left(\frac{1+\dfrac{a^{\frac{3}{2}}}{1-a^{\frac{3}{2}}}-\displaystyle\sum_{n=1}^{\infty}\left(\frac{a^{9\cdot 2^{n-2}}}{1-a^{3\cdot 2^{n}}}\left(1+\sum_{m\geq 1,1\leq j_1<_2 j_2<_2\ldots<_2 j_m\leq n-2}a^{3(2^{j_1-2}+\ldots+2^{j_m-2})}\right)\right)}{1+\dfrac{a^{3}}{1-a^{3}}-\displaystyle\sum_{n=1}^{\infty}\left(\frac{a^{9\cdot 2^{n-1}}}{1-a^{3\cdot 2^{n+1}}}\left(1+\sum_{m\geq 1,1\leq j_1<_2 j_2<_2\ldots<_2 j_m\leq n-2}a^{3(2^{j_1-1}+\ldots+2^{j_m-1})}\right)\right)}\right)$$

$$= a^{-\frac{1}{2}}+\frac{1}{a^{-2^0}+}\ \frac{1}{a^{-2^1}+}\ \frac{1}{a^{-2^2}+\ldots}, \quad \text{as required.} \qquad \square$$

Let v_n be the $(h,k)^{\text{th}}$ Fibonacci sequence where $k \geq 3$:

$$\begin{aligned} v_n &= n && \text{for } 1 \leq n \leq k,\\ v_n &= v_{n-1} + v_{n-h} && \text{for } k < n < h+k,\\ v_n &= v_{n-1} + v_{n-k} + (k-h) && \text{for } n \geq h+k. \end{aligned}$$

By Theorem 1.7, the $(h,k)^{\text{th}}$ Fibonacci representation of an integer $N \geq 1$ is

$$N = v_{i_1} + v_{i_2} + \ldots + v_{i_d}$$

where $1 \leq i_1 <_h i_2 <_k i_3 <_k \ldots <_k i_d$.

Theorem 3.2. Let a be a real number such that $|a| \leq \frac{1}{4}$. Then

$$a^{-1}+\frac{1}{a^{-v_1}+}\ \frac{1}{a^{-v_{1+(k-1)}}+}\ \frac{1}{a^{-v_{1+2(k-1)}}+}\ \frac{1}{a^{-v_{1+3(k-1)}}+\ldots} = a^{-1}\frac{F(a)}{G(a)}$$

where
$$F(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_d-1)(k-1)}}$$
$$+\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_d-1)(k-1)}}\right)$$
$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_d-1)(k-1)}}\right)\right)\right),$$

$$G(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{2+i_1(k-1)}+\ldots+v_{2+i_d(k-1)}}$$

$$+\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{2+i_1(k-1)}+\ldots+v_{2+i_d(k-1)}}\right)$$

$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d} a^{v_{2+i_1(k-1)}+\ldots+v_{2+i_d(k-1)}}\right)\right)\right)$$

and (v_n) is the $(h,k)^{\text{th}}$ Fibonacci sequence such that $h=k$, $k\geq 3$.

Proof. Let $C_n = a^{v_{2+(n-1)(k-1)}}$ $(n\geq 1)$ where a is a real number such that $\mid a\mid \leq \frac{1}{4}$. Substitute C_n into (2.1) to get

$$\begin{aligned}
1+\frac{a^{v_2}}{1+}\ \frac{a^{v_{2+(k-1)}}}{1+}\ \frac{a^{v_{2+2(k-1)}}}{1+\ldots} &= 1+\frac{C_1}{1+}\ \frac{C_2}{1+}\ \frac{C_3}{1+\ldots}\\
&= \frac{A_\infty(C_1,C_2,\ldots)}{A_\infty(C_2,C_3,\ldots)}\\
&= \frac{1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}}{1+\sum_{n\geq 1,\ 2\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}}\\
&= \frac{1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}}{1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{2+i_1(k-1)}+\ldots+v_{2+i_n(k-1)}}}.\\
&:= \frac{F(a)}{G(a)} \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
\text{Let}\quad A &= \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}\\
&\quad+\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+(h-1)<_k i_3<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}\\
&\quad+\ \ldots\ +\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+2<_k i_3<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}\\
&= \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\ldots+v_{2+(i_n-1)(k-1)}}
\end{aligned}$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$+ \sum_{m=2}^{h-1} \left(\sum_{d \geq 2,\ 1 \leq i_1 < i_2 = i_1 + m <_k i_3 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \right).$$

