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LIST ASSIGNMENT PROBLEMS

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การกำหนดค่ารายการแบบ-k ของกราฟ G คือฟังก์ชันจากเซตของจุดยอดของกราฟ G ไปเซตของเซตขนาด k ให้ L เป็นการกำหนดค่ารายการแบบ-k ของกราฟ G เรียก L ว่าเป็นการกำหนดค่ารายการแบบ-(k,t) เมื่อ $|\bigcup_{v\in V(G)} L(v)|=t$ และ G เป็นกราฟระบายสีได้แบบ-L เมื่อ G มีฟังก์ชันการระบายสี f ที่ $f(v)\in L(v)$ ทุก $v\in V(G)$ ถ้ากราฟ G เป็นกราฟระบายสีได้แบบ-L สำหรับทุก L ที่เป็นการกำหนดค่ารายการแบบ-(k,t) จะเรียก G ว่ากราฟเลือกได้แบบ-(k,t) และถ้า G เป็นกราฟเลือกได้แบบ-(k,t) สำหรับทุก t แล้วจะเรียก G ว่าเป็นกราฟเลือกได้แบบ-k

ในวิทยานิพนธ์ฉบับนี้เราหาเงื่อนไขเพียงพอที่ทำให้กราฟที่มีจุดยอด n จุดเป็นกราฟเลือกได้แบบ-(k,t) และหาเงื่อนไขที่เพียงพอที่ทำให้กราฟที่มีจุดยอด n จุดซึ่งไม่มี K_{k+1} เป็นกราฟย่อยเป็นกราฟเลือกได้แบบ-(k,t) นอกจากนั้นเราสร้างกลยุทธ์ใหม่เพื่อที่ได้ผลลัพธ์ทั้งหมดเกี่ยวกับสมบัติการเลือกได้แบบ-3 ของกราฟสองส่วนแบบบริบูรณ์ที่มีจุดยอดไม่เกิน 16 จุด และศึกษาสมบัติการเลือกได้แบบ-(k,t) ของกราฟสองส่วนแบบบริบูรณ์ $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$

ภาควิชา.............คณิตศาสตร์.............	ลายมือชื่อนิสิต...
สาขาวิชา ....คณิตศาสตร์.............	ลายมือชื่ออ.ที่ปรึกษาวิทยานิพนธ์หลัก............................
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A k-*list assignment* of a graph G is a function which assigns a set of size k to each vertex of G. Given a k-list assignment L of a graph G, L is called a (k,t)-*list assignment* when $|\bigcup_{v\in V(G)} L(v)| = t$ and G is L-*colorable* when G has a proper coloring f such that $f(v) \in L(v)$ for all $v \in V(G)$. If a graph G is L-colorable for every (k,t)-list assignment L, then G is called (k,t)-*choosable* and if G is (k,t)-choosable for each positive integer t then G is called k-*choosable*.

In this dissertation, we investigate a sufficient condition to be (k,t)-choosable of n-vertex graphs and n-vertex graphs not containing K_{k+1} as a subgraph. Moreover, we establish new strategies to obtain the complete result of 3-choosability of complete bipartite graphs with at most 16 vertices, and study the (k,t)-choosability of the complete bipartite graph $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$ for all positive integers t.

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Field of Study :	...Mathematics...	Advisor's Signature :...............................
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CHAPTER I

INTRODUCTION

1.1 Definitions and Notations

Recall some known definitions and notations here. For other terminologies, we follow West's book [22]. Unless we say otherwise, G denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. A *clique* is a set of pairwise adjacent vertices in a graph; a clique of size k is called a k*-clique.* A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle; the cycle with n vertices is denoted by C_n. A *complete graph* is a graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets called *partite sets.* A *complete bipartite graph* is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets; the complete bipartite graph with partite sets of size a and b is denoted by $K_{a,b}$. Given a graph G and $S \subseteq V(G)$, $G - S$ is the graph obtained from G by deleting all vertices of S. In case $S = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph induced by* S, denoted by $G[S]$ is the graph obtained from G by deleting all vertices of $V(G)$ outside S. Given a graph H, a graph is said to be H*-free* if H is not its induced subgraph. A graph is said to be a *triangle-free* if it does not contain a 3-clique. A *complement* of a graph G, denoted by $\overline{G}$, is the graph with the vertex set $V(G)$

defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. The *join* of graphs G and H, written $G \vee H$, is the graph obtained from G and H by adding the edges between all vertices of G and all vertices of H.

A *coloring* of a graph G is a mapping from $V(G)$ to a set of colors S such that adjacent vertices receive distinct colors. If $|S| = t$, then such coloring is called a t*-coloring.* A graph is t*-colorable* if it has a t-coloring. The *chromatic number* of G, denoted by $\chi(G)$ is the smallest positive integer t such that G is t-colorable. A *list assignment* of a graph G is a mapping which assigns a set of colors, called a *list* to each vertex $v \in V(G)$. A list assignment L of a graph G is said to be a k*-list assignment* if $|L(v)| = k$ for all $v \in V(G)$. A k-list assignment L of a graph G is said to be a (k,t)*-list assignment* if $|\bigcup_{v \in V(G)} L(v)| = t$. Given a list assignment L of a graph G, a coloring f of G is an L*-coloring* of G if $f(v)$ is chosen from $L(v)$ for each vertex $v \in V(G)$. A graph is L*-colorable* if it has an L-coloring. Particularly, if L is a (k,k)-list assignment of a graph G, then any L-coloring of G is a k-coloring of G. A graph G is (k,t)*-choosable* if G is L-colorable for every (k,t)-list assignment L of G. If a graph G is (k,t)-choosable for each positive integer t then G is called k*-choosable*, and the smallest positive integer k satisfying this property is called the *list chromatic number* of G denoted by $\chi_l(G)$.

Example 1.1.1. Let L be the 2-list assignment of C_5 as shown in Figure 1.1.1. That is, $L(v_1) = \{1,2\}$, $L(v_2) = \{1,3\}$, $L(v_3) = \{1,2\}$, $L(v_4) = \{2,3\}$ and $L(v_5) = \{1,3\}$. Because of $|\bigcup_{v \in V(C_5)} L(v)| = 3$, L is called a $(2,3)$-list assignment of C_5.

Let f be a coloring of C_5 as shown in Figure 1.1.1. That is, $f(v_1) = 2$, $f(v_2) = 3$, $f(v_3) = 1$, $f(v_4) = 3$ and $f(v_5) = 1$. Because of $f(v) \in L(v)$ for all $v \in V(C_5)$, f is an L-coloring of C_5.

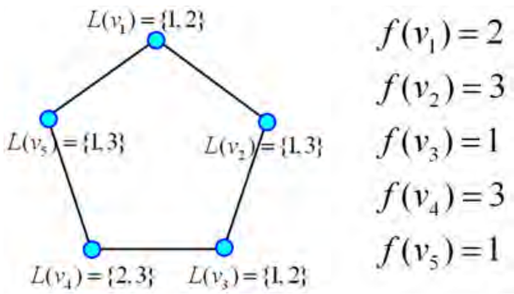


Figure 1.1.1: A $(2,3)$-list assignment of C_5.

If there is no ambiguous, each list is written without commas and braces; moreover, each box containing a color from each list represent its coloring in order to simplify our figure. Figure 1.1.2 is the simplify figure of Figure 1.1.1.

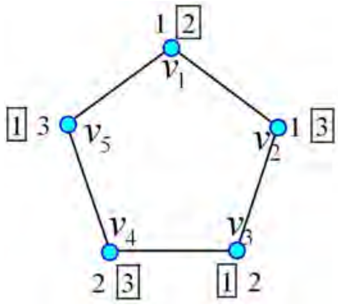


Figure 1.1.2: A $(2,3)$-list assignment of C_5.

Now, we consider $(2,3)$-choosability of C_5. The set of all $(2,3)$-list assignments of C_5 is divided into eight cases. $L_1, L_2, \ldots, L_8$ in Figure 1.1.3 represent a $(2,3)$-list assignments of C_5 in each case

The $(2,3)$-list assignment L_1 contains four vertices with the same list while L_2, L_3, L_4 and L_5 contain three vertices with the same list. The list assignments L_6, L_7 and L_8 contain only two vertices with the same list. It is shown in Figure 1.1.3 that C_5 is L_i-colorable for each $i = 1, 2, \ldots, 8$.

$\square$

Example 1.1.2. Let G be the graph with eight vertices in Figure 1.1.4. The minimum number of colors in a 3-list assignment of G occurs when all vertices

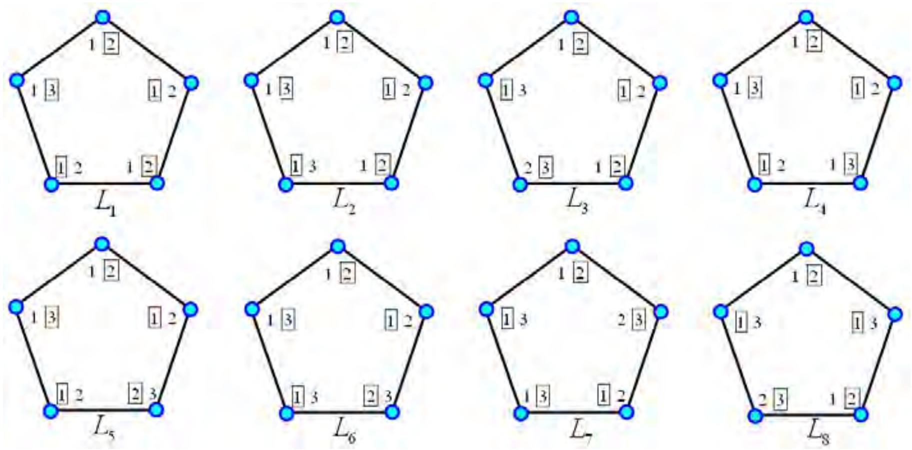


Figure 1.1.3: C_5 is $(2,3)$-choosable.

are assigned by the same list of size 3 while the maximum number of colors in a 3-list assignment of G occurs when all vertices are assigned by mutually disjoint lists as shown in Figure 1.1.4.

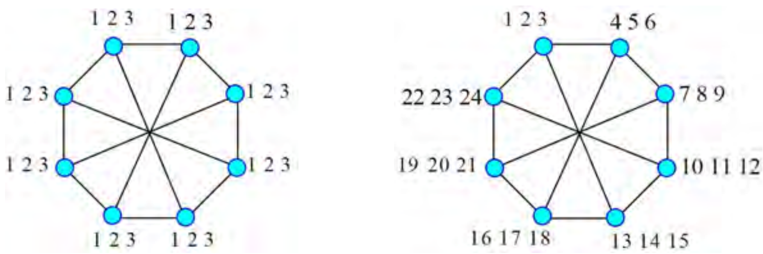


Figure 1.1.4: A $(3,3)$-list assignment and a $(3,24)$ list assignment

$\square$

Unless we say otherwise, our parameters k, n and t in this dissertation are always positive integers such that $t \geq k$ and $t \leq kn$ because when $t < k$ or $t > kn$, there is no (k,t)-list assignment of a graph with n vertices, so it is automatically (k,t)-choosable. If $k \geq n$ then all graphs with n vertices are (k,t)-choosable. Besides, when $k \geq \chi_l(G)$, a graph G is always (k,t)-choosable; therefore, we focus on a positive integer k such that $k < \chi_l(G)$.

Let $S \subseteq V(G)$. If L is a list assignment of G, we let $L|_S$ denote L restricted to S and $L(S)$ denote $\bigcup_{v \in S} L(v)$. For a color set A, let $L - A$ be the new list assignment obtained from L by deleting all colors in A from $L(v)$ for each $v \in V(G)$. When A has only one color a, we write $L - a$ instead of $L - \{a\}$. Examples are illustrated in Figures 1.1.5.

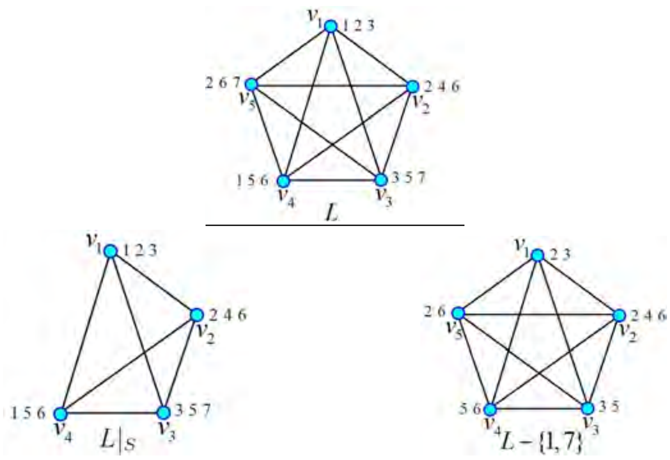


Figure 1.1.5: the list assignment $L|_S$ of $K_5[S]$ where $S = \{v_1, v_2, v_3, v_4\}$ and the list assignment $L - \{1, 7\}$ of K_5

1.2 History and Outline

The problem of list assignments was first studied in 1976 by Vizing [21] and by Erdős, Rubin and Taylor [4]. The authors gave a characterization of 2-choosable graphs. However, for a positive integer $k \geq 3$, there has been no literature giving a complete solution of k-choosable graphs, yet only some specific classes of graphs are investigated. For example, all planar graphs are 5-choosable, while some planar graphs are 3-choosable. (See [11],[20],[19],[23],[24],[25], [26] for more details.)

Finding the list chromatic number of a graph is considered to be a complicated problem. Even in the case of bipartite graphs, a characterization of complete bipartite graphs which are k-choosable is revealed only when $k \leq 3$. Let a and

b be positive integers such that $a \leq b$. Then the complete bipartite graph $K_{a,b}$ is 2-choosable if and only if $a = 1$ or $(a, b) \in \{(2, b)|b \leq 3\}$ and $K_{a,b}$ is 3-choosable if and only if $a \leq 2$ or $(a, b) \in \{(3, b)|b \leq 26\} \cup \{(4, b)|b \leq 20\} \cup \{(5, b)|b \leq 12\} \cup \{(6, b)|b \leq 10\}$. (See [4],[12],[17],[15].) Moreover, for $7 \leq a \leq b$, Erdős et al. showed in [4] that $K_{a,b}$ is not 3-choosable because $K_{7,7}$ is not 3-choosable. They defined a list assignment from the set of the seven lines in the Fano plane. Given $\mathcal{F} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$, let $L_{\mathcal{F}}$ be the 3-list assignment of $K_{7,7}$ such that all seven vertices in each partite set are assigned by distinct lists from $\mathcal{F}$. Erdős et al. proved that $K_{7,7}$ is not $L_{\mathcal{F}}$-colorable. Later, in 1996, Hanson, MacGillivray, and Toft [8] proved that every complete bipartite graph with at most 13 vertices is 3-choosable. Hence, the smallest complete bipartite graph which is not 3-choosable has 14 vertices. Fitzpatrick and MacGillivray [5] added that every complete bipartite graph with 14 vertices except $K_{7,7}$ is 3-choosable. Moreover, $L_{\mathcal{F}}$ is the unique list assignment up to renaming the colors which prevents $K_{7,7}$ from being 3-choosable. This result inspires us to study more on 3-choosability of complete bipartite graphs with fifteen vertices and sixteen vertices in Chapter III and Chapter IV.

Since k-choosability implies k-colorability, we have $\chi(G) \leq \chi_l(G)$ for every graph G. Note that for a tree T, $\chi(T) = \chi_l(T) = 2$; however, there exists a graph of which such two parameters are significantly different. These graphs was found in [21] and [4], for all positive integer k, the authors gave a non k-choosable complete bipartite graph $K_{m,m}$ where $m = \binom{2k-1}{k}$ with a list assignment L containing $2k - 1$ colors such that $K_{m,m}$ is not L-colorable. In other words, $K_{m,m}$ is not $(k, 2k-1)$-choosable. We then are interested in exploring more results when the total number of colors is not $2k - 1$. We investigate (k, t)-choosability of $K_{m,m}$ when $t \neq 2k - 1$ in Chapter V.

Ganjari et al. [6] first defined (k,t)-choosability in order to generalize a characterization of uniquely 2-list colorable graphs. Besides, Fitzpatrick and MacGillivray [5] proved 3-choosability of complete bipartite graphs with 14 vertices by showing the graphs is $(3,t)$-choosable for each positive integer t.

The dissertation has six chapters, including this introduction in Chapter I. Next, we start studying a (k,t)-list assignment of any graph in Chapter II. We obtain a sufficient condition of a (k,t)-choosable graph with n vertices; if $t \geq kn-k^2+1$ then every graph with n vertices is always (k,t)-choosable. Moreover, we prove that this bound is best possible because a graph with n vertices containing $k+1$-clique is not $(k, kn-k^2)$-choosable. However, we also improve this bound for a K_{k+1}-free graph; if $k \geq kn-k^2-2k+1$, then every K_{k+1}-free graph with n vertices is (k,t)-choosable and this bound is best possible for a K_{k+1}-free graph with n vertices.

Chapter III and Chapter IV are devoted to solve the problem of the 3-choosability of complete bipartite graphs with at most 16 vertices. In 2005, Fitzpatrick and MacGillivray [5] extend the result in [4] and [8] to obtain a stronger result that every complete bipartite graph with 14 vertices except $K_{7,7}$ is 3-choosable. Moreover, $L_{\mathcal{F}}$ is the unique list assignment up to renaming the colors which prevents $K_{7,7}$ from being 3-choosable. In order to keep extending this result to 16 vertices, we establish new strategies in Chapter III, which also lead to an alternative proof of [5].

Chapter V focuses on (k,t)-choosability of $K_{m,m}$ where $m = \binom{2k-1}{k}$. We give results of (k,t)-choosability of $K_{m,m}$ when $t \neq 2k-1$; if $t \leq 2k-2$ or $t \geq 2km-2k^2+2k$, then $K_{m,m}$ is L-colorable, while if $2k-1 \leq t \leq 17 \cdot 2^{k-2}-4k-4$ then $K_{m,m}$ is not (k,t)-choosable. In particularly, when $k=3$, we integrate the results in Chapters II, III and IV to conclude our results in Theorems 5.3.1 and

5.3.3. Finally, we summarize results from all chapters and introduce some future work in Chapter VI.

CHAPTER II

ON (k,t)-CHOOSABILITY OF GRAPHS

2.1 Basic Properties and Examples

In Example 2.1.1, we show that C_{2n} is $(2,t)$-choosable for every positive integer t and C_{2n+1} is $(2,t)$-choosable for every positive integer $t \geq 3$. Moreover, we show that $K_{2,3}$ is $(2,t)$-choosable for every positive integer t.

Example 2.1.1.

(i) *Choosability of cycles. The cycle* C_n *is* $(2,t)$*-choosable unless* n *is odd and* $t = 2$.

Note that a graph G is $(2,2)$-choosable if and only if G is 2-colorable. Hence, C_n is $(2,2)$-choosable if and only if n is even. It remains to show that all of the cycles are $(2,t)$-choosable for $t \geq 3$.

Let $t \geq 3$ and L be a $(2,t)$-list assignment of C_n. Thus there are two adjacent vertices $v_1, v_n \in V(G)$ such that $L(v_1) \neq L(v_n)$. Let $v_2, v_3 \ldots, v_{n-1}$ be remaining vertices along the cycle C_n where v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$. First, we label v_1 by a color c in $L(v_1)$ which is not in $L(v_n)$ and then we label vertex v_2 by a color in $L(v_2)$ different from c and so on. This algorithm guarantees that each pair of adjacent vertices receives distinct colors.

(ii) *Choosability of* $K_{2,3}$. *The complete bipartite graph* $K_{2,3}$ *is* $(2,t)$*-choosable for every positive integer* t.

Let $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ be the partite sets of $K_{2,3}$ and L be a $(2,t)$-list assignment of $K_{2,3}$. If $L(u_1) \cap L(u_2) \neq \varnothing$, then u_1 and u_2 can be colored by the same color; hence, the remaining vertices in another partite set can be easily colored. Otherwise, $L(u_1) \cap L(u_2) = \varnothing$. There are 4 possible ways to pick a color from each of $L(u_1)$ and $L(u_2)$. Thus, we can choose $c_1 \in L(u_1)$ and $c_2 \in L(u_2)$ such that $\{c_1, c_2\}$ is distinct from $L(v_i)$ for $i = 1, 2, 3$. Then, we can color v_i by a color which is neither c_1 nor c_2 in $L(v_i)$ for $i = 1, 2, 3$. □

When we try to color all vertices of a graph with some conditions, it tends to success and be easier if we have more colors. However, this is not true for a (k, t)-list assignment. It may not be true that (k, t)-choosability implies $(k, t+1)$-choosability. Example 2.1.2 illustrates this fact.

Example 2.1.2. Let X, Y be the bipartite sets of $K_{10,10}$. To show that $K_{10,10}$ is $(3, 4)$-choosable, let L be a $(3, 4)$-list assignment of $K_{10,10}$. For any $u \in X$, at least one of the numbers $1, 2$ is in $L(u)$. Hence, each vertex in X can be colored by only color 1 or 2. For all $v \in Y$, at least one of the numbers $3, 4$ is in $L(v)$. Hence, we can color each vertex in Y by only color 3 or 4.

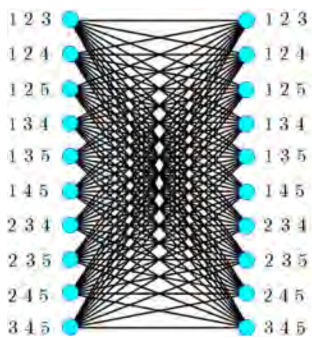


Figure 2.1.1: A $(3, 5)$-list assignment of $K_{10,10}$

To show that $K_{10,10}$ is not $(3, 5)$-choosable, let L be the $(3, 5)$-list assignment as shown in Figure 2.1.1. At least three colors must be used to color all vertices in each partite set of $K_{10,10}$. However, only five colors are available; hence, there are $u \in X$ and $v \in Y$ receiving the same color. It is a contradiction.

□

Although (k,t)-choosability does not imply $(k,t+1)$-choosability, if the number t is large enough, we can prove that (k,t)-choosability implies $(k,t+1)$-choosability. In Theorem 2.1.3, Hanson et.al. gives the number of colors we need to guarantee this statement.

Theorem 2.1.3. [8] *Let G be an n-vertex graph. If G is L_1-colorable for every k-list assignment L_1 such that $|\bigcup_{v\in V(G)} L_1(v)| = t$ and $n\binom{k}{2} < \binom{t+1}{2}$, then G is L_2-colorable for every k-list assignment L_2 such that $|\bigcup_{v\in V(G)} L_2(v)| \geq t$.*

Proof. Since $n\binom{k}{2} < \binom{t+1}{2}$, there exists a pair of colors which does not appear together in a list, say $1, 2$. Then we construct a k-list assignment L_1 defined by

$$L_1(v) = \begin{cases} L_2(v) & \text{if } 1 \in L_2(v), \\ L_2(v) \cup \{1\} - \{2\} & \text{if } 2 \in L_2(v). \end{cases}$$

Since G is not L_2-colorable, G is not L_1-colorable. $\square$

Definition 2.1.4. [22] Given a collection of sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, a *System of Distinct Representatives* (SDR) of $\mathcal{A}$ is a set of distinct elements $a_1, a_2, \ldots a_n$ such that $a_i \in A_i$ for all i.

The following theorem shows the well-known necessary and sufficient condition for the existence of an SDR. Indeed, Hall's Theorem [7] is originally proved in the language of an SDR and is equivalent to Manger's Theorem [13].

Theorem 2.1.5. [22] *Given a collection of sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, an SDR of $\mathcal{A}$ exists if and only if $|\bigcup_{i\in J} A_i| \geq |J|$ for all $J \subset \{1, 2, \ldots, n\}$.*

Corollary 2.1.6. *Let L be a list assignment of a graph G. If $|L(S)| \geq |S|$ for all $S \subset V(G)$, then G is L-colorable. Moreover, there exists an L-coloring such that each vertex of G assigned by distinct colors.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. From Theorem 2.1.5, there exist $c_1 \in L(v_1), c_2 \in L(v_2), \ldots, c_n \in L(v_n)$ such that $c_1, c_2, \ldots, c_n$ are distinct. Thus we define $f : V(G) \to \{1, 2, \ldots, n\}$ by $f(v_i) = c_i$; hence, f is an L-coloring. □

Theorem 2.1.7 studies a more profound condition than one in Corollary 2.1.6 to conclude an L-colorable graph. Kierstead [10] and He et al. [9] used it to investigate the list chromatic number on some complete multipartite graphs.

Theorem 2.1.7. [10] *Let L be a list assignment of a graph G and let $S \subset V(G)$ be a maximal non-empty subset such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$-colorable then G is L-colorable.*

To utilize Theorem 2.1.7 as well as simplify our proof, throughout the rest of the dissertation, we will prove a stronger assumption by considering all nonempty subsets $S \subset V(G)$ such that $|L(S)| < |S|$. We apply this theorem to obtain the expected results.

2.2 On (k, t)-choosability of K_{k+1}-free Graphs

In this section, we first find the parameters k and t such that an n-vertex graph is (k, t)-choosable. Theorem 2.2.2 states that if $t \geq kn - k^2 + 1$, then every n-vertex graph is always (k, t)-choosable and this bound is best possible because an n-vertex graph containing a k-clique is not $(k, kn-k^2)$-choosable. Fortunately, this bound can be improved for K_{k+1}-free graphs. Theorem 2.2.11 states that if $k \geq 3$ and $t \geq kn - k^2 - 2k + 1$, then every n-vertex graph which is K_{k+1}-free is always (k, t)-choosable and Theorem 2.2.12 states that if $t \geq 2n - 6$, then every n-vertex graph which is triangle-free is always $(2, t)$-choosable. Moreover, these bounds are best possible, as well.

The next lemma has a simple proof but quite powerful when we combine with Theorem 2.1.7 in order to obtain Theorem 2.2.2.

Lemma 2.2.1. *Let $A_1, A_2, \ldots, A_n$ be k-sets and $J \subset \{1, 2, \ldots, n\}$. If $|\bigcup_{i=1}^{n} A_i| \geq p$, then $|\bigcup_{i \in J} A_i| \geq p - (n - |J|)k$.*

Proof. Suppose that $|\bigcup_{i \in J} A_i| < p - (n - |J|)k$. Thus $|\bigcup_{i=1}^{n} A_i| \leq |\bigcup_{i \in J} A_i| + |\bigcup_{i \notin J} A_i| < p - nk + |J|k + k(n - |J|) = p$. It is a contradiction. □

Theorem 2.2.2. *For an n-vertex graph G, if $t \geq kn - k^2 + 1$ then G is (k,t)-choosable.*

Proof. Let L be a (k,t)-list assignment of G such that $t \geq kn - k^2 + 1$; that is, we obtain $|L(V(G))| = t \geq kn - k^2 + 1$. Let $S \subset V(G)$. If $|S| \leq k$, then, together with $|L(S)| \geq k$ always, $|L(S)| \geq |S|$. Otherwise, $|S| \geq k + 1$. By Lemma 2.2.1, $|L(S)| \geq kn - k^2 + 1 - (n - |S|)k = k|S| - k^2 + 1 = |S| + (k-1)|S| - k^2 + 1 \geq |S| + (k-1)(k+1) - k^2 + 1 = |S|$. Hence $|L(S)| \geq |S|$ for all $S \subset V(G)$; therefore, by Corollary 2.1.6, G is L-colorable. □

In particular, Theorem 2.2.2 can be rephrased in terms of a sufficient condition of the existence of an SDR on k-sets, concluded in Corollary 2.2.3.

Corollary 2.2.3. *Let $A_1, A_2, \ldots, A_n$ be k-sets. If $|\bigcup_{i=1}^{n} A_i| \geq kn - k^2 + 1$, then $A_1, A_2, \ldots, A_n$ have an SDR.*

Next, we will prove the bound in Theorem 2.2.2 is best possible by giving an n-vertex graph which is not $(k, kn - k^2)$-choosable.

Theorem 2.2.4. *An n-vertex graph containing a $(k+1)$-clique is not (k,t)-choosable where $k \leq t \leq kn - k^2$.*

Proof. Let G be an n-vertex graph containing $(k+1)$-clique K and $k \leq t \leq kn - k^2$. Consider a (k,t)-list assignment L of G such that $L(v) = \{1, 2, \ldots, k\}$

for each vertex v in K. Because $t-k \leq k(n-k-1)$, it is possible to construct a (k,t)-list assignment L in which the union of lists for the rest $n-k-1$ vertices outside K is $\{k+1, k+2, \ldots, t\}$. However, since every vertex in K receives the same list of size k, we cannot color all vertices in this $(k+1)$-clique. Therefore, G is not L-colorable. □

Theorem 2.2.4 shows the necessity of the first part in Theorem 2.2.14. The sufficiency will be held by Theorem 2.2.11. Besides, Theorems 2.2.12 and 2.2.13 are provided to claim the statement for the case $k = 2$. To simplify the proofs of our desired theorems, we prove a number of lemmas along the way.

Lemma 2.2.5. *Let G be an n-vertex graph. If $k \geq n-2$ and G is K_{k+1}-free, then G is (k,t)-choosable for every positive integer t.*

Proof. Let L be a (k,t)-list assignment of G where $t \geq k$. By Theorem 2.1.7, it suffices to show that $\forall S \subset V(G)$, if $|L(S)| < |S|$, then $G[S]$ is $L|_S$-colorable.

Let $S \subset V(G)$ such that $|L(S)| < |S|$. Recall that $|L(S)| \geq k$ and $|S| \leq n \leq k+2$; hence, $|S| = k+1$ or $|S| = k+2$.

Case 1. $|S| = k+1$. We obtain $|L(S)| = k$. Since G is K_{k+1}-free, $G[S]$ is k-colorable. Therefore, $G[S]$ is $L|_S$-colorable.

Case 2. $|S| = k+2$. Then $S = V(G)$, so $|L(S)| = k$ or $k+1$. Let u, v be nonadjacent vertices of G. If $L(u) \cap L(v) = \varnothing$, then $2k = |L(u) \cup L(v)| \leq t \leq k+1$. Hence $k \leq 1$, which is a trivial case. Suppose that $c \in L(u) \cap L(v)$.

Case 2.1 $G - \{u, v\}$ is not a complete graph. It is easy to check that a k-vertex graph which is not complete graph is always L'-colorable for every $(k-1)$-list assignment L'. Therefore, $G - \{u, v\}$ is $(L-c)$-colorable. Together with coloring u and v by c, we have that G is L-colorable.

Case 2.2. $G - \{u, v\}$ is a complete graph. Since $G - \{u, v\}$ has k vertices,

$G-\{u,v\}$ is $L|_{V(G-\{u,v\})}$-colorable. Since G does not contain K_{k+1}, each of vertices u,v is adjacent to at most $k-1$ vertices in $G-\{u,v\}$. Therefore, u,v can be colored. $\square$

Corollary 2.2.6 is obtained from Lemma 2.2.5. This gives a characterization of an upper bound of some graphs. It then suggests a simple proof to conclude that

$$\chi_l(K_n - e_1 - e_2) = \begin{cases} n-1 & \text{if } e_1, e_2 \in E(K_n) \text{ are incident;} \\ n-2 & \text{otherwise.} \end{cases}$$

Corollary 2.2.6. *Let G be an n-vertex graph. Then $\chi_l(G) \le n-2$ if and only if G contains two pairs of nonadjacent vertices or an independent set of size 3.*

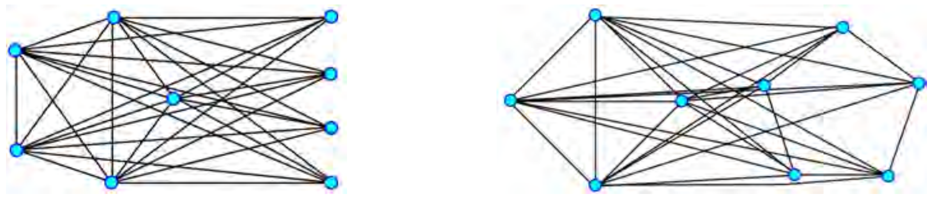
Proof. Let $k = |V(G)| - 2$. Assume that G contains two pairs of nonadjacent vertices or an independent set of size 3. Since G has $k+2$ vertices, it is K_{k+1}-free. By Lemma 2.2.5, G is (k,t)-choosable for every positive integer $t \ge k$, i.e. $\chi_l(G) \le k = n-2$.

Conversely, assume that $\chi_l(G) \le k$. Then G is k-colorable. Since $k = n-2$, there exist three vertices assigned the same color or two pairs of vertices such that each pair assigned the same color. $\square$

The *join* of graphs G and H, written $G \vee H$, is the graph obtained from G and H by adding the edges between all vertices of G and all vertices of H.

Lemma 2.2.7. *Let G be a K_{k+1}-free graph with $k+3$ vertices. Then G is isomorphic to either $K_{k-1} \vee \overline{K_4}$ or $K_{k-2} \vee C_5$ if and only if $G-\{u,v\}$ contains a k-clique for every pair of nonadjacent vertices u,v.*

Proof. It is easy to check that the necessity is true. For sufficiency, assume that $G-\{u,v\}$ contains a k-clique for every pair of nonadjacent vertices u,v.



$K_5 \vee \overline{K_4}$ $\qquad$ $K_4 \vee C_5$

Figure 2.2.1: Examples of $K_{k-1} \vee \overline{K_4}$ or $K_{k-2} \vee C_5$

Since G has $k+3$ vertices and does not contain any $(k+1)$-clique, G contains four distinct vertices u_1, u_2, v_1, v_2 such that u_i is not adjacent to v_i for $i = 1, 2$. Let $X = \{u_1, u_2, v_1, v_2\}$ and $H = G - X$. By the assumption, $G - \{u_1, v_1\}$ contains a k-clique. Since $G - \{u_1, v_1\}$ has $k+1$ vertices, exactly one vertex among nonadjacent vertices u_2, v_2 must be in such k-clique, say v_2. That is, $V(H) \cup \{v_2\}$ is a k-clique. Similarly, we may assume that $V(H) \cup \{v_1\}$ is a k-clique by considering $G - \{u_2, v_2\}$. As a consequence, v_1 is not adjacent to v_2; otherwise, G contains a $(k+1)$-clique. (See Figure 2.2.2.)

