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MULTIPLICATIONS OF COPULAS

SUITABLE FOR

MULTIVARIATE SHUFFLES OF COPULAS

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ในวิทยานิพนธ์ฉบับนี้ เราศึกษานิยามและสมบัติต่าง ๆ ของคอปูลาในหลายมิติรวมถึงศึกษา shuffles of Min หลายตัวแปรซึ่งเป็นหนึ่งในชนิดพิเศษของคอปูลาหลายตัวแปร เราได้นิยามของสองสิ่งขึ้นมา นั่นคือ คอปูลาเรียงสับเปลี่ยนหลายตัวแปร และตัวดำเนินการตัวใหม่ซึ่งมีชื่อว่า heart-product และเรายังมีสมบัติต่าง ๆ ของสองสิ่งนี้ด้วย โดยรายละเอียดคอปูลาเรียงสับเปลี่ยนหลายตัวแปรคือความหนาแน่นในเซตของคอปูลาหลายตัวแปรทั้งหมดที่วัดด้วย supnorm และ heart-product ได้ถูกนิยามมาเพื่อศึกษาความสัมพันธ์ระหว่างคอปูลาหลายตัวแปรนั่นเอง

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In this thesis, we study the definition and properties of d-dimensional copulas (shortly, d-copulas), including multivariate shuffles of Min, which is special class of d-copulas. We define two things; multivariate shuffles of copulas and a new product operation, namely the heart-product, as well as their properties. The heart-product gives an algebraic connection between shuffles of d-copulas and the corresponding shuffles of Min.

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CONTENTS

CHAPTER I

INTRODUCTION

Copulas are one of tools for modelling dependence between random variables regardless of their marginal distributions. Since dependence structure between random variables can be captured by copulas, copula theory has become one of widely studied subjects in probability theory and statistics.

In 1992, Darsow, Nguyen and Olsen introduced a binary operation on bivariate copulas, called $*$-product. In the same year, Mikusiński, Sherwood and Taylor studied a speacial class of bivariate copulas called shuffles of Min. A shuffle of Min is constructed by cutting and rearranging the probability mass of the Min copula M. Mikusiński et al.[4] showed that the copula of continuous random variables X, Y is a shuffle of Min if and only if $Y = f(X)$ almost surely for some piecewise continuous invertible function f. They also showed that any 2-copulas can be approximated pointwise, and hence uniformly, by shuffles of Min.

In 2009, a generalization of bivariate shuffles of Min was introduced by Durante, Sarkoci and Sempi. They defined how to shuffle the mass distribution of a 2-copula and called the resulting copula a shuffle of copula.

In 2012, it is shown by Ruankong, Santiwipanont and Sumetkijakan that every shuffle of a 2-copula C is the $*$-product of a shuffle of Min and C and vice versa.

In 2010, Mikusiński and Taylor studied various approximations of multivariate copulas, one of which is multivariate shuffle-of-Min approximations.

In this thesis, after a brief introduction to copulas in Chapter 2 and a review of Durante and Fernández-Sánchez's generalization of shuffles of Min [2] in Chapter 3, we introduce a definition of multivariate shuffles of copulas motivated by Mikusiński and

Taylor's multivariate shuffles of Min [5] and prove some of its properties including its relationship with the $*$-product in Chapter 4. In Chapter 5, we define a new product operation of d-copulas and investigate a connection between d-shuffles of copulas and d-shuffles of Min via the product.

CHAPTER II

PRELIMINARIES

2.1 Basic measure theory

In this section, we will recall necessary basic measure theory for this thesis.

Definition 1. *A collection $\mathcal{M}$ of subsets of a set X is called a* ***σ-algebra*** *on X if*

(i) $X \in \mathcal{M}$;

(ii) *if* $A \in \mathcal{M}$, *then* $A^c \in \mathcal{M}$; *and*

(iii) *if* $A_1, A_2, \ldots \in \mathcal{M}$, *then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

$(X, \mathcal{M})$ *is called a* ***measurable space*** *if $\mathcal{M}$ is a σ-algebra on X.*

Definition 2. *Let $(X, \mathcal{M})$ be a measurable space. A* ***measure*** *on $\mathcal{M}$ is a function $\mu\colon \mathcal{M} \to [0, \infty]$ such that*

(i) $\mu(\varnothing) = 0$ *and*

(ii) *μ is countably additive, viz., for any countable collection of disjoint sets $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{M}$,*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i\right).$$

Moreover, μ is called a ***probability measure*** *if*

(iii) $\mu(X) = 1$.

If μ is a measure on $\mathcal{M}$ then the triple $(X, \mathcal{M}, \mu)$ is called a **measure space**. Moreover, it is called a **probability space** if μ is a probability measure.

Definition 3. *Let X be a topological space. Then the **Borel σ-algebra** on X is the σ-algebra generated by the collection of all open sets on X, and denoted by $\mathcal{B}(X)$. Any element of $\mathcal{B}(X)$ is called a **Borel set**.*

Let λ denote Lebesgue measure on $\mathbb{R}$ and λ^d denote d-dimensional Lebesgue measure on $\mathbb{R}^d$. Let $\mathbb{I}$ denote the unit interval $[0,1]$, and so $\mathbb{I}^d$ is the unit d-dimensional hypercube(shortly, d-hypercube) ($d \geq 2$).

Next, we recall absolutely continuous measure, d-fold stochastic measure, push-forward measure and μ-decomposition, respectively.

Definition 4. *Let $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d), \mu)$ be a measure space. Then μ is **absolutely continuous** with respect to λ^d if $\lambda^d(E) = 0$ implies $\mu(E) = 0$ for all $E \in \mathcal{B}(\mathbb{I}^d)$.*

Definition 5. *Let $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d), \mu)$ be a measure space. Then μ is a **d-fold stochastic measure** if*

$$\mu(\mathbb{I}^{k-1} \times E \times \mathbb{I}^{d-k}) = \lambda(E),$$

for any $E \in \mathcal{B}(\mathbb{I})$ and $k = 1, \ldots, d$.
*In particular, μ is a **two-fold stochastic measure** (known as **doubly stochastic measure**) if and only if*

$$\mu(E \times \mathbb{I}) = \mu(\mathbb{I} \times E) = \lambda(E),$$

for any $E \in \mathcal{B}(\mathbb{I})$.

Remark 6. *If μ is a d-fold stochastic measure, then each hyperplane has measure zero with respect to μ, and hence can be ignored.*

Definition 7. *Let $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d), \mu)$ be a measure space and $T\colon \mathbb{I}^d \to \mathbb{I}^d$ be a measurable transformation. T is **measure-preserving** if for all $E \in \mathcal{B}(\mathbb{I}^d)$, $\mu(T^{-1}(E)) = \mu(E)$.*

Definition 8. *Let $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d), \mu)$ be a measure space and let $f\colon \mathbb{I}^d \to \mathbb{I}^d$ be a measurable function. A **push-forward** of μ under f is the function $f\#\mu\colon \mathcal{B}(\mathbb{I}^d) \to [0, \infty)$ defined* *by*

$$f\#\mu(E) = \mu(f^{-1}(E)),$$

for any $E \in \mathcal{B}(\mathbb{I}^d)$.

Proposition 9. *Let* $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d), \mu)$ *be a measure space and let* $f\colon \mathbb{I}^d \to \mathbb{I}^d$ *be a measurable function. Then a push-forward* $f\#\mu$ *is a measure on* $\mathcal{B}(\mathbb{I}^d)$.

Furthermore, if μ *is a probability measure then so is* $f\#\mu$.

Proof.

1. $(f\#\mu)(\varnothing) = \mu(f^{-1}(\varnothing)) = \mu(\varnothing) = 0$, since μ is a measure.

2. For any countable collection of disjoint sets $\{E_n\}_{n=1}^{\infty}$ in $\mathcal{B}(\mathbb{I}^d)$,

$$\begin{aligned}(f\#\mu)\left(\bigcup_{n=1}^{\infty}(E_n)\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty}(E_n)\right)\right)\\ &= \mu\left(\bigcup_{n=1}^{\infty} f^{-1}(E_n)\right)\\ &= \sum_{n=1}^{\infty}\mu(f^{-1}(E_n)) && (\mu \text{ is measure})\\ &= \sum_{n=1}^{\infty}(f\#\mu)(E_n).\end{aligned}$$

3. If μ is a probability measure, then we have $\mu(\mathbb{I}^d) = 1$. Then

$$(f\#\mu)(\mathbb{I}^d) = \mu(f^{-1}(\mathbb{I}^d)) = \mu(\mathbb{I}^d) = 1,$$

which is the desired assertion. □

Definition 10. *Let* $(X, \mathcal{B}(X), \mu)$ *be a probability space and* $\{E_i\}$ *be a finite or countably infinite collection of measurable subsets of* X. *Then* $\{E_i\}$ *is a* ***near-decomposition*** *of* X *if it satisfies*

1. $\mu(E_i \cap E_j) = 0$ *whenever* $i \neq j$.

2. $\sum_i \mu(E_i) = \mu(X)$.

We can also refer to such a decomposition as a ***μ-decomposition***.