Now,

$$\begin{aligned} F(a) &= 1 + \sum_{n \geq 1,\ 1 \leq i_1 <_2 i_2 <_2 \ldots <_2 i_n} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ &= 1 + A + B \end{aligned}$$

where B is a sum of the same form as in $F(a)$ but extends over $n \geq 1,\ 1 \leq i_1 <_2 i_2 <_2 \ldots <_2 i_{j-1} <_2 i_j <_k i_{j+1} <_k \ldots <_k i_n$ such that $i_2 = i_1 + m_1 \ \ (2 \leq m_1 \leq h)$ and there exists a pair $i_j = i_{j-1} + m_2 \ \ (2 \leq m_2 \leq k-1)$ for some $j \geq 3$. Then

$$\begin{aligned} B = \sum_{m_1=2}^{h} \Bigg(& \sum_{d \geq 3,\ 1 \leq i_1 < i_2 = i_1 + m_1 < i_3 = i_2 + (k-1) <_k i_4 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \sum_{d \geq 3,\ 1 \leq i_1 < i_2 = i_1 + m_1 < i_3 = i_2 + (k-2) <_k i_4 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \ldots + \sum_{d \geq 3,\ 1 \leq i_1 < i_2 = i_1 + m_1 < i_3 = i_2 + 2 <_k i_4 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \sum_{d \geq 4,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 < i_4 = i_3 + (k-1) <_k i_5 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \sum_{d \geq 4,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 < i_4 = i_3 + (k-2) <_k i_5 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \ldots + \sum_{d \geq 4,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 < i_4 = i_3 + 2 <_k i_5 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \sum_{d \geq 5,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 <_2 i_4 < i_5 = i_4 + (k-1) <_k i_6 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \sum_{d \geq 5,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 <_2 i_4 < i_5 = i_4 + (k-2) <_k i_6 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \ldots + \sum_{d \geq 5,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 <_2 i_4 < i_5 = i_4 + 2 <_k i_6 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \ldots \Bigg) \\ = \sum_{m_1=2}^{h} \Bigg(\sum_{m_2=2}^{k-1} \Bigg(& \sum_{d \geq 3,\ 1 \leq i_1 < i_2 = i_1 + m_1 < i_3 = i_2 + m_2 <_k i_4 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \\ & + \sum_{d \geq 4,\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 i_3 < i_4 = i_3 + m_2 <_k i_5 <_k \ldots <_k i_d} a^{v_{2+(i_1-1)(k-1)} + \ldots + v_{2+(i_n-1)(k-1)}} \end{aligned}$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$+ \ \dots \Bigg)\Bigg)$$

$$= \sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{\substack{d\geq n \\ 1\leq i_1<i_2=i_1+m_1<_2\dots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\dots<_k i_d}} a^{v_{2+(i_1-1)(k-1)}+\dots+v_{2+(i_n-1)(k-1)}}\right)\right)\right)\right)$$

Thus $F(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\dots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\dots+v_{2+(i_n-1)(k-1)}}$

$$+ \sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\dots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\dots+v_{2+(i_n-1)(k-1)}}\right)$$

$$+ \sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\dots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\dots<_k i_d} a^{v_{2+(i_1-1)(k-1)}+\dots+v_{2+(i_n-1)(k-1)}}\right)\right)\right)\right).$$

Let $A' = \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\dots<_k i_d} a^{v_{2+i_1(k-1)}+\dots+v_{2+i_n(k-1)}}$

$$+ \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+(h-1)<_k i_3<_k\dots<_k i_d} a^{v_{2+i_1(k-1)}+\dots+v_{2+i_n(k-1)}}$$

$$+ \ \dots \ + \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+2<_k i_3<_k\dots<_k i_d} a^{v_{2+i_1(k-1)}+\dots+v_{2+i_n(k-1)}}$$

$$= \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\dots<_k i_d} a^{v_{2+i_1(k-1)}+\dots+v_{2+i_n(k-1)}}$$

$$+ \sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\dots<_k i_d} a^{v_{2+i_1(k-1)}+\dots+v_{2+i_n(k-1)}}\right).$$

Now,

$$G(a) = 1 + \sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\dots<_2 i_n} a^{v_{2+i_1(k-1)}+\dots+v_{2+i_n(k-1)}}$$

$$= 1 + A' + B'$$

where B' is a sum of the same form as in $G(a)$ but extends over $n \geq 1,\ 1 \leq i_1 <_2 i_2 <_2 \dots <_2 i_{j-1} <_2 i_j <_k i_{j+1} <_k \dots <_k i_n$ such that $i_2 = i_1 + m_1$ $(2 \leq m_1 \leq h)$ and there exists a pair $i_j = i_{j-1} + m_2$ $(2 \leq m_2 \leq k-1)$ for some $j \geq 3$.