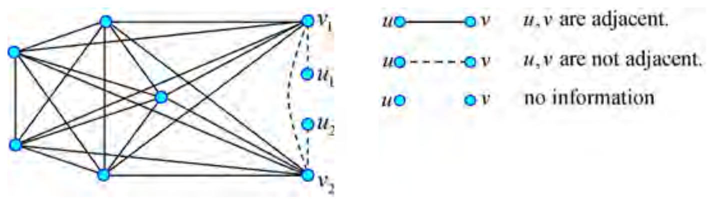


Figure 2.2.2: $V(H) \cup \{v_1\}$ and $V(H) \cup \{v_2\}$ are k-cliques while $v_1 \not\leftrightarrow u_1$, $v_2 \not\leftrightarrow u_2$ and $v_1 \not\leftrightarrow v_2$.

Suppose both u_1 and u_2 are adjacent to every vertex in H. If X is not an independent set, then G contains a $(k+1)$-clique which is a contradiction. If X is an independent set, then G is isomorphic to $K_{k-1} \vee \overline{K_4}$. Now, we can suppose that there is $w \in V(H)$ such that w is not adjacent to u_1.

We know that $G - \{u_1, w\}$ has $k+1$ vertices and contains a k-clique. Since v_2 is not adjacent to v_1 and u_2, the vertex v_2 cannot be in the k-clique. Therefore, $V(H-w) \cup \{v_1, u_2\}$ forms a k-clique. Besides, u_2 is not adjacent to w; otherwise,

$V(H) \cup \{v_1, u_2\}$ forms a $(k+1)$-clique. (See Figure 2.2.3.)

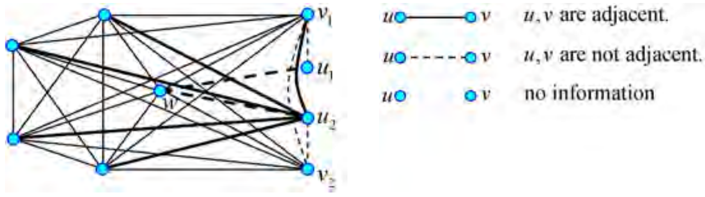


Figure 2.2.3: $G - \{w, u_1, v_2\}$ is a complete graph with k vertices.

Similarly, considering $G - \{w, u_2\}$, we obtain that $V(H - w) \cup \{v_2, u_1\}$ forms a k-clique.

Finally, we consider $G - \{v_1, v_2\}$. Then w cannot be in any k-clique of $G - \{v_1, v_2\}$ because w is not adjacent to both u_1 and u_2. Then $V(H - w) \cup \{u_1, u_2\}$ forms a k-clique. That is, u_1 is adjacent to u_2. (See Figure 2.2.4.) Therefore, $\{w, v_1, u_2, u_1, v_2\}$ forms a cycle of length 5 and $H - w$ is a complete graph with $k - 2$ vertices; moreover, all vertices of C_5 are adjacent to all vertices of $H - w$.

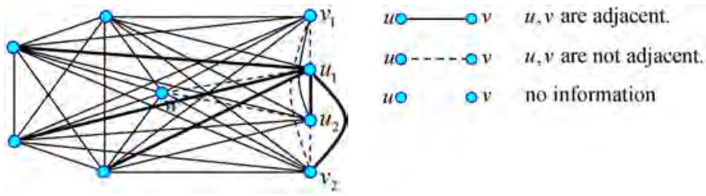


Figure 2.2.4: $\{w, v_1, u_2, u_1, v_2\}$ forms a cycle of length 5

□

Lemma 2.2.8. *If a $(k+3)$-vertex graph is K_{k+1}-free, then it is (k,t)-choosable for $t \geq k+1$.*

Proof. Let G be a graph with $k+3$ vertices and L be a (k,t)-list assignment of G. Assume that G does not contain K_{k+1} as a subgraph and $t \geq k+1$. Let $S \subset V(G)$ such that $|L(S)| < |S|$. It suffices to show by Theorem 2.1.7 that $G[S]$ is $L|_S$-colorable. If $k = 1$ then G has no edges. Therefore, it is $(1,t)$-choosable for every positive integer t. If $k = 2$, then G is triangle-free and has

5 vertices which could be only C_5 or a subgraph of $K_{2,3}$. By Example 2.1.1, G is $(2,t)$-choosable for $t \geq 3$. If $|S| = k+1, k+2$, then the statement holds by Lemma 2.2.5.

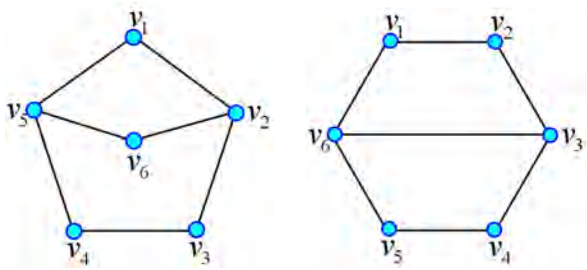
Now, assume that $k \geq 3$ and $|S| = k+3$; that is, $S = V(G)$.

Case 1. There exists a pair of nonadjacent vertices $u, v \in V(G)$ such that $G - \{u,v\}$ does not contain a k-clique. Since $t = |L(V(G))| < |V(G)| = k+3$, we obtain $t \leq k+2$. Moreover, $L(u) \cap L(v) \neq \varnothing$ since $k \geq 3$. Let $c \in L(u) \cap L(v)$. By Lemma 2.2.5, $G - \{u,v\}$ is $(L-c)|_{V(G-\{u,v\})}$-colorable. Extend this to an L-coloring of G by coloring vertices u, v with color c.

Case 2. $G - \{u,v\}$ contains a k-clique for every pair of nonadjacent vertices u, v. Apply Lemma 2.2.7; G can be only two possible graphs. If $G \cong K_{k-1} \vee \overline{K_4}$, then first color all vertices in K_{k-1} and then choose a remaining color in $L(v)$ to color v for each $v \in \overline{K_4}$. Otherwise, $G \cong K_{k-2} \vee C_5$. Begin with coloring all vertices of K_{k-2}; each vertex of C_5 has at least two remaining colors. The total number of remaining colors is at least $t - (k-2) \geq 3$. So, by Example 2.1.1, every vertex of C_5 can be colored. Therefore, G is L-colorable. $\square$

In the next two following lemmas, we focus on 2-list assignments. Both two lemmas are prepared for Theorem 2.2.12.

Lemma 2.2.9. *Graphs G_1 and G_2 in* Figure 2.2.5 *are* $(2,5)$*-choosable.*



G_1 G_2

Figure 2.2.5: $(2,5)$-choosable graphs

Proof. Let L be a $(2,5)$-list assignment of G_1. Since $|L(V(G-v_6))| \geq 3$, $G-v_6$ has an $L|_{V(G-v_6)}$-coloring, say ϕ_1. Now, ϕ_1 can be extend to be an L-coloring unless, without loss of generality, $L(v_6)=\{1,2\}$ and $\phi_1(v_2)=1, \phi_1(v_5)=2$.

In such case, let ϕ_2 be a new $L|_{V(G-v_6)}$-coloring such that $\phi_2(v_2) = A \in L(v_2)-\{\phi_1(v_2)\}$ and $\phi_2(v)=\phi_1(v)$ for the remaining vertices v. Notice that A can be any color from $\{2,3,4,5\}$. If ϕ_2 is a proper coloring then it can be extend to be an L-coloring. In the remaining case, suppose ϕ_2 is not a proper coloring. That is, $\phi_2(v_1)=A$ or $\phi_2(v_3)=A$. Both two cases have similar proof; hence, we suppose that $\phi_2(v_3)=A$.

Again, we let ϕ_3 be a new $L|_{V(G-v_6)}$-coloring such that $\phi_3(v_3)=B \in L(v_3)-\{\phi_2(v_3)\}$ and $\phi_3(v)=\phi_2(v)$ for the remaining vertices v. If ϕ_3 is a proper coloring then it can be extend to be an L-coloring. Otherwise, we define a new $L|_{V(G-v_6)}$-coloring and so on. Finally, if all new $L|_{V(G-v_6)}$-colorings are not proper then we know the list assignment L of G_1 shown in Figure 2.2.6. Since L have 5 colors, we know that $\{A,B,C\}=\{3,4,5\}$. Therefore, we easily investigate an L-coloring of G_1.

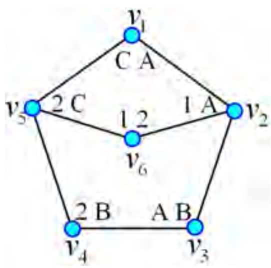


Figure 2.2.6: The list assignment L of G_1

Let L be a $(2,5)$-list assignment of G_2. Since $|L(V(G_2)|=5$, we obtain an L-coloring of $G-v_3v_6$, say ϕ_1. The L-coloring ϕ_1 is also an L-coloring of G_2 unless $\phi_1(v_3)=\phi_1(v_6)$. In such case, let ϕ_2 be a new L-coloring of $G-v_3v_6$ such that $\phi_2(v_3)=A \in L(v_3)-\{\phi_2(v_3)\}$ and $\phi_2(v)=\phi_1(v)$ for the remaining vertices

v. If ϕ_2 is proper, then it is an L-coloring of G_2. In case ϕ_2 is not proper, we define a new L-coloring of $G - v_3v_6$, We continue to define a new L-coloring of $G - v_2v_6$ similar to the proof of G_1. Finally, if all new L-colorings of $G - v_3v_6$ are not proper, we obtain the list assignment L of G_2 shown in Figure 2.2.7. Since L have 5 colors, we know that $\{A, B, C, D\} = \{2, 3, 4, 5\}$. Therefore, we easily investigate an L-coloring of G_2.

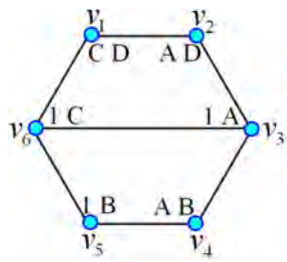


Figure 2.2.7: The list assignment L of G_2

□

Lemma 2.2.10. *A triangle-free graph with six vertices is* $(2,5)$*-choosable if and only if it is neither* $K_{3,3}$ *nor* $K_{3,3} - e$.

Proof. The $(2,5)$-list assignment L of $K_{3,3}$ or $K_{3,3} - e$ shown in Figure 2.2.8 has no proper coloring.

Figure 2.2.8: A $(2,5)$-list assignment of $K_{3,3}$ and $K_{3,3} - e$

Let G be a triangle-free graph with six vertices and L be a $(2,5)$-list assignment of G. Assume that G is neither $K_{3,3}$ nor $K_{3,3} - e$. If G has no cycle, G can be easily colored. If G contains only one cycle, then we can color the cycle, and the remaining vertices outside the cycle can be easily colored. Assume G contains at least 2 cycles. Since G is a triangle-free graph, G is one of the graphs in Lemma 2.2.9. Therefore, G is L-colorable. □

Now, we are ready to prove our theorems. Start with Theorem 2.2.11.

Theorem 2.2.11. *Let $k \geq 3$. A K_{k+1}-free graph with n vertices is (k,t)-choosable for $t \geq kn - k^2 - 2k + 1$.*

Proof. Let $k \geq 3$, $t \geq kn - k^2 - 2k + 1$ and G be a K_{k+1}-free graph with n vertices. Let $S \subset V(G)$ be such that $|L(S)| < |S|$. We will prove that $G[S]$ is $L|_S$-colorable in order to utilize Theorem 2.1.7. By Lemma 2.2.1, $|S|k - k^2 - 2k + 1 \leq |L(S)| < |S|$. Hence $|S| < k + 3 + \frac{2}{k-1}$; i.e. $|S| \leq k + 3$.

If $|S| \leq k + 2$, then $G[S]$ is $L|_S$-colorable by Lemma 2.2.5. If $|S| = k + 3$ and $|L(S)| = k$ then by Lemma 2.2.1 we obtain $t = |L(V(G))| \leq kn - k^2 - 2k$, a contradiction. Otherwise, $|S| = k + 3$ and $|L(S)| \geq k + 1$; hence $G[S]$ is also $L|_S$-colorable by Lemma 2.2.8. □

It is worth mentioning that Theorem 2.2.11 is not true when $k = 2$. However, the statement is correct if the bound is slightly improved. This is illustrated in Theorem 2.2.12. Furthermore, Theorem 2.2.13 reveals all graphs forbidding the case for which Theorem 2.2.11 fails when $k = 2$.

Theorem 2.2.12. *A triangle-free graph with n vertices is $(2,t)$-choosable where $t \geq 2n - 6$.*

Proof. Assume that G is a triangle-free graph with n vertices. Let $S \subset V(G)$ such that $|L(S)| < |S|$. Again, we will show that $G[S]$ is $L|_S$-colorable in order to utilize Theorem 2.1.7. By Lemma 2.2.1, $2|S| - 6 \leq |L(S)| < |S|$. Hence $|S| < 6$. If $|S| \leq 4$ then $G[S]$ is $L|_S$-colorable by Lemma 2.2.5. Now assume that

$|S| = 5$. By Lemma 2.2.1, $|L(S)| \geq 2n - 6 - 2(n - |S|) = 4$; therefore, $G[S]$ is $L|_S$-colorable by Lemma 2.2.8. □

Theorem 2.2.13. *A triangle-free graph with n vertices is $(2, 2n-7)$-choosable if and only if it does not contain $K_{3,3} - e$ as a subgraph.*

Proof. Let G be a triangle-free graph with n vertices.
Necessity. Assume that G contains $K_{3,3} - e$ as a subgraph. We will find a $(2, 2n-7)$-list assignment of G such that G is not L-colorable. First, assign lists of colors for vertices in $K_{3,3} - e$ shown in Figure 2.2.8. Assign disjoint sets of colors to each remaining $n - 6$ vertices; this uses $2n - 12$ colors. Thus we obtain $(2, 2n-7)$-list assignment L of G. Since $K_{3,3} - e$ is not $L|_{V(K_{3,3}-e)}$-colorable, G is not L-colorable.
Sufficiency. Assume that G does not contain $K_{3,3} - e$ as a subgraph. Let L be a $(2, 2n-7)$-list assignment of G. Let $S \subset V(G)$ such that $|L(S)| < |S|$. By Theorem 2.1.7, it suffices to show that $G[S]$ is $L|_S$-colorable.

By Lemma 2.2.1, $2|S| - 7 \leq |L(S)| < |S|$; therefore, $|S| \leq 6$. If $|S| = 6$, then $|L(S)| \geq 2 \cdot 6 - 7 = 5$; hence, the proof is done by Lemma 2.2.10. If $|S| = 5$, then $|L(S)| \geq 2 \cdot 5 - 7 = 3$, so the proof is done by Lemma 2.2.8. Otherwise, $|S| \leq 4$. Since $G[S]$ is triangle-free, it is a subgraph of $K_{2,3}$; hence, it is L-colorable by Example 2.1.1. Therefore, $G[S]$ is $L|_S$-colorable. □

Theorem 2.2.14. *Let n, k, t be positive integers such that $nk - k^2 - 2k + 1 \leq t \leq nk - k^2$ and $3 \leq k \leq n - 3$. An n-vertex graph is (k, t)-choosable if and only if it is K_{k+1}-free. Moreover, for $k = 2$ and $2n - 6 \leq t \leq 2n - 4$, an n-vertex graph is $(2, t)$-choosable if and only if G is triangle-free.*

Proof. Theorem 2.2.4 and Theorem 2.2.11 are necessity and sufficiency for the case $k \geq 3$ this theorem. Furthermore, Theorems 2.2.4, 2.2.12 and 2.2.13 prove the remaining case of the theorem. □

We next step further to the case $k \leq t \leq nk-k^2-2k$. Some K_{k+1}-free graphs with n vertices are (k,t)-choosable. Theorem 2.2.15 gives us forbidden graphs.

Theorem 2.2.15. *Let G be an n-vertex graph and $k \leq t \leq nk - k^2 - 2k$ where $k \geq 2$. If G contains $C_5 \vee K_{k-2}$ then G is not (k,t)-choosable.*

Proof. Consider a (k,t)-list assignment L of G such that $L(v) = \{1, 2, \ldots, k\}$ for every vertex v in $C_5 \vee K_{k-2}$. It is possible to construct such (k,t)-list assignment L because $t-k \leq k(n-k-3)$. Notice that the union of lists for the rest $n-k-3$ vertices outside $C_5 \vee K_{k-2}$ is $\{k+1, k+2, \ldots, t\}$. However, since every vertex in $C_5 \vee K_{k-2}$ receives the same list of size k, we cannot color all vertices in $C_5 \vee K_{k-2}$. Therefore, G is not L-colorable. $\square$

CHAPTER III

ON 3-CHOOSABILITY OF COMPLETE BIPARTITE GRAPHS

3.1 Background

In [4], the authors illustrated the list assignment L such that $K_{7,7}$ is not L-colorable. Such list assignment originated from the Fano plane which is defined in Notation 3.1.1.

Notation 3.1.1. Let $\mathcal{F} = \{\{1,2,3\},\ \{1,4,5\},\ \{1,6,7\},\ \{2,4,6\},\ \{2,5,7\},\ \{3,4,7\},\ \{3,5,6\}\}$ and $L_{\mathcal{F}}$ be the 3-list assignment of $K_{7,7}$ such that all seven vertices in each partite set are assigned by distinct lists from $\mathcal{F}$.

Later, in 1996, Hanson, MacGillivray, and Toft [8] proved that every complete bipartite graph with at most 13 vertices is 3-choosable. Hence, the smallest complete bipartite graph which is not 3-choosable has 14 vertices. Fitzpatrick and MacGillivray [5] added that every complete bipartite graph with 14 vertices except $K_{7,7}$ is 3-choosable. Moreover, $L_{\mathcal{F}}$ is the unique list assignment up to renaming the colors which prevents $K_{7,7}$ from being 3-choosable. We will give another proof of this statement in Theorem 3.3.6.

It is noticeable that renaming the colors in a list assignment does not affect its colorability. Thus, all results throughout the rest of dissertation does not depend on renaming the colors.

In Section 3.2, we first establish new strategies which can be utilized to verify

3-choosability of complete bipartite graphs. In Section 3.3, we use our strategies to obtain another proof of [5] and in Section 3.4, we also extend this result to a complete bipartite graph with 15 vertices. We prove that every complete bipartite graph with 15 vertices is 3-choosable except $K_{7,8}$. Besides, for a 3-list assignment L, $K_{7,8}$ is not L-colorable if and only if $L|_{V(K_{7,7})} = L_{\mathcal{F}}$. New notations and definitions used in Chapter 3 and chapter 4 are defined in Notations 3.1.2, 3.1.3 and Definition 3.1.4. Example 3.1.5 illustrate these notations and definitions.

Notation 3.1.2. Let L be a list assignment of the complete bipartite graph $K_{a,b}$. The notation L_a and L_b denote the collections of lists assigned to the vertices in the partite sets with a and b vertices, respectively. If $a = b$, we use the notation $L_{a(i)}$ and $L_{a(ii)}$.

Notation 3.1.3. For convenience, we write lists without commas and braces. For example, we write $\{123, 145, 167, 246, 257, 347, 356\}$ in stead of $\{\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 7\}$, $\{3, 5, 6\}\}$. For a list A, the notation $A - 1$ represents the list which is obtained from A by removing color 1 from A. Similarly, the notation $A - 12$ represents the list which is obtained from A by removing color 1 and color 2 from A.

Definition 3.1.4. Given a collection of lists $\mathbb{X} = \{X_1, X_2, \ldots, X_n\}$, a *coloring* of $\mathbb{X}$ is a set $C \subseteq X_1 \cup X_2 \cup \ldots \cup X_n$ such that $C \cap X_i \neq \varnothing$ for all $i = 1, 2, \ldots, n$. A coloring C of $\mathbb{X}$ is called a t*-coloring* if $|C| = t$.

Notice that a coloring of a collection of lists $\mathbb{X}$ is not necessary a coloring of a graph G; for example, if a graph G has $|\mathbb{X}|$ vertices and has no edge, and L is a list assignment of G such that all vertices of G are assigned by distinct lists from $\mathbb{X}$, then a coloring of $\mathbb{X}$ and an L-coloring of G are the same. We are interested in a collection of lists $\mathbb{X}$ when it is a collection of lists which are assigned to all vertices in a partite set of complete bipartite graphs. Example 3.1.5 illustrates these notations.

Example 3.1.5. Let L be the list assignment of $K_{3,3}$ as shown in Figure 3.1.1. Then $L_{3(i)} = \{12, 13, 45\}$ and $L_{3(ii)} = \{14, 15, 23\}$. Since $1 \in 12, 13$ and $4 \in 45$, we conclude that $\{1, 4\}$ is a 2-coloring of $L_{3(i)}$. Similarly, $\{1, 5\}$ is a 2-coloring of $L_{3(i)}$ while $\{2, 3, 4\}$ and $\{2, 3, 5\}$ are 3-colorings of $L_{3(i)}$.

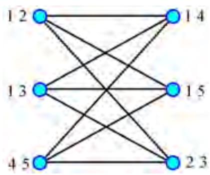


Figure 3.1.1: The list assignment L of $K_{3,3}$

□

3.2 Strategies

In order to prove our desire results, we may prove many similar cases. We group similar cases together and construct tools for each group. First, we introduce a lemma by Hanson, MacGillivray and Toft [8] which will be used throughout this section.

Lemma 3.2.1. [8] *Let L be a list assignment of the complete bipartite graph $K_{a,b}$. Then $K_{a,b}$ is not L-colorable if and only if every coloring of L_a (or L_b) has a subset that is a list in L_b (or L_a).*

Proof. Necessity. Assume that there is a coloring C of L_a which does not contain any lists in L_b. Then after we color L_a by C, each list in L_b still has an available color. Hence $K_{a,b}$ is L-colorable.

Sufficiency. Assume that every coloring of L_a has a subset that is a list in L_b.

Let C be any coloring of L_a. So, there exists B_i in L_b such that $B_i \subseteq C$. Then we cannot use C to color vertices in L_a because there is no color left to color B_i. □

We introduce six theorems, called *strategies*, which can be applied to prove 3-choosability complete of bipartite graphs. To begin with, we find a sufficient condition of $K_{a,b}$ to be L-colorable when all lists in L_a are mutually disjoint.

Theorem 3.2.2. (Strategy A) *Let L be a list assignment of $K_{a,b}$ with $L_a = \{A_1, A_2, \ldots, A_a\}, L_b = \{B_1, B_2, \ldots, B_b\}$ and all lists have size at most 3. If all lists in L_a are mutually disjoint and $\prod_{i=1}^{a} |A_i| > 3^{a-1}n_1 + \lfloor 3^{a-2} \rfloor n_2 + \lfloor 3^{a-3} \rfloor n_3$ where $n_i = |\{B \in L_b, |B| = i\}|$ for $i = 1, 2, 3$, then $K_{a,b}$ is L-colorable.*

Proof. Since there are $|A_i|$ possible ways to color the list A_i for each i and all A_i's are mutually disjoint, the number of a-colorings of L_a is $\prod_{i=1}^{a} |A_i|$. Now we count the number of those a-colorings containing each B_i of L_b for $i = 1, 2, \ldots, b$. Consider $B_i \in L_b$.

Case 1. $|B_i| = 1$, say $B_i = r$. If $r \notin A_j$ for all $j = 1, 2, \ldots, j$, then all a-colorings of L_a do not contain B_i. Without loss of generality, suppose that $r \in A_1$. To complete an a-coloring of L_a, we choose the other $a - 1$ colors each from the remaining A_j where $j = 2, 3, \ldots, a$. Thus the number of the a-colorings of L_a containing r is $\prod_{j=2}^{a} |A_j|$. That is, the number of the a-colorings of L_a which contain B_i as a subset is at most 3^{a-1}.

Case 2. $|B_i| = 2$, say $B_i = rs$. Consider an a-coloring of L_a containing both r and s. Without loss of generality, suppose that $r \in A_1$ and $s \in A_2$. To complete an a-coloring of L_a, we choose the other $a - 2$ colors each from the remaining A_j where $j = 3, 4, \ldots, a$. Thus the number of the a-colorings of L_a which contain B_i as a subset is $\prod_{j=3}^{a} |A_j|$. That is, the number of the a-colorings of L_a which contain B_i as a subset is at most 3^{a-2}. Note that in case $a = 1$, all a-colorings are 1-colorings; hence, the number of a-colorings contains B_i as a subset is $\lfloor 3^{a-2} \rfloor = 0$.

Case 3. $|B_i| = 3$, say $B_i = rst$. Consider an a-coloring of L_a containing r, s and t. Without loss of generality, suppose that $r \in A_1, s \in A_2, t \in A_3$. Again, we choose the other $a-3$ colors from each A_j where $j = 4, 5, \ldots, a$. Thus the number of the a-colorings of L_a which contain B_i as a subset is $\prod_{j=4}^{a} |A_j|$. That is, the number of the a-colorings of L_a which contain B_i as a subset is at most 3^{a-3}. Note that in case $a \leq 2$, all a-colorings are 1-colorings or 2-colorings; hence, the number of a-colorings contains B_i as a subset is $\lfloor 3^{a-3} \rfloor = 0$.

Hence L_a has at most $3^{a-1}n_1 + \lfloor 3^{a-2} \rfloor n_2 + \lfloor 3^{a-3} \rfloor n_3$ a-colorings containing some B_i. Since the number of a-colorings of L_a is $\prod_{j=1}^{a} |A_j|$ and $\prod_{j=1}^{a} |A_j| > 3^{a-1}n_1 + \lfloor 3^{a-2} \rfloor n_2 + \lfloor 3^{a-3} \rfloor n_3$, there exists a coloring of L_a which does not contain any list in L_b. Therefore, $K_{a,b}$ is L-colorable. $\square$

Notation 3.2.3. We can conclude the same result if we consider the other way around, that is, the assumption in Strategy A for a list assignment L of $K_{a,b}$ becomes *all lists in* L_b *are mutually disjoint and* $\prod_{i=1}^{b} |B_i| \leq 3^{b-1}n_1 + \lfloor 3^{b-2} \rfloor n_2 + \lfloor 3^{b-3} \rfloor n_3$, *where* $n_i = |\{A \in L_a, |A| = i\}|$ *for* $i = 1, 2, 3$. Then we call it *Strategy A for* L_b and we call the original version *Strategy A* for L_a.

A remark from Strategy A, if $\prod_{i=1}^{a} |A_i| > 3^{a-1}n_1 + \lfloor 3^{a-2} \rfloor n_2 + \lfloor 3^{a-3} \rfloor n_3$ where $n_i = |\{A \in L_a, |A| = i\}|$ for $i = 1, 2, 3$, then $K_{a,b}$ may not L-colorable. For example, let L be a 3-list assignment of $K_{3,27}$ such that $L_3 = \{123, 456, 789\}$ and $L_{27} = \{abc | a \in 123, b \in 456, c \in 789\}$.

The next five strategies, B, C, D, E and F, can be used to color $K_{a,b}$ with respect to a list assignment L in the case that a color appears in at least $a-1, a-2, a-3$ and $a-4$ in L_a, respectively. The next strategy is called *Strategy B* *for* L_a and we can define *Strategy B* for L_b, similarly.

Theorem 3.2.4. (Strategy B) *Let* L *be a* 3*-list assignment of* $K_{a,b}$*. If a color appears in* $a-1$ *lists in* L_a*, then* $K_{a,b}$ *is* L*-colorable.*

Proof. Notice that L_a can be labeled by at most two colors. Since every list in L_b has size 3, all list in L_b still have available colors. □

Remark 3.2.5. Let L be a list assignment of $K_{a,b}$ and C be a 2-coloring of L_a. Then,

(i) if L is a 3-list assignment then $K_{a,b}$ is L-colorable;

(ii) if all lists of size at most 2 in L_b have a color which is not in C, then $K_{a,b}$ is L-colorable.

Theorem 3.2.6. (Strategy C) *Let* L *be a* 3*-list assignment of* $K_{a,b}$ *such that each color appears in at most eight lists in* L_b*. If a color appears in* $a-2$ *lists in* L_a *then* $K_{a,b}$ *is* L*-colorable.*

Proof. Strategy B takes care the case that a color appears in more than $a-2$ lists in L_a. Assume that a color appears in exactly $a-2$ lists in L_a. If the two remaining lists in L_a have a common color, then there exists a 2-coloring of L_a. Since all lists in L_b are of size 3, $K_{a,b}$ is L-colorable by Remark 3.2.5. Suppose that the two remaining lists in L_a have no common color. Hence, L_a has at least nine 3-colorings containing color 1. However, by the assumption, color 1 appears in at most eight lists in L_b. Thus, at least one of such nine 3-colorings is not a list in L_b. Therefore, by Lemma 3.2.1 $K_{a,b}$ is L-colorable. □

Theorem 3.2.7. (Strategy D) *Let* L *be a* 3*-list assignment of* $K_{a,b}$ *such that each color appears in at most* r *lists in* L_b*. If a color appears in* $a-3$ *lists in* L_a *and* $(r,b) \in \{(r,b)|r \leq 2, b \leq 22\} \cup \{(3,b)|b \leq 14\} \cup \{(4,b)|b \leq 12\} \cup \{(5,b)|b \leq 9\}$*, then* $K_{a,b}$ *is* L*-colorable.*

Proof. Let $L_a = \{A_1, A_2, \ldots, A_a\}$ and $L_b = \{B_1, B_2, \ldots, B_b\}$.If a color appears in more than $a-3$ lists, we apply Strategy C. Assume that color 1 appears in exactly $a-3$ lists in L_a, and $(r,b) \in \{(r,b)|r \leq 2, b \leq 22\} \cup \{(3,b)|b \leq 14\} \cup \{(4,b)|b \leq 12\} \cup \{(5,b)|b \leq 9\}$. Without loss of generality, let $1 \in A_1, A_2, \ldots, A_{a-3}$. First, we label $A_1, A_2, \ldots, A_{a-3}$ by color 1. Now, we consider the remaining vertices which form $K_{3,b}$. For the worst case, we may suppose that $1 \in B_1, B_2, \ldots, B_r$. Let L' be the list assignment of $K_{3,b}$ which is obtained from L by removing color 1. Notice that $L'_3 = \{A_{a-2}, A_{a-1}, A_a\}$ and $L'_b = \{B_1 - 1, \ldots, B_r - 1, B_{r+1} \ldots, B_b\}$. If $A_{a-2} \cap A_{a-1} \cap A_a \neq \varnothing$ then there is a 2-coloring of L_a ;hence, $K_{a,b}$ is L-colorable by Remark 3.2.5. Suppose that $A_{a-2} \cap A_{a-1} \cap A_a = \varnothing$.

Case 1. $|A_{a-2} \cap A_{a-1}| = 2$.

Let $2,3 \in A_{a-2}, A_{a-1}$ and $A_a = 456$. Then L_a has at least six 3-colorings, called $\{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}$. Since $r \leq 5$, at least one of the six 3-colorings is not a list in L_b. By Lemma 3.2.1, $K_{a,b}$ is L-colorable.

Case 2. $|A_{a-2} \cap A_{a-1}| = 1$.

Let $A_{a-2} = 234, A_{a-1} = 256$ and $A_a = pqr$ where $p,q,r \notin \{1,2\}$. Then we divide this case into several subcases.

Case 2.1 $\{p,q,r\} \cap \{3,4,5,6\} \neq \varnothing$.

Without loss of generality, we let $p = 3$. Then L_a has at least five 3-colorings, called $\{1,2,3\}, \{1,2,q\}, \{1,2,r\}, \{1,3,5\}, \{1,3,6\}$. If one of such 3-colorings is not a list in L_b, then $K_{a,b}$ is L-colorable by Lemma 3.2.1. Suppose that such 3-colorings are lists in L_b. Thus $r = 5$ and $b \leq 9$. Let $B_1 = 123, B_2 = 12q, B_3 = 12r, B_4 = 135$ and $B_5 = 136$. We label B_1, B_2, B_3, B_4 and B_5 by color 2 and color 3. Now, the remaining vertices form $K_{3,b-5}$ where $b \leq 9$. For the worst case, we suppose $b = 9$. Let L'' be the list assignment of $K_{3,4}$ which is obtained from L' by removing color 2. Then $L''_3 = \{4, 56, qr\}$ and $L''_4 = \{B_6, B_7, \ldots, B_9\}$.

If L''_3 has a 2-coloring, then $K_{3,4}$ is L''-colorable by Remark 3.2.5. Hence, suppose that L''_3 has no 2-coloring. That is, $q, r \notin \{4,5,6\}$. We let $q = 7$ and $r = 8$. Then L''_3 has four 3-colorings, namely $\{4,5,7\}, \{4,5,8\}, \{4,6,7\}, \{4,6,8\}$. Again, we suppose that such 3-colorings are lists in L''_4. Now, $L_b = L_9 = \{123, 127, 128, 135, 136, 457, 458, 478, 468\}$. Hence, color 1 and color 4 form a 2-coloring of L_b. By Remark 3.2.5, $K_{a,b}$ is L-colorable.

Case 2.2 $p, q, r \notin \{3,4,5,6\}$.

Let $p = 7, q = 8$ and $r = 9$. Then $\{1,2,7\}, \{1,2,8\}$ and $\{1,2,9\}$ are 3-colorings of L_a. Again, by Lemma 3.2.1, $K_{a,b}$ is L-colorable unless the case that L_b contains $127, 128$ and 129. Let $B_1 = 127, B_2 = 128, B_3 = 129$. Thus $r \geq 3$. Next, we label B_1, B_2, B_3 by color 2. Let L'' be the list assignment of $K_{3,b-3}$ which is obtained from L' by removing color 2. Then $L''_3 = \{A_{a-2} - 2, A_{a-1} - 2, A_a\}$ and $L''_{b-3} = \{B_4 - 1, \ldots, B_r - 1, B_{r+1}, B_{r+2}, \ldots, B_b\}$. Now, we apply Strategy A for L''_3.

Case 2.2.1 $r = 3$. Then all lists in L''_{b-3} are of size 3. We apply Strategy A for L''_3 because $12 > 3^{3-3}(b-3)$.

Case 2.2.2 $r = 4$. For the worst case, we suppose that $1 \in B_4$. That is L''_{b-3} has exactly one lists of size 2 and the remaining lists are of size 3. Again, we apply Strategy A for L''_3 because $12 > 3^{3-2} \cdot 1 + 3^{3-3}(b-4)$.

Case 2.2.3 $r = 5$. For the worst case, we suppose that $1 \in B_4, B_5$. That is L''_{b-3} has exactly two lists of size 2 and the remaining lists are of size 3. Again, we apply Strategy A for L''_3 because $12 > 3^{3-2} \cdot 2 + 3^{3-3}(b-5)$.