Example 11. *Let $m \in \mathbb{N}$. Then $\left\{\left(\frac{p}{m}, \frac{p+1}{m}\right)\right\}_{p=0}^{m-1}$ is a near-decomposition of $\mathbb{I}$.*

Solution. 1. Let $i, j \in \{0, 1, \ldots, m-1\}$ be such that $i \neq j$. Then

$$\lambda\left(\left(\frac{i}{m}, \frac{i+1}{m}\right) \cap \left(\frac{j}{m}, \frac{j+1}{m}\right)\right) = \lambda(\varnothing) = 0.$$

2. $\displaystyle\sum_{p=0}^{m-1} \lambda\left(\left(\frac{p}{m}, \frac{p+1}{m}\right)\right) = \sum_{p=0}^{m-1} \frac{1}{m} = 1 = \lambda(\mathbb{I}).$

Thus, $\left\{\left(\frac{p}{m}, \frac{p+1}{m}\right)\right\}_{p=0}^{m-1}$ is a near-decomposition of $\mathbb{I}$. □

Set $\mathcal{I}_m = \{0, 1, \ldots, m-1\}$. In [5], Mikusiński and Taylor setup a **standard near-decomposition of** $\mathbb{I}^d$ defined by

$$\mathbb{I}^d_{m,\boldsymbol{i}} = \left(\frac{i_1}{m}, \frac{i_1+1}{m}\right) \times \cdots \times \left(\frac{i_d}{m}, \frac{i_d+1}{m}\right),$$

where $\boldsymbol{i} = (i_1, \ldots, i_d)$ ranges over $\mathcal{I}^d_m$. In this thesis, we shall denote the d-tuple $(i_1, \ldots, i_d)$ by $\boldsymbol{i}$.

Example 12. *Let $m \in \mathbb{N}$. Then $\{\mathbb{I}^d_{m,\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathcal{I}^d_m}$ is a μ-decomposition on $\mathbb{I}^d$ when μ is a d-fold stochastic measure.*

Solution. Let μ be a d-fold stochastic measure,

1. Let $i_k \neq j_k$ for some k. Then

$$\begin{aligned}
\mu(\mathbb{I}^d_{m,\boldsymbol{i}} \cap \mathbb{I}^d_{m,\boldsymbol{j}}) &= \mu\left(\left(\left(\frac{i_1}{m}, \frac{i_1+1}{m}\right) \cap \left(\frac{j_1}{m}, \frac{j_1+1}{m}\right)\right) \times\right. \\
&\quad \cdots \times \left(\left(\frac{i_k}{m}, \frac{i_k+1}{m}\right) \cap \left(\frac{j_k}{m}, \frac{j_k+1}{m}\right)\right) \times \\
&\quad \left.\cdots \times \left(\left(\frac{i_d}{m}, \frac{i_d+1}{m}\right) \cap \left(\frac{j_d}{m}, \frac{j_d+1}{m}\right)\right)\right) \\
&= \mu\left(\left(\left(\frac{i_1}{m}, \frac{i_1+1}{m}\right) \cap \left(\frac{j_1}{m}, \frac{j_1+1}{m}\right)\right) \times \cdots \times \varnothing \times \cdots\right. \\
&\quad \left.\times \left(\left(\frac{i_d}{m}, \frac{i_d+1}{m}\right) \cap \left(\frac{j_d}{m}, \frac{j_d+1}{m}\right)\right)\right) \\
&= \mu(\varnothing) \\
&= 0.
\end{aligned}$$

2. Since μ is a measure,

$$\begin{aligned}\sum_{\boldsymbol{i}\in\mathcal{I}_m^d}\mu(\mathbb{I}_{m,\boldsymbol{i}}^d) &= \sum_{\boldsymbol{i}\in\mathcal{I}_m^d}\mu\left(\left(\frac{i_1}{m},\frac{i_1+1}{m}\right)\times\cdots\times\left(\frac{i_d}{m},\frac{i_d+1}{m}\right)\right)\\ &= \mu\left(\bigcup_{\boldsymbol{i}\in\mathcal{I}_m^d}\left(\left(\frac{i_1}{m},\frac{i_1+1}{m}\right)\times\cdots\times\left(\frac{i_d}{m},\frac{i_d+1}{m}\right)\right)\right)\\ &= \mu\left(\bigcup_{\boldsymbol{i}\in\mathcal{I}_m^d}\left(\left[\frac{i_1}{m},\frac{i_1+1}{m}\right]\times\cdots\times\left[\frac{i_d}{m},\frac{i_d+1}{m}\right]\right)\right)\\ &= \mu(\mathbb{I}^d),\end{aligned}$$

where we have used Remark 6 in the last equality.

Hence, $\{\mathbb{I}_{m,\boldsymbol{i}}^d\}_{\boldsymbol{i}\in\mathcal{I}_m^d}$ is a μ-decomposition on $\mathbb{I}^d$. □

Next, we recall the definition of d-fold stochastic matrix and its relationship with d-fold stochastic measure.

Let $<$ be the lexicographic ordering on $\mathcal{I}_m^d$; that is

$$(a_1,\ldots,a_d) < (b_1,\ldots,b_d)$$

if there is $l\in\{1,\ldots,d\}$ such that $a_l<b_l$ and $a_k=b_k$ for $k=1,\ldots,l-1$. Set

$$\mu_{\boldsymbol{i}} = \mu(\mathbb{I}_{m,\boldsymbol{i}}^d),$$

where $\boldsymbol{i}\in\mathcal{I}_m^d$.

Definition 13. *Let $Q=[Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^d}$ be a matrix. Q is called **d-fold stochastic matrix** if*

$$\sum_{i_k=p} Q_{\boldsymbol{i}} = \frac{1}{m},$$

for all $k=1,\ldots,d$ and $p\in\mathcal{I}_m$.

Theorem 14. *If μ is a d-fold stochastic measure, then $[\mu_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^d}$ is a d-fold stochastic matrix, where $\mu_{\boldsymbol{i}}=\mu\left(\mathbb{I}_{m,\boldsymbol{i}}^d\right)$.*

Proof. Since μ is a d-fold stochastic measure, by Remark 6,

$$\begin{aligned}\sum_{i_k=p}\mu_{\boldsymbol{i}} &= \sum_{i_k=p}\mu\left(\left(\frac{i_1}{m},\frac{i_1+1}{m}\right)\times\cdots\times\left(\frac{i_k}{m},\frac{i_k+1}{m}\right)\times\cdots\times\left(\frac{i_d}{m},\frac{i_d+1}{m}\right)\right)\\ &= \sum_{\substack{i_n=0\\ \forall n\neq k}}^{m-1}\mu\left(\left(\frac{i_1}{m},\frac{i_1+1}{m}\right)\times\cdots\times\left(\frac{p}{m},\frac{p+1}{m}\right)\times\cdots\times\left(\frac{i_d}{m},\frac{i_d+1}{m}\right)\right)\\ &= \mu\left(\mathbb{I}^{k-1}\times\left(\frac{p}{m},\frac{p+1}{m}\right)\times\mathbb{I}^{d-k}\right) = \lambda\left(\left(\frac{p}{m},\frac{p+1}{m}\right)\right) = \frac{1}{m},\end{aligned}$$

so $[\mu_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^d}$ is a d-fold stochastic matrix. □

2.2 Copulas

Let $\boldsymbol{x} = (x_1,\ldots,x_d)$ and $\boldsymbol{y} = (y_1,\ldots,y_d)$ be any points in $\mathbb{I}^d$ for which $x_k \le y_k$ for all k. Define the closed d-hypercube $[\boldsymbol{x},\boldsymbol{y}]$ by

$$[\boldsymbol{x},\boldsymbol{y}] = [x_1,y_1]\times\cdots\times[x_d,y_d].$$

And we set,

$$\triangle^k_{(a,b)}f(x_1,\ldots,x_d) = f(x_1,\ldots,x_{k-1},b,x_{k+1},\ldots,x_d) - f(x_1,\ldots,x_{k-1},a,x_{k+1},\ldots,x_d),$$

$$\triangle^k_{(a,b)}\mu([0,x_1]\times\ldots\times[0,x_d]) = \mu([0,x_1]\times\ldots\times[0,x_{k-1}]\times[a,b]\times[0,x_{k+1}]\times\ldots\times[0,x_d]),$$

where f is a function and μ is a measure. A d-copula ($d \ge 2$) is the joint distribution function of d uniform $[0,1]$ random variables. One can also define as follows:

Definition 15. *A* ***d-dimensional copula*** *(or shortly, a* ***d-copula****) is a function* $C\colon \mathbb{I}^d \to \mathbb{I}$ *satisfying the following conditions:*

(C1) $C(x_1,\ldots,x_d) = 0$, *whenever* $x_k = 0$ *for some* k.

(C2) $C(x_1,\ldots,x_d) = x_k$, *whenever* $x_i = 1$ *for all* $i \neq k$.