Similarly,

$$G(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k \ldots <_k i_d} a^{v_{2+i_1(k-1)}+\ldots+v_{2+i_n(k-1)}}$$

$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{\substack{d\,\geq\,n,\\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d}} a^{v_{2+i_1(k-1)}+\ldots+v_{2+i_n(k-1)}}\right)\right)\right).$$

$$\text{Consider}\quad 1 + \frac{a^{v_2}}{1+}\ \frac{a^{v_{2+(k-1)}}}{1+}\ \frac{a^{v_{2+2(k-1)}}}{1+}\ \frac{a^{v_{2+3(k-1)}}}{1+\ldots}$$

$$= 1 + \frac{a^{-v_1}a^{v_2}}{a^{-v_1}+}\ \frac{a^{-v_{1+(k-1)}}a^{v_{2+(k-1)}}a^{-v_1}}{a^{-v_{1+(k-1)}}+}\ \frac{a^{-v_{1+2(k-1)}}a^{v_{2+2(k-1)}}a^{-v_{1+(k-1)}}}{a^{-v_{1+2(k-1)}}+\ldots}$$

$$= 1 + \frac{a}{a^{-v_1}+}\ \frac{1}{a^{-v_{1+(k-1)}}+}\ \frac{1}{a^{-v_{1+2(k-1)}}+}\ \frac{1}{a^{-v_{1+3(k-1)}}+\ldots}$$

By (3.2),

$$1 + \frac{a}{a^{-v_1}+}\ \frac{1}{a^{-v_{1+(k-1)}}+}\ \frac{1}{a^{-v_{1+2(k-1)}}+}\ \frac{1}{a^{-v_{1+3(k-1)}}+\ldots} = \frac{F(a)}{G(a)}.$$

$$a^{-1} + \frac{1}{a^{-v_1}+}\ \frac{1}{a^{-v_{1+(k-1)}}+}\ \frac{1}{a^{-v_{1+2(k-1)}}+}\ \frac{1}{a^{-v_{1+3(k-1)}}+\ldots} = a^{-1}\frac{F(a)}{G(a)}. \qquad \square$$

Theorem 3.3. Let a be a real number such that $\mid a \mid \leq \frac{1}{4}$. Then

$$a^{v_{2h}-v_{2h+1}+1} + \frac{1}{a^{-v_{2h}-1}+}\ \frac{1}{a^{-v_{3h}-1}+}\ \frac{1}{a^{-v_{4h}-1}+\ldots} = a^{v_{2h}-v_{2h+1}+1}\frac{F(a)}{G(a)}$$

where

$$F(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}$$

$$+\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}\right)$$

$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}\right)\right)\right),$$

$$G(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+2)h+1}+\ldots+v_{(i_d+2)h+1}}$$

$$+\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+2)h+1}+\ldots+v_{(i_d+2)h+1}}\right)$$
$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d} a^{v_{(i_1+2)h+1}+\ldots+v_{(i_d+2)h+1}}\right)\right)\right)\right)$$

and (v_n) is the $(h,k)^{\text{th}}$ Fibonacci sequence such that $k=h+1$, $k\geq 3$.

Proof. Let $C_n = a^{v_{(n+1)h+1}}$ $(n\geq 1)$ where a is a real number such that $|\,a\,| \leq \frac{1}{4}$. Substitute C_n into (2.1) to get

$$\begin{aligned}
1+\frac{a^{v_{2h+1}}}{1+}\ \frac{a^{v_{3h+1}}}{1+}\ \frac{a^{v_{4h+1}}}{1+\ldots} &= 1+\frac{C_1}{1+}\ \frac{C_2}{1+}\ \frac{C_3}{1+\ldots}\\
&= \frac{A_\infty(C_1,C_2,\ldots)}{A_\infty(C_2,C_3,\ldots)}\\
&= \frac{1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}}{1+\sum_{n\geq 1,\ 2\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}}\\
&= \frac{1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}}{1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{(i_1+2)h+2}+\ldots+v_{(i_d+2)h+2}}}\\
&:= \frac{F(a)}{G(a)}. \qquad (3.3)
\end{aligned}$$

Let
$$\begin{aligned}
A &= \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}\\
&\quad + \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+(h-1)<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}\\
&\quad + \ \ldots\ + \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+2<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}\\
&= \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}
\end{aligned}$$

$$+\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}\right).$$

Now,

$$F(a) \ = \ 1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}$$

$$= \ 1 \ + \ A \ + \ B$$

where B is a sum of the same form as in $F(a)$ but extends over $n\geq 1,\ 1\leq i_1 <_2 i_2 <_2 \ldots <_2 i_{j-1} <_2 i_j <_k i_{j+1} <_k \ldots <_k i_n$ such that $i_2 = i_1 + m_1 \ \ (2\leq m_1\leq h)$ and there exists a pair $i_j = i_{j-1}+m_2 \ \ (2\leq m_2\leq k-1)$ for some $j\geq 3$. Then