Case 3. A_{a-2}, A_{a-1}, A_a *are mutually disjoint.*

Then $|A_{a-2}| \cdot |A_{a-1}| \cdot |A| = 3^3$ Now, we use Strategy A for L'_3. Note that there are r lists in L_b containing color 1. So the number of lists of size 2 and size 3 in L'_b are $n_2 = r$ and $n_3 = b - r$, respectively. Thus $3 \cdot r + (b - r) < 3^3$. Hence $K_{3,b}$

is L'-colorable by Strategy A for L'_3. Therefore, $K_{a,b}$ is L-colorable. □

The next lemma is used only in Strategy E.

Lemma 3.2.8. *Let L be a 3-list assignment of $K_{a,b}$ and each color appears in at most three lists in L_b. If color 1 appears in exactly $a-4$ lists in L_a, and color 1 and color 2 appear together in three lists in L_b, then $K_{a,b}$ is L-colorable.*

Proof. Let $L_a = \{A_1, A_2, \ldots, A_a\}$, $L_b = \{B_1, B_2, \ldots, B_b\}$. Assume that $1 \in A_1, A_2, \ldots, A_{a-4}$ and $1, 2 \in B_1, B_2, B_3$. If $A_{a-3} \cap A_{a-2} \cap A_{a-1} \cap A_a$ is not empty, then L_a has a 2-coloring; hence, $K_{a,b}$ is L-colorable by Remark 3.2.5. Suppose that $A_{a-3} \cap A_{a-2} \cap A_{a-1} \cap A_a = \varnothing$. Then we label $A_1, A_2, \ldots, A_{a-4}$ by color 1 and label B_1, B_2, B_3 by color 2. Next, we consider the remaining vertices which form $K_{4,b-3}$. Let L' be the list assignment of $K_{4,b-3}$ which is obtain from L by removing color 1 and color 2. For the worst case, we suppose that $2 \in A_{a-3}, A_{a-2}, A_{a-1}$. That is, $L'_4 = \{A_{a-3} - 2, A_{a-2} - 2, A_{a-1} - 2, A_a\}$ and $L'_{b-3} = \{B_4, B_5, \ldots, B_b\}$. If any two lists in L'_4 have a common color, it can be verified that L'_4 has at least four 3-colorings of L'_4. Since every color appears in at most three lists in L'_{b-3}, at least one of these 3-colorings is not a list in L'_{b-3}. Then we suppose that all lists in L'_4 have no common color. Let $L'_4 = \{34, 56, 78, 9AB\}$. Since all lists in L'_4 are subsets of $\{3, 4, 5, 6, 7, 8, 9, A, B\}$, we may suppose that all lists in L'_{b-3} are subsets of $\{3, 4, 5, 6, 7, 8, 9, A, B\}$. Since every color appears in at most three lists in L'_b, we obtain $b - 3 \leq 9$.

Case 1. $b - 3 \leq 7$.

Then $K_{4,b-3}$ is L'-colorable by Strategy A for L'_4.

Case 2. $b - 3 = 8$. We consider the possibility of L'_8 such that $K_{4,8}$ is not L'-colorable. Then L'_8 must be $\{357, 358, 367, 368, 457, 458, 467, 468\}$. However, this case cannot occur because every color appears in at most three lists in L'_8.

Case 3. $b-3=9$. Then every color from $3,4,5,6,7,8,9,A,B$ must appear in three lists in L'_9. Then we label 34 in L'_4 by color 3 and label three lists containing color 4 in L'_9 by color 4. The remaining vertices form $K_{3,6}$. Let L'' be the lists assignment of $K_{3,6}$ which is obtained from L' by removing color 3 and color 4. Then $L''_3 = \{56, 78, 9AB\}$. For the worst case, we suppose that L''_6 has three lists of size 2 and three lists of size 3. Again, we consider the possibilities of L''_6 such that $K_{4,6}$ is not L''-colorable. Without loss of generality, L''_6 must be $\{57, 58, 67, 689, 68A, 68B\}$. However, this case cannot occur because every color appears in at most three lists in L''_6. □

Theorem 3.2.9. (Strategy E) *Let L be a 3-list assignment of $K_{a,b}$ such that each color appears in at most r lists in L_b. If color 1 appears in $a-4$ lists in L_a, and $(r,b) \in \{(r,b)|r \leq 2, b \leq 22\} \cup \{(3,b)|b \leq 14\}$, then $K_{a,b}$ is L-colorable unless $\mathcal{F} \subseteq L_b$ and the four remaining lists in L_a are $246, 257, 347, 356$ up to rename the colors.*

Proof. Let $L_a = \{A_1, A_2, \ldots, A_a\}$ and $L_b = \{B_1, B_2, \ldots, B_b\}$. If a color appears in more than $a-4$ lists in L_a, then we apply Strategy D. Assume that color 1 appears in exactly $a-4$ lists in L_a and $(r,b) \in \{(r,b)|r \leq 2, b \leq 22\} \cup \{(3,b)|b \leq 14\}$. Without loss of generality, we suppose that $1 \in A_1, A_2, \ldots, A_{a-4}$. Moreover, we suppose that the four remaining lists in L_a are not $246, 257, 347, 356$ or $\mathcal{F} \nsubseteq L_b$.

We first label $A_1, A_2, \ldots, A_{a-4}$ by color 1. Then the remaining vertices form $K_{4,b}$. For the worst case, we may suppose that $1 \in B_1, B_2, \ldots, B_r$. Let L' be the list assignment of $K_{4,b}$ which is obtained from L by removing color 1. Then $L'_4 = \{A_{a-3}, A_{a-2}, A_{a-1}, A_a\}$ and $L'_b = \{B_1 - 1, \ldots, B_r - 1, B_{r+1}, \ldots, B_b\}$.

Case 1. *A color appears in all lists in L'_4.*

Thus we use such color to label all lists in L'_4. It is easy to see that every list in

L'_b still has an available color. Then $K_{a,b}$ is L-colorable.

Case 2. *A color appears in three lists in L'_4.*

If a color appears in four lists, then it is done by case 1. Suppose that no color appears in four lists in L'_4. Let $2 \in A_{a-3} \cap A_{a-2} \cap A_{a-1}$ and $A_a = 345$. Now, we consider L of $K_{a,b}$. Then L_a has at least three 3-colorings, that is, $\{1,2,3\},\{1,2,4\},\{1,2,5\}$. If L_b does not contain all of these 3-colorings, $K_{a,b}$ is immediately L-colorable by Lemma 3.2.1. Otherwise, we suppose that $B_1 = 123, B_2 = 124, B_3 = 125$. By Lemma 3.2.8, $K_{a,b}$ is L-colorable.

Case 3. *A color appears in two lists in L'_4 and the remaining two lists have no common color.*

If a color appears in more than two lists, then the proof is done by Case 1 and Case 2. Suppose that each color appears in at most two lists in L'_4. Let $2 \in A_{a-3}, A_{a-2}$ and $A_{a-1} \cap A_a = \varnothing$. We next label A_{a-3} and A_{a-2} by color 2. Then, we focus on the remaining vertices which form $K_{2,b}$. Let L'' be the list assignment of $K_{2,b}$ which is obtained from L' by removing color 2. Since we use color 1 and color 2 to label lists in L_a, we may suppose that both 1 and color 2 appear in three lists in L''_b for the worse case. Thus, there are four possibilities of L''_b.

Case 3.1 L''_b has six lists of size 2 and $b-6$ lists of size 3. We see that $|A_{a-1}| \cdot |A_a| = 3^2 > 3^0 \cdot 6$. By Strategy A for L''_2, $K_{2,b}$ is L''-colorable. Then $K_{a,b}$ is L-colorable.

Case 3.2 L''_b has one list of size 1, four lists of size 2 and $b-5$ lists of size 3. We see that $|A_{a-1}| \cdot |A_a| = 3^2 > 3 \cdot 1 + 4$. By Strategy A for L''_2, $K_{2,b}$ is L''-colorable. Then $K_{a,b}$ is L-colorable.

Case 3.3 L''_b has two lists of size 1, two lists of size 2 and $b-4$ lists of size 3. We see that $|A_{a-1}| \cdot |A_a| = 3^2 > 3 \cdot 2 + 2$. By Strategy A, $K_{2,b}$ is L''-colorable. Then $K_{a,b}$ is L-colorable.

Case 3.4 L''_b has three lists of size 1, no list of size 2 and $b-3$ lists of size 3. That is, color 1 and color 2 appear together in exactly three lists of L_b. Then $K_{a,b}$ is L-colorable by Lemma 3.2.8.

Case 4. *A color appears in two lists in* L'_4 *and the remaining two lists have a common color.*

Similar to case 3, we suppose that no color appears in three lists in L'_4. Let $2 \in A_{a-3}, A_{a-2}$ and $3 \in A_{a-1} \cap A_a$. Hence, $\{1, 2, 3\}$ is a 3-coloring of L_a. If 123 is not a list in L_b, then $K_{a,b}$ is L-colorable by Lemma 3.2.1. Otherwise, we suppose that $B_1 = 123$.

Case 4.1 $|A_{a-3} \cap A_{a-2}| \geq 2$ *and* $|A_{a-1} \cap A_a| \geq 2$.

Let $4 \in A_{a-3} \cap A_{a-2}$ and $5 \in A_{a-1} \cap A_a$. We obtain at least four 3-colorings of L_a, that is, $\{1, 2, 3\}, \{1, 2, 5\}, \{1, 4, 3\}, \{1, 4, 5\}$. Since each color appears in at most three lists in L_b, at least one of such 3-colorings is not a list in L_b. Then $K_{a,b}$ is L-colorable by Lemma 3.2.1.

Case 4.2 $|A_{a-3} \cap A_{a-2}| \geq 2$ *and* $|A_{a-1} \cap A_a| = 1$.

We may suppose that $|A_{a-3} \cap A_{a-2}| = 2$. Let $A_{a-3} = 24x, A_{a-2} = 24y, A_{a-1} = 356$ and $A_a = 378$ where $x \neq y$ and $x, y \notin \{1, 2, 3, 4\}$. Then $\{1, 4, 3\}$ is a 3-coloring of L_a. If 143 is not a list in L_b, then $K_{a,b}$ is L-colorable by Lemma 3.2.1. Otherwise, suppose that $B_2 = 143$. Recall that we have already labeled $A_1, A_2, \ldots, A_{a-4}$ by color 1. Now, we label B_1, B_2 by color 3. Consider the uncolor vertices which form $K_{4,b-2}$. Let L'' be a list assignment of $K_{4,b-2}$ which is obtained from L by removing color 3. Then $L''_4 = \{24x, 24y, 56, 78\}$ and $L''_{b-2} = \{B_3 - 1, B_4, B_5, \ldots, B_b\}$. By the fact that L'_4 has at least eight 3-colorings and every color appears in at most three colors in L_b, it can be verified that $K_{4,b-2}$ is L''-colorable.

Case 4.3 $|A_{a-3} \cap A_{a-2}| = 1$ *and* $|A_{a-1} \cap A_a| \geq 2$.

It is similar to Case 4.2.

Case 4.4 $|A_{a-3} \cap A_{a-2}| = 1$ *and* $|A_{a-1} \cap A_a| = 1$.

Let $A_{a-1} = 345, A_a = 367$ and $A_{a-3} = 2ef, A_{a-2} = 2gh$ where e, f, g, h are distinct. Note that $\{1,2,4,6\}, \{1,2,4,7\}, \{1,2,5,6\}, \{1,2,5,7\}$ are 4-colorings of L_a. By Lemma 3.2.1, if one of these 4-colorings has no subset that is a list in L_b, then $K_{a,b}$ is L-colorable. Again, suppose that these 4-colorings have a subset that is a list in L_b. Without loss of generality, L_b can be verified that there are two possibilities of L_b.

Case 4.4.1 $B_2 = 124$ *and* $B_3 = 125$.

Then $K_{a,b}$ is L-colorable by Lemma 3.2.8.

Case 4.4.2 $B_2 = 146, B_3 = 147, B_4 = 256, B_5 = 257$.

Recall that we have already labeled $A_1, A_2, \ldots, A_{a-4}$ by color 1. Now, we label A_{a-1}, A_a by color 3 and label B_1, B_4, B_5 by color 2. Next, we consider the remaining vertices which forms $K_{2,b-3}$. Let L'' be the list assignment of $K_{2,b-3}$ which is obtained from L' by removing color 2 and color 3. That is, $L''_2 = \{ef, gh\}$ and $L''_{b-3} = \{46, 47, B_6, B_7, \ldots, B_b\}$. Then L''_2 has exactly four 2-colorings, namely $\{e,g\}, \{e,h\}, \{f,g\}$ and $\{f,h\}$. If one of such 2-colorings is not a list in L''_{b-3}, then $K_{2,b-3}$ is L''-colorable by Lemma 3.2.1. Suppose that such four 2-colorings are lists in L''_{b-3}. Then L''_{b-3} has at least four lists of size 2. Recall that $3 \in B_1$. Then color 3 appears in two lists in $B_6, B_7, \ldots, B_b$. Hence, we suppose that $3 \in B_6, B_7$. Then $L''_{b-3} = \{56, 57, B_6 - 3, B_7 - 3, B_8, B_9, \ldots, B_{b-3}\}$.

Let L^* be a 2-list assignment of $K_{2,4}$ such that $L^*_2 = \{ef, gh\}$ and $L^*_4 = \{56, 57, B_6 - 3, B_7 - 3\}$. By Remark 3.2.5, $K_{2,b-3}$ is L''-colorable if and only if $K_{2,4}$ is L^*-colorable. Moreover, $K_{2,4}$ is not L^*-colorable if and only if $L^*_2 = \{45, 67\}$ and $L^*_4 = \{46, 47, 56, 57\}$. Therefore, $K_{2,4}$ is not L^*-colorable if and only if $\{A_{a-3}, A_{a-2}, A_{a-1}, A_a\} \neq \{246, 257, 347, 356\}$ or $\mathcal{F} \not\subseteq L_b$.

Case 5. *All lists in L'_4 are mutually disjoint.*

Note that L'_b has $b-r$ lists of size 3, r lists of size 2 and no list of size 1. We have that $\prod_{i=a-3}^{a} |A_i| = 3^4 > 3^2 \cdot r + 3 \cdot (b-r)$. By Strategy A for L'_4, $K_{4,b}$ is L'-colorable. $\square$

The next lemma is used only in Strategy F.

Lemma 3.2.10. *Let L be a 3-list assignment of $K_{a,b}$ and each color appears in at most two lists in L_b. If a color appears in exactly $a-5$ lists in L_a and a color appears in exactly three of the five remaining lists, then $K_{a,b}$ is L-colorable.*

Proof. Let $L_a = \{A_1, A_2, \ldots, A_a\}$ and $L_b = \{B_1, B_2, \ldots, B_b\}$. Suppose that $1 \in A_1, A_2, \ldots, A_{a-5}$ and $2 \in A_{a-4}, A_{a-3}, A_{a-2}$. Then we first label $A_1, A_2, \ldots, A_{a-5}$ by color 1 and label $A_{a-4}, A_{a-3}, A_{a-2}$ by color 2. Consider the remaining vertices which form $K_{2,b}$. Let L' be the list assignment of $K_{2,b}$ which is obtained from L by removing color 1 and color 2. Note that $L'_2 = \{A_{a-1}, A_a\}$. Next, we divide the proof into four cases.

Case 1. $A_{a-1} \cap A_a = \varnothing$.

To apply Strategy A for L'_2, we count the number of lists of size 1, size 2 and size 3 in L'_b. We have three possibilities. Denote that n_i is the number of lists of size i in L'_b for $i = 1, 2, 3$.

1. $n_1 = 2, n_2 = 0$ and $n_3 = b-2$.

2. $n_1 = 1, n_2 = 2$ and $n_3 = b-3$.

3. $n_1 = 0, n_2 = 4$ and $n_3 = b-4$.

All possibilities satisfy conditions in Strategies A of L'_2. Therefore, $K_{2,b}$ is L'-colorable.

Case 2. $|A_{a-1} \cap A_a| = 1$.

Let $A_{a-1} = 345$ and $A_a = 367$. If 123 is not a list in L_b, then $K_{a,b}$ is L-colorable. Without loss of generality, suppose that $B_1 = 123$. Then we label all lists containing color 3 in L'_b by color 3. Now, we consider all uncolored vertices. For the worst case, we suppose that no other list except B_1 containing color 3. Thus the remaining vertices form $K_{2,b-1}$. Let L'' be the list assignment of $K_{2,b-1}$ which is obtained from L' by removing color 3. Then we can apply Strategy A for L'_2.

Case 3. $|A_{a-1} \cap A_a| = \varnothing$.

Let $A_{a-1} = 345$ and $A_a = 346$. If 123 and 124 are not lists in L_b, then $K_{a,b}$ is immediately L-colorable. Without loss of generality, suppose that $B_1 = 123$ and $B_2 = 124$. Then we label B_1, B_2, A_{a-1} and A_a by color 3, color 4, color 5 and color 6, respectively. Notice that every uncolored vertex in L'_b still has an available color. Therefore, $K_{2,b}$ is L'-colorable. □

Theorem 3.2.11. (Strategy F) *Let L be a 3-list assignment of $K_{a,b}$ and each color appears in at most two lists in L_b. If a color appears in $a-5$ lists in L_a and $a+b \leq 18$, then $K_{a,b}$ is L-colorable.*

Proof. Let $L_a = \{A_1, A_2, \ldots, A_a\}$ and $L_b = \{B_1, B_2, \ldots, B_b\}$. Since $a + b \leq 18$ and $a \geq 5$, we obtain $b \leq 13$. Since each color appears in at most two lists in L_b, we have $\mathcal{F} \not\subset L_b$. Then we can apply Strategy E if a color appears in more than $a-5$ lists. Suppose that a color appears in exactly $a-5$ lists. Without loss of generality, we assume $1 \in A_1, A_2, \ldots, A_{a-5}$. Then we label the $a-5$ lists by color 1. For the worst case, we assume that color 1 is in two list in L_b, say B_1, B_2. Next, consider the remaining vertices which form $K_{5,b}$. Let L' be the list assignment of $K_{5,b}$ which is obtained from L by removing color 1. Then $L'_5 = \{A_{a-4}, A_{a-3}, A_{a-2}, A_{a-1}, A_a\}$ and $L'_b = \{B_1 - 1, B_2 - 1, B_3, \ldots, B_b\}$.

Case 1. *A color appears in all lists in* L'_5.

Then L_a has a 2-coloring; hence, $K_{a,b}$ is L-colorable by Remark 3.2.5.

Case 2. *A color appears in four lists in* L'_5.

By case 1, we may suppose that color 2 appears in exactly four lists in L'_5. Let $2 \in A_{a-4}, A_{a-3}, A_{a-2}, A_{a-1}$ and $A_a = 345$. We obtain three 3-colorings of L_a, that is, $\{1,2,3\}, \{1,2,4\}, \{1,2,5\}$. Since every color appears in at most two lists in L_b, at least one of the 3-colorings is not a list in L_b. Therefore, $K_{a,b}$ is L-colorable by Lemma 3.2.1.

Case 3. *A color appears in three lists in* L'_5.

By Lemma 3.2.10, $K_{a,b}$ is L-colorable.

Case 4. *A color appears in two lists in* L'_5.

From Case 3, we may suppose that each color appears in at most two lists in L'_5. Since color 1 appears in at most two lists in L_b, at most four colors appears in the same lists with color 1 in L_b. We apply Theorem 2.1.3. Since $18 \cdot \binom{3}{2} \leq \binom{10+1}{2}$, we may suppose that $|\bigcup_{i=a-4}^{a} A_i| \leq |\bigcup_{v \in V(K_{a,b})} L(v)| \leq 10$. Since $|A_{a-4}| + |A_{a-3}| + |A_{a-2}| + |A_{a-1}| + |A_a| = 15$ and the number of colors is at most ten, at least five colors must appear in exactly two lists in L'_5. Recall that only B_1, B_2 contain color 1. Hence, at most four colors from the five colors appear in the same lists with color 1 in L_b. Hence, we can choose the remaining color such that no list in L_b contain both color 1 and this color, namely color 2. Let $2 \in A_{a-4}, A_{a-3}$ and then we label A_{a-4}, A_{a-3} by color 2. Let L'' be the list assignment of $K_{3,b}$ which is obtained from L' by removing color 2. For the worst case, we suppose $2 \in B_3, B_4$. Hence, $L''_b = \{B_1 - 1, B_2 - 1, B_3 - 2, B_4 - 2, B_5 \ldots, B_b$ and $L''_2 = \{A_{a-2}, A_{a-1}, A_a\}$.

If color 3 appears in exactly two lists in A_{a-2}, A_{a-1}, A_a, then L''_3 has at least three 2-colorings containing color 3. Since every color appears in at most two lists in L''_b, at least one 2-coloring is not a list in L''_b. Otherwise, we suppose that A_{a-2}, A_{a-1}, A_a are mutually disjoint. To apply Strategy A, we count the number of lists of size 1, size 2 and size 3 in L''_b. We obtain that L''_b has no list of size 1, four lists of size 2 and $b-4$ lists of size 3 where $b-4 \leq 6$. Then $|A_{a-2}| \cdot |A_{a-1}| \cdot |A_a| = 3^3 > 3 \cdot 4 + (b-4)$.

Case 5. $A_{a-4}, A_{a-3}, A_{a-2}, A_{a-1}, A_a$ *are mutually disjoint.*

Then L'_b has at most three lists of size $b-2$ and two lists of size 2. Since $\prod_{i=a-4}^{a} |A_i| = 3^5 > 3^3 \cdot 2 + 3^2 \cdot (b-2)$, $K_{a,b}$ is L-colorable by Strategy A for L'_5.

□

Notation 3.2.12. Strategies A,B,C,D,E and F show that there exists a coloring of L_a such that every list in L_b still has available colors. It is called *Strategy A(B,C,D,E,F) for* L_a. However, we can exchange the role between L_a and L_b for a list assignment L of $K_{a,b}$ and we call *Strategy A(B,C,D,E,F) for* L_b.

Strategy A can be applied for a list assignment whose all lists have size at most 3. However, we can imitate Strategy A to build a new strategy which can be applied when all lists have size at most 2.

Theorem 3.2.13. (Strategy A$'$) *Let L be a list assignment of $K_{a,b}$ with $L_a = \{A_1, A_2, \ldots, A_a\}, L_b = \{B_1, B_2, \ldots, B_b\}$ and all lists have size at most 2. If all lists in L_a are mutually disjoint and $2^a > \prod_{i=1}^{a} \lfloor 2^{a-1} \rfloor n_1 + \lfloor 2^{a-2} \rfloor n_2$ where $n_i = |\{B \in L_b, |B| = i\}|$ for $i = 1, 2$, then $K_{a,b}$ is L-colorable.*

Proof. Similar to Strategy A. □

According to the proof of Strategies B, C, D, E and F, if color 1 appears in $a-1, a-2, a-3, a-4$ and $a-5$ lists, respectively, then we label such lists by color 1. The size of lists containing color 1 is insignificant. Then we can prove Strategies B$'$, C$'$, D$'$, E$'$ and F$'$ similar to Strategies B, C, D, E and F, respectively.

Theorem 3.2.14. (Strategy B$'$) *Let L be a list assignment of $K_{a,b}$. If a color appears in $a-1$ lists in L_a and the remaining lists in L_a and L_b are of size 3, then $K_{a,b}$ is L-colorable.*

Proof. Similar to Strategy B. □

Theorem 3.2.15. (Strategy C$'$) *Let L be a list assignment of $K_{a,b}$ where every color appears in at most eight lists in L_b. If a color appears in $a-2$ lists in L_a and the remaining lists in L_a and L_b are of size 3, then $K_{a,b}$ is L-colorable.*

Proof. Similar to Strategy C. □

Theorem 3.2.16. (Strategy D$'$) *Let L be a list assignment of $K_{a,b}$ where every color appears in at most r lists in L_b. If a color appears in $a-3$ lists in L_a, the remaining lists in L_a and L_b are of size 3 and $(r,b) \in \{(r,b)|r \le 2, b \le 22\} \cup \{(3,b)|b \le 14\} \cup \{(4,b)|b \le 12\} \cup \{(5,b)|b \le 9\}$, then $K_{a,b}$ is L-colorable.*

Proof. Similar to Strategy D. □

Theorem 3.2.17. (Strategy E$'$) *Let L be a list assignment of $K_{a,b}$ where every color appears in at most r lists in L_b. If color 1 appears in $a-4$ lists in L_a, the remaining lists in L_a and L_b are of size 3 and and $(r,b) \in \{(r,b)|r \le 2, b \le 22\} \cup \{(3,b)|b \le 14\}$, then $K_{a,b}$ is L-colorable unless the four remaining lists in L_a are $246, 257, 347, 356$ and $\mathcal{F} \subseteq L_b$.*

Proof. Similar to Strategy E. □

Theorem 3.2.18. (Strategy F$'$) *Let L be a list assignment of $K_{a,b}$ where every color appears in at most two lists in L_b. If a color appears in $a-5$ lists in L_a, the remaining lists in L_a and L_b are of size 3, and $a+b \le 18$, then $K_{a,b}$ is L-colorable.*

Proof. Similar to Strategy F. □

3.3 Complete Bipartite Graphs with Fourteen Vertices: a New Proof

This section gives another proof of the result by Fitzpatrick and MacGillivray [5] which was stated that every complete bipartite graph with 14 vertices except $K_{7,7}$ is 3-choosable and $L_{\mathcal{F}}$ (see Notation 3.1.1) is the unique 3-list assignment such that $K_{a,b}$ is not $L_{\mathcal{F}}$-colorable. Their proof is a detailed case analysis which cannot be extended to verify 3-choosability of complete bipartite graphs with 15 vertices while our proof is obtained from Strategies A, B, C, D, E and F, and our proof can be applied to give results of 3-choosability of complete bipartite graphs with 15 vertices. Moreover, our strategies can be applied to verify complete bipartite graphs to be L-colorable for some 3-list assignments L.

Lemma 3.3.1. *The complete bipartite graph $K_{3,b}$ is 3-choosable if and only if $b \leq 26$.*

Proof. Let L be the 3-list assignment of $K_{3,27}$ defined by $L_3 = \{123, 456, 789\}$ and $L_{27} = \{\{a, b, c\} | a \in \{1, 2, 3\}, b \in \{4, 5, 6\}, c \in \{7, 8, 9\}\}$. Notice that every coloring of L_3 is a list in L_{27}. By Lemma 3.2.1, $K_{3,27}$ is not L-colorable.

Next, we will prove $K_{3,26}$ is 3-choosable. Let L be a 3-list assignment of $K_{3,26}$. If some lists in L_3 have a common color, $K_{3,26}$ is immediately L-colorable by Strategy B for L_3. Suppose that all lists in L_3 have no common color. To apply Strategy A for L_3, we count the number of 3-coloring of L_3 and count the number of lists of size 1, size 2 and size 3 in L_{26}. We see that the number of 3-coloring of L_3 is 27. Since L_{26} has only 26 lists of size 3, at least one of those 3-colorings is not a list in L_{26}. Hence, we can use such 3-coloring to color L_3 while every list in L_{26} still has an available color. □

Lemma 3.3.2. *The complete bipartite graph $K_{4,10}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{4,10}$. Let r_4 (and r_{10}) be the maximum number of lists in L_4 (and L_{10}) containing a common color. Note that $r_4 \leq 4$ and $r_{10} \leq 10$.

Case 1. $r_4 = 3, 4$ or $r_{10} = 9, 10$; apply Strategy B for L_4 or Strategy B for L_{10}, respectively.

Case 2. $r_4 = 2$ and $r_{10} \leq 8$; apply Strategy C for L_4.

Case 3. $r_4 = 1$ and $r_{10} \leq 8$; apply Strategy A for L_4. Notice that $\prod_{i=1}^{4} |A_i| = 3^4 > 3 \cdot 10 = 3^{4-3} n_3$. □

Lemma 3.3.3. *The complete bipartite graph $K_{5,9}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{5,9}$. Let r_5 (and r_9) be the maximum number of lists in L_5 (and L_9) containing a common color. Then $r_5 \leq 5$ and $r_9 \leq 9$.

Case 1. $r_5 = 4, 5$ or $r_9 = 8, 9$; apply Strategy B for L_5 or Strategy B for L_9, respectively.

Case 2. $r_5 = 3$ and $r_9 \leq 7$; apply Strategy C for L_5.

Case 3. $r_5 \leq 2$ and $r_9 = 7$; apply Strategy C for L_9.

Case 4. $r_5 \leq 2$ and $r_9 = 6$; apply Strategy D for L_9.

Case 5. $r_5 \leq 2$ and $r_9 = 5$; apply Strategy E for L_9. Notice that $\mathcal{F} \not\subset L_5$ because L_5 contains only five lists.

Case 6. $r_5 = 2$ and $r_9 \leq 4$; apply Strategy D for L_5.

Case 7. $r_5 = 1$ and $r_9 \leq 4$; apply Strategy A for L_5. Notice that $\prod_{i=1}^{5} |A_i| = 3^5 > 3^2 \cdot 9 = 3^{5-3} n_3$. □

Lemma 3.3.4. *The complete bipartite graph $K_{6,8}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{6,8}$. Let r_6 (and r_8) be the maximum number of lists in L_6 (and L_8) containing a common color. Then $r_6 \leq 6$ and

$r_8 \leq 8$.

Case 1. $r_6 = 5, 6$ or $r_8 = 7, 8$; apply Strategy B for L_6 or Strategy B for L_8, respectively.

Case 2. $r_6 = 4$ and $r_8 \leq 6$; apply Strategy C for L_6.

Case 3. $r_6 \leq 3$ and $r_8 = 6$; apply Strategy C for L_8.

Case 4. $r_6 \leq 3$ and $r_8 = 5$; apply Strategy D for L_8.

Case 5. $r_6 \leq 3$ and $r_8 = 4$; apply Strategy E for L_8. Notice that $\mathcal{F} \not\subset L_6$ because L_6 contains only six lists.

Case 6. $r_6 = 3$ and $r_8 \leq 3$; apply Strategy D for L_6.

Case 7. $r_6 = 2$ and $r_8 \leq 3$; apply Strategy E for L_6 unless $1 \in A_1, A_2$, $A_3 = 246, A_4 = 257, A_5 = 347, A_6 = 356$ and $\mathcal{F} \subset L_8$. In such case, color $1, 2, 3, 4, 5, 6, 7$ have already appeared in two lists in L_6, we have two new color $8, 9 \in A_1$ because $r_6 = 2$. Hence $3 \in A_{a-1}, A_a$ and the four remaining lists cannot rename the colors to $246, 257, 347, 356$ because the union of the four remaining lists contains eight colors. Therefore, we can apply Strategy E for L_6.

Case 8. $r_6 = 1$ and $r_8 \leq 3$; apply Strategy A for L_6. Notice that $\prod_{i=1}^{6} |A_i| = 3^6 > 3^3 \cdot 8$. $\square$

Lemma 3.3.5. *Let L be a 3-list assignment of $K_{7,7}$. The complete bipartite graph $K_{7,7}$ is L-colorable unless $L_{7(i)} = L_{7(ii)} = \mathcal{F}$.*

Proof. Let L be a 3-list assignment such that $\mathcal{F} \not\subset L_{7(i)}$ or $\mathcal{F} \not\subset L_{7(ii)}$. Let $r_{7(i)}$ (and $r_{7(ii)}$) be the maximum number of lists in $L_{7(i)}$ (and $L_{7(ii)}$) containing a common color. Then $r_{7(i)}, r_{7(ii)} \leq 7$.

Let $t = |\bigcup_{v \in V(K_{7,7})} L(v)|$. By Theorem 2.1.3, we may suppose that $t \leq 10$ because $14 \cdot 3 < \binom{10+1}{2}$. Since $\sum_{v \in L_{7(i)}} |L(v)| = 21$, we obtain $r_{7(i)} \geq 3$ by the pigeonhole principle. Similarly, $r_{7(ii)} \geq 3$.

Case 1. $r_{7(i)} = 6, 7$ or $r_{7(ii)} = 6, 7$; apply Strategy B for $L_{7(i)}$ or Strategy B for

$L_{7(ii)}$, respectively.

Case 2. $r_{7(i)} = 5$ and $r_{7(ii)} \leq 5$; apply Strategy C for $L_{7(i)}$.

Case 3. $r_{7(i)} \leq 4$ and $r_{7(ii)} = 5$; apply Strategy C for $L_{7(ii)}$.

Case 4. $r_{7(i)} = 4$ and $r_{7(ii)} \leq 4$; apply Strategy D for $L_{7(i)}$.

Case 5. $r_{7(i)} = 3$ and $r_{7(ii)} = 4$; apply Strategy D for $L_{7(ii)}$.

Case 6. $r_{7(i)} = 3$ and $r_{7(ii)} = 3$; apply Strategy E for $L_{7(i)}$ unless $1 \in A_1, A_2, A_3$, $A_4 = 246, A_5 = 257, A_6 = 347, A_7 = 356$ and $L_{7(ii)} = \mathcal{F}$. In such case, $\{1,2,3\}, \{1,4,5\}, \{1,6,7\}$ are 3-colorings of $L_{7(ii)}$. One of such 3-colorings is not a list in $L_{7(i)}$ because $L_{7(i)} \neq \mathcal{F}$. Then $K_{7,7}$ is L-colorable by Lemma 3.2.1. □

Theorem 3.3.6. *The complete bipartite graph with* 14 *vertices is* 3*-choosable if and only if it is not* $K_{7,7}$. *For a* 3*-list assignment* L, $K_{7,7}$ *is* L*-colorable unless* $L = L_{\mathcal{F}}$.

Proof. It follows from Lemmas 3.3.1, 3.3.2 3.3.3, 3.3.4 and 3.3.5. □

By Lemmas 3.3.1, 3.3.2, 3.3.3 and 3.3.4, we can easily verify that every complete bipartite graph is 3-choosable.

Theorem 3.3.7. *The complete bipartite graph with at most* 13 *vertices is* 3*-choosable.*

Proof. If $a + b \leq 13$, then $K_{a,b}$ is a subgraph of one of $K_{3,26}$, $K_{4,10}$, $K_{5,9}$, $K_{6,8}$ which are 3-choosable by Lemmas 3.3.1, 3.3.2, 3.3.3 and 3.3.4. Therefore, a complete bipartite graph with at most 13 vertices is 3-choosable. □

Since $K_{7,7}$ is not $L_{\mathcal{F}}$-colorable and $L_{\mathcal{F}}$ is a $(3,7)$-list assignment, $K_{7,7}$ is not $(3,7)$-choosable. However, $K_{7,7}$ is $(3,t)$-choosable if and only if $t \neq 7$. Theorem 3.3.8 gives all positive numbers t such that all complete bipartite graphs with 14 vertices are $(3,t)$-choosable.