(C3) C *is d-increasing, viz., for each d-hypercube* $[\boldsymbol{a},\boldsymbol{b}] \subseteq \mathbb{I}^d$,

$$V_C([\boldsymbol{a},\boldsymbol{b}]) = \triangle^d_{(a_d,b_d)}\cdots\triangle^1_{(a_1,b_1)}C(x_1,\ldots,x_d) \ge 0.$$

Remark 16. *A probability measure associated with d-copula C is the unique measure $\mu_C\colon \mathcal{B}(\mathbb{I}^d) \to \mathbb{I}$ defined for d-hypercubes $[0,x_1]\times\cdots\times[0,x_d]\subseteq \mathbb{I}^d$ by*

$$\mu_C([0,x_1]\times\cdots\times[0,x_d]) = C(x_1,\ldots,x_d).$$

Let $\mathscr{C}_d$ denote the set of all d-copulas on $\mathbb{I}^d$. Two useful d-copulas are M^d and Π^d where

$$M^d(x_1,\ldots,x_d) = \min\{x_1,\ldots,x_d\},$$
$$\Pi^d(x_1,\ldots,x_d) = x_1\cdots x_d,$$

for all $x_1,\ldots,x_d \in \mathbb{I}$. It is easy to prove that M^d and Π^d are d-copulas. In [5], a random vector $(X_1,\ldots,X_d)$ has the copula M^d if and only if each X_i is almost surely a monotone increasing function of every other X_j and a random vector $(X_1,\ldots,X_d)$ has the copula Π^d if and only if they are independent. We shall also denote M^2,Π^2 by M,Π, respectively. Another useful 2-copula is W where

$$W(x_1,x_2) = \max\{x_1+x_2-1, 0\},$$

for all $x_1,x_2\in\mathbb{I}$. It is easy to prove that W is 2-copula. In [3], a random vector (X_1,X_2) has the copula W if and only if each of X_1 and X_2 is almost surely a decreasing function of the other one.

Definition 17. *The **support** of a d-copula C is the complement of the union of all open subsets E of $\mathbb{I}^d$ with $V_C(E)=0$; that is*

$$supp(C) = \left[\bigcup\{E \text{ is open } : V_C(E)=0\}\right]^c.$$

Example 18.

1. *$supp(M^d)$ is uniformly distributed along the main diagonal (the line segment $x_1=\cdots=x_d$) in $\mathbb{I}^d$. In particular, $supp(M^d)=\{(x,\ldots,x): x\in\mathbb{I}\}$.*
2. *$supp(\Pi^d)=\mathbb{I}^d$.*
3. *$supp(W)$ is uniformly distributed along the diagonal from $(0,1)$ to $(1,0)$ (the line segment $x_2=1-x_1$). In particular, $supp(W)=\{(x,1-x): x\in\mathbb{I}\}$.*

Definition 19. *A d-copula C is said to be* ***absolutely continuous*** *if μ_C is absolutely continuous with respect to λ^d.*

Next, we shall show that a probability measure associated with d-copula C is a d-fold stochastic measure.

Theorem 20. *Let $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d), \mu)$ be a probability space. Then μ is a d-fold stochastic measure if and only if there exists a d-copula C such that $\mu = \mu_C$.*

Proof. Assume that μ is a d-fold stochastic measure. Define

$$C(x_1, \ldots, x_d) = \mu([0, x_1] \times \cdots \times [0, x_d]),$$

for all $x_1, \ldots, x_d \in \mathbb{I}$.

(C1) By Remark 6, $C(x_1, \ldots, x_d) = \mu([0, x_1] \times \cdots \times [0, x_d]) = 0$ whenever $x_k = 0$ for some k.

(C2) Since μ is a d-fold stochastic measure, for all $k = 1, \ldots, d$,

$$C(\overbrace{1, \ldots, 1}^{k-1 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}) = \mu(\mathbb{I}^{k-1} \times [0, x_k] \times \mathbb{I}^{d-k}) = \lambda([0, x_k]) = x_k.$$

(C3) For any $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{I}^d$,

$$\begin{aligned} V_C([\boldsymbol{a}, \boldsymbol{b}]) &= \triangle^d_{(a_d, b_d)} \cdots \triangle^1_{(a_1, b_1)} C(x_1, \ldots, x_d) \\ &= \triangle^d_{(a_d, b_d)} \cdots \triangle^1_{(a_1, b_1)} \mu([0, x_1] \times \cdots \times [0, x_d]) \\ &= \mu([a_1, b_1] \times \cdots \times [a_d, b_d]) \\ &\geq 0, \text{ since } \mu \text{ is a measure.} \end{aligned}$$

Thus, C is a d-copula.

Conversely, let C be a d-copula. Claim that μ_C is a d-fold stochastic measure. Consider, for any $k = 1, \ldots, d$,

$$\mu_C(\mathbb{I}^{k-1} \times [0, x_k] \times \mathbb{I}^{d-k}) = C(\overbrace{1, \ldots, 1}^{k-1 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}) = x_k = \lambda([0, x_k]),$$

which is the desired assertion. $\square$

And, we have an essential lemma for later use.

Lemma 21. *For* $f\colon \mathbb{I}^d \to \mathbb{I}$ *and* $a_k, b_k \in \mathbb{I}$, $a_k < b_k$, *define* $f_k\colon \mathbb{I}^{d-1} \to \mathbb{I}$ *by*

$$f_k(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d) = \triangle^k_{(a_k,b_k)} f(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_d).$$

If f *is* d*-increasing, then* f_k *is* $(d-1)$*-increasing.*

Proof. Since f is d-increasing, for any $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{I}^d$,

$$\begin{aligned} 0 &\le V_f([\boldsymbol{a}, \boldsymbol{b}]) \\ &= \triangle^d_{(a_d,b_d)} \cdots \triangle^1_{(a_1,b_1)} f(t_1, \ldots, t_d) \\ &= \triangle^d_{(a_d,b_d)} \cdots \triangle^{k+1}_{(a_{k+1},b_{k+1})} \triangle^{k-1}_{(a_{k-1},b_{k-1})} \triangle^1_{(a_1,b_1)} [\triangle^k_{(a_k,b_k)} f(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_d)] \\ &= \triangle^d_{(a_d,b_d)} \cdots \triangle^{k+1}_{(a_{k+1},b_{k+1})} \triangle^{k-1}_{(a_{k-1},b_{k-1})} \triangle^1_{(a_1,b_1)} [f_k(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_d)] \\ &= V_{f_k}([a_1, b_1] \times \cdots \times [a_{k-1}, b_{k-1}] \times [a_{k+1}, b_{k+1}] \times \cdots \times [a_d, b_d]) \end{aligned}$$

Thus, f_k is $(d-1)$-increasing. □

Remark 22. *By Lemma 21, given a* d*-copula* C*, for all* $k = 1, \ldots, d$,

$$\triangle^k_{(a_k,b_k)} C(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_d)$$

is $(d-1)$*-increasing.*

For each $k = 1, \ldots, d$, *by using Lemma 21 repeatedly,*

$$\triangle^d_{(a_d,b_d)} \cdots \triangle^{k+1}_{(a_{k+1},b_{k+1})} \triangle^{k-1}_{(a_{k-1},b_{k-1})} \cdots \triangle^1_{(a_1,b_1)} C(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_d)$$

is an increasing function of x_k*, and hence*

$$\partial_k [\triangle^d_{(a_d,b_d)} \cdots \triangle^{k+1}_{(a_{k+1},b_{k+1})} \triangle^{k-1}_{(a_{k-1},b_{k-1})} \cdots \triangle^1_{(a_1,b_1)} C(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_d)] \ge 0$$

almost everywhere.

2.3 Bivariate shuffles of Min

Mikusiński, Sherwood and Taylor [4] had defined a special class of 2-copulas, called shuffles of Min.

Definition 23 (Mikusiński, Sherwood and Taylor [4]). *A 2-copula C is a **shuffle of Min** if there is a natural number n, two partitions $0 = s_0 < s_1 < \ldots < s_n = 1$ and $0 = t_0 < t_1 < \ldots < t_n = 1$ of $\mathbb{I}$, and a permutation σ of $\{1, 2, \ldots, n\}$ such that each $[s_{k-1}, s_k] \times [t_{\sigma(k)-1}, t_{\sigma(k)}]$ is a square in which C distributes a mass $s_k - s_{k-1}$ uniformly spread along one of the diagonals.*

.

Remark 24. *For any two uniformly distributed random variables X and Y, the 2-copula of X and Y is a shuffle of Min if and only if there exists a bijective piece-wise continuous function $f\colon \mathbb{I} \to \mathbb{I}$ such that $X = f(Y)$.*

An important application of shuffles of Min is that any 2-copula can be approximated by certain shuffles of Min in the uniform norm given by $||f|| = \sup\limits_{x\in\mathbb{I}^2} |f(x)|$ for $f \in span\mathscr{C}_2$.

Theorem 25 (Mikusiński, Sherwood and Taylor [4]). *Shuffles of Min are dense in $\mathscr{C}_2$ with respect to the uniform norm.*

2.4 Bivariate shuffles of copulas

A shuffle of 2-copula C is a 2-copula whose mass distribution is obtained from shuffling vertical strips of the mass of C. Durante, Sarkoci and Sempi [3] defined shuffles of 2-copulas starting from a measure-preserving bijective transformation $T\colon \mathbb{I} \to \mathbb{I}$ and a map $S_T\colon \mathbb{I}^2 \to \mathbb{I}^2$ defined by

$$S_T(u, v) = (T(u), v),$$

for all $u, v \in \mathbb{I}$.