$$\begin{aligned}
B \ = \ \sum_{m_1=2}^{h}\Bigg(& \sum_{d\geq 3,\ 1\leq i_1<i_2=i_1+m_1<i_3=i_2+(k-1)<_k i_4<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \sum_{d\geq 3,\ 1\leq i_1<i_2=i_1+m_1<i_3=i_2+(k-2)<_k i_4<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& +\ldots+ \sum_{d\geq 3,\ 1\leq i_1<i_2=i_1+m_1<i_3=i_2+2<_k i_4<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \sum_{d\geq 4,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+(k-1)<_k i_5<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \sum_{d\geq 4,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+(k-2)<_k i_5<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& +\ldots+ \sum_{d\geq 4,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+2<_k i_5<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \sum_{d\geq 5,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<_2 i_4<i_5=i_4+(k-1)<_k i_6<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \sum_{d\geq 5,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<_2 i_4<i_5=i_4+(k-2)<_k i_6<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& +\ldots+ \sum_{d\geq 5,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<_2 i_4<i_5=i_4+2<_k i_6<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \ldots \Bigg) \\
= \ \sum_{m_1=2}^{h}\Bigg(\sum_{m_2=2}^{k-1}\Bigg(& \sum_{d\geq 3,\ 1\leq i_1<i_2=i_1+m_1<i_3=i_2+m_2<_k i_4<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}} \\
& + \sum_{d\geq 4,\ 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+m_2<_k i_5<_k\ldots<_k i_d} a^{v_{(i_1+1)h+1}+\ldots+v_{(i_d+1)h+1}}
\end{aligned}$$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$+ \; \dots \Bigg) \Bigg)$$

$$= \sum_{m_1=2}^{h} \left(\sum_{m_2=2}^{k-1} \left(\sum_{n \geq 3} \left(\sum_{\substack{d \geq n \\ 1 \leq i_1 < i_2 = i_1 + m_1 <_2 \dots <_2 i_{n-1} < i_n = i_{n-1} + m_2 <_k i_{n+1} <_k \dots <_k i_d}} a^{v_{(i_1+1)h+1} + \dots + v_{(i_d+1)h+1}} \right) \right) \right)$$

Thus $F(a) = 1 + \sum_{d \geq 1,\; 1 \leq i_1 <_h i_2 <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+1)h+1} + \dots + v_{(i_d+1)h+1}}$

$$+ \sum_{m=2}^{h-1} \left(\sum_{d \geq 2,\; 1 \leq i_1 < i_2 = i_1 + m <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+1)h+1} + \dots + v_{(i_d+1)h+1}} \right)$$

$$+ \sum_{m_1=2}^{h} \left(\sum_{m_2=2}^{k-1} \left(\sum_{n \geq 3} \left(\sum_{d \geq n,\; 1 \leq i_1 < i_2 = i_1 + m_1 <_2 \dots <_2 i_{n-1} < i_n = i_{n-1} + m_2 <_k i_{n+1} <_k \dots <_k i_d} a^{v_{(i_1+1)h+1} + \dots + v_{(i_d+1)h+1}} \right) \right) \right).$$

Let $A' = \sum_{d \geq 1,\; 1 \leq i_1 <_h i_2 <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+2)h+1} + \dots + v_{(i_d+2)h+1}}$

$$+ \sum_{d \geq 2,\; 1 \leq i_1 < i_2 = i_1 + (h-1) <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+2)h+1} + \dots + v_{(i_d+2)h+1}}$$

$$+ \; \dots \; + \sum_{d \geq 2,\; 1 \leq i_1 < i_2 = i_1 + 2 <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+2)h+1} + \dots + v_{(i_d+2)h+1}}$$

$$= \sum_{d \geq 1,\; 1 \leq i_1 <_h i_2 <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+2)h+1} + \dots + v_{(i_d+2)h+1}}$$

$$+ \sum_{m=2}^{h-1} \left(\sum_{d \geq 2,\; 1 \leq i_1 < i_2 = i_1 + m <_k i_3 <_k \dots <_k i_d} a^{v_{(i_1+2)h+1} + \dots + v_{(i_d+2)h+1}} \right).$$

Now,

$$G(a) = 1 + \sum_{n \geq 1,\; 1 \leq i_1 <_2 i_2 <_2 \dots <_2 i_n} a^{v_{(i_1+2)h+1} + \dots + v_{(i_d+2)h+1}}$$

$$= 1 + A' + B'$$

where B' is a sum of the same form as in $G(a)$ but extends over $n \geq 1,\; 1 \leq i_1 <_2 i_2 <_2 \dots <_2 i_{j-1} <_2 i_j <_k i_{j+1} <_k \dots <_k i_n$ such that $i_2 = i_1 + m_1$ $(2 \leq m_1 \leq h)$ and there exists a pair $i_j = i_{j-1} + m_2$ $(2 \leq m_2 \leq k-1)$ for some $j \geq 3$.

Similarly,

$$G(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k \ldots <_k i_d} a^{v_{(i_1+2)h+1}+\ldots+v_{(i_d+2)h+1}}$$

$$+ \sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{\substack{d\,\geq\, n,\\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d}} a^{v_{(i_1+2)h+1}+\ldots+v_{(i_d+2)h+1}}\right)\right)\right).$$