Theorem 3.3.8. *A complete bipartite graph with* 14 *vertices is* $(3,t)$*-choosable unless* $t = 7$*.*

Proof. Let a, b be positive integers such that $a \leq b$ and $a + b = 14$. Then $a \leq 7$.

Case 1. $a \leq 3$.

Then $K_{a,b}$ is a subgraph of $K_{3,26}$ which is 3-choosable by Lemma 3.3.1.

Case 2. $a = 4, 5, 6$.

Then $K_{a,b}$ is one of $K_{4,10}, K_{5,9}, K_{6,8}$ which is 3-choosable by Lemma 3.3.2, Lemma 3.3.3 and Lemma 3.3.4.

Case 3. $a = 7$ Since $L_{\mathcal{F}}$ is the unique 3-list assignment such that $K_{7,7}$ is not $L_{\mathcal{F}}$-colorable, $K_{7,7}$ is $(3,t)$-choosable for all $t \neq 7$. □

3.4 Complete Bipartite Graphs with Fifteen Vertices

In this section, we keep utilizing our strategies to extend the result in the previous section to 15 vertices. We first show that $K_{4,11}, K_{5,10}, K_{6,9}$ are 3-choosable and then we prove that for a 3-list assignment L, $K_{7,8}$ is L-colorable unless $L|_{V(K_{7,7})} = L_{\mathcal{F}}$.

Lemma 3.4.1. *The complete bipartite graph* $K_{4,11}$ *is* 3*-choosable.*

Proof. Let L be a 3-list assignment of $K_{4,11}$ and r_4 (and r_{11}) be the maximum number of lists in L_4 (and L_{11}) containing a common color. Then $r_4 \leq 4$ and $r_{11} \leq 11$.

Case 1. $r_4 = 3, 4$ or $r_{11} = 10, 11$; apply Strategy B for L_4 or Strategy B for L_{11}, respectively.

Case 2. $r_4 \leq 2$ and $r_{11} = 9$; apply Strategy C for L_{11}.

Case 3. $r_4 = 2$ and $r_{11} \leq 8$; apply Strategy C for L_4.

Case 4. $r_4 = 1$ and $r_{11} \leq 8$; apply Strategy A for L_4. Notice that $\prod_{i=1}^{4} |A_i| = 3^4 > 3 \cdot 11 = 3^{4-3} n_3$. $\square$

Lemma 3.4.2. *The complete bipartite graph $K_{5,10}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{5,10}$ and r_5 (and r_{10}) be the maximum number of lists in L_5 (and L_{10}) containing a common color. Then $r_5 \leq 5$ and $r_{10} \leq 10$.

Case 1. $r_5 = 4, 5$ or $r_{10} = 9, 10$; apply Strategy B for L_5 or Strategy B for L_{10}, respectively.

Case 2. $r_5 = 3$ and $r_{10} \leq 8$; apply Strategy C for L_5.

Case 3. $r_5 \leq 2$ and $r_{10} = 8$; apply Strategy C for L_{10}.

Case 4. $r_5 \leq 2$ and $r_{10} = 7$; apply Strategy D for L_{10}.

Case 5. $r_5 \leq 2$ and $r_{10} = 6$; apply Strategy E for L_{10}. Notice that $\mathcal{F} \not\subset L_5$ because L_5 contains only five lists.

Case 6. $r_5 \leq 2$ and $r_{10} = 5$; apply Strategy F for L_{10}.

Case 7. $r_5 = 2$ and $r_{10} \leq 4$; apply Strategy D for L_5.

Case 8. $r_5 = 1$ and $r_{10} \leq 4$; apply Strategy A for L_5. Notice that $\prod_{i=1}^{5} |A_i| = 3^5 > 3^2 \cdot 10 = 3^{5-3} n_3$. $\square$

Lemma 3.4.3. *The complete bipartite graph $K_{6,9}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{6,9}$ and r_6 (and r_9) be the maximum number of lists in L_6 (and L_9) containing a common color. Then $r_6 \leq 6$ and $r_9 \leq 9$.

Case 1. $r_6 = 5, 6$ or $r_9 = 8, 9$; apply Strategy B for L_6 or Strategy B for L_9, respectively.

Case 2. $r_6 = 4$ and $r_9 \leq 7$; apply Strategy C for L_6.

Case 3. $r_6 \leq 3$ and $r_9 = 7$; apply Strategy C for L_9.

Case 4. $r_6 \leq 3$ and $r_9 = 6$; apply Strategy D for L_9.

Case 5. $r_6 \leq 3$ and $r_9 = 5$; apply Strategy E for L_9. Notice that $\mathcal{F} \not\subset L_6$ because L_6 contains only six lists.

Case 6. $r_6 = 3$ and $r_9 \leq 4$; apply Strategy D for L_6.

Case 7. $r_6 \leq 2$ and $r_9 = 4$; apply Strategy F for L_9.

Case 8. $r_6 = 2$ and $r_9 \leq 3$; apply Strategy E for L_6 unless $1 \in A_1, A_2$ and $A_3 = 246, A_4 = 257, A_5 = 347, A_6 = 356$. In such case, we obtain that $4, 5, 6, 7 \notin A_1, A_2$ because $r_6 = 2$. Let $A_1 = 178$. Then $3 \in A_5, A_6$ and the four remaining lists cannot rename the colors to be $246, 257, 347, 356$. Hence, we still apply Strategy D for L_6.

Case 9. $r_6 = 1$ and $r_9 \leq 3$; apply Strategy A for L_6. Notice that $\prod_{i=1}^{6} |A_i| = 3^6 > 3^3 \cdot 9 = 3^{6-3} n_3$. □

Lemma 3.4.4. *Let L be a 3-list assignment of $K_{7,8}$. The complete bipartite graph $K_{7,8}$ is L-colorable unless $\mathcal{F} \subset L_7, L_8$.*

Proof. Let L be a 3-list assignment of $K_{7,8}$ such that $\mathcal{F} \not\subset L_7$ or $\mathcal{F} \not\subset L_8$. Let r_7 (and r_8) be the maximum number of lists in L_7 (and L_8) containing a common color. Then $r_7 \leq 7$ and $r_8 \leq 8$.

Case 1. $r_7 = 6, 7$ or $r_8 = 7, 8$; apply Strategy B for L_7 or Strategy B for L_8, respectively.

Case 2. $r_7 = 5$ and $r_8 \leq 6$; apply Strategy C for L_7.

Case 3. $r_7 \leq 4$ and $r_8 = 6$; apply Strategy C for L_8.

Case 4. $r_7 \leq 4$ and $r_8 = 5$; apply Strategy D for L_8.

Case 5. $r_7 = 4$ and $r_8 \leq 4$; apply Strategy D for L_7.

Case 6. $r_7 \leq 3$ and $r_8 = 4$; apply Strategy E for L_8 unless $1 \in B_1, B_2, B_3, B_4$, $B_5 = 246, B_6 = 257, B_7 = 347, B_8 = 356$ and $L_7 = \mathcal{F}$. Since $L_7 = \mathcal{F}$, $\{1,2,3\}, \{1,4,5\}$ and $\{1,6,7\}$ are 3-colorings of L_7. Since $\mathcal{F} \not\subset L_8$, one of such 3-colorings is not a list in L_8. Hence $K_{7,8}$ is L-colorable by Lemma 3.2.1,

Case 7. $r_7 = 3$ and $r_8 \leq 3$; apply Strategy E for L_7 unless $1 \in A_1, A_2, A_3$, $A_4 = 246, A_5 = 257, A_6 = 347, A_7 = 356$ and $\mathcal{F} \subset L_8$. In such case, let $B_1 = 123, B_2 = 145, B_3 = 146, B_4 = 246, B_5 = 257, B_6 = 347, B_7 = 356$. Suppose that $B_8 = 89A$ because $r_8 \leq 3$ and color 1 to color 7 are appears in three lists in $B_1, B_2, \ldots, B_7$. Since $L_7 \neq \mathcal{F}$, we obtain that $123, 145$ or 167 are not a list in L_7. Suppose that $123 \notin L_7$. Notice that $\{1,2,3,8\}, \{1,2,3,9\}$ and $\{1,2,3,A\}$ are 4-colorings of L_8. Since color 2 appears in at most two lists in A_1, A_2, A_3, $128, 129$ or $12A$ is not a list in L_7. Suppose that $128 \notin L_7$. Then $\{1,2,3,8\}$ is a 4-coloring of L_8 which has no subset that is a list in L_7. Then $K_{7,8}$ is L-colorable by Lemma 3.2.1.

Case 8. $r_7 \leq 2$ and $r_8 = 3$; apply Strategy F for L_8.

Case 9. $r_7 = 2$ and $r_8 \leq 2$; apply Strategy F for L_7.

Case 10. $r_7 = 1$ and $r_8 \leq 2$; apply Strategy A for L_7. Notice that $\prod_{i=1}^{7} |A_i| = 3^7 \geq 3^4 \cdot 8 = 3^{7-3} n_3$. □

Theorem 3.4.5. *The complete bipartite graph with* 15 *vertices is* 3*-choosable if and only if it is not* $K_{7,8}$*. For a* 3*-list assignment* L*,* $K_{7,8}$ *is* L*-colorable unless* $L|_{V(K_{7,7})} = L_{\mathcal{F}}$*.*

Proof. It follows from Lemmas 3.3.1, 3.4.1, 3.4.2, 3.4.3 and 3.4.4. □

Since $K_{7,8}$ is not L-colorable when $L|_{V(K_{7,7})}$, $K_{7,8}$ is not $(3,t)$-choosable for $t = 7, 8, 9, 10$. However, $K_{7,8}$ is $(3,t)$-choosable if and only if $t \neq 7, 8, 9, 10$. Theorem 3.4.6 gives all positive numbers t such that all complete bipartite graphs with 15 vertices are $(3,t)$-choosable.

Theorem 3.4.6. *A complete bipartite graph with* 15 *vertices is* $(3,t)$*-choosable unless* $t = 7, 8, 9, 10$*.*

Proof. Let a, b be positive integers such that $a \leq b$ and $a + b = 15$. Then $a \leq 7$.

Case 1. $a \leq 3$.

Then $K_{a,b}$ is a subgraph of $K_{3,26}$ which is 3-choosable by Lemma 3.3.1.

Case 2. $a = 4, 5, 6$.

Then $K_{a,b}$ is one of $K_{4,11}, K_{510}$ or $K_{6,9}$ which is 3-choosable by Lemma 3.4.1, Lemma 3.4.2 and Lemma 3.4.3, respectively.

Case 3. $a = 7$ When $t \leq 6$ or $t \geq 11$, we obtain that $\mathcal{F} \not\subset L_7$ or $\mathcal{F} \not\subset L_8$. Then $K_{7,8}$ is L-colorable by Lemma 3.4.4. $\square$

CHAPTER IV

ON 3-CHOOSABILITY OF COMPLETE BIPARTITE GRAPHS WITH 16 VERTICES

In this chapter, we keep studying about 3-choosability of complete bipartite graphs. The main result of this chapter is Theorem 4.3.10 which is stated that every complete bipartite graph with 16 vertices is $(3,t)$-choosable for $t \leq 6$ or $t \geq 14$. We will apply this result to prove Theorem 5.3.1 in Chapter 5.

In Section 4.1, we study 3-choosability of complete bipartite graphs by using strategies from Section 3.1. Unlikely, some cases of $K_{6,10}$, $K_{7,9}$ and $K_{8,8}$ cannot be proved by our strategies. For $K_{6,10}$, we claim that $K_{6,10}$ is 3-choosable by referring to [15]. For $K_{8,8}$ and its 3-list assignment L, we prove that $K_{8,8}$ is L-colorable unless $L|_{V(K_{7,7})} = L_{\mathcal{F}}$ (See Notation 3.1.1) in Section 4.2. For $K_{7,9}$ which is more difficult than $K_{8,8}$, we prove that $K_{7,9}$ is $(3,t)$-choosable if and only if $t \leq 6$ or $t \geq 14$ in Section 4.3.

4.1 Consequence of the Strategies

We apply our strategies to study 3-choosability of complete bipartite graphs with 16 vertices.

Lemma 4.1.1. *The complete bipartite graph $K_{4,12}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{4,12}$. Let r_4 (and r_{12}) be the maximum number of lists in L_4 (and L_{12}) containing a common color. Then $r_4 \leq 4$ and $r_{12} \leq 12$.

Case 1. $r_4 = 3, 4$ or $r_{12} = 11, 12$; apply Strategy B for L_4 or Strategy B for L_{12}, respectively.

Case 2. $r_4 \leq 2$ and $r_{12} = 10$; apply Strategy C for L_{12}.

Case 3. $r_4 \leq 2$ and $r_{12} = 9$; apply Strategy D for L_{12}.

Case 4. $r_4 = 2$ and $r_{12} \leq 8$; apply Strategy C for L_4.

Case 5. $r_4 = 1$; apply Strategy A for L_4. □

Lemma 4.1.2. *The complete bipartite graph $K_{5,11}$ is 3-choosable.*

Proof. Let L be a 3-list assignment of $K_{5,11}$. Let r_5 (and r_{11}) be the maximum number of lists in L_5 (and L_{11}) containing a common color. Then $r_5 \leq 5$ and $r_{11} \leq 11$.

Case 1. $r_5 = 4, 5$ or $r_{11} = 10, 11$; apply Strategy B for L_5 or Strategy B for L_{11}, respectively.

Case 2. $r_5 \leq 3$ and $r_{11} = 9$; apply Strategy C for L_{11}.

Case 3. $r_5 = 3$ and $r_{11} \leq 8$; apply Strategy C for L_5.

Case 4. $r_5 \leq 2$ and $r_{11} = 8, 7, 6$; apply Strategies D,E and F for L_{11}, respectively.

Case 5. $r_5 = 2$ and $r_{11} \leq 5$; apply Strategy D for L_5.

Case 6. $r_5 = 1$; apply Strategy A for L_5. □

To study 3-choosability of $K_{6,10}$, we divide the proof into several cases. However, our strategies cannot be applied for a case as shown in Lemma 4.1.3. We do not prove the missing case here because O'Donnell[15] has done it.

Lemma 4.1.3. *Let L be a 3-list assignment of $K_{6,10}$. Let r_6 (and r_{10}) be the maximum number of lists in L_6 (and L_{10}) containing a common color. If $(r_6, r_{10}) \neq (2, 4)$, then $K_{6,10}$ is L-colorable.*

Proof. **Case** 1. $r_6 = 5, 6$ or $r_{10} = 9, 10$; apply Strategy B for L_6 or Strategy B for L_{10}, respectively.

Case 2. $r_6 = 4$ and $r_{10} \leq 8$; apply Strategy C for L_6.

Case 3. $r_6 \leq 3$ and $r_{10} = 8, 7, 6$; apply Strategy C,D,E for L_{10}, respectively.

Case 4. $r_6 = 3$ and $r_{10} \leq 5$; apply Strategy D for L_6.

Case 5. $r_6 \leq 2$ and $r_{10} = 5$; apply Strategy F for L_{10}.

Case 6 $r_6 = 2$ and $r_{10} \leq 3$; apply Strategy E for L_6.

Case 7. $r_6 = 1$; apply Strategy A for L_6. □

Lemma 4.1.4. [15] *The complete bipartite graph $K_{6,b}$ is 3-choosable if and only if $b \leq 16$.*

To study 3-choosability of $K_{7,9}$, we cannot use our strategies to prove all cases of the proof as shown in Lemma 4.1.5. However, we prove that $K_{7,9}$ is $(3, t)$-choosable if and only if $t \leq 6$ or $t \geq 14$ in Section 4.3.

Lemma 4.1.5. *Let L be a 3-list assignment of $K_{7,9}$. Let r_7 (and r_9) be the maximum number of lists in L_7 (and L_9) containing a common color. If $(r_7, r_9) \neq (3, 4), (2, 3)$ and $L|_{V(K_{7,7})} \neq L_{\mathcal{F}}$, then $K_{7,9}$ is L-colorable.*

Proof. **Case** 1. $r_7 = 6, 7$ or $r_9 = 8, 9$; apply Strategy B for L_7 or Strategy B for L_9, respectively.

Case 2. $r_7 = 5$ and $r_9 \leq 7$; apply Strategy C for L_7.

Case 3. $r_7 \leq 4$ and $r_9 = 6$; apply Strategy D for L_9.

Case 4. $r_7 = 4$ and $r_9 \leq 5$; apply Strategy D for L_7.

Case 5. $r_7 \leq 3$ and $r_9 = 5$; apply Strategy E for L_9. In this case, $K_{7,9}$ is L-colorable unless $\mathcal{F} \subset L_7, L_9$.

Case 6. $r_7 \leq 2$ and $r_9 = 4$; apply Strategy F for L_9.

Case 7. $r_7 = 3$ and $r_9 \leq 3$; apply Strategy E for L_7. In this case, $K_{7,9}$ is L-colorable unless $\mathcal{F} \subset L_7, L_9$.

Case 8. $r_7 = 2$ and $r_9 \leq 2$; apply Strategy F for L_7.

Case 9. $r_7 = 1$; apply Strategy A for L_7. □

Again, to study 3-choosability of $K_{8,8}$, we cannot use our strategies to prove all cases as shown in Lemma 4.1.6. However, we have a complete proof in Section 4.3; for a 3-list assignment L of $K_{8,8}$, it is L-colorable if and only if $L|_{V(K_{7,7})} \neq L_{\mathcal{F}}$.

Lemma 4.1.6. *Let L be a 3-list assignment of $K_{8,8}$. Let $r_{8(i)}$ (and $r_{8(ii)}$) be the maximum number of lists in $L_{8(i)}$ (and $L_{8(ii)}$) containing a common color. If $(r_{8(i)}, r_{8(ii)}) \neq (4,4), (4,3), (3,4), (3,3), (2,2)$, then $K_{8,8}$ is L-colorable.*

Proof. **Case** 1. $r_{8(i)} = 7, 8$ or $r_{8(ii)} = 7, 8$; apply Strategy B for $L_{8(i)}$ or Strategy B for $L_{8(ii)}$, respectively.

Case 2. $r_{8(i)} = 6$ and $r_{8(ii)} \leq 6$; apply Strategy C for $L_{8(i)}$.

Case 3. $r_{8(i)} \leq 5$ and $r_{8(ii)} = 6$; apply Strategy C for $L_{8(ii)}$.

Case 4. $r_{8(i)} = 5$ and $r_{8(ii)} \leq 5$; apply Strategy D for $L_{8(i)}$.

Case 5. $r_{8(i)} \leq 4$ and $r_{8(ii)} = 5$; apply Strategy D for $L_{8(ii)}$.

Case 6. $r_{8(i)} = 4$ and $r_{8(ii)} \leq 2$; apply Strategy E for $L_{8(i)}$.

Case 7. $r_{8(i)} \leq 2$ and $r_{8(ii)} = 4$; apply Strategy E for $L_{8(ii)}$.

Case 8. $r_{8(i)} = 3$ and $r_{8(ii)} \leq 2$; apply Strategy F for $L_{8(i)}$.

Case 9. $r_{8(i)} \leq 2$ and $r_{8(ii)} = 3$; apply Strategy F for $L_{8(ii)}$.

Case 10. $r_{8(i)} = 1$; apply Strategy A for $L_{8(i)}$.

Case 11. $r_{8(ii)} = 1$; apply Strategy A for $L_{8(ii)}$. □

Theorem 4.1.7. *A complete graph with 16 vertices is 3-choosable unless it is $K_{7,9}$ or $K_{8,8}$.*

Proof. It follows from Lemma 3.3.1, Lemma 4.1.1, Lemma 4.1.2 and Lemma 4.1.4. □

4.2 On 3-choosability of $K_{8,8}$

Recall that $r_{8(i)}$ (and $r_{8(ii)}$) be the maximum number of lists in $L_{8(i)}$ (and $L_{8(ii)}$) containing a common color and Lemma 4.1.6 has results of 3-choosability of $K_{8,8}$ except when $(r_{8(i)}, r_{8(ii)}) = (4,4), (4,3), (3,4), (3,3), (2,2)$. Then lemmas and theorems can be classified into three groups. The first group, which is from Lemma 4.2.1 to Theorem 4.2.18, deals with $(r_{8(i)}, r_{8(ii)}) = (4,4), (4,3), (3,4)$. The second group, which is from Theorem 4.2.19 to Theorem 4.2.34, deals with $(r_{8(i)}, r_{8(ii)}) = (3,3)$ The last group, which is only Theorem 4.2.35, deals with $(r_{8(i)}, r_{8(ii)}) = (2,2)$. The conclusion is in Theorem 4.2.36 which is stated that, for a 3-list assignment L of $K_{8,8}$, it is L-colorable if and only if $\{123,\ 145,\ 167,\ 246,\ 257,\ 347,\ 356\} \subset L_{8(i)}, L_{8(ii)}$.

Lemma 4.2.1. *Let L be a list assignment of $K_{4,5}$ such that $L_4 = \{A_1, A_2, A_3, A_4\}$ and $L_5 = \{B_1, B_2, \ldots, B_5\}$. If $|A_1| = |A_2| = |A_3| = |B_1| = 2$ and $|A_4| = |B_2| = |B_3| = |B_4| = |B_5| = 3$, then $K_{4,5}$ is L-colorable unless $L_4 = \{1p, 1q, 23, 245\}$ and $L_5 = \{12, 134, 135, 267, 367\}$ where $p \neq q$ and $p, q \neq 1, 2, 3$ up to renaming the colors.*

Proof. **Case** 1. *A_1, A_2, A_3, A_4 are mutually disjoint.*

Hence, we apply Strategy A for L_4.

Case 2. *A_1, A_2, A_3, A_4 are not mutually disjoint but A_1, A_2, A_3 are mutually disjoint.*

Suppose $A_1 = 12, A_2 = 34, A_3 = 56$ and $1 \in A_4$. It is easy to verify that if a color appears in four lists of L_5 then $K_{4,5}$ is L-colorable. Hence, we suppose that each color appears in at most three lists of L_5.

Then L_4 has at least four 3-colorings, namely, $\{1,3,5\}, \{1,3,6\}, \{1,4,5\}$ and $\{1,4,6\}$. If one of such 3-colorings has no subset that is a list in L_5, then $K_{4,5}$

is L-colorable by Lemma 3.2.1. Hence, we suppose that such 3-colorings have a subset that is a list in L_5. Without loss of generality, let $B_1 = 13, B_2 = 145$ and $B_3 = 146$ because each color appears in at most three lists of L_5. Hence, we label B_1, B_2, B_3 by color 1 and label A_1 by color 2. Now, the remaining vertices form $K_{3,2}$. Let L' be the list assignment of $K_{3,2}$ which is obtained from L by removing color 1 and color 2. For the worst case, we suppose that $2 \in B_4, B_5$. Then $L'_3 = \{34, 56, A_4 - 1\}$ and $L'_2 = \{B_4 - 2, B_5 - 2\}$. Since $K_{3,2}$ is 2-choosable, $K_{3,2}$ is L'-colorable. Hence, $K_{8,8}$ is L-colorable.

Case 3. A_1, A_2, A_3 *are not mutually disjoint and* $A_3 \cap A_4 = \varnothing$.

Let $1 \in A_1, A_2$ and $A_3 = 23, A_4 = 456$. Thus, $L_{8(i)}$ has at least six 3-colorings containing color 1. Again, we suppose that such 3-colorings has subset that is a list in $L_{8(ii)}$ by Lemma 3.2.1. Without loss of generality, we let $B_1 = 12, B_2 = 134, B_3 = 135, B_4 = 136$. Hence, we label B_1, B_2, B_3, B_4 by color 1 and the remaining vertices are easily labeled.

Case 4. A_1, A_2, A_3 *are not mutually disjoint and* $A_3 \cap A_4 \neq \varnothing$.

If L_4 has a coloring which is has no subset that is a list in L_5, then $K_{4,5}$ is L-colorable by Lemma 3.2.1. Suppose that each coloring of L_4 has a subset that is a list in L_5. Since L_5 has only one list of size 2, L_4 has at most one 2-coloring. That is, $|A_1 \cap A_2| = |A_3 \cap A_4| = 1$. Let $A_1 = 1p, A_2 = 1q, A_3 = 23, A_4 = 245$ where $p, q \neq 1$. We consider possibility of p, q. Since L_4 has at most one 2-coloring, we have $p, q \neq 2$, $p \neq q$ and if $p = 3$, then $q \neq 4, 5$.

Case 4.1 $p = 3$ or $q = 3$.

Suppose that $p = 3$ and $q = 6$. Thus we swap A_2 and A_3. That is, $A_1 = 13, A_2 = 23, A_3 = 16$ and $A_4 = 245$. The case $|A_1 \cap A_2| = 1$ and $A_3 \cap A_4 = \varnothing$ is Case 3 that we have already done.

Case 4.2 $p, q \neq 3$.

Since $\{1,3,4\},\{1,3,5\}$ and $\{2,p,q\}$ are 3-colorings of L_4, we let $B_2 = 134, B_3 = 135$ and $B_4 = 2pq$. Since $\{p,q,3,4\}$ and $\{p,q,3,5\}$ are 4-colorings of L_4 and we have only one list of size 3 left, we let $B_5 = 3pq$.

It can be directly verified that if $L_4 = \{1p, 1q, 23, 245\}$ and $L_5 =$ {12, 134, 135, 267 ,367}, then $K_{4,5}$ is not L-colorable. □

Lemma 4.2.2. *Let L be a list assignment of $K_{4,5}$ such that $L_4 = \{A_1, A_2, A_3, A_4\}$ and $L_5 = \{B_1, B_2, \ldots, B_5\}$. If $|A_1| = |A_2| = |A_3| = 2$ and $|A_4| = |B_1| = |B_2| = |B_3| = |B_4| = |B_5| = 3$, then $K_{4,5}$ is L-colorable.*

Proof. If L_4 has a 2-coloring then $K_{4,5}$ is L-colorable by Lemma 3.2.1. Suppose that L_4 has no 2-coloring. Let $p \in B_1$ and L^* be the list assignment of $K_{4,5}$ such that $L_4^* = L_4$ and $L_5^* = \{B_1 - p, B_2, B_3, B_4\}$. Since $L_4^* = L_4$ has no 2-coloring, we have $L_4^* \neq \{1p, 1q, 23, 245\}$. Hence, $K_{4,5}$ is L^*-colorable by Lemma 4.2.1. Therefore, $K_{4,5}$ is L-colorable. □

Theorem 4.2.3. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$ and $2 \in A_5, A_6, A_7$ then $K_{8,8}$ is L-colorable.*

Proof. If $2 \in A_8$ then we label A_1, A_2, A_3, A_4 by color 1 and label A_5, A_6, A_7, A_8 by color 2 and every remaining list still has an available color. Suppose that $A_8 = 345$. Then $L_{8(i)}$ has at least three 3-colorings of , namely $\{1,2,3\},\{1,2,4\},\{1,2,5\}$. If one of such 3-colorings is not a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that $B_1 = 123, B_2 = 124, B_3 = 125$.

We first label A_1, A_2, A_3, A_4 by color 1 and label B_1, B_2, B_3 by color 2. The remaining vertices form $K_{4,5}$. Let L' be the list assignment which is obtained from L by removing color 1 and color 2.

Case 1. $1 \notin B_4$.

Then $K_{4,5}$ is L'-colorable by Lemma 4.2.2.

Case 2. $1 \in B_4$ *and* B_5, B_6, B_7, B_8 *have at least five* 2*-colorings.*

Then $L_{8(ii)}$ has at least five 3-colorings. Since color 1 appears in at most four lists in $L_{8(i)}$, at least one of such 3-colorings is not a list in $L_{8(i)}$. Hence, $K_{8,8}$ is L-colorable by Lemma 3.2.1.

Case 3. $1 \in B_4$ *and* B_5, B_6, B_7, B_8 *have at most four* 2*-colorings.*

That is, $L'_4 = \{A_5 - 2, A_6 - 2, A_7 - 2, A_8\}$ and $L'_5 = \{B_4 - 1, B_5, B_6, B_7, B_8\}$. Since B_5, B_6, B_7, B_8 have at most four 2-colorings, lists in L'_5 cannot be renamed to lists in $\{12, 134, 135, 267, 367\}$. By Lemma 4.2.1, $K_{4,5}$ is L'-colorable; hence, $K_{8,8}$ is L-colorable. □

Lemma 4.2.4. *Let* L *be a list assignment of* $K_{4,5}$ *such that* $L_4 = \{A_1, A_2, A_3, A_4\}$ *and* $L_5 = \{B_1, B_2, \ldots, B_5\}$*. If* $|A_1| = |A_2| = |B_1| = 2$*,* $|A_3| = |A_4| = |B_2| = |B_3| = |B_4| = |B_5| = 3$*, then* $K_{4,5}$ *is* L*-colorable.*

Proof. Let $r \in A_3$ and L^* be the list assignment of $K_{4,5}$ such that $L^*_4 = \{A_1, A_2, A_3 - r, A_4\}$ and $L^*_5 = L_5$.

By Lemma 4.2.1, $K_{4,5}$ is L^*-colorable unless $L^*_4 = \{1p, 1q, 23, 245\}$ and $L^*_5 = \{12, 134, 135, 267, 367\}$ where $p \neq q$ and $p, q \neq 1, 2, 3$. Suppose that $L_4 = \{1p, 1q, 23r, 245\}$ and $L_5 = \{12, 134, 135, 267, 367\}$ where $p \neq q$ and $p, q \neq 1, 2, 3$. Then we label A_1, A_2 by color 1, label A_3 by color r and label A_4 by color 4. All remaining vertices still have available colors; hence, $K_{4,5}$ is L-colorable. □

Theorem 4.2.5. *Let* L *be a* 3*-list assignment of* $K_{8,8}$ *such that every color appears in at most four lists of each partite set where* $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ *and* $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$*. If* $1 \in A_1, A_2, A_3, A_4$ *and* $1, 2 \in B_1, B_2, B_3$ *then* $K_{8,8}$ *is* L*-colorable.*

Proof. We first label A_1, A_2, A_3, A_4 by color 1 and label B_1, B_2, B_3 by color 2. The remaining vertices form $K_{4,5}$. Let L' be the list assignment of $K_{4,5}$ which is obtained from L by removing color 1 and color 2. If color 2 appears in three lists of L'_4, then $K_{8,8}$ is L-colorable by Theorem 4.2.3. Hence, suppose that color 2 appears in at most two lists of L'_4. Moreover, for the worst case, we let $2 \in A_5, A_6$. That is, $L'_4 = \{A_5 - 2, A_6 - 2, A_7, A_8\}$ and $L'_5 = \{B_4 - 1, B_5, B_6, B_7, B_8\}$. By Lemma 4.2.4, $K_{4,5}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable. □

Lemma 4.2.6. *Let L be a list assignment of $K_{2,4}$ such that $L_2 = \{A_1, A_2\}$ and $L_4 = \{B_1, B_2, B_3, B_4\}$. If $|A_1| = |A_2| = 3$ and $|B_1|, |B_2|, |B_3|, |B_4| \leq 3$ and $|B_1| + |B_2| + |B_3| + |B_4| \geq 8$, then $K_{2,4}$ is L-colorable.*

Proof. For the worst case, we suppose that $|B_1|+|B_2|+|B_3|+|B_4| = 8$. Without loss of generality, let $|B_1| \leq |B_2| \leq |B_3| \leq |B_4|$.

Case 1. $|B_1| = |B_2| = |B_3| = |B_4| = 2$.

If A_1 and A_2 have a common color, we use this color to color A_1 and A_2; hence, all lists in L_4 still have an available color. Otherwise, we suppose that $A_1 \cap A_2 = \varnothing$. Thus we apply Strategy A for L_2.

Case 2. $|B_1| = 1, |B_2| = |B_3| = 2$ *and* $|B_4| = 3$.

Let $B_1 = 1$. Since B_1 has only one color, we must use color 1 to color lists in L_4. For the worst case, we suppose $1 \in A_1, A_2$ but $1 \notin B_2, B_3, B_4$. After we color B_1, the remaining vertices form $K_{2,3}$. Let L' be the list assignment of $K_{2,3}$ which is obtained from L by removing color 1. Thus $L'_2 = \{A_1 - 1, A_2 - 1\}$ and $L'_3 = \{B_2, B_3, B_4\}$.

If $A_1 - 1$ and $A_2 - 1$ have a common color, we can use the color to color $A_1 - 1$ and $A_2 - 1$ and all lists in L'_3 still have an available color. Otherwise, we suppose that $A_1 - 1, A_2 - 1$ are disjoint. Hence, L'_2 has four 2-colorings. Since L'_3 has two lists of size 2, at least one of such 2-colorings of L'_2 is not a list in L'_3. By Lemma 3.2.1, $K_{2,3}$ is L'-colorable. Therefore, $K_{2,4}$ is L-colorable.

Case 3. $|B_1| = |B_2| = 1$ *and* $|B_3| = |B_4| = 3$.

Let $B_1 = 1$ and $B_2 = 2$. Since each of B_1 and B_2 has only one color, we must use color 1 and color 2 to label lists in L_4. For the worst case, we suppose $1, 2 \in A_1, A_2$ but $1, 2 \notin B_2, B_3, B_4$. Let $A_1 = 123$ and $A_2 = 124$ After we color B_1 and B_2, the remaining vertices form $K_{2,2}$. Let L' be the list assignment of $K_{2,2}$ which is obtained from L by removing color 1 and color 2. Thus $L'_{2(i)} = \{3, 4\}$ and $L'_{2(ii)} = \{B_3, B_4\}$. We have to label lists $L'_{2(i)}$ by color 3 and color 4. Since each list in $L'_{2(ii)}$ still have an available color, $K_{2,2}$ is L'-colorable. Therefore, $K_{2,4}$ is L-colorable. $\square$

Lemma 4.2.7. *Let L be a list assignment of $K_{3,4}$ such that $L_3 = \{A_1, A_2, A_3\}$ and $L_4 = \{B_1, B_2, B_3, B_4\}$. If $|A_1| = |A_2| = |B_1| = |B_2| = 2$ and $|A_3| = |B_3| = |B_4| = 3$, then $K_{3,4}$ is L-colorable.*

Proof. **Case** 1. *A_1 and A_2 have a common color.*
Thus L_3 has at least three 2-colorings. Since L_4 has only two lists of size 2, at least one of such 2-colorings is not a list in L_4. By Lemma 3.2.1, $K_{3,4}$ is L-colorable.