Let $\mathcal{T}$ denote the set of all measure-preserving bijections of the measure space

$(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda)$.

Remark 26. *Since $T \in \mathcal{T}$, for all $E_1, E_2 \in \mathcal{B}(\mathbb{I})$,*

$$\lambda^2(S_T(E_1 \times E_2)) = \lambda^2((T(E_1) \times E_2)) = \lambda(T(E_1)) \times \lambda(E_2) = \lambda(E_1) \times \lambda(E_2) = \lambda^2(E_1 \times E_2),$$

so S_T is a measure preserving on the space $(\mathbb{I}^2, \mathcal{B}(\mathbb{I}^2), \lambda^2)$.

Definition 27 (Durante, Sarkoci and Sempi [3]). *A 2-copula D is a **shuffle of 2-copula** C if there exists $T \in \mathcal{T}$ such that $\mu_D = S_T \# \mu_C$.*

*In this case, D is also called the T**-shuffle of** C and denoted by C_T.*

And they have an interesting proposition.

Proposition 28. *If a 2-copula C is absolutely continuous, then so are all its shuffles.*

Proof. Let a 2-copula C be an absolutely continuous copula and let $T \in \mathcal{T}$. Let E be a Borel set of $\mathbb{I}^2$ with $\lambda^2(E) = 0$. Then $\lambda^2(S_T^{-1}(E)) = \lambda^2(E) = 0$. So, by the absolute continuity of C, $S_T \# \mu_C(E) = \mu_C(S_T^{-1}(E)) = 0$. Thus $S_T \# \mu_C$ is absolutely continuous, as asserted. □

2.5 The $*$-product

The $*$-product is a binary operation of 2-copulas. So we recall its definition, some properties and a theorem relating the $*$-product and measures.

Let ∂_k denote the partial derivative with respect to the k^{th} coordinate and let $\partial_{i_1,\ldots,i_d}$ denote the partial derivative with respect to the $i_d^{th}, \ldots, i_1^{th}$ coordinates, respectively.

Definition 29 (Darsow, Nguyen and Olsen [1]). *Let $A, B \in \mathscr{C}_2$. The **$*$-product** of A and B is the function $A * B \colon \mathbb{I}^2 \to \mathbb{I}$ defined by, for $x, y \in \mathbb{I}$,*

$$A * B(x, y) = \int_0^1 \partial_2 A(x, t) \partial_1 B(t, y)\, dt.$$

Theorem 30. *The $*$-product is a binary operation on $\mathscr{C}_2$.*

Proof. Let $A, B \in \mathscr{C}_2$. Since (C1) is obvious, we only prove (C2) and (C3).

(C2) For all $x, y \in \mathbb{I}$,

$$A * B(x,1) = \int_0^1 \partial_2 A(x,t)\partial_1 B(t,1)\,dt = \int_0^1 \partial_2 A(x,t)(1)\,dt = A(x,1) = x.$$
$$A * B(1,y) = \int_0^1 \partial_2 A(1,t)\partial_1 B(t,y)\,dt = \int_0^1 (1)\partial_1 B(t,y)\,dt = B(1,y) = y.$$

(C3) For any $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{I}^2$,

$$\begin{aligned}
V_{A*B}([\boldsymbol{a},\boldsymbol{b}]) &= A*B(b_1,b_2) - A*B(b_1,a_2) - A*B(a_1,b_2) + A*B(a_1,a_2) \\
&= \int_0^1 \partial_2 A(b_1,t)\partial_1 B(t,b_2)\,dt - \int_0^1 \partial_2 A(b_1,t)\partial_1 B(t,a_2)\,dt \\
&\quad - \int_0^1 \partial_2 A(a_1,t)\partial_1 B(t,b_2)\,dt + \int_0^1 \partial_2 A(a_1,t)\partial_1 B(t,a_2)\,dt \\
&= \int_0^1 \partial_2[A(b_1,t) - A(a_1,t)]\partial_1[B(t,b_2) - B(t,a_2)]\,dt \\
&\geq 0, \text{ by Remark 22.}
\end{aligned}$$

Hence, $A * B$ is a 2-copula. $\square$

Proposition 31. *Let $C \in \mathscr{C}_2$. Then*

$$M * C = C = C * M,$$
$$\Pi * C = \Pi = C * \Pi,$$
$$\textit{and} \quad W * W = M.$$

Proof. For any $x, y \in \mathbb{I}$,

$$M * C(x,y) = \int_0^1 \partial_2 M(x,t)\partial_1 C(t,y)\,dt = \int_0^x (1)\partial_1 C(t,y)\,dt = C(x,y),$$
$$C * M(x,y) = \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y)\,dt = \int_0^y \partial_2 C(x,t)(1)\,dt = C(x,y),$$

and

$$\Pi * C(x,y) = \int_0^1 \partial_2 \Pi(x,t)\partial_1 C(t,y)\,dt = \int_0^1 (x)\partial_1 C(t,y)\,dt = xy = \Pi(x,y).$$

$$C * \Pi(x,y) = \int_0^1 \partial_2 C(x,t)\partial_1 \Pi(t,y)\,dt = \int_0^1 \partial_2 C(x,t)(y)\,dt = xy = \Pi(x,y).$$

and

$$\begin{aligned} W * W(x,y) &= \int_0^1 \partial_2 W(x,t)\partial_1 W(t,y)\,dt \\ &= \int_{1-x}^1 (1)\partial_1 W(t,y)\,dt \\ &= W(1,y) - W(1-x,y) \\ &= y - \max\{-x+y,0\} \\ &= y + \min\{x-y,0\} = \min\{x,y\} = M(x,y), \end{aligned}$$

which is the desired assertion. □

Theorem 32 (Ruankong, Santiwipanont and Sumetkijakan [7]). *If μ and ν are doubly stochastic measures on $\mathbb{I}^2$, then*

$$\mu * \nu(I \times J) = \int_0^1 \frac{d}{dt}\mu(I \times [0,t])\frac{d}{dt}\nu([0,t] \times J)\,dt$$

is a doubly stochastic measure. Moreover, if A and B are 2-copulas, then $\mu_{A*B} = \mu_A * \mu_B$.

CHAPTER III

A GENERALIZATION OF SHUFFLES OF MIN

In this chapter, we recall a method for constructing multivariate shuffles of copulas as introduced by Durante and Fernández-Sánchez [2]. This can be considered as a generalization of shuffles of Min.

Definition 33. *Given $N \subseteq \mathbb{N}$, let $\mathscr{J}^1, \ldots, \mathscr{J}^d$ be systems of closed and non-empty intervals of $\mathbb{I}$, where each $\mathscr{J}^k = \{J_n^k = [a_n^k, b_n^k] : n \in N\}$ is a near-decomposition of $\mathbb{I}$ such that $\lambda(J_n^1) = \lambda(J_n^2) = \ldots = \lambda(J_n^d)$ for every $n \in N$. A system $\mathscr{J} = (\mathscr{J}^k)_{k=1}^d$ satisfying the above properties is called a* ***shuffling structure*** *(shortly,* ***$\mathscr{J}$-structure****). The set of all $\mathscr{J}$-structures based on an index set N is indicated by $\mathcal{J}_N$, while $\mathcal{J} = \bigcup\limits_{N \subseteq \mathbb{N}} \mathcal{J}_N$.*

Remark 34. *Since for each k, $\mathscr{J}^k = \{J_n^k = [a_n^k, b_n^k] : n \in N\}$ is a near-decomposition of $\mathbb{I}$, $\bigcup\limits_{(n_1,\ldots,n_d) \in N^d} (J_{n_1}^1 \times \cdots \times J_{n_d}^d) = \mathbb{I}^d$*

Definition 35. *Let $N \subseteq \mathbb{N}$. Let $(\mu_n)_{n \in N}$ be a system of probability measure on $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d))$. Let a system $\mathscr{J} = (\mathscr{J}^k)_{k=1}^d$ be an $\mathscr{J}$-structure. Let $\mu \colon \mathcal{B}(\mathbb{I}^d) \to [0, \infty)$ be the set-function defined, for every $E \in \mathcal{B}(\mathbb{I}^d)$, by*

$$\mu(E) = \sum_{(n_1,\ldots,n_d) \in N^d} \mu_{n_1,\ldots,n_d}(E \cap (J_{n_1}^1 \times \cdots \times J_{n_d}^d)),$$

where, for all $(n_1, \ldots, n_d) \in N^d$, $\mu_{n_1,\ldots,n_d}$ is the set-function defined on the Borel sets of $J_{n_1}^1 \times \cdots \times J_{n_d}^d$ in the following way:

(M1) *$\mu_{n_1,\ldots,n_d} = 0$ if $n_k \neq n_{k'}$ for some k, k';*

(M2) *for every Borel set $E \subseteq J_n^1 \times \cdots \times J_n^d$,*

$$\mu_{n,\ldots,n}(E) = \lambda(J_n^1)\mu_n(\varphi_n(E)),$$

where $\varphi_n \colon J_n^1 \times \cdots \times J_n^d \to \mathbb{I}^d$ *is given by*

$$\varphi_n(x_1, \ldots, x_d) = \left(\frac{x_1 - a_n^1}{\lambda(J_n^1)}, \ldots, \frac{x_d - a_n^d}{\lambda(J_n^1)} \right).$$

The set-function μ *is called a* ***shuffling set-function*** *correlated with the* $\mathscr{J}$*-structure* $(\mathscr{J}^k)_{k=1}^d$ *and to* $(\mu_n)_{n\in N}$*. It is indicated by the symbol* $\mu = \left\langle (\mathscr{J}^k)_{k=1}^d, (\mu_n)_{n\in N} \right\rangle$.