Consider $\quad 1 + \dfrac{a^{v_{2h+1}}}{1+}\ \dfrac{a^{v_{3h+1}}}{1+}\ \dfrac{a^{v_{4h+1}}}{1+}\ \dfrac{a^{v_{5h+1}}}{1+\ldots}$

$$= 1 + \frac{a^{-v_{2h}}a^{v_{2h+1}}}{a^{-v_{2h}}+}\ \frac{a^{-v_{3h}}a^{v_{3h+1}}a^{-v_{2h}}}{a^{-v_{3h}}+}\ \frac{a^{-v_{4h}}a^{v_{4h+1}}a^{-v_{3h}}}{a^{-v_{4h}}+\ldots}$$

$$= 1 + \frac{a^{-v_{2h}+v_{2h+1}}}{a^{-v_{2h}}+}\ \frac{a}{a^{-v_{3h}}+}\ \frac{a}{a^{-v_{4h}}+}\ \frac{a}{a^{-v_{5h}}+\ldots}$$

$$= 1 + \frac{a^{-v_{2h}+v_{2h+1}-1}}{a^{-v_{2h}-1}+}\ \frac{1}{a^{-v_{3h}-1}+}\ \frac{1}{a^{-v_{4h}-1}+}\ \frac{1}{a^{-v_{5h}-1}+\ldots}\ .$$

By (3.3), $\quad 1 + \dfrac{a^{-v_{2h}+v_{2h+1}-1}}{a^{-v_{2h}-1}+}\ \dfrac{1}{a^{-v_{3h}-1}+}\ \dfrac{1}{a^{-v_{4h}-1}+}\ \dfrac{1}{a^{-v_{5h}-1}+\ldots} = \dfrac{F(a)}{G(a)}.$

$$a^{v_{2h}-v_{2h+1}+1} + \frac{1}{a^{-v_{2h}-1}+}\ \frac{1}{a^{-v_{3h}-1}+}\ \frac{1}{a^{-v_{4h}-1}+}\ \frac{1}{a^{-v_{5h}-1}+\ldots} = a^{v_{2h}-v_{2h+1}+1}\frac{F(a)}{G(a)}.$$

□

3.2 Explicit continued fractions whose partial denominators are 1

Let v_n be the $(1,2)^{\text{th}}$ Fibonacci sequence:

$$v_1 = 1,\quad v_2 = 2 \text{ and } \quad v_n = v_{n-1} + v_{n-2} + 1 \qquad \text{where} \qquad n \geq 3.$$

Set $v_0 = 0$ and $v_{-1} = 0$. Then $v_2 = v_1 + v_0 + 1$ and $v_1 = v_0 + v_{-1} + 1$. and

$$v_n = v_{n-1} + v_{n-2} + 1 \quad (n \geq 1).$$

By Theorem 1.7, the $(1,2)^{\text{th}}$ Fibonacci representation of an integer $N \geq 1$ is

$$N = v_{i_1} + v_{i_2} + \ldots + v_{i_d}$$

where $1 \leq i_1 < i_2 <_2 i_3 <_2 \ldots <_2 i_d$.

Theorem 3.4. Let a be a real number such that $| a |< 1$. Then

$$1 + \frac{a^{v_1}}{1+} \; \frac{a^{v_2}}{1+} \; \frac{a^{v_3}}{1+\ldots} = \frac{F(a)}{G(a)}$$

where $F(a) = 1+\dfrac{a}{1-a} - \displaystyle\sum_{n\geq 2}\left(a^{v_{n-1}-v_n}\left(1+\sum_{d\geq 3, 1\leq j_3<_2\ldots<_2 j_d} a^{v_{j_3+n+1}+v_{j_4+n+1}+\ldots+v_{j_d+n+1}}\right)\right)$,

$$G(a) = 1 + \sum_{d\geq 1, 1\leq i_1<i_2<_2 i_3<_2\ldots<_2 j_d} a^{v_{i_1+1}+\ldots+v_{i_d+1}} - \sum_{n\geq 2}\left(a^{v_n - v_{n+1}}\left(1+\sum_{d\geq 3, 1\leq j_3<_2\ldots<_2 j_d} a^{v_{j_3+n+2}+v_{j_4+n+2}+\ldots+v_{j_d+n+2}}\right)\right)$$

and (v_n) is the $(1,2)^{\text{th}}$ Fibonacci sequence.

Proof. Let $C_n = a^{v_n}$ where a is a real number such that $| a | < 1$. Substitute C_n into (2.1) to get

$$\begin{aligned}
1 + \frac{a^{v_1}}{1+} \; \frac{a^{v_2}}{1+} \; \frac{a^{v_3}}{1+\ldots} &= 1 + \frac{C_1}{1+}\,\frac{C_2}{1+}\,\frac{C_3}{1+\ldots} \\
&= \frac{A_\infty(C_1, C_2, \ldots)}{A_\infty(C_2, C_3, \ldots)} \\
&= \frac{1+\displaystyle\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+\ldots+v_{i_n}}}{1+\displaystyle\sum_{n\geq 1,\ 2\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+\ldots+v_{i_n}}} \\
&= \frac{1+\displaystyle\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+\ldots+v_{i_n}}}{1+\displaystyle\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1+1}+\ldots+v_{i_n+1}}}\,.
\end{aligned}$$

$$:= \frac{F(a)}{G(a)} \tag{3.4}$$

Let $A = \sum_{d\geq 1,\ 1\leq i_1<i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$.