Case 2. *A_1 and A_2 are disjoint.*
Let $A_1 = 12$ and $A_2 = 34$. If A_1, A_2, A_3 are mutually disjoint, then we apply Strategy A for L_3. Otherwise, we suppose that $1 \in A_3$. Thus, $\{1, 3\}$ and $\{1, 4\}$ are 2-colorings of L_3. If 13 and 14 are not lists in L_4, then $K_{3,4}$ is L-colorable by Lemma 3.2.1. Otherwise, we let $B_1 = 13$ and $B_2 = 14$.

Now, we label A_2 by color 2 and label B_1, B_2 by color 1. The remaining vertices form $K_{2,2}$. Let L' be the list assignment of $K_{2,2}$ which is obtained from L by removing color 1 and color 2. For the worst case, we suppose that $2 \in B_3, B_4$. Then $L'_{2(i)} = \{A_2, A_3 - 1\}$ and $L'_{2(ii)} = \{B_3 - 2, B_4 - 2\}$. We can directly verify that $K_{2,2}$ is L'-colorable. Hence, $K_{3,4}$ is L-colorable. $\square$

Theorem 4.2.8. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, A_4$, $1, 2 \in B_1, B_2$ and $1, 3 \in B_3, B_4$, then $K_{8,8}$ is L-colorable.*

Proof. We first label A_1, A_2, A_3, A_4 by color 1, label B_1, B_2 by color 2 and label B_3, B_4 by color 3. Then the remaining vertices form $K_{4,4}$. Let L' be the list assignment of $K_{4,4}$ which is obtained from L by removing color 1, color 2 and color 3.

If a color appears in three lists in A_5, A_6, A_7, A_8 or three lists in B_5, B_6, B_7, B_8, then $K_{8,8}$ is L-colorable by Theorem 4.2.3. Suppose that each color appears in at most two lists in A_5, A_6, A_7, A_8 and at most two lists in B_5, B_6, B_7, B_8.

For the worst case, we suppose that both color 2 and color 3 appear in two lists in A_5, A_6, A_7, A_8.

Case 1. $2 \in A_5, A_6$ *and* $3 \in A_7, A_8$.

Then $L_{8(i)}$ has a 3-coloring, namely $\{1, 2, 3\}$. If 123 is not a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Without loss of generality, we suppose $B_1 = 123$. Hence, $K_{8,8}$ is L-colorable by Theorem 4.2.5.

Case 2. $2 \in A_5, A_6$ *and* $3 \in A_5, A_7$.

Let $A_5 = 234$. Then $A_5 - 23$ which is a list in $L'_{4(i)}$ has only one color left. Hence, we label A_5 by color 4. The remaining vertices forms $K_{3,4}$. Let L'' be the list assignment of $K_{3,4}$ which is obtained form L' by removing color 4. Since each color appears in at most two lists of B_5, B_6, B_7, B_8, we let $4 \in B_5, B_6$ for the worst case. Then $L''_3 = \{A_6 - 2, A_7 - 3, A_8\}$ and $L''_4 = \{B_5 - 4, B_6 - 4, B_7, B_8\}$. Thus, $K_{3,4}$ is L''-colorable by Lemma 4.2.7. Therefore, $K_{8,8}$ is L-colorable.

Case 3. $2 \in A_5, A_6$ *and* $3 \in A_5, A_6$.

Let $A_5 = 234$ and $A_6 = 235$. Then $A_5 - 23$ and $A_6 - 23$ which are lists in $L'_{4(i)}$

have only one color left. Hence, we label A_5 and A_6 by color 4 and color 5, respectively. The remaining vertices form $K_{2,4}$. Let L'' be the list assignment of $K_{2,4}$ which is obtained form L' by removing color 4 and color 5.

Since each color appears in at most two lists of B_5, B_6, B_7, B_8, we obtain that $|B_5 - 23| + |B_6 - 23| + |B_7 - 23| + |B_8 - 23| \geq |B_5| + |B_6| + |B_7| + |B_8| - 2 \cdot 2 = 4 \cdot 3 - 2 \cdot 2 = 8$. Thus, $K_{2,4}$ is L''-colorable by Lemma 4.2.6. Therefore, $K_{8,8}$ is L-colorable. $\square$

Lemma 4.2.9. *Let L be a list assignment of $K_{4,4}$ such that $L_{4(i)} = \{A_1, A_2, A_3, A_4\}$ and $L_{4(ii)} = \{B_1, B_2, B_3, B_4\}$. If $|A_1| = |A_2| = |B_1| = |B_2| = 2$, $|A_3| = |A_4| = |B_3| = |B_4| = 3$ and $A_1 \cap A_2 = A_3 \cap A_4 = \varnothing$, then $K_{4,4}$ is L-colorable.*

Proof. If A_1, A_2, A_3, A_4 are mutually disjoint, then we apply Strategy A for $L_{4(i)}$. Suppose that A_1, A_2, A_3, A_4 are not mutually disjoint. Without loss of generality, suppose that $p = 1$.

If $L_{4(i)}$ has a coloring which has no subset that is a list in $L_{4(ii)}$, then $K_{4,4}$ is L-colorable by Lemma 3.2.1. Suppose that every coloring of $L_{4(i)}$ has a subset that is a list in $L_{4(i)}$. Then $r, s \neq 1$. In the next three cases, we will prove that $1 \in B_1, B_2$.

Case 1. $r, s \in \{4, 5, 6\}$.

Then $\{1, r\}$ and $\{1, s\}$ are 2-colorings of $L_{4(i)}$. Hence, we suppose that $B_1 = 1r, B_2 = 1s$.

Case 2. $r \in \{4, 5, 6\}$ but $s \notin \{4, 5, 6\}$.

Since $\{1, r\}$ is a 2-coloring of $L_{4(i)}$, let $B_1 = 1r$. Since $\{1, s, 4\}, \{1, s, 5\}$ and $\{1, s, 6\}$ are 3-colorings of $L_{4(i)}$ but we have only one list of size 2 and two lists of size 3, let $B_2 = 1s$.

Case 3. $r, s \notin \{4, 5, 6\}$.

Then $L_{4(i)}$ has at least six 3-colorings, namely $\{1, r, 4\}$, $\{1, r, 5\}$, $\{1, r, 6\}$, $\{1, s, 4\}$,

$\{1, s, 5\}$ and $\{1, s, 6\}$. If all six 3-colorings have a subset that is a list in $L_{4(ii)}$, then $B_1 = 1r, B_2 = 1, s$.

We label B_1, B_2 by color 1 and the remaining vertices can be colored by Lemma 4.2.6; hence, $K_{4,4}$ is L-colorable. $\square$

Theorem 4.2.10. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$, $2 \in A_5, A_6$ and $A_7 \cap A_8 = \varnothing$, then $K_{8,8}$ is L-colorable.*

Proof. If $2 \in A_7$ or $2 \in A_8$, then $K_{8,8}$ is L-colorable by Theorem 4.2.3. Let $A_7 = 345$ and $A_8 = 678$.

Case 1. *No list in $L_{8(ii)}$ contains both color 1 and color 2.*

Then we label A_1, A_2, A_3, A_4 by color 1 and label A_5, A_6 by color 2. The remaining vertices form $K_{2,8}$. Let L' be the list assignment of $K_{2,8}$ which is obtained from L by removing color 1 and color 2. Then we apply Strategy A for L'_2.

Case 2. *Only one list in $L_{8(ii)}$ contains both color 1 and color 2.*

Let $1, 2 \in B_1$. Then $L_{8(i)}$ has at least nine 4-colorings, namely $\{1, 2, 3, 6\}$, $\{1, 2, 3, 7\}$, $\{1, 2, 3, 8\}$, $\{1, 2, 4, 6\}$, $\{1, 2, 4, 7\}$, $\{1, 2, 4, 8\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 5, 7\}$ and $\{1, 2, 5, 8\}$. If one of such 4-colorings has no list that is a subset in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable. Suppose that such 4-colorings has subset that is a list in $L_{8(ii)}$.

Case 2.1 $B_1 \cap \{3, 4, 5, 6, 7, 8\} = \varnothing$.

Then we label A_1, A_2, A_3, A_4 by color 1, label A_5, A_6 by color 2 and label B_1 by the remaining color. Then we apply Strategy A similar to case 1. Thus, the proof is done.

Case 2.2 $B_1 \cap \{3, 4, 5, 6, 7, 8\} \neq \varnothing$.

Then we suppose that $3 \in B_1$. That is, $B_1 = \{1, 2, 3\}$. Consider such 4-colorings, $L_{8(i)}$ has six 4-colorings not containing 3. Since the remaining seven lists do not contain both color 1 and color 2, we need six lists to be a subset of six 4-colorings. That is, we may suppose that $1 \in B_2, B_3, B_4$ and $2 \in B_5, B_6, B_7$. Hence, $K_{8,8}$ is L-colorable by Theorem 4.2.3.

Case 3. *Exactly two lists in $L_{8(ii)}$ contain both color 1 and color 2.*

Let $1, 2 \in B_1, B_2$. Similar to Case 2, we suppose that nine 4-colorings of $L_{8(i)}$ have a subset that is a list in $L_{8(ii)}$.

Case 3.1 $B_1 \cap A_7 = \varnothing$ *and* $B_1 \cap A_8 = \varnothing$.

Then we label A_1, A_2, A_3, A_4 by color 1, label A_5, A_6 by color 2 and label B_1 by the remaining color. Similar to Case 2, we can prove that the remaining vertices can be colored.

Case 3.2 $B_2 \cap A_7 = \varnothing$ *and* $B_2 \cap A_8 = \varnothing$.

Similar to Case 3.1.

Case 3.3 $B_1 \cap A_7 \neq \varnothing$ *and* $B_2 \cap A_7 \neq \varnothing$.

Suppose $B_1 = \{1, 2, 3\}$ and $B_2 = \{1, 2, 4\}$. Then $\{1, 2, 5, 6\}, \{1, 2, 5, 7\}$ and $\{1, 2, 5, 8\}$ do no contain B_1 or B_2 as a subset. Hence, we need three more lists to be a subset of such three 4-colorings. If $B_3 = 156, B_4 = 157$ or $B_3 = 157, B_4 = 158$ or $B_3 = 157, B_4 = 158$, then the proof is done by Theorem4.2.8. Hence, we suppose that $B_3 = 156$ and $B_4 = 257, B_5 = 258$. Then we label A_1, A_2, A_3, A_4 by color 1 and label B_1, B_2, B_4, B_5 by color 2. Lemma 4.2.9 guarantee that the remaining vertices can be colored.

Case 3.4 $B_1 \cap A_7 \neq \varnothing$ *and* $B_2 \cap A_8 \neq \varnothing$.

Suppose $B_1 = \{1, 2, 3\}$ and $B_2 = \{1, 2, 6\}$. Then $\{1, 2, 4, 7\}$, $\{1, 2, 4, 8\}$, $\{1, 2, 5, 7\}$ and $\{1, 2, 5, 8\}$ do no contain B_1 or B_2 as a subset. Hence, we need four more lists to be a subset of such three 4-colorings. That is, we may suppose that $1 \in B_3, B_4$ and $2 \in B_5, B_6$. Then we label A_1, A_2, A_3, A_4 by color 1 and label B_1, B_2, B_4, B_5 by color 2. Lemma 4.2.9 guarantee that the remaining vertices can be colored.

Case 3.5 *$B_1 \cap A_8 \neq \varnothing$ and $B_2 \cap A_8 \neq \varnothing$.*

Similar to Case 3.3.

Case 3.6 *$B_1 \cap A_8 \neq \varnothing$ and $B_2 \cap A_7 \neq \varnothing$.*

Similar to Case 3.4.

Case 4. *Exactly three lists in $L_{8(ii)}$ contain both color 1 and color 2.*

Then $K_{8,8}$ is L-colorable by Theorem 4.2.5. □

Theorem 4.2.11. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$, $2, 3 \in A_5, A_6$ and $4, 5 \in A_7, A_8$, then $K_{8,8}$ is L-colorable.*

Proof. Notice that $L_{8(i)}$ has at least four 3-colorings, namely $\{1, 2, 4\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{1, 3, 5\}$. If one of such 4-colorings is not a list in $L_{8(ii)}$ then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Hence, we suppose that $B_1 = 124, B_2 = 125, B_3 = 134, B_4 = 135$. Therefore $K_{8,8}$ is L-colorable by Theorem 4.2.8. □

Lemma 4.2.12. *Let L be a list assignment of $K_{3,6}$ such that $L_3 = \{A_1, A_2, A_3\}$ and $L_6 = \{B_1, B_2, B_3, B_4, B_5, B_6\}$. If $A_1 = \{1, 2\}, A_2 = \{3, 4\}, A_3 = \{5, 6\}$ and $|B_1| = |B_2| = 2$, $|B_3| = |B_4| = |B_5| = |B_6| = 3$, then $K_{3,6}$ is L-colorable or $L_6 = \{13, 14, 235, 236, 245, 246\}$ or $L_6 = \{13, 24, 145, 146, 235, 236\}$ or $L_6 = \{13, 45, 146, 235, 236, 246\}$ up to renaming the colors.*

Proof. Suppose that $K_{8,8}$ is not L-colorable.

Notice that L_3 has at least eight 3-colorings, namely $\{1, 3, 5\}$, $\{1, 3, 6\}$, $\{1, 4, 5\}$, $\{1, 4, 6\}$, $\{2, 3, 5\}$, $\{2, 3, 6\}$, $\{2, 4, 5\}$, $\{2, 4, 6\}$. Since $K_{3,6}$ is not L-colorable, all 3-coloring of L_3 have a subset that is a list in L_6 by Lemma 3.2.1.

If a list in L_6 contains a color which is not in any A_i for $i = 1, 2, 3$, then at most seven 3-colorings of L_3 have a subset that is a list in L_6. Hence, we suppose that every list in L_6 is a subset of $\{1, 2, 3, 4, 5, 6\}$.

If $A_1 = B_1$ then B_1 is not a subset of any 3-coloring of L_3; hence, at most six 3-colorings of L_3 has a subset that is a list in L_6. Hence, we suppose that $A_i \neq A_j$ for $i = 1, 2, 3$ and $j = 1, 2$.

Since B_1 must be a subset of a 3-coloring of L_3, we may suppose that $B_1 = 13$. That is, two 3-colorings of L_3, namely $\{1, 3, 5\}$ and $\{1, 3, 6\}$, contain B_1 as a subset. Without loss of generality, we divide the possibility of B_2 into four cases.

Case 1. $B_2 = 14$.

That is, two 3-colorings of L_3, namely $\{1, 4, 5\}$ and $\{1, 4, 6\}$, contain B_2 as a subset. Hence, the remaining four 3-colorings of L_3 must be B_3, B_4, B_5, B_6. Therefore, $L_6 = \{13, 14, 235, 236, 245, 246\}$.

Case 2. $B_2 = 15$.

That is, two 3-colorings of L_3, namely $\{1, 3, 5\}$ and $\{1, 4, 5\}$, contain B_2 as a subset. Now, the remaining five 3-colorings do not contain B_1 or B_2 as a subset but we have only four lists of size 3 left in L_6. It is a contradiction to the assumption that $K_{3,6}$ is not L-colorable.

Case 3. $B_2 = 24$.

That is, two 3-colorings of L_3, namely $\{2, 4, 5\}$ and $\{2, 4, 6\}$, contain B_2 as a subset. Hence, the remaining four 3-colorings of L_3 must be B_3, B_4, B_5, B_6. Therefore, $L_6 = \{13, 24, 145, 146, 235, 236\}$.

Case 4. $B_2 = 25$.

That is, two 3-colorings of L_3, namely $\{2, 3, 5\}$ and $\{2, 4, 5\}$, contain B_2 as a subset. Hence, the remaining four 3-colorings of L_3 must be B_3, B_4, B_5, B_6. Therefore, $L_6 = \{13, 45, 146, 235, 236, 246\}$. □

Theorem 4.2.13. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$ and $2, 3 \in A_5 \cap A_6$, then $K_{8,8}$ is L-colorable.*

Proof. If $A_7 \cap A_8$ is an empty set, the proof is done by Theorem 4.2.10 and if $|A_7 \cap A_8| \geq 2$, the proof is done by Theorem 4.2.11. Hence, we suppose that $|A_7 \cap A_8| = 1$. Let $A_7 = \{6, p, q\}$ and $A_8 = \{6, r, s\}$. If a color appears in three lists of A_5, A_6, A_7, A_8, then the proof is done by Theorem 4.2.3. We suppose each color appears in at most two lists of A_5, A_6, A_7, A_8. Hence, $2, 3 \notin A_7, A_8$.

If there exists a coloring in $L_{8(i)}$ such that has no subset that is a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that every coloring in $L_{8(i)}$ has a subset that is a list in $L_{8(ii)}$. Since $\{1, 2, 6\}$ and $\{1, 3, 6\}$ are 3-colorings of $L_{8(i)}$, we may suppose that $B_1 = 126$ and $B_2 = 136$. Since $\{1, 4, 5, 6\}$ is a 4-coloring of $L_{8(i)}$, we may suppose that 145 or 146 or 156 is a list in $L_{8(ii)}$. If 146 or 156 is a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Theorem 4.2.5. Hence, we suppose that $B_3 = 145$. We first label A_1, A_2, A_3, A_4 by color 1 and label B_1 and B_2 by color 6. Then the remaining vertices form $K_{4,6}$. Let L' be the list assignment of $K_{4,6}$ which is obtained from L by removing color 1 and color 6. Then we define the new list assignment L^* of $K_{3,6}$ such that $L^*_3 = \{23, pq, rs\}$ and $L^*_6 = L'_6$. It is easy to see that if $K_{3,6}$ is L^*-colorable, then $K_{4,6}$ is L'-colorable.

Case 1. *Color* 1 *appears only in* B_1, B_2, B_3.

Then $L'_4 = \{234, 235, pq, rs\}$ and $L'_6 = \{45, B_4, B_5, B_6, B_7, B_8\}$. Then we apply Strategy A for L^*_3 to prove that $K_{3,6}$ is L^*-colorable; hence, $K_{4,6}$ is L'-colorable.

Case 2. *Color* 1 *appears in one of* B_4, B_5, B_6, B_7, B_8.

Suppose that $1 \in B_4$. Then $L'_4 = \{234, 235, pq, rs\}$ and $L'_6 = \{45,\ B_4 - 1,\ B_5,\ B_6,\ B_7,\ B_8\}$. If a color appears in three lists in B_5, B_6, B_7, B_8 then the proof is done by Theorem 4.2.3. If $|B_5 \cap B_6| \geq 2$ and $|B_7 \cap B_8| \geq 2$ then the proof is done by Theorem 4.2.11. Hence, we suppose that each color appears in at most two lists in B_5, B_6, B_7, B_8 and ($|B_5 \cap B_6| \leq 1$ or $|B_7 \cap B_8| \leq 1$). Then L^*_6 cannot rename color to be $\{13, 14, 235, 236, 245, 246\}$ or $\{13, 24, 145, 146, 235, 236\}$ or

$\{13, 45, 146, 235, 236, 246\}$. By Lemma 4.2.12, $K_{3,6}$ is L^*-colorable; hence, $K_{8,8}$ is L-colorable. $\square$

For the next lemma, the alphabet A represents 10 and the alphabet B represents 11.

Lemma 4.2.14. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$ and $A_5 = 245, A_6 = 267, A_7 = 389, A_8 = 3AB$, then $K_{8,8}$ is L-colorable.*

Proof. Notice that $\{1, 2, 3\}$ is a 3-coloring of $L_{8(i)}$. If 123 is not a list in $L_{8(i)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that $B_1 = 123$.

Case 1. *At least two lists in $L_{8(ii)}$ contain both color 1 and color 2.*

If three lists in $L_{8(ii)}$ contains both color 1 and color 2, then the proof is done by Theorem 4.2.5. Suppose that $1, 2 \in B_2$ and no list from $B_3, B_4, \ldots, B_8$ contains both color 1 and color 2. We label A_1, A_2, A_3, A_4 by color 1. Let L' be the list assignment of $K_{4,8}$ which is obtained from L by removing color 1. For the worst case, suppose that $1 \in B_1, B_2, B_3, B_4$. Then $L'_4 = \{A_5, A_6, A_7, A_8\}$ and $L'_8 = \{B_1 - 1, B_2 - 1, B_3 - 1, B_4 - 1, B_5, B_6, B_7, B_8\}$.

Case 1.1 $3 \in B_3 \cup B_4$.

Suppose that $3 \in B_3$. Then we label B_1, B_2 by color 2 and label B_3 by color 3. Let L'' be the list assignment of $K_{4,5}$ which is obtained from L by removing color 2 and color 3. Then we apply Strategy A$'$ for L'_4.

Case 1.2 $3 \in B_5 \cup B_6 \cup B_7 \cup B_8$.

Suppose that $3 \in B_5$. Then we label B_1, B_2 by color 2 and label B_5 by color 3. Let L'' be the list assignment of $K_{4,5}$ which is obtained from L by removing color 2 and color 3. Then we apply Strategy A$'$ for L'_4.

Case 1.3 $3 \notin B_3 \cup B_4 \cup \ldots \cup B_8$.

Then we label A_1, A_2, A_3, A_4 by color 1, label B_1, B_2 by color 2 and label A_7, A_8 by color 3. The remaining vertices can be easily labeled.

Case 2. *At least two lists in $L_{8(ii)}$ contain both color 1 and color 3.*

Similar to Case 1.

Case 3. *No list from $B_2, B_3 \ldots, B_8$ contains both color 1 and color i where $i = 1, 2$.*

Case 3.1 *Color 1 appears in four lists in $L_{8(ii)}$.*

Suppose that $1 \in B_1, B_2, B_3, B_4$. If $L_{8(i)}$ has a coloring which has no subset that is a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that every coloring of $L_{8(i)}$ has a subset that is a list in $L_{8(ii)}$. Notice that $\{1, 2, 8, A\}$, $\{1, 2, 8, B\}$, $\{1, 2, 9, A\}$ and $\{1, 2, 9, B\}$ are 4-colorings of $L_{8(i)}$. If two lists of B_5, B_6, B_7, B_8 have two common colors, then the proof is done by Theorem 4.2.13. Without loss of generality, suppose that $28A, 29B, 18B, 19A \in L_{8(ii)}$. Notice that $\{1, 3, 4, 6\}$, $\{1, 3, 4, 7\}$, $\{1, 3, 5, 6\}$ and $\{1, 3, 5, 7\}$ are 4-colorings of $L_{8(i)}$. Similarly, $346, 357, 147, 156 \in L_{8(ii)}$. It is a contradiction to the fact that $L_{8(ii)}$ contains exactly eight lists.

Case 3.2 *Color 1 appears in at most three lists in $L_{8(ii)}$ and color 2 appears in at most two lists in $L_{8(ii)}$.*

We first label A_1, A_2, A_3, A_4 by color 1. For the worst case, suppose that $1 \in B_1, B_2, B_3$ and $2 \in B_1, B_4$. Then we label A_5, A_6 by color 2 and label B_1 by color 3. The remaining vertices are easily labeled.

Case 3.3 *Color 1 appears in at most three lists in $L_{8(ii)}$ and color 2 appears in at least three lists in $L_{8(ii)}$.*

We first label A_1, A_2, A_3, A_4 by color 1. For the worst case, suppose that $1 \in B_1, B_2, B_3$ and $2 \in B_1, B_4, B_5$. Then we label B_1 by color 3 and label B_4, B_5

by color 2. The remaining vertices form $K_{4,5}$. Let L' be the list assignment of $K_{4,5}$ which is obtained from L by removing color 2 and color 3. Then we apply Strategy A$'$ for L'_4. □

Lemma 4.2.15. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$ and $A_5 = 246, A_6 = 247, A_7 = 358, A_8 = 359$, then $K_{8,8}$ is L-colorable.*

Proof. If $L_{8(i)}$ has a coloring which has no subset that is a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that every coloring of $L_{8(i)}$ has a subset that is a list in $L_{8(ii)}$. Since $\{1,2,3\}$ and $\{1,4,5\}$ are 3-colorings of $L_{8(i)}$, we suppose that $B_1 = 123$ and $B_2 = 145$.

Case 1. *A list from $B_3, B_4, \ldots, B_8$ contains both color 1 and color x for some $x \in \{2,3,4,5\}$.*

Suppose that $1, 2 \in B_3$.

Case 1.1 $1 \notin B_4 \cup B_5 \cup \ldots \cup B_8$.

Notice that $\{1,3,4,7\}$, $\{1,3,5,6\}$, $\{1,3,6,7\}$, $\{1,4,7,8\}$ and $\{1,5,6,9\}$ are 4-colorings of $L_{8(i)}$. Then we suppose that $B_4 = 347$, $B_5 = 356$, $B_6 = 367$, $B_7 = 478$, $B_8 = 569$. Then we label B_1, B_4, B_5, B_6 by color 3, label B_2, B_7 by color 4, label B_3 by color 2 and label B_8 by color 5. Since no list in $L_{8(i)}$ is a subset of $\{2,3,4,5\}$, $K_{8,8}$ is L-colorable.

Case 1.2 $1 \in B_4 \cup B_5 \cup \ldots \cup B_8$.

Suppose that $1 \in B_4$. Notice that $\{1,3,4,7\}$, $\{1,3,5,6\}$ and $\{1,3,6,7\}$ are 4-colorings of $L_{8(i)}$. If $347, 356, 367 \in B_8$, then we apply Theorem 4.2.3. If $4 \in B_4$ or $5 \in B_4$, then we apply Theorem 4.2.8. Otherwise, we suppose that $347, 356$ and 167 are lists in $L_{8(ii)}$. Again, since $\{1,4,7,8\}$ and $\{1,5,6,9\}$ are 4-colorings of $L_{8(i)}$, we suppose that $478, 659 \in B_8$. Since $347, 478 \in L_{8(ii)}$, we apply Theorem 4.2.13.

Case 2. *No list* $B_3, B_4, \ldots, B_8$ *contains both color* 1 *and color* x *for all* $x \in \{2,3,4,5\}$.

Notice that $\{1,2,4,8\}, \{1,2,5,9\}, \{1,3,4,7\}, \{1,3,5,6\}$ are 4-colorings of $L_{8(i)}$. Then each of such 4-coloring has a subset that is a list in $L_{8(ii)}$. Suppose that $B_5 = 348$, $B_6 = 259$, $B_7 = 347$, $B_8 = 356$. Again, $\{1,2,8,9\}$ and $\{1,3,6,7\}$ are 4-colorings of $L_{8(i)}$; hence, we suppose that $B_3 = 189$ and $B_4 = 167$. Finally, we label all lists in $L_{8(i)}$ by color 1, color 4, color 7 and color 8 and all lists in $L_{8(ii)}$ still have available colors. □

Theorem 4.2.16. *Let* L *be a* 3*-list assignment of* $K_{8,8}$ *such that every color appears in at most four lists of each partite set where* $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$*. If* $1 \in A_1, A_2, A_3, A_4$ *and* $A_5 \cap A_7 = \varnothing$*, then* $K_{8,8}$ *is* L*-colorable.*

Proof. If A_5, A_6, A_7 and A_8 are mutually disjoint, we apply Strategy A for L'_4 to guarantee that $K_{4,8}$ is L'-colorable. Suppose that color 2 appears at least two lists in A_5, A_6, A_7, A_8. If $2 \in A_6, A_8$, then $K_{8,8}$ is L-colorable by Theorem 4.2.10. Without loss of generality, let $2 \in A_5, A_6$. Again, if $A_7 \cap A_8 = \varnothing$ then $K_{8,8}$ is L-colorable by Theorem 4.2.10. Hence, we suppose that $3 \in A_7, A_8$, as well.

If $|A_5 \cap A_6| \geq 2$ or $|A_7 \cap A_8| \geq 2$ then $K_{8,8}$ is L-colorable by Theorem 4.2.13. Suppose that $|A_5 \cap A_6| = 1$ and $|A_7 \cap A_8| = 1$. Let $A_5 = 246$, $A_6 = 2pr$, $A_7 = 357$ and $A_8 = 3qs$ where p, q, r, s are distinct colors.

If $q = 2$, $s = 2$, $p = 3$ or $r = 3$, then $K_{8,8}$ is L-colorable by Theorem 4.2.3. Suppose that $q, s \neq 2$ and $p, r \neq 3$.

Case 1. $\{p,r\} \cap \{4,6,\} \neq \varnothing$ or $\{q,s\} \cap \{5,7\} \neq \varnothing$.

Then $K_{8,8}$ is L-colorable by Theorem 4.2.13.

Case 2. $\{p,r\} \cap \{4,6,\} = \{q,s\} \cap \{5,7\} = \varnothing$ *but* $\{p,q,r,s\} \cap \{4,5,6,7\} \neq \varnothing$.

Suppose that $p = 5$.

Case 2.1 $q \notin \{4, 6\}$.

Then $K_{8,8}$ is L-colorable by Theorem 4.2.10.

Case 2.2 $q \in \{4, 6\}$.

Let $q = 4$. If $r = 7$, or $s = 6$ then $K_{8,8}$ is L-colorable by Theorem 4.2.13. Suppose that $r \neq 7$ and $s \neq 6$. Then r, s must be new colors. Let $r = 8$ and $s = 9$. Hence, the proof is done by Lemma 4.2.15.

Case 3. $\{p, q, r, s\} \cap \{4, 5, 6, 7\} = \varnothing$.

The proof is done by Lemma 4.2.14.

Case 4. $p \in \{5, 7\}$ *but* $r \notin \{1, 2, 3, 4, 5, 6, 7\}$.

Suppose that $p = 5$ and $r = 8$. Then $5 \in A_6, A_7$. If $A_5 \cap A_8 = \varnothing$, then $K_{8,8}$ is L-colorable by Theorem 4.2.10. Suppose that $A_5 \cap A_8 \neq \varnothing$. Let $q = 4$ □

Corollary 4.2.17. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$ and there are $A_i, A_j \in \{A_5, A_6, A_7, A_8\}$ such that $|A_i \cap A_j| \neq 1$, then $K_{8,8}$ is L-colorable.*

Proof. It follows from Theorem 4.2.13 and Theorem 4.2.16. □

Theorem 4.2.18. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most four lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$. If $1 \in A_1, A_2, A_3, A_4$ then $K_{8,8}$ is L-colorable unless $\{123, 145, 167, 246, 257, 347, 356\} \subset L_{8(i)}, L_{8(ii)}$ up to renaming colors.*

Proof. If $|A_i \cap A_j| \neq 1$ for some $i, j \in \{5, 6, 7, 8\}$ then $K_{8,8}$ is L-colorable by Corollary 4.2.17. If a color appears in at least three lists in A_5, A_6, A_7, A_8 then $K_{8,8}$ is L-colorable by Theorem 4.2.3. Hence, we suppose that $A_5 = 246, A_6 = 257, A_7 = 347, A_8 = 356$.

If $L_{8(i)}$ has a coloring which has no subset that is a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that each coloring in $L_{8(i)}$ has a subset that is a list in $L_{8(ii)}$. Since $\{1,2,3\},\{1,4,5\},\{1,6,7\}$ are 3-colorings of $L_{8(i)}$, we suppose that $B_1 = 123, B_2 = 145, B_3 = 167$.

Case 1. *Color* 1 *appears in at most three lists in* $L_{8(ii)}$.

Consider 4-colorings of $L_{8(i)}$, $\{1,2,4,6\},\{1,2,7,5\},\{1,3,4,7\}$ and $\{1,3,6,5\}$. Note that they do not contain B_1, B_2 or B_3 as a subset. Hence, $246, 257, 347, 356$ must be lists in $L_{8(ii)}$, say $B_5 = 246, B_6 = 257, B_7 = 347, B_8 = 356$.

If $123, 145, 167 \in L_{8(i)}$, then the proof is done because we have $\mathcal{F} \subset L_{8(i)}, L_{8(ii)}$. Suppose that 123 is not a list in $L_{8(i)}$.

Case 1.1 $2 \in B_4$ or $3 \in B_4$.

Then $\{1,2,3\}$ is a 3-coloring of $L_{8(ii)}$. Since 123 is not a list in $L_{8(i)}$, $K_{8,8}$ is L-colorable by Lemma 3.2.1.

Case 1.2 $2, 3 \notin B_3, B_4$.

Then we label B_1, B_2, B_3 by color 1, label B_5, B_6 by color 2 and label B_7, B_8 by color 3; hence, the remaining vertices can be easily labeled.

Case 2. *Color* 1 *appears in exactly four lists in* $L_{8(ii)}$.

Let $1 \in B_4$. Similar to Case 1, $\{1,2,4,6\}$, $\{1,2,7,5\}$, $\{1,3,4,7\}$, $\{1,3,6,5\}$ are 4-colorings of $L_{8(i)}$ which do not contain B_1 or B_2 or B_3 as a subset. The list B_4 is a subset of at most one of such 4-colorings. Hence, at least three of such 4-colorings do not contain B_1, B_2, B_3, B_4 as a subset. Without loss of generality, suppose that $\{1,2,4,6\}$, $\{1,2,7,5\}$, $\{1,3,4,7\}$ do not contain B_1, B_2, B_3, B_4 as a subset. Then we suppose that $B_5 = 246, B_6 = 257, B_7 = 347$.

Again, if $|B_i \cap B_j| \neq 1$ for some $i, j \in \{5,6,7,8\}$ then $K_{8,8}$ is L-colorable by Corollary 4.2.17. If a color appears in at least three lists in B_5, B_6, B_7, B_8 then $K_{8,8}$ is L-colorable by Theorem 4.2.3. Hence, we suppose that $|B_i \cap B_j| = 1$ for all $i, j \in \{5,6,7,8\}$ and each color appears in at most two lists in B_5, B_6, B_7, B_8. Hence, $B_8 = 356$. Therefore, $\mathcal{F} \subset L_{8(i)}, L_{8(ii)}$. $\square$

Next, we focus on a list assignment L of $K_{8,8}$ such that each color appears in at most three lists in each partite set.

Theorem 4.2.19. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1$ but $1 \notin B_2, B_3 \ldots, B_8$, then $K_{8,8}$ is L-colorable.*

Proof. We first label A_1, A_2, A_3 by color 1. Then the remaining vertices form $K_{5,8}$. Let L' be the list assignment of $K_{5,8}$ which is obtained from L by removing color 1. Then $L'_5 = \{A_4, A_5, A_6, A_7, A_8\}$ and $L'_8 = \{23, B_2, B_3, \ldots, B_8\}$.