Proposition 36. *Let* $\mu = \left\langle (\mathscr{J}^k)_{k=1}^d, (\mu_n)_{n\in N} \right\rangle$ *be a shuffling set-function. Then* μ *is a probability measure.*

Proof. 1. Since $\mu_{n_1,\ldots,n_d}$ is a measure,

$$\mu(\varnothing) = \sum_{(n_1,\ldots,n_d)\in N^d} \mu_{n_1,\ldots,n_d}(\varnothing \cap (J_{n_1}^1 \times \cdots \times J_{n_d}^d)) = \sum_{(n_1,\ldots,n_d)\in N^d} \mu_{n_1,\ldots,n_d}(\varnothing) = 0.$$

2. Since $\mu_{n_1,\ldots,n_d}$ is a measure, for any disjoint sets $E_i \in \mathcal{B}(\mathbb{I}^d)$,

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu(E_i) &= \sum_{i=1}^{\infty} \sum_{(n_1,\ldots,n_d)\in N^d} \mu_{n_1,\ldots,n_d}(E_i \cap (J_{n_1}^1 \times \cdots \times J_{n_d}^d)) \\
&= \sum_{(n_1,\ldots,n_d)\in N^d} \mu_{n_1,\ldots,n_d}\left(\bigcup_{i=1}^{\infty} E_i \cap (J_{n_1}^1 \times \cdots \times J_{n_d}^d) \right) \\
&= \mu\left(\bigcup_{i=1}^{\infty} E_i \right).
\end{aligned}$$

3. Since $\mu_n(\mathbb{I}^d) = 1$ and, by (M1) and (M2), we have

$$\begin{aligned}
\mu(\mathbb{I}^d) &= \sum_{(n_1,\ldots,n_d)\in N^d} \mu_{n_1,\ldots,n_d}(\mathbb{I}^d \cap (J_{n_1}^1 \times \cdots \times J_{n_d}^d)) \\
&= \sum_{(n_1,\ldots,n_d)\in N^d} \mu_{n_1,\ldots,n_d}(J_{n_1}^1 \times \cdots \times J_{n_d}^d) \\
&= \sum_{n\in N} \mu_{n,\ldots,n}(J_n^1 \times \cdots \times J_n^d) \\
&= \sum_{n\in N} \lambda(J_n^1)\mu_n(\varphi_n(J_n^1 \times \cdots \times J_n^d)) \\
&= \sum_{n\in N} \lambda(J_n^1)\mu_n(\mathbb{I}^d) = \sum_{n\in N} \lambda(J_n^1) = 1.
\end{aligned}$$

Hence, μ is a probability measure on $(\mathbb{I}^d, \mathcal{B}(\mathbb{I}^d))$. $\square$

Proposition 37. *Let* $\mu = \left\langle (\mathscr{J}^k)_{k=1}^d, (\mu_n)_{n\in N} \right\rangle$ *be a shuffling set-function. If every* μ_n *is d-fold stochastic, then* μ *is d-fold stochastic.*

Proof. Assume that each μ_n is d-fold stochastic. Claim that μ is d-fold stochastic. Suppose that $i = 1$ (the other cases are analogous). Let $x_1 \in \mathbb{I}$. First, we want to show $\mu(\{x_1\} \times \mathbb{I}^{d-1}) = 0$. Consider,

$$\begin{aligned}\mu(\{x_1\} \times \mathbb{I}^{d-1}) &= \sum_{(n_1,\dots,n_d)\in N^d} \mu_{n_1,\dots,n_d}((\{x_1\} \times \mathbb{I}^{d-1}) \cap (J^1_{n_1} \times \cdots \times J^d_{n_d})) \\ &= \sum_{(n_1,\dots,n_d)\in N^d} \mu_{n_1,\dots,n_d}((\{x_1\} \cap J^1_{n_1}) \times J^2_{n_2} \times \cdots \times J^d_{n_d}).\end{aligned}$$

Cases 1. If $x_1 \notin J^1_{n_1}$, then

$$\mu(\{x_1\} \times \mathbb{I}^{d-1}) = \sum_{(n_1,\dots,n_d)\in N^d} \mu_{n_1,\dots,n_d}(\varnothing \times J^2_{n_2} \times \cdots \times J^d_{n_d}) = 0.$$

Cases 2. If $x_1 \in J^1_{n_1}$, then

$$\begin{aligned}\mu(\{x_1\} \times \mathbb{I}^{d-1}) &= \sum_{(n_1,\dots,n_d)\in N^d} \mu_{n_1,\dots,n_d}(\{x_1\} \times J^2_{n_2} \times \cdots \times J^d_{n_d}) \\ &= \sum_{n\in N} \mu_{n,\dots,n}(\{x_1\} \times J^2_n \times \cdots \times J^d_n) && \text{(By (M1))} \\ &= \sum_{n\in N} \lambda(J^1_n)\mu_n(\varphi_n(\{x_1\} \times J^2_n \times \cdots \times J^d_n)) && \text{(By (M2))} \\ &= \sum_{n\in N} \lambda(J^1_n)\mu_n\left(\left\{\frac{x_1 - a^1_n}{\lambda(J^1_n)}\right\} \times \mathbb{I}^{d-1}\right) \\ &= \sum_{n\in N} \lambda(J^1_n)\lambda\left(\left\{\frac{x_1 - a^1_n}{\lambda(J^1_n)}\right\}\right) && \text{(By assumption)} \\ &= \sum_{n\in N} \lambda(J^1_n)(0) = 0.\end{aligned}$$

Since $\mu(\{x_1\} \times \mathbb{I}^{d-1}) = 0$, without loss of generality, we can suppose that $x_1 \notin \bigcup_{n\in N} \{a^1_n, b^1_n\}$. Let $N_1 = \{n \in N : J^1_n \subseteq [0, x_1] \text{ and } x_1 \notin J^1_n\}$. Let $\hat{n} \in N$ be such that $x_1 \in J^1_{\hat{n}} = [a^1_{\hat{n}}, b^1_{\hat{n}}]$. By (M1) and (M2),

$$\mu([0, x_1] \times \mathbb{I}^{d-1}) = \sum_{n\in N} \lambda(J^1_n)\mu_n(\varphi_n(([0, x_1] \times \mathbb{I}^{d-1}) \cap (J^1_n \times J^2_n \times \cdots \times J^d_n)))$$

$$= \sum_{n \in N_1} \lambda(J_n^1)\mu_n(\varphi_n(J_n^1 \times \cdots \times J_n^d)))$$

$$+ \lambda(J_{\hat{n}}^1)\mu_{\hat{n}}(\varphi_{\hat{n}}(([0, x_1] \cap J_{\hat{n}}^1) \times J_{\hat{n}}^2 \times \cdots \times J_{\hat{n}}^d))$$

$$= \sum_{n \in N_1} \lambda(J_n^1)\mu_n(\mathbb{I}^d) + \lambda(J_{\hat{n}}^1)\mu_{\hat{n}}\left(\left[0, \frac{x_1 - a_{\hat{n}}^1}{\lambda(J_{\hat{n}}^1)}\right] \times \mathbb{I}^{d-1}\right)$$

$$= \sum_{n \in N_1} \lambda(J_n^1) + \lambda(J_{\hat{n}}^1)\lambda\left(\left[0, \frac{x_1 - a_{\hat{n}}^1}{\lambda(J_{\hat{n}}^1)}\right]\right) \qquad \text{(By assumption)}$$

$$= \sum_{n \in N_1} \lambda(J_n^1) + \lambda(J_{\hat{n}}^1)\left(\frac{x_1 - a_{\hat{n}}^1}{\lambda(J_{\hat{n}}^1)}\right)$$

$$= \sum_{n \in N_1} \lambda(J_n^1) + (x_1 - a_{\hat{n}}^1)$$

$$= \lambda([0, a_{\hat{n}}^1]) + \lambda([a_{\hat{n}}^1, x_1]) = \lambda([0, x_1]).$$

Hence, μ is d-fold stochastic. □

A d-copula C is a **shuffling copula** (shortly, **$\mathscr{J}$-copula**) if μ_C can be represented as a shuffling set-function that is a d-fold stochastic. It is indicated by the symbol $C = \left\langle (\mathscr{J}^k)_{k=1}^d, (C_n)_{n \in N} \right\rangle = \left\langle (\mathscr{J}^k)_{k=1}^d, (\mu_n)_{n \in N} \right\rangle$, where $\mu_n = \mu_{C_n}$.