By Theorem 1.7, $A = \sum_{k\in\mathbb{N}} a^k = \frac{a}{1-a}$.

Now,

$$F(a) = 1 + \sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2 \ldots <_2 i_n} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$

$$= 1 + A - B$$

where B is a sum of the same form as in $F(a)$ but extends over $n \geq 1,\ 1 \leq i_1 < i_2 <_2 \ldots <_2 i_n$ such that $i_2 = i_1 + 1$. To evaluate B, we split the sum into subsums corresponding to all pair of consecutive values of each possible consecutive indices, namely, $B = B_{1,2} + B_{2,3} + B_{3,4} + \ldots$, where $B_{m,m+1}$ denoted the subsum taken over all possible consecutive indices $i_1,\ i_2$ with value pair $(m, m+1)$.

Thus for value-pair(1,2), possible indices are $i_1 = 1 < i_2 = 2$ and the subsum is

$$B_{1,2} = \sum_{d\geq 2,\ 1=i_1<2=i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$

$$= \sum_{d\geq 2,\ 2=i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_1+v_{i_2}}(a^{v_{i_3}+v_{i_4}+\ldots+v_{i_d}}).$$

Thus for value-pair(2,3), possible indices are $i_1 = 2 < i_2 = 3$ and the subsum is

$$B_{2,3} = \sum_{d\geq 2,\ 2=i_1<3=i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$

$$= \sum_{d\geq 2,\ 3=i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_2+v_{i_2}}(a^{v_{i_3}+v_{i_4}+\ldots+v_{i_d}})$$

Thus for value-pair(3,4), possible indices are $i_1 = 3 < i_2 = 4$ and the subsum is

$$B_{3,4} = \sum_{d\geq 2,\ 3=i_1<4=i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$

$$= \sum_{d\geq 2,\ 4=i_2<_2 i_3<_2 \ldots <_2 i_d} a^{v_3+v_{i_2}}(a^{v_{i_3}+v_{i_4}+\ldots+v_{i_d}})$$

The computations of other $B_{m,m+1}$ are carried out in a similar manner and we arrive at

$$\begin{aligned} F(a) &= 1 + A - B \\ &= 1+\frac{a}{1-a}-\sum_{n\geq 2}\left(\sum_{d\geq 2,\ n=i_2<_2 i_3<_2\ldots<_2 i_d} a^{v_{i_2-1}+v_{i_2}}\left(a^{v_{i_3}+v_{i_4}+\ldots+v_{i_d}}\right)\right) \\ &= 1+\frac{a}{1-a}-\sum_{n\geq 2}\left(a^{v_{n-1}-v_n}\left(1+\sum_{d\geq 3,\ n<_2 i_3<_2\ldots<_2 i_d} a^{v_{i_3}+v_{i_4}+\ldots+v_{i_d}}\right)\right) \\ &= 1+\frac{a}{1-a}-\sum_{n\geq 2}\left(a^{v_{n-1}-v_n}\left(1+\sum_{d\geq 3,\ n+2\leq i_3<_2\ldots<_2 i_d} a^{v_{i_3}+v_{i_4}+\ldots+v_{i_d}}\right)\right) \\ &= 1+\frac{a}{1-a}-\sum_{n\geq 2}\left(a^{v_{n-1}-v_n}\left(1+\sum_{d\geq 3, 1\leq j_3<_2\ldots<_2 j_d} a^{v_{j_3+n+1}+v_{j_4+n+1}+\ldots+v_{j_d+n+1}}\right)\right). \end{aligned}$$

Let $A' = \sum_{d\geq 1,\ 1\leq i_1<i_2<_2 i_3<_2\ldots<_2 i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$.

Now,

$$\begin{aligned} G(a) &= 1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}} \\ &= 1 + A' - B' \end{aligned}$$

where B' is a sum of the same form as in $G(a)$ but extends over $n \geq 1,\ 1 \leq i_1 < i_2 <_2 \ldots <_2 i_n$ such that $i_2 = i_1 + 1$. To evaluate B', we split the sum into subsums corresponding to all pair of consecutive values of each possible consecutive indices, namely, $B' = B'_{1,2} + B'_{2,3} + B'_{3,4} + \ldots$, where $B'_{m,m+1}$ denoted the subsum taken over all possible consecutive indices $i_1,\ i_2$ with value pair $(m, m+1)$.