Let L^* be the list assignment of $K_{6,10}$ such that $L^*_6 = L'_5 \cup \{xyz\}$ where x, y, z are new colors and $L^*_{10} = \{x23, y23, z23, B_2, B_3, \ldots, B_{10}\}$. It is easy to see that if $K_{6,10}$ is L^*-colorable, then $K_{5,8}$ is L'-colorable By Lemma 3.3.4, $K_{6,10}$ is L^*-colorable; hence, $K_{5,8}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable. $\square$

Lemma 4.2.20. *Let L be a list assignment of $K_{4,5}$ where $L_4 = \{A_1, A_2, A_3, A_4\}$ and $L_5 = \{B_1, B_2, B_3, B_4, B_5\}$. If $|A_i| = 2$ for $i = 1, 2, 3, 4$ and $|B_j| = 3$ for $j = 1, 2, 3, 4, 5$, then $K_{4,5}$ is L-colorable.*

Proof. If all lists in L_4 are mutually disjoint, then we apply Strategy A$'$ for L_4. Hence, we suppose that $1 \in A_1, A_2$. If L_4 has a coloring which has no subset that is list in L_5, then the proof is finished by Lemma 3.2.1. Suppose that every coloring of L_4 has a subset that is a list in L_5. Since L_5 has no list of size 2, we suppose L_4 has no 2-coloring. Then we suppose that $A_3 = 23, A_4 = 45$. Moreover, we may suppose that $B_1 = 124, B_2 = 125, B_3 = 134, B_4 = 135$. Then we label B_1, B_2, B_3, B_4 by color 1 and the remaining vertices are easily colored. Therefore, $K_{4,5}$ is L-colorable. $\square$

Lemma 4.2.21. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1, B_2$ and $2 \in A_4, A_5, A_6$ but $1 \notin B_3, B_4, \ldots, B_8$, then $K_{8,8}$ is L-colorable.*

Proof. **Case** 1. $A_7 \cap A_8 = \varnothing$.

Then we label A_1, A_2, A_3 by color 1 and label A_4, A_5, A_6 by color 2. The remaining vertices form $K_{2,8}$. Let L' be the list assignment of $K_{2,8}$ which is obtained from L by removing color 1 and color 2. Since $A_7 \cap A_8 = \varnothing$, we apply Strategy A for L'_2.

Case 2. $|A_7 \cap A_8| = 1$.

Let $3 \in A_7 \cap A_8$. Then $\{1, 2, 3\}$ is a 3-coloring of $L_{8(i)}$. If 123 is not a list in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that $B_1 = 123$. Then we label A_1, A_2, A_3 by color 1, label A_4, A_5, A_6 by color 2 and label B_1 by color 3. If color 3 is in other lists in $L_{8(ii)}$, then we label the lists by color 3. For the worst case, we suppose that $3 \notin B_2, B_3, \ldots, B_8$. Let L' be the list assignment of $K_{2,7}$ which is obtained from L by removing color 1, color 2 and color 3. Now, we apply Strategy A for L'_2.

Case 3. $|A_7 \cap A_8| = 2$.

Let $A_7 = 345$ and $A_8 = 346$. Then $\{1, 2, 3\}$ and $\{1, 2, 4\}$ are 3-colorings of $L_{8(i)}$. If 123 or 124 is not a list in $L_{8(ii)}$ then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Suppose that $B_1 = 123$ and $B_2 = 124$. Again, since $\{1, 2, 5, 6\}$ is a 4-coloring of $L_{8(i)}$, we suppose that $\{1, 2, 5, 6\}$ has a list that is a subset in $L_{8(ii)}$. Since color 1 appears in exactly two lists of $L_{8(ii)}$, we suppose that $B_3 = 256$.

Then we label A_1, A_2, A_3 by color 1 and label B_1, B_2, B_3 by color 2. The remaining vertices form $K_{5,5}$. Let L' be the list assignment of $K_{5,5}$ which is obtained from L by removing color 1 and color 2. Then $L'_{5(i)} = \{A_4 - 2, A_5 -$

$2, A_6 - 2, 345, 346\}$ and $L'_{5(ii)} = \{B_4, B_5, B_6, B_7, B_8\}$.

Let L^* be the list assignment of $K_{4,5}$ such that $L^*_4 = \{A_4 - 2, A_5 - 2, A_6 - 2, 34\}$ and $L^*_5 = L'_{5(ii)}$. It is easy to see that if $K_{4,5}$ is L^*-colorable, then $K_{5,5}$ is L'-colorable. By Lemma 4.2.20, $K_{4,5}$ is L^*-colorable; hence, $K_{5,5}$ is L'-colorable. That is, $K_{8,8}$ is L-colorable. □

Theorem 4.2.22. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1, B_2$ but $1 \notin B_3, B_4 \ldots, B_8$, then $K_{8,8}$ is L-colorable.*

Proof. **Case** 1. *A color appears in three lists in A_4, A_5, A_6, A_7, A_8.*
Then $K_{8,8}$ is L-colorable by Lemma 4.2.21.

Case 2. *A color appears in two lists in A_4, A_5, A_6, A_7, A_8 but it is not in $B_1 \cup B_2$.*
Let L^* be the new list assignment of $K_{8,8}$ which is obtained from L by changing color 2 to color 1. It is easy to see that if $K_{8,8}$ is L^*-colorable, then $K_{8,8}$ is L-colorable. By Strategy D, $K_{8,8}$ is L^*-colorable; hence, $K_{8,8}$ is L-colorable.

Case 3. *Every color which appears in two lists in A_4, A_5, A_6, A_7, A_8 must be in $B_1 \cup B_2$ and no color appears in three lists in A_4, A_5, A_6, A_7, A_8.*
Notice that $B_1 \cup B_2 - \{1\}$ has at most four colors. Thus, at most four colors appear in two lists of A_4, A_5, A_6, A_7, A_8. Since $|A_4| + |A_5| + |A_6| + |A_7| + |A_8| = 15$, we have at least 11 colors in A_4, A_5, A_6, A_7, A_8. Since $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 - \{1\}$ is a set of size 10, there exists a color which is not a color in $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2$. Let $2 \in A_4$ but $2 \notin A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2$. Similar to Case 2, we define the new list assignment L^* of $K_{8,8}$ which is obtained from L by changing color 2 to color 1. Then color 1 appears in four lists in $L^*_{8(i)}$. By Theorem 4.2.16, $K_{8,8}$ is L^*-colorable. That is, $K_{8,8}$ is L-colorable. □

Lemma 4.2.23. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1, B_2, B_3$ and $2 \in A_4$, A_5, A_6, B_4, B_5, B_6 then $K_{8,8}$ is L-colorable.*

Proof. We define the new list assignment L^* of $K_{8,8}$ which is obtained from L by changing color 2 to color 1. It is easy to see that if $K_{8,8}$ is L^*-colorable, then $K_{8,8}$ is L-colorable. By Strategy C for $L_{8(i)}$, $K_{8,8}$ is L^*-colorable. That is, $K_{8,8}$ is L-colorable. □

Lemma 4.2.24. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1, B_2, B_3$ and $2 \in A_4$, A_5, A_6, B_1, B_4, B_5 then $K_{8,8}$ is L-colorable.*

Proof. If a coloring of $L_{8(i)}$ (or $L_{8(ii)}$) has no subset that is a list in $L_{8(ii)}$ (or $L_{8(i)}$), then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Hence, we suppose that every coloring of $L_{8(i)}$ (or $L_{8(ii)}$) has a subset that is a list in $L_{8(ii)}$ (or $L_{8(i)}$).

Case 1. $|A_7 \cap A_8| \geq 2$.

Let $3, 4 \in A_7, A_8$. Since $\{1,2,3\}$ and $\{1,2,4\}$ are 3-colorings of $L_{8(i)}$, we have $123, 124 \in L_{8(ii)}$. It is contradiction to the fact that only one list in $L_{8(ii)}$ contains both color 1 and color 2.

Case 2. $|A_7 \cap A_8| = 1$.

Let $A_7 = 345$ and $A_8 = 367$. Since $\{1,2,3\}$ is a 3-coloring of $L_{8(i)}$, we suppose that $B_1 = 123$. Since $\{1,2,4,6\}, \{1,2,4,7\}, \{1,2,5,6\}$ and $\{1,2,5,7\}$ are 4-colorings of $L_{8(i)}$, we obtain that $\{B_2 - 1, B_3 - 1, B_4 - 2, B_5 - 2\} = \{46, 47, 56, 57\}$.

Case 2.1. *B_6, B_7, B_8 are not mutually disjoint.*

We suppose that $w \in B_6, B_7$. If $w \in B_8$ then we label all lists in $L_{8(ii)}$ by color

1, color 2 and color w; hence, the proof is done. Suppose that $B_8 = xyz$ where w, x, y, z are distinct colors. Then $\{1, 2, w, x\}, \{1, 2, w, y\}$ and $\{1, 2, w, z\}$ are 4-colorings of $L_{8(ii)}$. Since no list in $L_{8(i)}$ contains both color 1 and color 2, we have $\{wx, wy, wz\} \subset \{A_1 - 1, A_2 - 1, A_3 - 1, A_4 - 2, A_5 - 2, A_6 - 2\}$. That is, color w appears in at least three lists in $L_{8(i)}$. Since each color appears in at most three lists in $L_{8(i)}$, w is not color 3. Since each color appears in at most three lists in $L_{8(ii)}$, w is not an element in $\{1, 2, 4, 5, 6, 7\}$. Hence, w appears in exactly two lists in $L_{8(ii)}$ and in exactly three lists in $L_{8(i)}$. By Theorem 4.2.22, $K_{8,8}$ is L-colorable.

Case 2.2. B_6, B_7, B_8 *are mutually disjoint.*

We label B_1, B_2, B_3 by color 1 and label B_4, B_5 by color 2. The remaining vertices form $K_{8,3}$. Let L' be the list assignment of $K_{8,3}$ which is obtained from L by removing color 1 and color 2. Since B_6, B_7, B_8 are mutually disjoint, we apply Strategy A for L'_3.

Case 3. $A_7 \cap A_8 = \varnothing$.

We label A_1, A_2, A_3 by color 1 and label A_4, A_5, A_6 by color 2. The remaining vertices form $K_{2,8}$. Let L'' be the list assignment of $K_{2,8}$ which is obtained from L by removing color 1 and color 2. Since A_7 and A_8 are disjoint, we apply Strategy A for L''_2. $\square$

Lemma 4.2.25. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1, B_2, B_3$ and $2 \in A_4$, A_5, A_6, B_1, B_2, B_4, then $K_{8,8}$ is L-colorable.*

Proof. If a coloring of $L_{8(i)}$ (or $L_{(ii)}$) has no subset that is a list in $L_{8(ii)}$ (or $L_{8(i)}$), then $K_{8,8}$ is L-colorable by Lemma 3.2.1. Hence, we suppose that every coloring of $L_{8(i)}$ (or $L_{8(ii)}$) has a subset that is a list in $L_{8(ii)}$ (or $L_{8(i)}$). Notice that if color 3 appears in three lists in $L_{8(ii)}$ and color 3 appears in two lists in $L_{8(i)}$, then the proof is finished by Theorem 4.2.22 and if color 3 appears in three lists in $L_{8(ii)}$ and color 3 appears in three lists in $L_{8(i)}$, then the proof is finished by Lemma 4.2.24. Suppose that color 3 appears in two lists in $L_{8(ii)}$.

Case 1. $A_7 \cap A_8 = \varnothing$.

We label A_1, A_2, A_3 by color 1 and label A_4, A_5, A_6 by color 2. The remaining vertices form $K_{2,8}$. Let L' be the list assignment of $K_{2,8}$ which is obtained from L by removing color 1 and color 2. Since A_7 and A_8 are disjoint, we apply Strategy A for L'_2.

Case 2. $|A_7 \cap A_8| = 1$.

Let $A_7 = 345$ and $A_8 = 367$. Since $\{1,2,3\}$ is a 3-coloring of $L_{8(i)}$, we suppose that $B_1 = 123$. Since $\{1,2,4,6\}, \{1,2,4,7\}, \{1,2,5,6\}$ and $\{1,2,5,7\}$ are 4-colorings of $L_{8(i)}$, we suppose that $B_2 = 124, B_3 = 156$ and $B_4 = 257$.

We label A_1, A_2, A_3 by color 1 and label B_1, B_2, B_4 by color 2. The remaining vertices form $K_{5,5}$. Let L' be the list assignment of $K_{5,5}$ which is obtained from L by removing color 1 and color 2. That is, $L'_{5(i)} = \{A_4-2, A_5-2, A_6-2, 345, 367\}$ and $L'_{5(ii)} = \{56, B_5, B_6, B_7, B_8\}$.

Case 2.1 $A_4 - 2, A_5 - 2, A_6 - 2$ *have a common color, say* p.

Then we label $A_4 - 2, A_5 - 2, A_6 - 2$ by color p and label A_7, A_8 by color 3. Since the remaining vertices in another partite set still have available colors, $K_{5,5}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable.

Case 2.2 $A_4 - 2, A_5 - 2, A_6 - 2$ *have no common color and not mutually disjoint.*

Let $p \in A_4 - 2, A_5 - 2$ and $A_7 = 2qr$.

Case 2.2.1 $p = 5$ and $6 \in \{q, r\}$.

Suppose that $q = 6$. Notice that $\{3, 5, r\}$ and $\{7, 5, r\}$ are 3-colorings of $L_{5(i)}$.

However, at most one of such 3-colorings is a list of $L_{5(ii)}$ because each color appears in at most three lists in $L_{8(ii)}$. Hence, $K_{5,5}$ is L'-colorable by Lemma 3.2.1. Therefore, $K_{8,8}$ is L-colorable.

Case 2.2.2 $p \neq 5$ or $6 \notin \{q, r\}$.

Then $\{3, p, q\}$ and $\{3, q, r\}$ are 3-colorings of $L'_{5(i)}$. Since color 3 appears in at most two lists in $L_{8(ii)}$ and $3 \in B_1$, at least one of such 3-colorings is not a list in $L_{5(ii)}$. Again, by Lemma 3.2.1, $K_{5,5}$ is L'-colorable by Lemma 3.2.1. Therefore, $K_{8,8}$ is L-colorable.

Case 2.3 $A_4 - 2, A_5 - 2, A_6 - 2$ *are mutually disjoint.*

We label A_7, A_8 by color 3. Then the remaining vertices form $K_{3,5}$. Recall that color 3 appears in at most two lists in $L_{8(ii)}$; suppose that $3 \in B_5$. Let L'' be the list assignment of $K_{3,5}$ which is obtained from L' by removing color 3. Then $L''_3 = \{A_4 - 2, A_5 - 2, A_6 - 2\}$ and $L''_5 = \{56, B_5 - 3, B_6, B_7, B_8\}$. Then we apply Strategy A for L''_3.

Case 3. $|A_7 \cap A_8| = 2$.

Let $A_7 = 345$ and $A_8 = 346$. We label A_1, A_2, A_3 by color 1 and label B_1, B_2, B_4 by color 2. Then the remaining vertices form $K_{5,5}$. Let L' be the list assignment of $K_{5,5}$ which is obtained from L by removing color 1 and color 2. Then $L'_{5(i)} = \{A_4 - 2, A_5 - 2, A_6 - 2, 345, 346\}$ and $L'_{5(ii)} = \{B_3 - 1, B_5, B_6, B_7, B_8\}$.

We define the new list assignment L^* of $K_{4,5}$ such that $L^*_4 = \{A_4 - 2, A_5 - 2, A_6 - 2, 34\}$ and $L^*_5 = L'_{5(ii)}$. It is easy to see that if $K_{4,5}$ is L^*-colorable, then $K_{5,5}$ is L'-colorable. By Lemma 4.2.20, $K_{4,5}$ is L^*-colorable; hence, $K_{5,5}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable. □

Lemma 4.2.26. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3, B_1, B_2, B_3$ and $2 \in A_4$, A_5, A_6, B_1,*

B_2, B_3, then $K_{8,8}$ is L-colorable.

Proof. We label A_1, A_2, A_3 by color 1 and label B_1, B_2, B_3 by color 2. The remaining vertices form $K_{5,5}$. Let L' be the list assignment of $K_{5,5}$ which is obtained from L by removing color 1 and color 2. That is, $L'_{5(i)} = \{A_4 - 2, A_5 - 2, A_6 - 2, A_7, A_8\}$ and $L'_{5(ii)} = \{B_4, B_5 \ldots, B_8\}$.

Case 1. *A color appears in exactly three lists in $L'_{5(ii)}$, say color 3.*
If color 3 appears in at most two lists in $L_{8(i)}$, then $K_{8,8}$ is L-colorable by Theorem 4.2.19 and Theorem 4.2.22. Suppose that color 3 appears in exactly three lists in $L_{8(i)}$.

Then at most two lists in $L_{8(i)}$ contains both color 1 and color 3, or at most two lists in $L_{8(i)}$ contains both color 2 and color 3. Hence, $K_{8,8}$ is L-colorable by Lemma 4.2.23 and Lemma 4.2.24.

Case 2. *Every color appears in at most two lists in $L'_{5(ii)}$.*
Let $x_iy_iz_i$ be a list in $L'_{5(ii)}$ such that $x_iy_iz_i \cap (A_{i+3} - 2) = \varnothing$ for $i = 1, 2, 3$. Let L^* be the 3-list assignment of $K_{11,5}$ such that $L^*_1 1 = \{\{x_i\} \cup A_{i+3} - 2 | i = 1, 2, 3\}$ $\cup\{\{y_i\} \cup A_{i+3} - 2 | i = 1, 2, 3\}$ $\cup\{\{z_i\} \cup A_{i+3} - 2 | i = 1, 2, 3\}$ $\cup\{A_4, A_5\}$ and $L^*_5 = L'_5$. Notice that if $K_{11,5}$ is L^*-colorable, then $K_{5,5}$ is L'-colorable. According to [17], $K_{11,5}$ is 3-choosable. Hence, $K_{5,5}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable. $\square$

Theorem 4.2.27. *Let L be a 3-list assignment of $K_{8,8}$ such that every color appears in at most three lists of each partite set where $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$. If $1 \in A_1, A_2, A_3$ and $2 \in A_4, A_5, A_6$ then $K_{8,8}$ is L-colorable.*

Proof. If color 1 or color 2 appear in at most two lists in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Theorem 4.2.19 and Theorem 4.2.22. Suppose that color 1 and color

2 appear in exactly three lists of $L_{8(ii)}$. Hence, the proof is done by Lemma 4.2.23, Lemma 4.2.24, Lemma 4.2.25 and Lemma 4.2.26. $\square$

Lemma 4.2.28. *Let L be a 3-list assignment of $K_{5,6}$ such that $L_5 = \{A_1,\ A_2,\ \ldots,\ A_5\}$ and $L_6 = \{B_1, B_2, \ldots, B_6\}$. If $|B_1| = 2$ and $|A_1| = \ldots = |A_5| = |B_2| = \ldots = |B_6| = 3$, then $K_{5,6}$ is L-colorable.*

Proof. Let $A_6 = xyz$ where $x, y, z \notin \bigcup_{v \in K_{5,6}} L(v)$. Let L^* be a 3-list assignment of $K_{6,8}$ such that $L^*_6 = \{A_1, A_2, \ldots, A_6\}$ and $L^*_8 = \{12x, 12y, 12z, B_2, B_3, \ldots, B_6\}$. Notice that if $K_{6,8}$ is L^*-colorable, then $K_{5,6}$ is L-colorable. By Lemma 3.3.4, $K_{6,8}$ is L^*-colorable; hence, $K_{5,6}$ is L-colorable. $\square$

Theorem 4.2.29. *Let L be a 3-list assignment of $K_{8,8}$ such that $L_{8(i)} = \{A_1,\ A_2,\ \ldots,\ A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$ where each color appears in at most three lists in each partite set. If $1 \in A_1, A_2, A_3$ and $2 \in A_1, A_2$ then $K_{8,8}$ is L-colorable.*

Proof. If color 1 appears at most two lists in $L_{8(ii)}$, then the proof is done by Theorem 4.2.19 and Theorem 4.2.22. Suppose that $1 \in B_1, B_2, B_3$.

We label B_1, B_2, B_3 by color 1 and label A_1, A_2 by color 2. The remaining vertices form $K_{6,5}$. Let L' be the list assignment of $K_{6,5}$ which is obtained from L by removing color 1 and color 2. By Lemma 4.2.28, $K_{6,5}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable. $\square$

Theorem 4.2.30. *Let L be a 3-list assignment of $K_{8,8}$ such that $L_{8(i)} = \{A_1,\ A_2,\ \ldots,\ A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$ where each color appears in at most three lists in each partite set. If color 1 and color 2 appear in exactly two lists in each partite set such that $1 \in A_1, A_2, B_1, B_2$, and $2 \in A_3, A_4, B_3, B_4$, then $K_{8,8}$ is L-colorable.*

Proof. We define the new list assignment L^* of $K_{8,8}$ which is obtained from L by changing color 8 to color 1. If $K_{8,8}$ is L^*-colorable, then $K_{8,8}$ is also L-colorable and the proof is done. Hence, we suppose that $K_{8,8}$ is not L^*-colorable. By Corollary 4.2.18, the remaining four lists are $246, 257, 347, 356$. That is, $A_5 = B_5 = 246, A_6 = B_6 = 257, A_7 = B_7 = 347, A_8 = B_8 = 356$.

Since $\{1, 8, 2, 3\}$ is a 4-coloring of $L_{8(i)}$, we may suppose that it has a subset that is a list in $L_{8(ii)}$ by Lemma 3.2.1. That is, there is a list from B_1, B_2, B_3, B_4 containing both color 2 and color 3. Similarly, a list from B_1, B_2, B_3, B_4 contains both color 4 and color 5 and another list in B_1, B_2, B_3, B_4 contains both color 6 and color 7. Since each color appears in at most three lists, the remaining list in B_1, B_2, B_3, B_4 contains two new colors, say color 9 and color A. With out loss of generality, let $B_1 = 123, B_2 = 145, B_3 = 167$ and $B_4 = 19A$. Similarly, we can prove that $23, 45, 67, 9A$ are a subset of a list in A_1, A_2, A_3, A_4.

Case 1. $9A \subset A_1$ or $9A \subset A_2$.

Suppose that $A_1 = 19A$. Then we use color 2, color 3, color 8 and color 9 to label lists in $L_{8(i)}$ and use color 1 and color A to label lists in $L_{8(ii)}$. The remaining vertices from $K_{1,5}$ which is easily colored.

Case 2. $9A \subset A_3$ or $9A \subset A_4$.

Suppose that $A_3 = 89A$. Then we use color 1, color 9, color 6 and color 7 to label lists in $L_{8(i)}$ and use color 8 to label lists in $L_{8(ii)}$. Then the remaining vertices form $K_{1,6}$ which are easily labeled. □

Theorem 4.2.31. *Let L be a 3-list assignment of $K_{8,8}$ such that $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$ where each color appears in at most three lists in each partite set. If $1 \in A_1, A_2, B_1, B_2$, and $2 \in A_3, A_4, B_3$ and no other list contains 1 or 2, then $K_{8,8}$ is L-colorable.*

Proof. The proof is similar to Theorem 4.2.30. □

Lemma 4.2.32. *Let L be a 3-list assignment of $K_{8,8}$ such that $L_{8(i)} = \{A_1,\ A_2,\ \ldots,\ A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$ and each color appears in at most three lists in each partite set . If $1 \in A_1, A_2, B_1, B_2$ but $1 \notin A_3, \ldots, A_8, B_3, \ldots, B_8$ $2 \in A_3, A_4, A_5, B_1, B_3, B_4$ and $x \in A_6$ and $x \notin A_7, A_8, B_1, B_2, \ldots, B_4$, then $K_{8,8}$ is L-colorable.*

Proof. **Case** 1. *Color x appears in exactly three lists in $L_{8(i)}$.*
If color x appears in exactly one list, two lists, three lists, then we apply Theorem 4.2.19, Theorem 4.2.22 and Theorem 4.2.27, respectively.

Case 2. *Color x appears in exactly two lists in $L_{8(i)}$.*
If color x appears in three lists in $L_{8(ii)}$, then the proof is done by Theorem 4.2.27. Then we suppose that color x appears in at most two lists in $L_{8(ii)}$. If $x \in A_1$ or $x \in A_2$, then we define a new list assignment of $K_{8,8}$ which is obtained from L by changing color x to color 2 and then we apply Strategy D. If $x \in A_3, x \in A_4$ or $x \in A_5$, then we define a new list assignment of $K_{8,8}$ which is obtained from L by changing color x to color 1 and then we apply Theorem 4.2.30 and Theorem 4.2.31.

Case 3. *Color x appears in exactly one list in $L_{8(i)}$.*
If color x appears in three lists in $L_{8(ii)}$, then the proof is done by Theorem 4.2.27. Then we suppose that color x appears in at most two lists in $L_{8(ii)}$. If x appears in exactly one list in $L_{8(ii)}$ then we define a new list assignment of $K_{8,8}$ which is obtained from L by changing color x to color 1 and then we apply Theorem 4.2.27. If x appears in exactly two list in $L_{8(ii)}$ then we define a new list assignment of $K_{8,8}$ which is obtained from L by changing color x to color 2 and then we apply Strategy D for $L_{8(ii)}$. □

Theorem 4.2.33. *Let L be a 3-list assignment of $K_{8,8}$ such that $L_{8(i)} = \{A_1,\ A_2,\ \ldots,\ A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$ and each color appears in at most* *three lists in each partite set. If* $1 \in A_1, A_2$ *but* $1 \notin A_3, \ldots, A_8$ $2 \in A_3, A_4, A_5$, *then* $K_{8,8}$ *is* L*-colorable.*

Proof. If a color appears in three lists in A_1, A_2, A_6, A_7, A_8, then $K_{8,8}$ is L-colorable by Theorem 4.2.27. We can suppose that each color appears in at most two lists in A_1, A_2, A_6, A_7, A_8.

If color 2 appears in at most two lists in $L_{8(ii)}$, then $K_{8,8}$ is L-colorable by Theorem 4.2.19 and Theorem 4.2.22. Suppose that color 2 appears in exactly three lists in $L_{8(ii)}$.

Case 1. *Color* 1 *is not in any list in* $L_{8(ii)}$.

Hence, we color A_1, A_2 by color 1. The remaining vertices from $K_{6,8}$ which is 3-choosable by Lemma 3.3.4.

Case 2. *Color* 1 *appears in exactly one list in* $L_{8(ii)}$.

Let $1 \in B_1$. If $2 \notin B_1$, then we define a new list assignment L^* by changing color 2 to color 1 and then we apply Strategy D. Suppose that $2 \in B_1, B_2, B_3$. Let 3 be the remaining color in B_1. Notice that color 3 appears in at most two list in A_6, A_7, A_8.

We label A_1, A_2 by color 1, label A_3, A_4, A_5 by color 2 and label B_1 by color 3. For the worst case, we suppose that $3 \in A_7, A_8$. The remaining vertices form $K_{3,7}$ Let L' be the list assignment of $K_{3,7}$ which is obtained from L by removing color 1, color 2 and color 3. That is, $L_3 = \{A_6 - 3, A_7 - 3, A_8\}$ and $L_7 = \{B_2 - 2, B_3 - 2, B_4, B_5 \ldots, B_8\}$.

We may suppose that B_2 and B_2 have only one common color because if B_2 and B_3 have more than one common color then the proof is done by Theorem 4.2.29. Hence, if L_3 has two 2-colorings which is not disjoint or has at least three 2-colorings, then at least one of such 2-colorings is not a list in L_7; hence, $K_{8,8}$ is L-colorable by Lemma 3.2.1.

Suppose that L_3 has at most one 2-coloring or has two 2-coloring which are disjoint. Hence, $A_6 - 2, A_7 - 2, A_8$ are mutually disjoint. Then we apply Strategy A for L_3 to prove that $K_{3,7}$ is L'-colorable. Therefore, $K_{8,8}$ is L-colorable.

Case 3. *Color* 1 *appears in exactly two lists in* $L_{8(ii)}$.

Let $1 \in B_1, B_2$. Recall that color 2 appears in exactly three lists in $L_{8(ii)}$. If $2 \in B_1, B_2$, then the proof is done by Theorem 4.2.29. If color $2 \notin B_1, B_2$, we define a new list assignment L^* of $K_{8,8}$ by changing color 2 to color 1 and then we apply Strategy D. Suppose that $2 \in B_1, B_3, B_4$.

We label A_1, A_2 by color 1 and label A_3, A_4, A_5 by color 2. The remaining vertices form $K_{3,8}$. Let L' be the list assignment of $K_{3,8}$ which is obtained from L by removing color 1 and color 2. If L'_3 has a coloring that is not a list in L'_8, then the proof is done by Lemma 3.2.1. Suppose that every coloring of L'_3 has a subset that is a list in L'_8.

Case 3.1 $|A_6 \cap A_7| \geq 2$ *or* $|A_6 \cap A_8| \geq 2$ *or* $|A_7 \cap A_8| \geq 2$.

Without loss of generality, suppose that $3, 4 \in A_6, A_7$ and $A_8 = 567$. Hence, L'_3 has at least six 2-colorings, namely $\{3,5\}, \{3,6\}, \{3,7\}, \{4,5\}, \{4,6\}, \{4,7\}$. Since every coloring in L'_3 must have a subset that is a list in L'_8. We have $3 \in B_1$ or $4 \in B_1$. Without loss of generality, suppose that $3 \in B_1$. Moreover, $45, 46, 47$ must be a list in L'_8. Hence, we suppose $B_2 = 245, B_3 = 246$ and $B_4 = 247$. Hence, there are two lists containing both color 2 and color 4. Then $K_{8,8}$ is L-colorable by Theorem 4.2.29.

Case 3.2 $|A_6 \cap A_7| = 1$ *and* $|A_6 \cap A_8| = 1$ *and* $|A_7 \cap A_8| = 1$.

Suppose that $A_6 = 345, A_7 = 367$ and $A_8 = 468$. Similar to Case 3.1, we may suppose that $B_1 = \{123, B_2 = 146, B_3 = 247$ and $B_4 = 256$. Since color 8 appears in exactly one list in A_6, A_7, A_8 and $8 \notin B_1, B_2, B_3, B_4, B_5$, $K_{8,8}$ is L-colorable by Lemma 4.2.32.

Case 3.3 $|A_6 \cap A_7| = 1$ *and* $|A_6 \cap A_8| = 1$ *and* $|A_7 \cap A_8| = 0$.
Suppose that $A_6 = 345, A_7 = 367$ and $A_8 = 489$. Similar to Case 3.1, we may suppose that $B_1 = 123, B_2 = 146$ and $B_3 = 247$.

If $8 \notin B_4$ or $9 \notin B_4$, then the proof is finished by Lemma 4.2.32. Suppose that $B_4 = 289$.

Since $\{1,2,5,7,8\}$ and $\{1,2,5,7,9\}$ are 5-colorings of $L_{8(i)}$, we may suppose that such 5-colorings has a subset that is a list in $L_{8(ii)}$ by Lemma 3.2.1. We suppose that $B_6 = 578$ and $B_7 = 579$. Therefore, $K_{8,8}$ is L-colorable by Theorem 4.2.29.

Case 3.4 $|A_6 \cap A_7| = 1$ *and* $|A_6 \cap A_8| = 0$ *and* $|A_7 \cap A_8| = 0$.
Then $A_6 \cup A_7 \cup A_8$ has at least eight colors; hence, there is a color $x \in A_6 \cup A_7 \cup A_8$ such that no list in L containing both color 1 and color x because color 1 appears only in four lists in L and B_1 has already contained color 1 and color 2. Thus we define a new list assignment L^* of $K_{8,8}$ by changing color x to color 1. If x appears in exactly two lists in A_6, A_7, A_8, then we apply Theorem 4.2.3 for L^*. If x appears in exactly one list in A_6, A_7, A_8, then we apply Theorem 4.2.27 for L^*.

Case 3.5 $|A_6 \cap A_7| = 0$ *and* $|A_6 \cap A_8| = 0$ *and* $|A_7 \cap A_8| = 0$.
The proof is similar to case 3.4.

Case 4. *Color* 1 *appears in exactly three lists in* $L_{8(ii)}$.
Since color 1 only appears in exactly two lists in $L_{8(i)}$, $K_{8,8}$ is L-colorable by Theorem 4.2.22. □

Theorem 4.2.34. *Let L be a 3-list assignment of $K_{8,8}$ such that $L_{8(i)} = \{A_1, A_2, \ldots, A_8\}$ and $L_{8(ii)} = \{B_1, B_2, \ldots, B_8\}$ and each color appears in at most three lists in each partite set. If $1 \in A_1, A_2, A_3$ then $K_{8,8}$ is L-colorable unless $\mathcal{F} \subset L_{8(i)}, L_{8(ii)}$.*

Proof. By Theorem 4.2.19 and Theorem 4.2.22, we may suppose that $1 \in B_1, B_2, B_3$.

By Theorem 4.2.27, we suppose that each color appears in at most two lists in A_4, A_5, A_6, A_7, A_8. Similarly, we suppose that each color appears in at most two lists in B_4, B_5, B_6, B_7, B_8. We will prove that if L has one of the following three properties, then $K_{8,8}$ is L-colorable and finally we prove that L must have one of these three properties,

Property 1. *There is a color $x \in A_4, A_5$ but $x \notin A_1, A_2, A_3$ or there is a color $x \in B_4, B_5$ but $x \notin B_1, B_2, B_3$.*

The proof is done by Theorem 4.2.33.

Property 2. *There is a color $x \in A_4, A_5$ but $x \notin B_1, B_2, B_3$ or there is a color $x \in B_4, B_5$ and $x \notin B_1, B_2, B_3$.*

We define the new list assignment of $K_{8,8}$ which is obtained from L by changing color x to color 1 and then we apply Strategy D.