First example, I want to show an easy example.

Example 38. *Let* $J_1^1 = [0, \frac{1}{2}]$, $J_2^1 = [\frac{1}{2}, 1]$, $J_1^2 = [\frac{1}{2}, 1]$, $J_2^2 = [0, \frac{1}{2}]$, $N = \{1, 2\}$, $d = 2$ *and* $C' \in \mathscr{C}_d$. *Let* $\mu_{C'} : \mathcal{B}(\mathbb{I}^2) \to [0, \infty)$ *be the set-function defined for every Borel set* $F \subseteq \mathbb{I}^2$,

$$\mu_{C'}(F) = \sum_{(n_1, n_2) \in N^2} \mu_{n_1, n_2}(F \cap (J_{n_1} \times J_{n_2})).$$

Solution. For every $u, v \in \mathbb{I}$, let $E = [0, u] \times [0, v]$,

$$C'(u, v) = \mu_{C'}([0, u] \times [0, v]) = \mu_{C'}(E)$$

$$= \sum_{(n_1, n_2) \in N^2} \mu_{n_1, n_2}(E \cap (J_{n_1}^1 \times J_{n_2}^2))$$

$$= \mu_{1,1}(E \cap (J_1^1 \times J_1^2)) + \mu_{1,2}(E \cap (J_1^1 \times J_2^2))$$

$$+ \mu_{2,1}(E \cap (J_2^1 \times J_1^2)) + \mu_{2,2}(E \cap (J_2^1 \times J_2^2))$$

$$= \mu_{1,1}(E \cap (J_1^1 \times J_1^2)) + \mu_{2,2}(E \cap (J_2^1 \times J_2^2)) \qquad \text{(By (M1))}$$

$$
\begin{aligned}
&= \lambda(J_1^1)\mu_1(\varphi_1((E \cap (J_1^1 \times J_1^2)))) \\
&\quad + \lambda(J_2^1)\mu_2(\varphi_2((E \cap (J_2^1 \times J_2^2)))) \qquad \text{(By (M2))} \\
&= \frac{1}{2}\mu_1(\varphi_1(([0,u] \cap J_1^1) \times ([0,v] \cap J_1^2))) \\
&\quad + \frac{1}{2}\mu_2(\varphi_2(([0,u] \cap J_2^1) \times ([0,v] \cap J_2^2))).
\end{aligned}
$$

Case 1. If $u \in J_1^1 = [0, \frac{1}{2}]$ and $v \in J_2^2 = [0, \frac{1}{2}]$, then

$$C'(u,v) = 0 + 0 = 0, \text{ by Remark 6.}$$

Case 2. If $u \in J_1^1 = [0, \frac{1}{2}]$ and $v \in J_1^2 = [\frac{1}{2}, 1]$, then

$$
\begin{aligned}
C'(u,v) &= \frac{1}{2}\mu_1\left(\left[0, \frac{u-0}{\frac{1}{2}}\right] \times \left[0, \frac{v-\frac{1}{2}}{\frac{1}{2}}\right]\right) + 0 \qquad \text{(By Remark 6)} \\
&= \frac{1}{2}\mu_1([0,2u] \times [0,2v-1]) \\
&= \frac{1}{2}C_1(2u, 2v-1).
\end{aligned}
$$

Case 3. If $u \in J_2^1 = [\frac{1}{2}, 1]$ and $v \in J_2^2 = [0, \frac{1}{2}]$, then

$$
\begin{aligned}
C'(u,v) &= 0 + \frac{1}{2}\mu_2\left(\left[0, \frac{u-\frac{1}{2}}{\frac{1}{2}}\right] \times \left[0, \frac{v-0}{\frac{1}{2}}\right]\right) \qquad \text{(By Remark 6)} \\
&= \frac{1}{2}\mu_2([0,2u-1] \times [0,2v]) \\
&= \frac{1}{2}C_2(2u-1, 2v).
\end{aligned}
$$

Case 4. If $u \in J_2^1 = [\frac{1}{2}, 1]$ and $v \in J_1^2 = [\frac{1}{2}, 1]$, then

$$
\begin{aligned}
C'(u,v) &= \frac{1}{2}\mu_1\left(\mathbb{I} \times \left[0, \frac{v-\frac{1}{2}}{\frac{1}{2}}\right]\right) + \frac{1}{2}\mu_2\left(\left[0, \frac{u-\frac{1}{2}}{\frac{1}{2}}\right] \times \mathbb{I}\right) \\
&= \frac{1}{2}\mu_1(\mathbb{I} \times [0,2v-1]) + \frac{1}{2}\mu_2([0,2u-1] \times \mathbb{I}) \\
&= \frac{1}{2}\lambda([0,2v-1]) + \frac{1}{2}\lambda([0,2u-1]) \\
&= \frac{1}{2}(2v-1) + \frac{1}{2}(2u-1) \\
&= u + v - 1.
\end{aligned}
$$

Thus,

$$C'(u,v) = \begin{cases} 0 & , \text{ if } u \in [0, \frac{1}{2}], v \in [0, \frac{1}{2}]; \\ \frac{1}{2}C_1(2u, 2v-1) & , \text{ if } u \in [0, \frac{1}{2}], v \in [\frac{1}{2}, 1]; \\ \frac{1}{2}C_2(2u-1, 2v) & , \text{ if } u \in [\frac{1}{2}, 1], v \in [0, \frac{1}{2}]; \\ u+v-1 & , \text{ if } u \in [\frac{1}{2}, 1], v \in [\frac{1}{2}, 1], \end{cases}$$

where C_1 and C_2 are given copulas. Since we want to shuffle of Min, setting $C_1 = M = C_2$, so

$$M_{T,a=\frac{1}{2}}(u,v) = \begin{cases} 0 & , \text{ if } u \in [0, \frac{1}{2}], v \in [0, \frac{1}{2}]; \\ \frac{1}{2}M(2u, 2v-1) & , \text{ if } u \in [0, \frac{1}{2}], v \in [\frac{1}{2}, 1]; \\ \frac{1}{2}M(2u-1, 2v) & , \text{ if } u \in [\frac{1}{2}, 1], v \in [0, \frac{1}{2}]; \\ u+v-1 & , \text{ if } u \in [\frac{1}{2}, 1], v \in [\frac{1}{2}, 1], \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } u \in [0, \frac{1}{2}], v \in [0, \frac{1}{2}]; \\ \frac{1}{2}\min\{2u, 2v-1\} & , \text{ if } u \in [0, \frac{1}{2}], v \in [\frac{1}{2}, 1]; \\ \frac{1}{2}\min\{2u-1, 2v\} & , \text{ if } u \in [\frac{1}{2}, 1], v \in [0, \frac{1}{2}]; \\ u+v-1 & , \text{ if } u \in [\frac{1}{2}, 1], v \in [\frac{1}{2}, 1], \end{cases}$$

$$= \begin{cases} 0 & , \text{ if } u \in [0, \frac{1}{2}], v \in [0, \frac{1}{2}]; \\ \min\{u, v-\frac{1}{2}\} & , \text{ if } u \in [0, \frac{1}{2}], v \in [\frac{1}{2}, 1]; \\ \min\{u-\frac{1}{2}, v\} & , \text{ if } u \in [\frac{1}{2}, 1], v \in [0, \frac{1}{2}]; \\ u+v-1 & , \text{ if } u \in [\frac{1}{2}, 1], v \in [\frac{1}{2}, 1], \end{cases}$$

$$= \begin{cases} v-v & , \text{ if } u \in [0, \frac{1}{2}], v \in [0, \frac{1}{2}]; \\ \min\{u+\frac{1}{2}, v\} - \frac{1}{2} & , \text{ if } u \in [0, \frac{1}{2}], v \in [\frac{1}{2}, 1]; \\ \min\{u-\frac{1}{2}, v\} + v - v & , \text{ if } u \in [\frac{1}{2}, 1], v \in [0, \frac{1}{2}]; \\ (u-\frac{1}{2}) + v - \frac{1}{2} & , \text{ if } u \in [\frac{1}{2}, 1], v \in [\frac{1}{2}, 1], \end{cases}$$

$$= \begin{cases} \min\{u+\frac{1}{2}, v\} - \min\{\frac{1}{2}, v\} & \text{, if } u \in [0, \frac{1}{2}], v \in [0, \frac{1}{2}]; \\ \min\{u+\frac{1}{2}, v\} - \min\{\frac{1}{2}, v\} & \text{, if } u \in [0, \frac{1}{2}], v \in [\frac{1}{2}, 1]; \\ \min\{u-\frac{1}{2}, v\} + \min\{1, v\} - \min\{\frac{1}{2}, v\} & \text{, if } u \in [\frac{1}{2}, 1], v \in [0, \frac{1}{2}]; \\ \min\{u-\frac{1}{2}, v\} + \min\{1, v\} - \min\{\frac{1}{2}, v\} & \text{, if } u \in [\frac{1}{2}, 1], v \in [\frac{1}{2}, 1], \end{cases}$$