Similarly,

$$\begin{aligned} G(a) = 1 &+ \sum_{d\geq 1,\ 1\leq i_1<i_2<_2 i_3<_2\ldots<_2 i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}} \\ &-\sum_{n\geq 2}\left(a^{v_n-v_{n+1}}\left(1+\sum_{d\geq 3, 1\leq j_3<_2\ldots<_2 j_d} a^{v_{j_3+n+2}+v_{j_4+n+2}+\ldots+v_{j_d+n+2}}\right)\right). \end{aligned}$$

By (3.4), $1+\frac{a^{v_1}}{1+}\ \frac{a^{v_2}}{1+}\ \frac{a^{v_3}}{1+\ldots} = \frac{F(a)}{G(a)}$. □

Let v_n be the $(h,k)^{\text{th}}$ Fibonacci sequence where $k \geq 3$:

$$\begin{aligned} v_n &= n && \text{for } 1 \leq n \leq k, \\ v_n &= v_{n-1} + v_{n-h} && \text{for } k < n < h+k, \\ v_n &= v_{n-1} + v_{n-k} + (k-h) && \text{for } n \geq h+k. \end{aligned}$$

By Theorem 1.7, the $(h,k)^{\text{th}}$ Fibonacci representation of an integer $N \geq 1$ is

$$N = v_{i_1} + v_{i_2} + \dots + v_{i_d}$$

where $1 \leq i_1 <_h i_2 <_k i_3 <_k \dots <_k i_d$.

Theorem 3.5. Let a be a real number such that $\mid a \mid \leq \frac{1}{4}$. Then

$$1 + \frac{a^{v_1}}{1+}\,\frac{a^{v_2}}{1+}\,\frac{a^{v_3}}{1+\dots} = \frac{F(a)}{G(a)}$$

where $F(a) = 1 + \dfrac{a}{1-a} + \displaystyle\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\dots<_k i_d} a^{v_{i_1}+v_{i_2}+\dots+v_{i_d}}\right)$

$$+ \sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\dots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\dots<_k i_d} a^{v_{i_1}+v_{i_2}+\dots+v_{i_d}}\right)\right)\right),$$

$$G(a) = 1 + \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\dots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\dots+v_{i_d+1}}$$

$$+ \sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\dots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\dots+v_{i_d+1}}\right)$$

$$+ \sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\dots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\dots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\dots+v_{i_d+1}}\right)\right)\right)$$

and (v_n) is the $(h,k)^{\text{th}}$ Fibonacci sequence, $k \geq 3$, $h = k-1$ or k.

Proof. Let $C_n = a^{v_n}$ where a is a real number such that $\mid a \mid \leq \frac{1}{4}$. Substitute C_n into (2.1) to get

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$$1+\frac{a^{v_1}}{1+}\ \frac{a^{v_2}}{1+}\ \frac{a^{v_3}}{1+\ldots} \;=\; 1+\frac{C_1}{1+}\ \frac{C_2}{1+}\ \frac{C_3}{1+\ldots}$$

$$= \frac{A_\infty(C_1, C_2, \ldots)}{A_\infty(C_2, C_3, \ldots)}$$

$$= \frac{1+\displaystyle\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+\ldots+v_{i_n}}}{1+\displaystyle\sum_{n\geq 1,\ 2\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+\ldots+v_{i_n}}}$$

$$= \frac{1+\displaystyle\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+\ldots+v_{i_n}}}{1+\displaystyle\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1+1}+\ldots+v_{i_n+1}}}\ .$$

$$:= \frac{F(a)}{G(a)} \tag{3.5}$$

Let
$$A = \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$
$$+ \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+(h-1)<_k i_3<_k\ldots<_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$
$$+\ \ldots\ + \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+2<_k i_3<_k\ldots<_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}.$$

By Theorem 1.7,
$$A = \sum_{k\in\mathbb{N}} a^k + \sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}\right)$$
$$= \frac{a}{1-a} + \sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}\right).$$

Now,
$$F(a) = 1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}$$
$$= 1 + A + B$$

where B is a sum of the same form as in $F(a)$ but extends over $n\geq 1,\ 1\leq i_1 <_2$

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

$i_2 <_2 \ldots <_2 i_{j-1} <_2 i_j <_k i_{j+1} <_k \ldots <_k i_n$ such that $i_2 = i_1 + m_1 \;\; (2 \leq m_1 \leq h)$ and there exists a pair $i_j = i_{j-1} + m_2 \;\; (2 \leq m_2 \leq k-1)$ for some $j \geq 3$. Then

$$
\begin{aligned}
B \; = \; \sum_{m_1=2}^{h} \Bigg(& \sum_{d\geq 3,\;\; 1\leq i_1<i_2=i_1+m_1<i_3=i_2+(k-1)<_k i_4<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \sum_{d\geq 3,\;\; 1\leq i_1<i_2=i_1+m_1<i_3=i_2+(k-2)<_k i_4<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \ldots + \sum_{d\geq 3,\;\; 1\leq i_1<i_2=i_1+m_1<i_3=i_2+2<_k i_4<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \sum_{d\geq 4,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+(k-1)<_k i_5<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \sum_{d\geq 4,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+(k-2)<_k i_5<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \ldots + \sum_{d\geq 4,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+2<_k i_5<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \sum_{d\geq 5,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<_2 i_4<i_5=i_4+(k-1)<_k i_6<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \sum_{d\geq 5,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<_2 i_4<i_5=i_4+(k-2)<_k i_6<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \ldots + \sum_{d\geq 5,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<_2 i_4<i_5=i_4+2<_k i_6<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \ldots \Bigg) \\
= \; \sum_{m_1=2}^{h} \Bigg(\sum_{m_2=2}^{k-1} \Bigg(& \sum_{d\geq 3,\;\; 1\leq i_1<i_2=i_1+m_1<i_3=i_2+m_2<_k i_4<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \sum_{d\geq 4,\;\; 1\leq i_1<i_2=i_1+m_1<_2 i_3<i_4=i_3+m_2<_k i_5<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \\
& + \ldots \Bigg)\Bigg) \\
= \sum_{m_1=2}^{h} \Bigg(\sum_{m_2=2}^{k-1} \Bigg(\sum_{n\geq 3} \Bigg(& \sum_{\substack{d\geq n \\ 1\leq i_1<i_2=i_1+m_1<_2 \ldots <_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k \ldots <_k i_d}} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \Bigg)\Bigg)\Bigg)
\end{aligned}
$$