Property 3. *There is a color $x \in A_4, B_4$ and the remaining lists do not contain x.*

We define the new list assignment L^* of $K_{8,8}$ which is obtained from L by changing color x to color 1. By Theorem 4.2.18, $K_{8,8}$ is L^*-colorable unless the remaining four lists are $246, 257, 347, 356$. Hence, we suppose that $A_5 = B_5 = 246, A_6 = B_6 = 257, A_7 = B_7 = 347$ and $A_8 = B_8 = 356$. If 123 is not a list in $L_{8(ii)}$, then we label $A_1, A_2, A_3, A_5, A_6, A_7$ by color $1, 2, 3$. The remaining vertices form $K_{1,8}$ which is easily colored. Hence, we suppose that $A_1 = 123, A_2 = 145$ and $A_3 = 167$. That is, $\{123, 145, 167, 246, 257, 247, 256\} \subset L_{8(ii)}$. Similarly, we can prove that $\{123, 145, 167, 246, 257, 247, 256\} \subset L_{8(i)}$. It can be directly verified that if $\{123, 145, 167, 246, 257, 247, 256\} \subset L_{8(i)}, L_{8(ii)}$ then $K_{8,8}$ is not L-colorable.

Finally, we will prove that L must have a color x with one of the properties. Suppose that no color x with the properties in Property 1 and Property 2. Let

$x_1, x_2, \ldots, x_k$ be the colors which appears in three lists in A_4, A_5, A_6, A_7, A_8. Since $A_1 \cup A_2 \cup A_3 - \{1\}$ (and $B_1 \cup B_2 \cup B_3 - \{1\}$) contains at most six colors, we have $k \leq 6$. Thus at least one list from A_1, A_2, A_3 and another list from B_1, B_2, B_3 contains x_i for each i. Hence, $A_1 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 - \{1, x_1, x_2, \ldots, x_k\}$ has at most $12 - 2k$ elements. Since $15 - 2k$ is the number of colors which appears once in A_4, A_5, A_6, A_7, A_8, there is a color $x \in A_5 \cup A_6 \cup A_7 \cup A_8$ which is not in $A_1, A_2, A_3, B_1, B_2, B_3$. If color x appears in two lists in B_5, B_6, B_7, B_8, then it is in Property 1. Hence, color x appears in exactly one list in $L_{8(ii)}$ which is in Property 3.. □

Theorem 4.2.35. *Let L be a 3-list assignment of $K_{8,8}$ such that each color appears in at most two lists in each partite set. Then $K_{8,8}$ is L-colorable*

Proof. If all lists in $L_{8(i)}$ are mutually disjoint, then we apply Strategy A. Otherwise, we suppose that $1 \in A_1, A_2$. For the six remaining lists in $L_{8(i)}$, we have at least nine colors because each color appears in at most two lists. However, color 1 appears in at most two lists in each partite set. At most eight colors are in the lists containing color 1. Without loss of generality, suppose that $2 \in B_3$ and no list containing both color 1 and color 2. Hence, we define the new list assignment L^* of $K_{8,8}$ which is obtained from L by changing color 2 to color 1. By Theorem 4.2.18 and Theorem 4.2.34, $K_{8,8}$ is L^*-colorable unless $246, 257, 347, 356$ are the lists in both partite set. If $K_{8,8}$ is L^*-colorable, then $K_{8,8}$ is L-colorable; hence, we suppose that $A_5 = B_5 = 246$, $A_6 = B_6 = 257$, $A_7 = B_7 = 347$ and $A_8 = B_8 = 356$.

Since every color appears in at most two lists in each partite set, the remaining lists do not contain color $2, 3, 4, 5, 6, 7$. That is, we can split graph $K_{8,8}$ is to two copies of $K_{4,4}$. Let L' be the 3-list assignment of $K_{4,4}$ such that $L'_{4(i)} = \{A_1, A_2, A_3, A_4\}$ and $L'_{4(ii)} = \{B_1, B_2, B_3, B_4\}$. Let L'' be the 3-list assignment of $K_{4,4}$ such that $L''_{4(i)} = \{A_5, A_6, A_7, A_8\}$ and $L''_{4(ii)} = \{B_5, B_6, B_7, B_8\}$. Since $K_{4,4}$ is 3-choosable, $K_{8,8}$ is L-colorable. $\square$

Theorem 4.2.36. *Let L be a 3-list assignment of $K_{8,8}$. Then $K_{8,8}$ is L-colorable if and only if $\mathcal{F} \subset L_{8(i)}, L_{8(ii)}$.*

Proof. Assume that $\mathcal{F} \not\subset L_{8(i)}$ or $\mathcal{F} \not\subset L_{8(ii)}$.
If $r_{8(i)} \geq 5$ or $r_{8(ii)} \geq 5$, then we apply Lemma 4.1.6. If $r_{8(i)} \leq 4$ and $r_{8(ii)} \leq 4$; apply Theorem 4.2.18, Theorem 4.2.34 and Theorem 4.2.35. In this case, $K_{8,8}$ is L-colorable unless $\mathcal{F} \subset L_{8(i)}, L_{8(ii)}$. $\square$

4.3 On $(3, t)$-choosability of $K_{7,9}$

Study 3-choosability of $K_{7,9}$ is difficult than $K_{8,8}$ because $K_{7,9}$ is not symmetric. That is, $K_{7,9}$ requires more cases. It is clear that, for a 3-list assignment L, $K_{7,9}$ is not L-colorable if $L|V(K_{7,7}) = L_{\mathcal{F}}$. We conjecture that, for a 3-list assignment L, $K_{7,9}$ is L-colorable if and only if $L|_{V(K_{7,7})} \neq L_{\mathcal{F}}$.

Here, we prove that $K_{7,9}$ is $(3, t)$-choosable if and only if $t \leq 6$ or $t \geq 14$. We still left a characterization of all 3-list assignments of L such that $K_{7,9}$ is not L-colorable for future work. (See Chapter 6.) We introduce remarks which are used several times in this section.

For the following remarks, let L be a $(3, t)$-list assignment of $K_{7,9}$ where $L_7 = \{A_1, A_2 \ldots, A_7\}$ and $L_9 = \{B_1, B_2, \ldots, B_9\}$ and r_7 (and r_9) be the maximum number of lists in L_7 (and L_9) containing a common color.

Suppose that $(r_7, r_9) = (3, 4)$; this is one of two missing cases of Lemma 4.1.5.

Remark 4.3.1. If $1 \in A_1, A_2, A_3, B_1, B_2, B_3, B_4$ and 2 is in some lists in L but no lists contains both color 1 and color 2, then we can define the new list assignment L^* from L by changing color 2 to color 1. It is easy to see that if $K_{7,9}$ is L^*-

colorable then $K_{7,9}$ is L-colorable. Notice that color 1 appears at least in four lists in L_7^* or at least five lists in L_9^*. Hence, $K_{7,9}$ is L^*-colorable by Lemma 4.1.5. Therefore, $K_{7,9}$ is L-colorable.

Remark 4.3.2. Suppose that $t \geq 14$ and color 1 appears in at most six lists in L. Since a list containing color 1 has another two colors, at most 12 colors appear in the lists which contain color 1. Since we have 14 color, we have one color left. Then there exists a color, say color 2 such that no list contains both color 1 and color 2. Then we construct new 3-list assignment l^* by changing color 2 to color 1.

Remark 4.3.3. Suppose that $t \geq$. If a color appears in lists in only one partite set, then we can label such vertices by this color and the remaining vertices can be labeled by Theorem 3.3.7, Theorem 3.3.8 and Theorem 3.4.6. Suppose that every color appears in lists of both partite sets.

If color 1 appears in three lists in L_7 but color 1 appears in at most three lists in L_9, then we construct a new 3-list assignment l^* as in Remark 4.3.2. Hence, color 1 appears in at least four lists in L_7^*, then $K_{7,9}$ is L^*-colorable by Lemma 4.1.5. Therefore, $K_{7,9}$ is L-colorable.

If color 1 appears in exactly four lists in L_9 but color 1 appears in at most two lists in L_7, then we can conclude that $K_{7,9}$ is L-colorable, similarly.

Remark 4.3.4. Suppose that $t \geq 14$. Let $\mathbb{X} = \{A_1, A_2, A_3, B_1, B_2, B_3, B_4\}$ and color 1 is in all lists in $\mathbb{X}$. If a color not including color 1 appears in at least three lists in $\mathbb{X}$ or at least two colors not including color 1 appear in two lists in $\mathbb{X}$, then there exists a color $x \notin \bigcup_{x\in\mathbb{X}} X$ because L contains at least 14 colors. Then $K_{7,9}$ is L-colorable by Remark 4.3.1.

Lemma 4.3.5. *Let L be a $(3,t)$-list assignment of $K_{7,9}$ such that $L_7 = \{A_1,$ $A_2, \ldots, A_7\}$ and $L_9 = \{B_1, B_2, \ldots, B_9\}$ where $t = 14, 15$. Let r_7 (and r_9)*

be the maximum number of lists in L_7 *(and* L_9*) containing a common color. If* $1 \in A_1, A_2, A_3$*,* $2 \in A_4, A_5, A_6$ *and* $(r_7, r_9) = (3, 4)$*, then* $K_{7,9}$ *is* L*-colorable.*

Proof. **Case** 1. *All of* $123, 124, 125$ *are lists in* L_9.

Then $K_{7,9}$ is L-colorable by Remark 4.3.4.

Case 2. *One of* $123, 124, 125$ *is not a list in* L_9.

Since $\{1,2,3\}, \{1,2,4\}$ and $\{1,2,5\}$ are 3-colorings of L_7, there exists a 3-coloring of L_7 which has no subset that is a list in L_9. Then $K_{7,9}$ is L-colorable by Lemma 3.2.1. □

Lemma 4.3.6. *Let* L *be a* $(3,t)$*-list assignment of* $K_{7,9}$ *such that* $L_7 = \{A_1, A_2, \ldots, A_7\}$ *and* $L_9 = \{B_1, B_2, \ldots, B_9\}$ *where* $t = 14, 15$*. Let* r_7 *(and* r_9*) be the maximum number of lists in* L_7 *(and* L_9*) containing a common color. If* $1 \in B_1, B_2, B_3, B_4$*,* $2 \in B_5, B_6, B_7$ *and* $(r_7, r_9) = (3, 4)$*, then* $K_{7,9}$ *is* L*-colorable.*

Proof. By Remark 4.3.3, we may suppose that $1 \in A_1, A_2, A_3$.

Case 1. $2 \in A_1, A_2, A_3$.

Notice that $A_1 - 12, A_2 - 12, A_2 - 12$ contain a color and $B_1 - 1, B_2 - 1, B_3 - 1$ contain two colors. At most nine colors (including color 2) are in the same lists with color 1. Since we have at least 14 colors, there exists a color x such that no list in L contain both color 1 and color x. Then the proof is done by Remark 4.3.1.

Case 2. $2 \in A_1, A_2$ *but* $2 \notin A_3$.

If $2 \in A_4$, then color 2 appears in three lists in L_7. By Remark 4.3.3, we suppose that color 2 appears in four lists in L_9. If $2 \in B_8$ or $2 \in B_9$, then we label B_1, B_2, B_3, B_4 by color 1 and label B_5, B_6, B_7, B_8 by color 2 and the remaining vertices can be easily colored. If $2 \in B_1 \cup B_2 \cup B_3 \cup B_4$, then there exists a color x such that no list in L contain both color 1 and color x. Then the proof is done by Remark 4.3.1.

Case 3. $2 \in A_1$ *but* $2 \notin A_2, A_3$.

If $|B_8 \cap B_9| \neq 1$, then we label $B_1, B_2 \ldots, B_7$ by color 1 and color 2. The remaining vertices can be directly labeled. Suppose that $B_8 = 346$ and $B_9 = 357$. If L_9 has a coloring which has no subset that is a list in L_7, then $K_{7,9}$ is L-colorable by Lemma 3.2.1. Suppose that each coloring of L_9 has a subset that is a list in L_7. Since $\{1,2,3\}$ is a 3-coloring of L_9, let $A_1 = 123$. Since $\{1,2,4,5\}, \{1,2,4,7\}, \{1,2,6,5\}$ and $\{1,2,6,7\}$ are 4-colorings of L_7, we suppose that $2 \in A_4, A_5$ and $\{A_2 - 1, A_3 - 1, A_4 - 2, A_5 - 2\} = \{45, 47, 65, 67\}$. Since $(B_2 - 1) \cup (B_3 - 1) \cup (B_4 - 1)$ is of size 6, one of colors $8, 9, A, B, C, D, E$ is not in $(B_2 - 1) \cup (B_3 - 1) \cup (B_4 - 1)$. Suppose that $8 \notin B_2, B_3, B_4$. Then color 8 is not in the same list with lists containing color 1. Then this case is done by Remark 4.3.1.

Case 4. $2 \notin A_1, A_2, A_3$.

If $2 \notin A_1, A_2, A_3$, then we define the new list assignment L^* by changing color 2 to color 1 and then we apply Strategy C for L_9^*. $\square$

Lemma 4.3.7. *Let* L *be a* $(3,t)$*-list assignment of* $K_{7,9}$ *such that* $L_7 = \{A_1, A_2, \ldots, A_7\}$ *and* $L_9 = \{B_1, B_2, \ldots, B_9\}$ *where* $t = 14, 15$. *Let* r_7 *(and* r_9*) be the maximum number of lists in* L_7 *(and* L_9*) containing a common color. If* $1 \in A_1, A_2, A_3$, $(r_7, r_9) = (3, 4)$ *and there exists another color which appears in three lists in* L_7, *then* $K_{7,9}$ *is* L*-colorable.*

Proof. Let color 2 be another color which appears in three lists in L_7.

Case 1. $2 \notin A_1 \cup A_2 \cup A_3$.

Then color 2 appears in three lists in A_4, A_5, A_6; hence, $K_{7,9}$ is L-colorable by Lemma 4.3.5.

Case 2. $2 \in A_1 \cup A_2 \cup A_3$.

By Remark 4.3.3, we suppose that both color 1 and color 2 appear in exactly four lists in L_9. If at least two lists in L_9 contains both color 1 and color 2, then $K_{7,9}$ is L-colorable by Remark 4.3.4. If at most one list in L_9 contains both color 1 and color 2, then $K_{7,9}$ is L-colorable by Lemma 4.3.6. □

Lemma 4.3.8. *Let L be a $(3,t)$-list assignment of $K_{7,9}$ such that $L_7 = \{A_1, A_2, \ldots, A_7\}$ and $L_9 = \{B_1, B_2, \ldots, B_9\}$ where $t = 14, 15$. Let r_7 (and r_9) be the maximum number of lists in L_7 (and L_9) containing a common color. If $1 \in B_1, B_2, B_3, B_4$ and $(r_7, r_9) = (3, 4)$, then $K_{7,9}$ is L-colorable.*

Proof. By Remark 4.3.3, suppose that $1 \in A_1, A_2, A_3$. By Lemma 4.3.7, suppose that each color appears in at most two lists in L_7.

We first label B_1, B_2, B_3, B_4 by color 1; hence, the remaining vertices form $K_{7,5}$. Let L' be the list assignment of $K_{7,5}$ which is obtained from L by removing color 1 and color 2.

Case 1. *No color appears in two lists in L'_5.*

Then we apply Strategy A for L'_5.

Case 2. *Exactly one color appears in two lists in L'_5.*

Let $2 \in B_5, B_6$. Then we label B_5, B_6 by color 2; hence, the remaining vertices form $K_{7,3}$. Let L'' be the list assignment of $K_{7,3}$ which is obtained from L' by removing color 2. Since color 2 appears in at most two lists in L_7, we can apply Strategy A for L''_3.

Case 3. *Exactly two colors appear in two lists in L'_5.*

Let $2 \in B_5, B_6$. If $3 \in B_5$ or $3 \in B_6$, then B_7, B_8, B_9 are still mutually disjoint; hence, the proof is similar to Case 2. Next, suppose that $3 \in B_7, B_8$.

Case 3.1 2 *or* $3 \notin A_1 \cup A_2 \cup A_3$

Define the new list assignment L^* of $K_{7,9}$ which is obtained from L by changing such color to color 1. Then we apply Strategy D$'$.

Case 3.2 $2 \in A_1 \cup A_2 \cup A_3$ *and color* 2 *appears in two lists in* L_7.

Recall that we have labeled B_1, B_2, B_3, B_4 by color 1. Then we label such two lists in L_7 by color 2 and label B_7 and B_8 by color 3. The remaining vertices form $K_{5,3}$. Let L'' be the list assignment of $K_{5,3}$ which is obtained from L by removing color 2 and color 3. Then we apply Strategy A for L_3''.

Case 3.3 $3 \in A_1 \cup A_2 \cup A_3$ *and color* 3 *appears in two lists in* L_7.

Similar to Case 3.2.

Case 3.4 $2, 3 \in A_1$ *and no other list in* L_7 *contains color* 2 *or color* 3.

We label B_5, B_6 by color 2 and label A_1 by color 3. For the worst case, we suppose that $2, 3 \notin A_2, A_3 \ldots, A_7$. Then the remaining vertices form $K_{6,3}$. Let L'' be the list assignment of $K_{6,3}$ which is obtained from L' by removing color 2 and color 3. Notice that L_3'' contains two lists of size 2 and one list of size 3. Define the new list assignment L^* of $K_{6,3}$ by deleting a color from only such list of size 3. It is obvious that if $K_{6,3}$ is L^*-colorable, then $K_{6,3}$ is L''-colorable. Then we apply Strategy A$'$ for L_3^* to guarantee that $K_{6,3}$ is L^*-colorable.

Case 3.5 $2 \in A_1, 3 \in A_2$ *and no other list in* L_7 *contains color* 2 *or color* 3.

We label B_5, B_6 by color 2 and label B_7, B_8 by color 3. The remaining vertices can be easily labeled.

Case 4. *At least three colors appear in exactly two lists in* L_5'.

Since $|B_5|+|B_6|+|B_7|+|B_8|+|B_9| = 15$, exactly nine colors appear in exactly one list. Since $t \geq 14$, there is a color, say color 2 which is not in B_5, B_6, B_7, B_8, B_9.

Case 4.1 $2 \notin B_1, B_2, B_3, B_4$.

Then color 2 only appears in L_7; hence, we label some lists in L_7 by color 2. The remaining vertices form a complete bipartite graph with at most 15 vertices which can be labeled by Theorem 3.3.7, Theorem 3.3.8 and Theorem 3.4.6.

Case 4.2 $2 \in B_1, B_2, B_3, B_4$.

Suppose that $2 \in B_1$. Similar to Case 5.1, we suppose that color 2 is in a list of

L_7. Then we label some lists in L_7 by color 2. For the worst case, suppose $2 \in A_4$ and no other list in L_7 contains color 2. The remaining vertices form $K_{6,5}$. Let L'' be the list assignment of $K_{6,5}$ such that $L''_6 = \{A_1 - 1, A_2 - 1, A_3 - 1, A_5, A_6, A_7\}$ and $L''_5 = \{B_5, B_6, B_7, B_8, B_9\}$.

Since each color appears in at most two lists in L''_5, there exists a list $x_1y_1z_1 \in L''_5$ such that $x_1y_1z_1 \cap (A_1 - 1) = \varnothing$. Similarly, there exist lists $x_2y_2z_2$ and $x_3y_3z_3$ such that $x_2y_2z_2 \cap (A_2 - 1) = x_3y_3z_3 \cap (A_3 - 1) = \varnothing$. Hence, we define the new list assignment L^* of $K_{12,5}$ such that $L^*_12 = \{A_5, A_6, a_7\} \cup \{\{x_i\} \cup (A_i - 1)|i = 1, 2, 3\} \cup \{\{y_i\} \cup (A_i - 1)|i = 1, 2, 3\} \cup \{\{z_i\} \cup (A_i - 1)|i = 1, 2, 3\}$. It is easy to see that if $K_{12,5}$ is L^*-colorable, then $K_{6,5}$ is L''-colorable. According to Shende[17], $K_{5,12}$ is 3-choosable. Hence, $K_{7,9}$ is L-colorable. □

Theorem 4.3.9. *The complete bipartite graph $K_{7,9}$ is $(3,t)$-choosable if and only if $t \leq 6$ or $t \geq 14$.*

Proof. If L is a 3-list assignment of $K_{7,9}$ such that $L_7 = \mathcal{F}$ and $L_9 = \mathcal{F} \cup \{x_1x_2x_3, y_1y_2y_3\}$ where $x_1, x_2, x_3, y_1, y_2, y_3$ are any colors, then $K_{7,9}$ is not L-colorable. Depending on such six colors, t may be $7, 8, 9, \ldots, 13$. Hence, $K_{7,9}$ is not $(3,t)$-choosable for $7, 8, \ldots, 13$.

Case 1. $t \leq 6$.

Then a color in L_9 appears in at least $\lceil \frac{9 \cdot 3}{6} \rceil = 5$ lists. Hence, $K_{7,9}$ is $(3,t)$-choosable for $t \leq 6$ by Lemma 4.1.5.

Case 2. $t \geq 16$.

Let $S \subset V(K_{7,9})$. If $|S| \leq 13$, then $K_{7,9}[S]$ is $L|_S$-colorable by Theorems 3.3.7; if $|S| = 14$, then $K_{7,9}[S]$ is $L|_S$-colorable by Theorem 3.3.8 and if $|S| = 15$ then $K_{7,9}[S]$ is $L|_S$-colorable by Theorem3.4.6. Then $K_{7,9}$ is L-colorable by Theorem 2.1.7.

Case 3. $t = 14, 15$.

By Lemma 4.1.5, $K_{7,9}$ is always L-colorable unless $(r_7, r_9) = (2,3), (3,4)$. If $(r_7, r_9) = (3,4)$, then $K_{7,9}$ is L-colorable by Lemma 4.3.8. Suppose that $(r_7, r_9) = (2,3)$. Let $1 \in B_1, B_2, B_3$. Notice that 1 appears in at most two lists in L_7. Since we have at least 14 colors, there exists a color, say color 2 which is not in the same list with color 1. Then we define the new 3-list assignment L^* of $K_{7,9}$ by changing color 2 to color 1. Then color 1 appears in at least four lists in L_7^*; hence, $K_{7,9}$ is L^*-colorable by Lemma 4.3.8 and Lemma 4.1.5. □

Theorem 4.3.10. *A complete bipartite graph with* 16 *vertices is* $(3,t)$*-choosable for* $t \leq 6$ *or* $t \geq 14$.

Proof. It follows from Theorem 4.1.7, Theorem 4.2.36 and Theorem 4.3.9. □

CHAPTER V

ON (k,t)-CHOOSABILITY of $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$

5.1 Background

Since k-choosability implies k-colorability, $\chi(G) \leq \chi_l(G)$ for every graph G. This bound is sharp because $\chi(G) = \chi_l(G) = 2$ when G is a tree. However, there exists a graph G such that $\chi(G)$ and $\chi_l(G)$ is significantly different. In [4], Erdős , Rubin, and Taylor gave an example of bipartite graphs which is not k-choosable for each positive integer k. Such graph is the complete bipartite graph $K_{m,m}$ when $m = \binom{2k-1}{k}$. They gave a k-list assignment L such that $K_{m,m}$ is not L-colorable. Example 5.1.1 shows a special case when $k = 3$.

Example 5.1.1. When $k = 3$, we have $m = \binom{5}{3} = 10$. Figure 2.1.1 shows the $(3,5)$-list assignment L such that $K_{10,10}$ is not L-colorable.

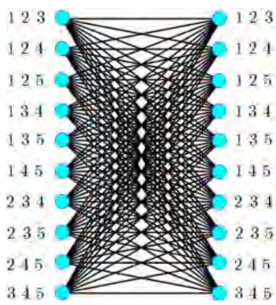


Figure 5.1.1: A $(3,5)$-list assignment L of $K_{10,10}$

The complete bipartite graph $K_{10,10}$ is not L-choosable because each partite set requires three colors but there are only five available colors. $\square$

In general, we assign distinct k-subsets of $\{1, 2, \dots, 2k-1\}$ to each vertex in each partite set of $K_{m,m}$ where $m = \binom{2k-1}{k}$ to form a k-list assignment L. If we use only $k-1$ colors to label lists in a partite set, then the remaining k colors form a list which we are not labeled. That is, we need at least k colors to color all vertices in each partite set. However, we have only $2k-1$ colors. Hence, we cannot label all vertices in both partite sets. Notice that the k-list assignment contains exactly $2k-1$ colors; in other words, $K_{m,m}$ is not $(k, 2k-1)$-choosable. Particularly, $K_{10,10}$ is not $(3,5)$-choosable. Next, we also show that $K_{10,10}$ is not $(3,t)$-choosable for $t = 6, 7, 8$.

Example 5.1.2 shows how to obtain a $(3,t)$-list assignment L of $K_{10,10}$ for $t = 5, 6, 7$ such that $K_{10,10}$ is not L-colorable.

Example 5.1.2. Let L be the $(3,5)$-list assignment L of $K_{10,10}$ in Figure 5.1.2. We will construct new list assignments L^1, L^2, L^3 of $K_{10,10}$ such that $K_{10,10}$ is not L^i-colorable for $i = 1, 2, 3$. The list assignments L^1, L^2 and L^3 are obtained from L by changing colors in boxes as shown in Figures 5.1.2, 5.1.3 and 5.1.4.

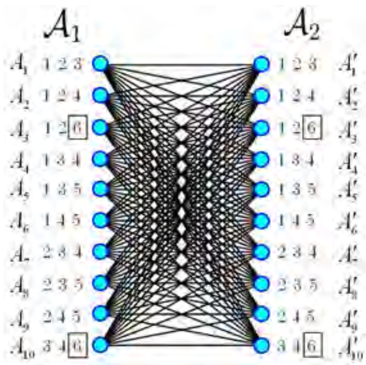


Figure 5.1.2: A $(3,6)$-list assignment L^1 of $K_{10,10}$

We show that $K_{10,10}$ is not L^1-colorable. Let $\mathcal{A}_1 = \{A_1, A_2, \ldots, A_{10}\}$ and $\mathcal{A}_2 = \{A'_1, A'_2, \ldots, A'_{10}\}$ be the lists of vertices in the left partite set and the right partite set in Figure 5.1.2, respectively. We show that $K_{10,10}$ is not L^1-colorable by dividing the proof into several cases.

Case 1. If we use both color 1 and color 2 to label some lists in $\mathcal{A}_1$, then we cannot label all of A_{10}, A'_1, A'_2, A'_3.

Case 2. Similar to Case 1, if we use both color 1 and color 2 to label some lists in $\mathcal{A}_2$, then we cannot label all of A_1, A_2, A_3, A'_{10}.

Case 3. If we use color 1 to label some lists in $\mathcal{A}_1$ and use color 2 to label lists in $\mathcal{A}_2$, then we cannot label all of $A_7, A_8, A_9, A'_4, A'_5, A'_6$.

Case 4. Similarly to Case 2, if we use color 2 to label some lists in $\mathcal{A}_1$ and use color 1 to label lists in $\mathcal{A}_2$, then we cannot label all of $A_4, A_5, A_6, A'_7, A'_8, A'_9$.

Case 5. If we use neither color 1 or color 2, we cannot color all of A_4, A_5, A_6, A'_7, A'_8, A'_9. Hence, $K_{10,10}$ is not L^1-colorable.

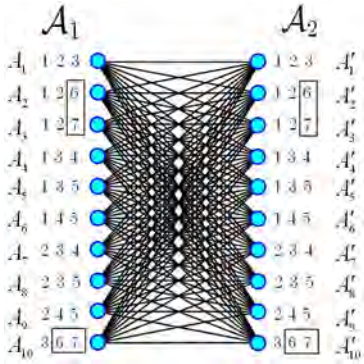


Figure 5.1.3: A $(3,7)$-list assignment L^2 of $K_{10,10}$

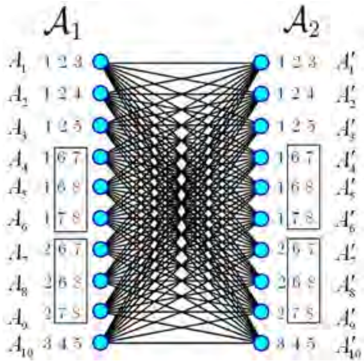


Figure 5.1.4: A $(3,8)$-list assignment L^3 of $K_{10,10}$

It can be proved similarly that $K_{10,10}$ is neither L^2-colorable nor L^3-colorable.

□

In this chapter, k, t and m are always positive integers such that $m = \binom{2k-1}{k}$. We have seen that $K_{m,m}$ is not (k,t)-choosable for $t = 2k-1$. We next are interested in $t < 2k-1$ or $t > 2k-1$ which is studied in Section 5.2. Given a positive integer k, we reveal all (k,t)-choosability of the complete bipartite graph $K_{m,m}$ except when $17 \cdot 2^{k-2} - 4k - 4 < t < 2km - k^2 + 2k$; in such a case, the problem still unsolved. In particular, Section 5.3 contains the complete results when $k = 3$. We combine the tool in Theorem 2.1.7 with and the main results from Chapter 3 and Chapter 4 to obtain these complete results when $k = 3$.

5.2 On (k,t)-choosability of $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$.

In this section, we focus on general cases. We prove that if $t \leq 2k-2$ or $t \geq 2km - 2k^2 + 2k$, then $K_{m,m}$ is (k,t)-choosable, and if $2k-1 \leq t \leq 17 \cdot 2^{k-2} - 4k - 4$ then $K_{m,m}$ is not (k,t)-choosable.

Theorem 5.2.1. *Let k, t, m be positive integers such that $k \geq 3$ and $m = \binom{2k-1}{k}$. If $t \leq 2k-2$, then $K_{m,m}$ is (k,t)-choosable.*

Proof. Let L be a (k,t)-list assignment of $K_{m,m}$. We can use $\lfloor \frac{t}{2} \rfloor$ colors to color all vertices in each partite set because $\lfloor \frac{t}{2} \rfloor + k \geq \lfloor \frac{t}{2} \rfloor + \frac{t+1}{2} \geq t+1$. Hence, we label vertices in one partite set by color $1, 2, \ldots, \lfloor \frac{t}{2} \rfloor$ and label vertices in the other partite set by color $\lfloor \frac{t}{2} \rfloor + 1, \lfloor \frac{t}{2} \rfloor + 2, \ldots, t$. □

In Theorem 5.2.2, we will show that if the number t is large enough, then $K_{m,m}$ is (k,t)-choosable.

Theorem 5.2.2. *Let k, t, m be positive integers such that $m = \binom{2k-1}{k}$. If $t \geq 2km - 2k^2 + 2k$, then $K_{m,m}$ is (k,t)-choosable.*

Proof. Let L be a (k,t)-list assignment of $K_{m,m}$. For every $S \subset V(G)$, $|L(S)| \geq t - L(V(G) - S) \geq 2km - 2k^2 + 2k - k(2m - |S|) = k|S| - 2k^2 + 2k$.

To apply Theorem 2.1.7, let $S \subset V(K_{m,m})$ be such that $|L(S)| < |S|$. Then $|S| > k|S| - 2k^2 + 2k$. Hence, $|S| < 2k$. It is easy to see that a bipartite graph with less than $2k$ vertices is k-choosable. Therefore, $K_{m,m}$ is (k,t)-choosable. $\square$

Before we prove our main result in Theorem 5.2.10, we need Lemma 5.2.4 as a basis step for mathematical induction.

Definition 5.2.3. Let L be a list assignment of a graph G. Then L is called a *colorable list assignment* of G if G is L-colorable; otherwise, L is called a *non-colorable list assignment* of G.

Recall Notation 3.1.2 that if L is a list assignment of $K_{a,a}$, then $L_{a(i)}$ and $L_{a(ii)}$ are collections of lists assigned to the vertices in each partite set.

Lemma 5.2.4. *Let t be a positive integer. $K_{3,3}$ is $(2,t)$-choosable if and only if $t \leq 2$ or $t \geq 6$. Moreover, for a 3-list assignment L of $K_{3,3}$, the complete bipartite graph $K_{3,3}$ is L-colorable if and only if $L \neq L^1, L^2, L^3$ where $L^1_{3(i)} = \{12, 13, 23\}, L^1_{3(ii)} = \{12, 13, 23\}$, $L^2_{3(i)} = \{12, 13, 24\}, L^2_{3(ii)} = \{12, 14, 23\}$, and $L^3_{3(i)} = \{12, 13, 45\}, L^3_{3(ii)} = \{14, 15, 23\}$.*

Proof. Let L be a $(2,t)$-list assignment of $K_{3,3}$.

Case 1. $t \geq 2$ or $t \geq 6$.

If $t = 2$, then $K_{3,3}$ is $(2,t)$-choosable because $K_{3,3}$ is 2-colorable. Suppose that $t \geq 6$. To apply Theorem 2.1.7, let $S \subset V(K_{3,3})$ be such that $|L(S)| < |S|$. Then $|S| = 5$. Hence, $K_{3,3}[S]$ is a subgraph of $K_{2,3}$ which is 2-choosable by Example 2.1.1(ii). Hence, $K_{3,3}$ is $L|_S$-colorable. Therefore, $K_{3,3}$ is L-colorable.

Case 2. $t = 3, 4, 5$.

Define a $(2,3)$-list assignment, a $(2,4)$-list assignment, and a $(2,5)$-list assignment L^1, L^2 and L^3 as follows. $L^1_{3(i)} = \{12, 13, 23\}, L^1_{3(ii)} = \{12, 13, 23\}$, $L^2_{3(i)} = \{12, 13, 24\}, L^2_{3(ii)} = \{12, 14, 23\}$, and $L^3_{3(i)} = \{12, 13, 45\}, L^3_{3(ii)} = \{14, 15, 23\}$. Since $K_{3,3}$ is not L^i-colorable for $i = 1, 2, 3$, $K_{3,3}$ is not $(2, t)$-choosable for $t = 3, 4, 5$.

Next, we characterize all non-colorable 2-list assignments L of $K_{3,3}$. If all lists in $L_{3(i)}$ are mutually disjoint, then $K_{3,3}$ is L-colorable by Strategy A$'$. Suppose that $L_{3(i)} = \{1a, 1b, cd\}$ where a, b, c, d are positive integers. By Lemma 3.2.1, if L is a non-colorable list assignment, then such colorings has a subset that is a list in $L_{3(ii)}$. Since $\{1, c\}$, $\{1, d\}$, $\{a, b, c\}$ and $\{a, b, d\}$ are colorings of $L_{3(i)}$, we obtain $L_{3(ii)} = \{1c, 1d, ab\}$. Next, we consider possibility of L.

Case 2.1. a, b, c, d are distinct.

Suppose that $a = 2, b = 3, c = 4$ and $d = 5$. Hence, $t = 5$ and $L_{3(i)} = \{12, 13, 45\}, L_{3(ii)} = \{14, 15, 23\}$.

Case 2.2. $a = c$ but a, b, d are distinct.

Suppose that $a = c = 2$, $b = 3$ and $d = 4$. Hence $t = 4$ and $L_{3(i)} = \{12, 13, 24\}, L_{3(ii)} = \{12, 14, 23\}$.