$$= \begin{cases} \min\{u+\frac{1}{2}, v\} - \min\{\frac{1}{2}, v\} & \text{, if } u \in [0, \frac{1}{2}]; \\ \min\{u-\frac{1}{2}, v\} + \min\{1, v\} - \min\{\frac{1}{2}, v\} & \text{, if } u \in [\frac{1}{2}, 1], \end{cases}$$

$$= \begin{cases} M(0, v) + M(u+\frac{1}{2}, v) - M(\frac{1}{2}, v) & \text{, if } u \in [0, \frac{1}{2}]; \\ M(u-\frac{1}{2}, v) + M(1, v) - M(\frac{1}{2}, v) & \text{, if } u \in [\frac{1}{2}, 1], \end{cases}$$

$$= M(\max\{u - \frac{1}{2}, 0\}, v) + M(\min\{u + \frac{1}{2}, 1\}, v) - M(\frac{1}{2}, v),$$

that is shuffle of Min at point $\frac{1}{2}$. $\square$

Remark 39. *An $\mathscr{J}$-copula C is obtained by the following processes:*

1. *define a partition $\{J^1_{n_1} \times \cdots \times J^d_{n_d}\}_{n_1,\ldots,n_d \in N^d}$ of $\mathbb{I}^d$ formed by d-hypercube (namely, $\mathscr{J}$-structure).*
2. *given a system of d-copula $(C_n)_{n\in N}$, fill a transformation of μ_{C_n} in $J^1_n \times \cdots \times J^d_n$.*

Proposition 40 (Durante and Fernández-Sánchez [2])**.** *Let $C = \left\langle (\mathscr{J}^k)^d_{k=1}, (C_n)_{n\in N} \right\rangle$ be an $\mathscr{J}$-copula. Then, for any point $(u_1, \ldots, u_d) \in \mathbb{I}^d$,*

$$C(u_1, \ldots, u_d) = \sum_{n\in N} \lambda(J^1_n) C_n \left(\frac{u_1 - a^1_n}{\lambda(J^1_n)}, \ldots, \frac{u_d - a^d_n}{\lambda(J^1_n)} \right). \tag{1}$$

Remark 41. *Since $\sum_{n\in N} \lambda(J^1_n) = 1$, by (1), $\mathscr{J}$-copula C can be interpreted as a generalization of ordinal sums.*

CHAPTER IV

MULTIVARIATE SHUFFLES OF COPULAS

In the first section, we recall Mikusiński and Taylor's approach to multivariate shuffles of Min. A definition of shuffles of d-copulas is given in the second section. And in the last section, we will focus on 2-copulas and obtain a relationship between shuffles of Min and shuffles of 2-copulas via the $*$-product.

4.1 Multivariate shuffles of Min

In [5], Mikusiński and Taylor studied various approximations of d-copulas, one of which is shuffles of M^d. They showed that every d-copula can be approximated in the uniform norm by shuffles of M^d. In this section, we shall summarize their definition of shuffle of M^d and how they can approximate any d-copula.

Now, let $m \in \mathbb{N}$ and $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i} \in \mathcal{I}_m^d}$ be a d-fold stochastic matrix which, by Theorem 14, can be constructed from a d-fold stochastic measure. Then, for each $\boldsymbol{i} \in \mathcal{I}_m^d$ and $k = 1, \ldots, d$, let $q_{\boldsymbol{i}}^k$ be cumulative of $Q_{\boldsymbol{j}}$ for which $\boldsymbol{j} < \boldsymbol{i}$ and $i_k = j_k$; that is,

$$q_{\boldsymbol{i}}^k = \sum \{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}_m^d, \boldsymbol{j} < \boldsymbol{i} \text{ and } j_k = i_k\}.$$

In the k^{th} dimension, fix $p \in \mathcal{I}_m$, they subdivide $\left(\frac{p}{m}, \frac{p+1}{m}\right)$ into $J_{\boldsymbol{i}}^k$, where $\boldsymbol{i} \in \mathcal{I}_m^d$ and $i_k = p$, so that each $J_{\boldsymbol{i}}^k$ has length $Q_{\boldsymbol{i}}$ and $J_{\boldsymbol{\alpha}}^k$ precedes $J_{\boldsymbol{\beta}}^k$ whenever $\boldsymbol{\alpha} < \boldsymbol{\beta}$ in the lexicographic order; that is,

$$J_{\boldsymbol{i}}^k = \left(\frac{p}{m} + q_{\boldsymbol{i}}^k, \frac{p}{m} + q_{\boldsymbol{i}}^k + Q_{\boldsymbol{i}}\right),$$

where $\boldsymbol{i} \in \mathcal{I}_m^d$, $k = 1, \ldots, d$ and $p \in \mathcal{I}_m$. Note that, for each $k = 1, \ldots, d$, $p \in \mathcal{I}_m$, $\{J_{\boldsymbol{i}}^k : \boldsymbol{i} \in \mathcal{I}_m^d \text{ and } i_k = p\}$ is a near-decomposition of $\left(\frac{p}{m}, \frac{p+1}{m}\right)$. In fact,

1. Let $\boldsymbol{i} \neq \boldsymbol{l}$ be such that $i_k = l_k = p$. Notice that lexicographic order is totally ordered, so WLOG, let $\boldsymbol{i} < \boldsymbol{l}$. Then

$$\begin{aligned}
\lambda(J_{\boldsymbol{i}}^k \cap J_{\boldsymbol{l}}^k) &= \lambda\left(\left(\frac{i_k}{m} + q_{\boldsymbol{i}}^k, \frac{i_k}{m} + q_{\boldsymbol{i}}^k + Q_{\boldsymbol{i}}\right) \cap \left(\frac{l_k}{m} + q_{\boldsymbol{l}}^k, \frac{l_k}{m} + q_{\boldsymbol{l}}^k + Q_{\boldsymbol{l}}\right)\right) \\
&= \lambda\left(\left(\frac{p}{m} + \sum_{\substack{\boldsymbol{j}<\boldsymbol{i} \\ j_k=p}} Q_{\boldsymbol{j}}, \frac{p}{m} + \sum_{\substack{\boldsymbol{j}\le\boldsymbol{i} \\ j_k=p}} Q_{\boldsymbol{j}}\right) \cap \left(\frac{p}{m} + \sum_{\substack{\boldsymbol{j}<\boldsymbol{l} \\ j_k=p}} Q_{\boldsymbol{j}}, \frac{p}{m} + \sum_{\substack{\boldsymbol{j}\le\boldsymbol{l} \\ j_k=p}} Q_{\boldsymbol{j}}\right)\right) \\
&= \lambda(\varnothing) = 0, \text{ since } \sum_{\substack{\boldsymbol{j}\le\boldsymbol{i} \\ j_k=p}} Q_{\boldsymbol{j}} \le \sum_{\substack{\boldsymbol{j}<\boldsymbol{l} \\ j_k=p}} Q_{\boldsymbol{j}}.
\end{aligned}$$

2. Since λ is measure, by Remark 6,

$$\sum_{i_k=p} \lambda\left(J_{\boldsymbol{i}}^k\right) = \lambda\left(\bigcup_{i_k=p} J_{\boldsymbol{i}}^k\right) = \lambda\left(\left(\frac{p}{m}, \frac{p+1}{m}\right)\right).$$

Next, they define a measure-preserving transformation $\phi_k \colon \mathbb{I} \to \mathbb{I}$ by letting ϕ_k map $J_{\boldsymbol{i}}^1$ onto $J_{\boldsymbol{i}}^k$:

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q_{\boldsymbol{i}}^k - q_{\boldsymbol{i}}^1 \text{ for } t \in J_{\boldsymbol{i}}^1,$$

and defined $\phi \colon \mathbb{I}^d \to \mathbb{I}^d$ by

$$\phi(t_1, \ldots, t_d) = (\phi_1(t_1), \ldots, \phi_d(t_d)).$$

Notice that ϕ is one-to-one and onto (except finite hyperplanes). Since each finite hyperplane has measure zero with respect to probability measure generated by d-copulas, we can ignore them.

In [5], they defined the shuffle of M^d by

$$M_Q^d(x_1, \ldots, x_d) = \mu_{M^d}(\phi^{-1}([0, x_1] \times \cdots \times [0, x_d])),$$

for all $x_1, \ldots, x_d \in \mathbb{I}$. M_Q^d is called the **Q-shuffle of** M^d.