Thus $F(a) \; = \; 1 \; + \; \dfrac{a}{1-a} \; + \; \displaystyle\sum_{m=2}^{h-1} \Bigg(\sum_{d\geq 2,\;\; 1\leq i_1<i_2=i_1+m<_k i_3<_k \ldots <_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}} \Bigg)$

$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{d\geq n,\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d} a^{v_{i_1}+v_{i_2}+\ldots+v_{i_d}}\right)\right)\right).$$

Let

$$A' = \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$$

$$+ \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+(h-1)<_k i_3<_k\ldots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$$

$$+\ \ldots\ + \sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+2<_k i_3<_k\ldots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$$

$$= \sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$$

$$+\sum_{m=2}^{h-1}\left(\sum_{d\geq 2,\ 1\leq i_1<i_2=i_1+m<_k i_3<_k\ldots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}\right).$$

Now,

$$G(a) = 1+\sum_{n\geq 1,\ 1\leq i_1<_2 i_2<_2\ldots<_2 i_n} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$$

$$= 1 + A' + B'$$

where B' is a sum of the same form as in $G(a)$ but extends over $n\geq 1,\ 1\leq i_1<_2 i_2<_2 \ldots <_2 i_{j-1}<_2 i_j<_k i_{j+1}<_k \ldots <_k i_n$ such that $i_2=i_1+m_1$ $(2\leq m_1\leq h)$ and there exists a pair $i_j=i_{j-1}+m_2$ $(2\leq m_2\leq k-1)$ for some $j\geq 3$.

Similarly,

$$G(a)=1+\sum_{d\geq 1,\ 1\leq i_1<_h i_2<_k i_3<_k\ldots<_k i_d} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}$$

$$+\sum_{m_1=2}^{h}\left(\sum_{m_2=2}^{k-1}\left(\sum_{n\geq 3}\left(\sum_{\substack{d\,\geq\, n,\\ 1\leq i_1<i_2=i_1+m_1<_2\ldots<_2 i_{n-1}<i_n=i_{n-1}+m_2<_k i_{n+1}<_k\ldots<_k i_d}} a^{v_{i_1+1}+v_{i_2+1}+\ldots+v_{i_d+1}}\right)\right)\right).$$

By (3.5), $\quad 1+\dfrac{a^{v_1}}{1+}\ \dfrac{a^{v_2}}{1+}\ \dfrac{a^{v_3}}{1+\ldots} = \dfrac{F(a)}{G(a)}.$ $\square$

REFERENCES

[1] Adams W.W. and Davison J.L. A remarkable class of continued fractions, **Proc. Amer. Math. Soc.** 65(1977): 194-198.

[2] Bowman D. A new generalization of Davison's Theorem, **Fibonacci Quart.** 26(1986): 40-45.

[3] Brown T.C. Descriptions of the characteristic sequence of an irrationsl, **Canad. Math. Bull.** 36(1993): 15-21.

[4] Davison J.L. A series and its associated continued fractions, **Proc. Amer. Math. Soc.** 63(1977): 29-32.

[5] Daykin D.E. Representation of natural numbers as sums of generalised fibonacci numbers, **J. London Math. Soc.** 35(1960): 143-160.

[6] Fraenkel A.S., Levitt J. and Shimshoni M., Characterization of the set of values $f(n) = [n\alpha],\ n = 1, 2, ...$, **Discrete Math.** 2(1972): 335-345.

[7] Jones W.B. and Thron W.J. Continued fractions analytic theory and applications, **Addison-Wesley** 1980.

[8] Lorentzen L. and Waadeland H. Continued fractions with applications , **North-Holland** 1992.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย

VITA

Miss Monrudee Yodphothong was born on December 29, 1978 in Bangkok, Thailand. She graduated with a Bachelor Degree of Science in Mathematics with first class honor from Kasetsart University in 1999. She has got a scholarship from University Development Commission(U.D.C.) since 2002. For her Master degree, she has studied Mathematics at the Department of Mathematics, Faculty of Science, Chulalongkorn University.

สถาบันวิทยบริการ
จุฬาลงกรณ์มหาวิทยาลัย