Case 2.3. $a = c$ and $b = d$.

Suppose that $a = c = 2$ and $b = d = 5$. Hence, $t = 3$ and $L_{3(i)} = \{12, 13, 23\}, L_{3(ii)} = \{12, 13, 23\}$. □

Notation 5.2.5. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be collections of lists. The notation $[\mathcal{A}_1, \mathcal{A}_2]$ represents the list assignment, say L, of $K_{|\mathcal{A}_1|,|\mathcal{A}_2|}$ such that $L_{|\mathcal{A}_1|} = \mathcal{A}_1$ and $L_{|\mathcal{A}_2|} = \mathcal{A}_2$.

Definition 5.2.6. Let S be a set and i a positive integer. Define the collection of sets $\binom{S}{i} = \{A \subset S | A \text{ has size } i\}$. Let $\mathbb{X}$ be a collection of sets and c be an element which is not in any set in $\mathbb{X}$. Define $c\mathbb{X} = \{\{c\} \cup X | X \in \mathbb{X}\}$.

Example 5.2.7. Let $S = \{1, 2, 3, 4\}$ and $\mathbb{X} = \binom{S}{2}$. Then $\mathbb{X} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$ and $5\mathbb{X} = \{\{1, 2, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}$. □

Remark 5.2.8 introduces an idea to construct a non-colorable list assignment of a complete bipartite graph from an existing non-colorable list assignment of a smaller complete bipartite graph. Example 5.2.9 illustrates the idea.

Remark 5.2.8. Let $[\mathcal{A}_1, \mathcal{A}_2]$ be a non-colorable list assignment of the complete bipartite graph $K_{|\mathcal{A}_1|,|\mathcal{A}_2|}$.
(i) If $[p\mathcal{A}_1, q\mathcal{A}_2]$ is a colorable list assignment of $K_{|\mathcal{A}_1|,|\mathcal{A}_2|}$, then any $[p\mathcal{A}_1, q\mathcal{A}_2]$-coloring of $K_{|\mathcal{A}_1|,|\mathcal{A}_2|}$ must use color p or color q.
(ii) If $[pq\mathcal{A}_1, \mathcal{A}_2]$ is a colorable list assignment of $K_{|\mathcal{A}_1|,|\mathcal{A}_2|}$, then any $[pq\mathcal{A}_1, \mathcal{A}_2]$-coloring of $K_{|\mathcal{A}_1|,|\mathcal{A}_2|}$ must use color p or color q.

Example 5.2.9. Let $\mathcal{A}_1 = \{34, 35, 45\}$, $\mathcal{A}_2 = \{34, 35, 45\}$, $\mathcal{B}_1 = \{34, 35, 46\}$ and $\mathcal{B}_2 = \{34, 46, 45\}$. Then $[\mathcal{A}_1, \mathcal{A}_2]$ and $[\mathcal{B}_1, \mathcal{B}_2]$ are a non-colorable $(2, 3)$-list assignment and a non-colorable $(2, 4)$-list assignment of $K_{3,3}$, respectively, by Lemma 5.2.4.

We will construct a non-colorable $(3, 9)$-list assignment of $K_{10,10}$ from $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$. First, let $C = \{6, 7, 8\}$ and $D = \{7, 8, 9\}$. Define a 3-list assignment L of $K_{10,10}$ as follows:

$$L_{10(i)} = 1\mathcal{A}_1 \cup 2\mathcal{B}_1 \cup 12\binom{C}{1} \cup \binom{D}{3}$$
$$L_{10(ii)} = 1\mathcal{B}_2 \cup 2\mathcal{A}_2 \cup 12\binom{D}{1} \cup \binom{C}{3}$$

That is,

$$L_{10(i)} = \{134, 135, 145\} \cup \{234, 235, 246\} \cup \{126, 127, 128\} \cup \{789\}$$

$$L_{10(ii)} = \{134, 136, 146\} \cup \{234, 235, 245\} \cup \{127, 128, 129\} \cup \{678\}$$

.

Consider the subgraph of $K_{10,10}$ induced by vertices labeled by $12\binom{C}{1} \subset L_{10(i)}$ and $\binom{C}{3} \subset L_{10(ii)}$. Since $[\binom{C}{1}, \binom{C}{3}]$ is a non-colorable list assignment of $K_{1,3}$, color 1 or color 2 is used to label lists in $12\binom{C}{1} \subset L_{10(i)}$ by Remark 5.2.8. Similarly, consider the subgraph of $K_{10,10}$ induced by vertices labeled by $12\binom{D}{1} \subset L_{10(ii)}$ and $\binom{D}{3} \subset L_{10(i)}$. Since $[\binom{D}{1}, \binom{D}{3}]$ is a non-colorable list assignment of $K_{1,3}$, color 1 or color 2 is used to label lists in $12\binom{D}{1} \subset L_{10(ii)}$, by Remark 5.2.8.

Case 1. *Color* 1 *is used to label lists in* $L_{10(i)}$ *and color* 2 *is used to label lists in* $L_{10(ii)}$.

It follows that lists in $L_{10(ii)}$ cannot be labeled by color 1 and lists in $L_{10(i)}$ cannot be labeled by color 2. Then consider the subgraph of $K_{10,10}$ induced by vertices labeled by $1\mathcal{B}_2 \subset L_{10(ii)}$ and $2\mathcal{B}_1 \subset L_{10(i)}$. Vertices of this induced subgraph cannot be labeled because $[\mathcal{B}_1, \mathcal{B}_2]$ is a non-colorable list assignment of $K_{3,3}$.

Case 2. *Color* 1 *is used to label lists in* $L_{10(ii)}$ *and color* 2 *is used to label lists in* $L_{10(i)}$.

Similar to Case 1, consider the subgraph of $K_{10,10}$ induced by vertices labeled by $1\mathcal{A}_1 \subset L_{10(i)}$ and $2\mathcal{A}_2 \subset L_{10(ii)}$. Vertices of this induced subgraph cannot be labeled because $[\mathcal{A}_1, \mathcal{A}_2]$ is a non-colorable list assignment of $K_{3,3}$.

Hence, we conclude that $K_{10,10}$ is not L-colorable.

Note further that the construction starts from two non-colorable list assignments of $K_{3,3}$, say $[\mathcal{A}_1, \mathcal{A}_2]$ and $[\mathcal{B}_1, \mathcal{B}_2]$. By Lemma 5.2.4, the number of colors in $\mathcal{A}_1 \cup \mathcal{A}_2$ ($\mathcal{B}_1 \cup B_2$) can possibly be three, four or five. Notice that $[\mathcal{A}_1, \mathcal{A}_2]$ can be the same as $[\mathcal{B}_1, \mathcal{B}_2]$ while C and D can be any sets of three colors. The set of colors in L consists of colors from $\mathcal{A}_1 \cup \mathcal{A}_2$, $\mathcal{B}_1 \cup \mathcal{B}_2$, C, D and two new colors. Then the total number of colors in L is smallest, which is five, when $\mathcal{A}_1 \cup \mathcal{A}_2$ and

$\mathcal{B}_1 \cup \mathcal{B}_2$ contains the same three colors and C, D are the set of such three colors. The total number of colors in L is largest, which is 18, when $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ contains five different colors and C, D are the disjoint sets of new three colors. It is easy to see that the total number of colors in L can possibly be any numbers from 3 to 18. Hence, $K_{10,10}$ is not $(3,t)$-choosable for $t = 3, 4, \ldots, 18$. $\square$

Theorem 5.2.10. *Let k, t, m be positive integers such that $m = \binom{2k-1}{k}$. If $2k-1 \le t \le 17 \cdot 2^{k-2} - 4k - 4$ then $K_{m,m}$ is not (k,t)-choosable.*

Proof. We will prove by mathematical induction on k. The basis step is shown in Lemma 5.2.4 for the case $k = 2$. We prove the induction step similar to Example 5.2.9.

Let C and D be any sets of size $2k-3$ and $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ be collections of sets of size $k-1$ such that $|\mathcal{A}_1| = |\mathcal{A}_2| = |\mathcal{B}_1| = |\mathcal{B}_2| = \binom{2k-3}{k-1}$. Suppose that $[\mathcal{A}_1, \mathcal{A}_2]$ and $[\mathcal{B}_1, \mathcal{B}_2]$ are non-colorable list assignments of $K_{\binom{2k-3}{k-1},\binom{2k-3}{k-1}}$ and suppose that C, D and all lists in $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ do not contain color 1 and color 2.

Define a (k,t)-list assignment L of $K_{m,m}$ by

$$L_{m(i)} = 1\mathcal{A}_1 \cup 2\mathcal{B}_1 \cup 12\binom{C}{k-2} \cup \binom{D}{k}$$

$$L_{m(ii)} = 1\mathcal{B}_2 \cup 2\mathcal{A}_2 \cup 12\binom{D}{k-2} \cup \binom{C}{k}$$

.

Since $2\binom{2k-3}{k-1} + \binom{2k-3}{k-2} + \binom{2k-3}{k} = \binom{2k-1}{k} = m$, all vertices of $K_{m,m}$ are assigned.

Consider the subgraph of $K_{m,m}$ induced by vertices labeled by $12\binom{C}{k-2} \subset L_{m(i)}$ and $\binom{C}{k} \subset L_{m(ii)}$. Since $[\binom{C}{k-2}, \binom{C}{k}]$ is a non-colorable list assignment of $K_{\binom{2k-3}{k-2},\binom{2k-3}{k}}$, color 1 or color 2 is used to label lists in $12\binom{C}{k-2} \subset L_{m(i)}$ by Remark 5.2.8. Similarly, consider the subgraph of $K_{m,m}$ induced by vertices labeled by $12\binom{D}{k-2} \subset L_{m(ii)}$ and $\binom{D}{k} \subset L_{m(i)}$. Since $[\binom{D}{k-2}, \binom{D}{k}]$ is a non-colorable list assignment of $K_{\binom{2k-3}{k-2},\binom{2k-3}{k}}$, color 1 or color 2 is used to label lists in $12\binom{D}{k-2} \subset L_{m(ii)}$, by Remark 5.2.8.

Case 1. *Color* 1 *is used to label lists in* $L_{m(i)}$ *and color* 2 *is used to label lists in* $L_{m(ii)}$.

It follows that lists in $L_{m(ii)}$ cannot be labeled by color 1 and lists in $L_{m(i)}$ cannot be labeled by color 2. Then consider the subgraph of $K_{m,m}$ induced by vertices labeled by $1\mathcal{B}_2 \subset L_{m(ii)}$ and $2\mathcal{B}_1 \subset L_{m(i)}$. Vertices of this induced subgraph cannot be labeled because $[\mathcal{B}_1, \mathcal{B}_2]$ is a non-colorable list assignment of $K_{\binom{2k-3}{k-1},\binom{2k-3}{k-1}}$.

Case 2. *Color* 1 *is used to label lists in* $L_{m(ii)}$ *and color* 2 *is used to label lists in* $L_{m(i)}$.

Similar to Case 1, consider the subgraph of $K_{m,m}$ induced by vertices labeled by $1\mathcal{A}_1 \subset L_{m(i)}$ and $2\mathcal{A}_2 \subset L_{m(ii)}$. Vertices of this induced subgraph cannot be labeled because $[\mathcal{A}_1, \mathcal{A}_2]$ is a non-colorable list assignment of $K_{\binom{2k-3}{k-1},\binom{2k-3}{k-1}}$.

Hence, we conclude that $K_{m,m}$ is not L-colorable.

Note further that the construction starts from two non-colorable list assignments of $K_{\binom{2k-3}{k-1},\binom{2k-3}{k-1}}$, say $[\mathcal{A}_1, \mathcal{A}_2]$ and $[\mathcal{B}_1, \mathcal{B}_2]$. By the induction hypothesis, the number of colors in $[\mathcal{A}_1, \mathcal{A}_2]$ and $[\mathcal{B}_1, \mathcal{B}_2]$ can possibly be any number from $2k-3$ to $17 \cdot 2^{k-3} - 4k$. The set of colors in L consists of colors in $\mathcal{A}_1 \cup \mathcal{A}_2$, $\mathcal{B}_1 \cup \mathcal{B}_2$, C, D and two new colors. Then the total number of colors in L is smallest, which is $2k-3+2 = 2k-1$, when $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ contains the same $2k-3$ colors and C, D are the set of such $2k-3$ colors. The total number of colors in L is largest, which is $2(17 \cdot 2^{k-3} - 4k) + 2(2k-3) + 2 = 17 \cdot 2^{2k-2} - 4k - 4$, when $\mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{B}_1 \cup \mathcal{B}_2$ contains $2^{k-3} - 4k$ different colors and C, D are the disjoint sets of new $2k-3$ colors. It is easy to see that the total number of colors in L can possibly be any numbers from $2k-1$ to $17 \cdot 2^{2k-2} - 4k - 4$. Hence,

$K_{m,m}$ is not $(3,t)$-choosable for all $2k-1 \le t \le 2^{2k-2} - 4k - 4$. $\square$

5.3 On $(3,t)$-choosability of $K_{10,10}$

Let k and m be positive integers such that $m = \binom{2k-1}{k}$. In Section 5.2, we have proved that $K_{m,m}$ is (k,t)-choosable for $t \le 2k-2$ and $t \ge 2km - 2k^2 + 2k$, and $K_{m,m}$ is not (k,t)-choosable for $2k-1 \le t \le 17 \cdot 2^{k-2} - 4k - 4$. When $17 \cdot 2^{k-2} - 4k - 4 < t < 2km - 2k^2 + 2k$, the problem is still unsolved. Now, we focus on a specific positive integer k. When $k = 2$, we get $m = \binom{2 \cdot 2 - 1}{2} = 3$. The complete result is proved in Lemma 5.2.4 that $K_{3,3}$ is $(2,t)$-choosable if and only if $t \neq 3, 4, 5$. Thus, here, we focus on $k = 3$, and then $m = 10$. Hence, we determine each positive integer t such that $K_{10,10}$ is $(3,t)$-choosable, in other words, we investigate the 3-choosability of complete bipartite graphs with 20 vertices. However, only results of complete bipartite graphs with at most 14 vertices are revealed. In 1996, Hanson, MacGillivray, and Toft [8] proved that all complete bipartite graphs with 13 vertices are 3-choosable, and in 2005 Fitzpatrick and MacGillivray [5] proved that all complete bipartite graphs with 14 vertices except $K_{7,7}$ is 3-choosable and $L_{\mathcal{F}}$ is the unique 3-list assignment such that $K_{7,7}$ is not $L_{\mathcal{F}}$-colorable. To extend their result from 14 vertices to 20 vertices, our work in Chapters II, III and IV are devoted to solve this problem. Applying Theorem 2.1.7, the problem can be solved for complete bipartite graphs with 17, 18, 19 and 20 vertices. Results in Chapters III and IV take care the rest.

The desired main result is concluded in Theorem 5.3.1 and Theorem 5.3.3.

Theorem 5.3.1. *Let t be a positive integer. The complete bipartite graph $K_{10,10}$ is $(3,t)$-choosable if and only if $t \neq 5, 6, \ldots, 25$.*

Proof. By Theorem 5.2.1, if $t \leq 4$, then $K_{10,10}$ is $(3,t)$-choosable. By Theorem 5.2.10, if $5 \leq t \leq 18$, then $K_{10,10}$ is not $(3,t)$-choosable. From Chapter 3, we have known that $K_{7,7}$ is not $(3,7)$-choosable and $L_{\mathcal{F}}$ is a non-colorable $(3,7)$-list assignment of $K_{7,7}$. We define a new assignment L^* of $K_{10,10}$ which is obtained from $L_{\mathcal{F}}$ by adding three new lists in each partite set. Then it follows that L^* is a non-colorable list assignment of $K_{10,10}$. The number of colors in L^* depends on the lists that we add. The minimum number of colors in L^* is 7 and the maximum number in L^* is 25. Hence, $K_{10,10}$ is not $(3,t)$-choosable for $t = 7, 8, \ldots, 25$.

Next, suppose that $t \geq 26$ and let L be any $(3,t)$-list assignment of $K_{10,10}$. We will prove that $K_{10,10}$ is L-colorable by Theorem 2.1.7. Let $S \subset V(K_{10,10})$ be such that $|S| > |L(S)|$. Since $|L(S)| \geq t - 3 \cdot |V(K_{10,10}) \smallsetminus S| \geq 3|S| - 34$, we have $|S| > |L(S)| \geq 3|S| - 34$. That is ,$|S| < 17$.

If $|S| = 16$, then $|L(S)| \geq 48 - 34 = 14$. Hence, $K_{10,10}[S]$ is $L|_S$-colorable by Theorem 4.3.10. If $|S| = 15$, then $|L(S)| \geq 45 - 34 = 11$. Hence, $K_{10,10}[S]$ is $L|_S$-colorable by Theorem 3.4.6. If $|S| = 14$, then $|L(S)| \geq 42 - 34 = 8$. Hence, $K_{10,10}[S]$ is $L|_S$-colorable by Theorem 3.3.8. If $|S| \leq 13$, then $K_{10,10}[S]$ is $L_{|S}$-colorable because every complete bipartite graph with at most 13 vertices is 3-choosable by Theorem 3.3.7. Therefore, by Theorem 2.1.7, $K_{10,10}$ is L-colorable. Hence, $K_{10,10}$ is $(3,t)$-choosable for $t \geq 26$. $\square$

Lemma 5.3.2. *The complete bipartite graph $K_{9,b}$ is always $(3,5)$-choosable.*

Proof. Let L be a $(3,5)$-list assignment of $K_{9,b}$.

Part 1. *All lists in L_9 can be color by only two colors.*

Because of $\binom{5}{3} = 10$, there exists a set $S \subset \{1,2,3,4,5\}$ such that $S \notin L_9$. Suppose that $S = \{1,2,3\}$. Hence, we use color $4,5$ label all lists in L_9.

Part 2. *All lists in L_b can be color by the remaining three colors.*

Since each list in L_b has size 3 and is a subset of $\{1,2,3,4,5\}$, it contains at least one color from $\{1, 2, 3\}$. That is, we all lists in L_b can be labeled by color 1, color 2, or color 3. $\square$

Theorem 5.3.3. *Let a, b, t be positive integers such that $a, b \geq 7$, $a+b \leq 20$ and $t \neq 6$ and $(a, b, t) \neq (10, 10, 5)$. Then $K_{a,b}$ is $(3, t)$-choosable if and only if $t \leq 5$ or $t \geq 3(a+b) - 34$. Moreover, $K_{10,10}$ is not $(3, 5)$-choosable.*

Proof. Let L be a $(3, t)$-list assignment of $K_{a,b}$. Then $t \geq 3$.

Case 1. $t = 3, 4$.

If $t = 3$, then $K_{a,b}$ is L-colorable because $K_{a,b}$ is 3-colorable. If $t = 4$, then we can use any two color label all vertices in each partite set, then $K_{a,b}$ is L-colorable, too.

Case 2. $t = 5$.

If $(a, b) = (10, 10)$, then $K_{10,10}$ is not $(3, 5)$-choosable by Example 5.1.1. Suppose that $a \leq b$ and $(a, b) \neq (10, 10)$. Since $a + b = 20$, we have $a \leq 9$. Hence, $K_{a,b}$ is $(3, 5)$-choosable by Lemma 5.3.2.

Case 3. $7 \leq t \leq 3(a+b) - 35$.

Notice that if $L|_{V(K_{7,7})} = L_{\mathcal{F}}$, then $K_{a,b}$ is not L-colorable. Hence, we construct a non-colorable 3-list assignment of $K_{a,b}$ by adding new $a+b-14$ lists to $L_{\mathcal{F}}$. The number of colors in such $a+b-14$ lists possibly be any number from 3 to $3(a+b) - 42$. Moreover, such $a+b-14$ lists may contain the same colors as colors in $L_{\mathcal{F}}$. Hence, L possibly contains $7, 8, \ldots, 3(a+b) - 35$ colors. That is, $K_{a,b}$ is not $(3, t)$-choosable for $7, 8, \ldots, 3(a+b) - 35$.

Case 4. $t \geq 3(a+b) - 34$.

We will prove that $K_{a,b}$ is L-colorable by Theorem 2.1.7. Let $S \subset V(K_{a,b})$ be such that $|S| > |L(S)|$. Since $|L(S)| \geq t - 3 \cdot (a + b - |S|) \geq 3S - 34$, we have $|S| > |L(S)| \geq 3S - 34$. That is ,$|S| < 17$.

If $|S| = 16$, then $|L(S)| \geq 26 - 12 = 14$. Hence, $K_{a,b}[S]$ is $L|_S$-colorable by

Theorem 4.3.10. If $|S| = 15$, then $|L(S)| \geq 26 - 15 = 11$. Hence, $K_{a,b}[S]$ is $L|_S$-colorable by Theorem 3.4.6. If $|S| = 14$, then $|L(S)| \geq 26 - 18 = 8$. Hence, $K_{a,b}[S]$ is $L|_S$-colorable by Theorem 3.3.8. If $|S| \leq 13$, then $K_{a,b}[S]$ is $K_{a,b}[S]$ is $L_{|S}$-colorable because every complete bipartite graph with at most 13 vertices is 3-choosable by Theorem 3.3.7. By Theorem 2.1.7, $K_{a,b}$ is L-colorable. $\square$

CHAPTER VI

CONCLUSIONS AND FUTURE WORK

6.1 Conclusions

In this dissertation, we have studied three main problems. Firstly, find a sufficient condition of (k,t)-choosable graphs and a sufficient condition of (k,t)-choosable graphs not containing K_{k+1}. Secondly, give a complete result on 3-choosability of complete bipartite graphs with 15 vertices by establishing new strategies; moreover, obtain some partial results on 3-choosability of complete bipartite graphs with 16 vertices. Lastly, study (k,t)-choosability of the complete bipartite graph $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$.

All results in this dissertation are listed as follows:

Sufficient conditions of (k,t)-choosable graphs

Let k,t and n be positive integers.

Theorem 2.2.2: If $t \geq kn - k^2 + 1$, then every graph with n vertices is (k,t)-choosable.

Theorem 2.2.4: If $k \leq t \leq kn - k^2$, then every graph with n vertices containing a $(k+1)$-clique is not (k,t)-choosable.

Lemma 2.2.5: If $k \geq n-2$, then a K_{k+1}-free graph with n vertices is (k,t)-choosable. In other words, the list chromatic number of a K_{k+1}-free graph with n vertices is at most $n-2$.

Lemma 2.2.8: If $t \geq k+1$, then a K_{k+1}-free graph with $k+3$ vertices is (k,t)-choosable.

Theorem 2.2.11: Let $k \geq 3$. If $t \geq kn - k^2 - 2k + 1$, then a K_{k+1}-free graph with n vertices is (k, t)-choosable.

Theorem 2.2.12: If $t \geq 2n - 6$, then a triangle-free graph with n vertices is $(2, t)$-choosable.

Theorem 2.2.13: A triangle-free graph with n vertices is $(2, 2n - 7)$-choosable if and only if it is $(K_{3,3} - e)$-free.

Theorem 2.2.14: Let $nk - k^2 - 2k + 1 \leq t \leq nk - k^2$ and $3 \leq k \leq n - 3$. A graph with n vertices is (k, t)-choosable if and only if it is K_{k+1}-free. Moreover, for $k = 2$ and $2n - 6 \leq t \leq 2n - 4$. A graph with n vertices is $(2, t)$-choosable if and only if it is triangle-free.

Theorem 2.2.15: If $k \leq t \leq kn - k^2 - 2k$, then every K_{k+1}-free graph with n vertices containing $C_5 \vee K_{k-2}$ is not (k, t)-choosable.

Strategies for 3-list assignments

Let L be a 3-list assignment of $K_{a,b}$ with $L_a = \{A_1, A_2, \ldots, A_a\}$ and $L_b = \{B_1, B_2, \ldots, B_b\}$. Let r be the maximum number of lists in L_b containing a common color.

Strategy A: If all lists in L_a are mutually disjoint and $\prod_{i=1}^{a} |A_i| > 3^{a-1}n_1 + \lfloor 3^{a-2} \rfloor n_2 + \lfloor 3^{a-3} \rfloor n_3$ where $n_i = |\{B \in L_b, |B| = i\}|$ for $i = 1, 2, 3$, then $K_{a,b}$ is L-colorable.

Strategy B: If a color appears in $a - 1$ lists in L_a, then $K_{a,b}$ is L-colorable.

Strategy C: If a color appears in $a - 2$ lists in L_a and $r \leq 8$, then $K_{a,b}$ is L-colorable.

Strategy D: If a color appears in $a - 3$ lists in L_a and $(r, b) \in \{(r, b) | r \leq 2, b \leq 22\} \cup \{(3, b) | b \leq 14\} \cup \{(4, b) | b \leq 12\} \cup \{(5, b) | b \leq 9\}$, then $K_{a,b}$ is L-colorable.

Strategy E: If a color appears in $a - 4$ lists in L_a, say color 1 and $(r, b) \in \{(r, b) | r \leq 2, b \leq 22\} \cup \{(3, b) | b \leq 14\}$, then $K_{a,b}$ is L-colorable unless the four remaining lists of L_a are 246, 257, 347, 356 and $\{123, 145, 167, 246, 257, 347, 356\} \subset L_b$.

Strategy F: If a color appears in $a - 5$ lists in L_a, $r \leq 2$ and $a + b \leq 18$, then $K_{a,b}$ is L-colorable.

On 3-choosability of complete bipartite graphs

Recall that $\mathcal{F} = \{123, 145, 167, 246, 257, 347, 356\}$ be the collection of all lines in the Fano plane and $L_{\mathcal{F}}$ be the 3-list assignment of $K_{7,7}$ which seven vertices in each partite set are assigned by distinct elements from $\mathcal{F}$.

Theorem 3.3.6: A complete bipartite graph with 14 vertices except $K_{7,7}$ is 3-choosable. Moreover, for a 3-list assignment L of $K_{7,7}$, it is not L-colorable if and only if $L = L_{\mathcal{F}}$.

Theorem 3.4.5: A complete bipartite graph with 15 vertices except $K_{7,8}$ is 3-choosable. Moreover, for a 3-list assignment L of $K_{7,8}$, it is not L-colorable if and only if $L|_{V(K_{7,7})} = L_{\mathcal{F}}$.

Theorem 3.4.6: A complete bipartite graph with 15 vertices is $(3,t)$-choosable for $t \leq 6$ and $t \geq 11$.

Theorem 4.1.7: A complete bipartite graph with 16 vertices except $K_{7,9}$ and $K_{8,8}$ is 3-choosable.

Theorem 4.2.36: For a 3-list assignment L of $K_{8,8}$, it is not L-colorable if and only if $L|_{V(K_{7,7})} = L_{\mathcal{F}}$.

Theorem 4.3.10: A complete bipartite graph with 16 vertices is $(3,t)$-choosable for $t \leq 6$ or $t \geq 14$.

On (k,t)-choosability of $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$.

Let k and t be positive integers.

Theorem 5.2.1, Theorem 5.2.2: If $t \leq 2k-2$ or $t \geq 2k \cdot \binom{2k-1}{k} - 2k^2 + 2k$, then $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$ is (k,t)-choosable.

Theorem 5.2.10: If $2k-1 \leq t \leq 17 \cdot 2^{k-2} - 4k - 4$, then $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$ is not (k,t)-choosable.

Lemma 5.2.4: $K_{3,3}$ is $(2,t)$-choosable if and only if $t \leq 2$ or $t \geq 6$.

Theorem 5.3.1: $K_{10,10}$ is $(3,t)$-choosable if and only if $t \leq 4$ or $t \geq 26$.

Theorem 5.3.3: Let a, b, t be positive integers such that $a, b \geq 7$, $a + b \leq 20$ and $t \neq 6$ and $(a, b, t) \neq (10, 10, 5)$. Then $K_{a,b}$ is $(3,t)$-choosable if and only if $t \leq 5$ or $t \geq 3(a+b) - 34$. Moreover, $K_{10,10}$ is not $(3,5)$-choosable.

6.2 Future Work

We propose some ideas for further research as follows:

1. Let G be a graph with n vertices which is K_{k+1}-free and $C_5 \vee K_{k-2}$-free. What is the smallest number t_0 in terms of k and n such that G is (k,t)-choosable for each positive integer $t \geq t_0$? We conjecture that if $t \geq kn - k^2 - 4k + 1$, then G is (k,t)-choosable and if G contains $C_7 \vee K_{k-2}$ as a subgraph, then G is not (k,t)-choosable for $k \leq t \leq kn - k^2 - 4k$.

2. Establish strategies for 4-choosable graphs. For example, let L be a 4-list assignment of $K_{a,b}$ such that $L_a = \{A_1, \dots, A_a\}$ and $L_b = \{B_1, \dots, B_b\}$. The following can be proved similar to Strategies A, B and C, respectively.

- If all lists in L_a are mutually disjoint and $\prod_{i=1}^{a} |A_i| > 4^{a-1}n_1 + \lfloor 4^{a-2} \rfloor n_2 + \lfloor 4^{a-3} + \rfloor n_3 + \lfloor 4^{a-4} + \rfloor n_4$ where $n_i = |\{B \in L_b, |B| = i\}|$ for $i = 1, 2, 3, 4$, then $K_{a,b}$ is L-colorable.

- If a color appears in $a - 2$ lists, then $K_{a,b}$ is L-colorable.

- If a color appears in $a - 3$ lists and each color appears in at most 63 lists in L_b, then $K_{a,b}$ is L-colorable.

3. Find all 3-list assignments L of $K_{7,9}$ such that it is not L-colorable. In chapter IV, we have proved that for a 3-list assignment L of $K_{8,8}$, it is L-colorable if and only if $L|_{V(K_{7,7})} = L_{\mathcal{F}}$. It leads to a conjecture that for a 3-list assignment L of $K_{7,9}$, it is L-colorable if and only if $L|_{V(K_{7,7})} = L_{\mathcal{F}}$.

4. Find the positive integer t_0 such that every complete bipartite graph with 17 vertices is $(3,t)$-choosable for all $t \geq t_0$.

5. Study 3-list assignments of complete bipartite graphs with more than 16 vertices. Notice that 3-choosability of complete bipartite graphs with 18 vertices is difficult to study because each of them has many 3-list assignments L such that it is not L-colorable. For example, we found at least three different 3-list assignments L such that $K_{9,9}$ is not L-colorable and five different 3-list assignments L such that $K_{7,11}$ is not L-colorable

The following are non-colorable list assignments L of $K_{a,b}$ where L_a and L_b are the collection of lists assigned to vertices in the partite set of size a and b, respectively.

$L_8 = \{158, 168, 159, 169, 278, 279, 345, 346\}$

$L_9 = \{158, 168, 159, 169, 278, 279, 345, 346\}$

$L_{9(i)} = \{127, 128, 129, 347, 348, 349, 567, 568, 569\}$

$L_{9(ii)} = \{135, 136, 145, 146, 235, 236, 245, 246, 789\}$

$L_{9(i)} = \{124, 135, 19A, 236, 237, 238, 456, 457, 458\}$

$L_{9(ii)} = \{678, 124, 125, 134, 135, 925, 934, A25, A34\}$

$L_{9(i)} = \{134, 156, 157, 189, 234, 256, 257, 289, ABC\}$

$L_{9(ii)} = \{12A, 12B, 12C, 367, 467, 358, 359, 358, 359\}$

$L_7 = \{678, 123, 124, 125, 134, 135, 925, 934\}$

$L_{11} = \{124, 135, 196, 197, 198, 236, 237, 238, 456, 457, 458\}$

$L_7 = \{167, 189, 18A, 267, 289, 28A, 345\}$

$L_{11} = \{123, 124, 125, 69A, 79A, 683, 684, 685, 783, 784, 785\}$

$L_7 = \{19B, 1AB, 236, 237, 238, 459, 45A\}$

$L_{11} = \{678, 29A, 39A, 124, 125, 134, 135, B24, B25, B34, B35\}$

$L_7 = \{167, 189, 1AB, 267, 289, 2AB, 345\}$

$L_{11} = \{123, 124, 125, 68A, 68B, 69A, 69B, 78A, 78B, 79A, 79B\}$

$L_7 = \{127, 128, 347, 348, 567, 568, 9AB\}$

$L_{11} = \{135, 136, 145, 146, 235, 236, 245, 246, 789, 78A, 78B\}$

$L_5 = \{345, 167, 189, 267, 289\}$

$L_{15} = \{123, 124, 125, 683, 684, 685, 693, 694, 695, 783, 784, 785, 793, 794, 795\}$

$L_8 = \{1DE, 1FG, 123, 124, 125, 29A, 2BC, 678\}$

$L_{12} = \{345, 126, 127, 128, 19B, 19C, 1AB, 1AC, 2DF, 2DG, 2EF, 2EG\}$

$L_9 = \{125, 126, 127, 345, 346, 347, 138, 248, 9AB\}$

$L_{12} = \{567, 138, 248, 149, 14A, 14B, 239, 23A, 23B, 813, 814, 823, 824\}$

$L_6 = \{123, 124, 125, 67B, 67C, 89A\}$

$L_{15} = \{345, 168, 169, 16A, 178, 179, 17A, 1BC, 268, 269, 26A, 278, 279, 27A, 2BC\}$

6. Study 4-choosability of complete bipartite graphs. Since it is easy to prove that $K_{4,b}$ is 4-choosable if and only if $b \leq 63$, an open problem is to find the maximum number of b such that $K_{5,b}$ $(K_{6,b}, K_{7,b}, \ldots)$ is 4-choosable.

7. Find the smallest number n such that there exists a non 4-choosable complete bipartite graph with n vertices. Recall that the smallest non 3-choosable complete bipartite graph has 14 vertices; this statement is proved by Hanson, MacGillivray, and Toft [8]. (See [1] [2] [3] [14] [16] [18] for more information.)

8. Find each positive integer t such that $K_{\binom{2\cdot 4-1}{4},\binom{2\cdot 4-1}{4}}$ is $(4,t)$-choosable.

9. In chapter V, we prove that if $2k-1 \leq t \leq 17\cdot 2^{k-2}-4k-4$, then $K_{\binom{2k-1}{k},\binom{2k-1}{k}}$ is not (k,t)-choosable. A possible future work is to improve the upper bound.

10. Theorem 5.3.3 gives results about $(3,t)$-choosability of complete bipartite graphs with at most 20 vertices except the case $t=6$. Another future work is to study the $(3,6)$-choosability of complete bipartite graphs with at most 20 vertices.

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