Theorem 42. *Given a d-fold stochastic matrix $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i} \in \mathcal{I}_m^d}$, the function M_Q^d defined above is a d-copula.*

Proof. For every Borel set $E \subseteq \mathbb{I}$,

$$\begin{aligned}\mu_{M_Q^d}(\mathbb{I}^{k-1} \times E \times \mathbb{I}^{d-k}) &= \mu_{M^d}(\phi^{-1}(\mathbb{I}^{k-1} \times E \times \mathbb{I}^{d-k})) \\ &= \mu_{M^d}(\mathbb{I}^{k-1} \times \phi_k^{-1}(E) \times \mathbb{I}^{d-k}) \\ &= \lambda(\phi_k^{-1}(E)) \qquad (\mu_{M^d} \text{ is a } d\text{-fold stochastic}) \\ &= \lambda(E). \qquad (\phi_k \text{ is measure-preserving})\end{aligned}$$

Thus, $\mu_{M_Q^d}$ is a d-fold stochastic measure. By Theorem 20, M_Q^d is a d-copula. □

Theorem 43. *Let M_Q^d be the Q-shuffle of M^d and let $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^d}$ be a d-fold stochastic matrix associated with a d-copula A. Then M_Q^d converges to A uniformly as $m \to \infty$.*

Proof. By Theorem 14, Q can be constructed from a d-fold stochastic measure μ_A. That is,

$$Q_{\boldsymbol{i}} = \mu_A\left(\left(\frac{i_1}{m}, \frac{i_1+1}{m}\right) \times \cdots \times \left(\frac{i_d}{m}, \frac{i_d+1}{m}\right)\right) = \mu_A(\mathbb{I}^d_{m,\boldsymbol{i}}).$$

Since $\{J_{\boldsymbol{i}}^k : \boldsymbol{i} \in \mathcal{I}_m^d \text{ and } i_k = p\}$ is near-decomposition of $\left(\frac{p}{m}, \frac{p+1}{m}\right)$, $J_{\boldsymbol{i}}^k \subseteq \left(\frac{i_k}{m}, \frac{i_k+1}{m}\right)$. Now, we set

$$J_{\boldsymbol{i}} = J_{\boldsymbol{i}}^1 \times \cdots \times J_{\boldsymbol{i}}^d \subseteq \left(\frac{i_1}{m}, \frac{i_1+1}{m}\right) \times \cdots \times \left(\frac{i_d}{m}, \frac{i_d+1}{m}\right) = \mathbb{I}^d_{m,\boldsymbol{i}}.$$

Furthermore, we have $J_{\boldsymbol{i}} = \phi_1(J_{\boldsymbol{i}}^1) \times \cdots \times \phi_d(J_{\boldsymbol{i}}^1) = \phi(J_{\boldsymbol{i}}^1 \times \cdots \times J_{\boldsymbol{i}}^1)$. Then,

$$\phi^{-1}(J_{\boldsymbol{i}}) = J_{\boldsymbol{i}}^1 \times \cdots \times J_{\boldsymbol{i}}^1.$$

By Example 18,

$$\mu_{M_Q^d}(J_{\boldsymbol{i}}) = \mu_{M^d}(\phi^{-1}(J_{\boldsymbol{i}})) = \mu_{M^d}(J_{\boldsymbol{i}}^1 \times \cdots \times J_{\boldsymbol{i}}^1) = \lambda(J_{\boldsymbol{i}}^1) = Q_{\boldsymbol{i}} = \mu_A(\mathbb{I}^d_{m,\boldsymbol{i}}).$$

Since $\mathbb{I}^d_{m,\boldsymbol{i}}$ is a μ-decomposition of $\mathbb{I}^d$,

$$1 = \sum_{\boldsymbol{i}} \mu_A(\mathbb{I}^d_{m,\boldsymbol{i}}) = \sum_{\boldsymbol{i}} \mu_{M_Q^d}(J_{\boldsymbol{i}}) \leq \sum_{\boldsymbol{i}} \mu_{M_Q^d}(\mathbb{I}^d_{m,\boldsymbol{i}}) = 1,$$

so $\sum_{\boldsymbol{i}} \mu_{M_Q^d}(\mathbb{I}^d_{m,\boldsymbol{i}}) = \sum_{\boldsymbol{i}} \mu_A(\mathbb{I}^d_{m,\boldsymbol{i}})$. It follows that for all $\boldsymbol{i} \in \mathcal{I}^d_m$,

$$M_Q^d(\frac{\boldsymbol{i}}{m}) = A(\frac{\boldsymbol{i}}{m}).$$

Consider, let $\boldsymbol{j} \in \mathbb{I}^d_{m,\boldsymbol{i}}$ some $\boldsymbol{i}$,

$$\begin{aligned}
\sup_{\boldsymbol{j}} |M_Q^d(\boldsymbol{j}) - A(\boldsymbol{j})| &= \sup_{\boldsymbol{j}} |M_Q^d(\boldsymbol{j}) - M_Q^d(\boldsymbol{i}) + M_Q^d(\boldsymbol{i}) - A(\boldsymbol{i}) + A(\boldsymbol{i}) - A(\boldsymbol{j})| \\
&= \sup_{\boldsymbol{j}}[|M_Q^d(\boldsymbol{j}) - M_Q^d(\boldsymbol{i})| + |M_Q^d(\boldsymbol{i}) - A(\boldsymbol{i})| + |A(\boldsymbol{i}) - A(\boldsymbol{j})|] \\
&\le \sup_{\boldsymbol{j}} |M_Q^d(\boldsymbol{j}) - M_Q^d(\boldsymbol{i})| + \sup_{\boldsymbol{j}} |M_Q^d(\boldsymbol{i}) - A(\boldsymbol{i})| + \sup_{\boldsymbol{j}} |A(\boldsymbol{i}) - A(\boldsymbol{j})| \\
&\le \frac{d}{m} + 0 + \frac{d}{m} = \frac{2d}{m} \to 0 \text{ as } m \to \infty.
\end{aligned}$$

Therefore, $M_Q^d \to A$ uniformly as $m \to \infty$. □

4.2 Multivariate shuffles of copula

In this section, by imitating Mikusiński and Taylor's definition of shuffle of M^d in [5], we introduce a definition of shuffle of d-copulas.

Definition 44. *Let $C \in \mathscr{C}_d$ and $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i} \in \mathcal{I}^d_m}$ a d-fold stochastic matrix. Then we define the following.*

1. $q^k_{\boldsymbol{i}} = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}^d_m, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}^d_m$ *and* $k = 1, 2, \ldots, d$.
2. $J^k_{\boldsymbol{i}} = \left(\frac{i_k}{m} + q^k_{\boldsymbol{i}}, \frac{i_k}{m} + q^k_{\boldsymbol{i}} + Q_{\boldsymbol{i}}\right)$.
3. *For any* $k = 1, 2, \ldots, d$,

   $$\phi_k(t) = t + \frac{i_k - i_1}{m} + q^k_{\boldsymbol{i}} - q^1_{\boldsymbol{i}} \text{ for } t \in J^1_{\boldsymbol{i}}$$

   and

   $$\phi(t_1, \ldots, t_d) = (\phi_1(t_1), \ldots, \phi_d(t_d)),$$

   for all $t_1, \ldots, t_d \in \mathbb{I}$.

4. *Define the d-variate shuffle of copula C by*

$$C_Q(x_1, \dots, x_d) = \mu_C(\phi^{-1}([0, x_1] \times \cdots \times [0, x_d]))$$

for any $(x_1, \dots, x_d) \in \mathbb{I}^d$.

C_Q *is called the* Q***-shuffle of*** C.

Remark 45. 1. $\phi_1(t) = t$ *for all* $t \in \mathbb{I}$.

2. *For all* k, $\phi_k \colon \mathbb{I} \to \mathbb{I}$ *is measure preserving because it is piecewise linear bijection with coefficients* $= 1$*(slope* $= 1$*). It follows that* $\phi \colon \mathbb{I}^d \to \mathbb{I}^d$ *is measure preserving.*

Theorem 46. *Let* $C \in \mathscr{C}_d$ *and* Q *be a d-fold stochastic matrix. Then* $\phi\#\mu_C$ *is a d-fold stochastic measure and hence* C_Q *is a d-copula.*

Proof. First, we shall show that $\phi\#\mu_C$ is d-fold stochastic measure. By Theorem 20, μ_C is a d-fold stochastic measure. So, for every Borel set $E \subseteq \mathbb{I}$,

$$\begin{aligned} \phi\#\mu_C(\mathbb{I}^{k-1} \times E \times \mathbb{I}^{d-k}) &= \mu_C(\phi^{-1}(\mathbb{I}^{k-1} \times E \times \mathbb{I}^{d-k})) \\ &= \mu_C(\mathbb{I}^{k-1} \times \phi_k^{-1}(E) \times \mathbb{I}^{d-k}) \\ &= \lambda(\phi_k^{-1}(E)) \qquad \text{(by assumption)} \\ &= \lambda(E), \end{aligned}$$

where we used the fact that ϕ_k is measure preserving transformation. Thus, $\phi\#\mu_C$ is d-fold stochastic.

Next, we shall show that C_Q is a d-copula. For every Borel set $F \subseteq \mathbb{I}^d$,

$$\mu_{C_Q}(F) = \mu_C(\phi^{-1}(F)) = \phi\#\mu_C(F).$$

Hence μ_{C_Q} is a d-fold stochastic measure. Then, by Theorem 20, there exists a d-copula D, $\mu_{C_Q} = \mu_D$. Therefore, C_Q is a d-copula. □

Clearly, C_Q is a generalization of Q-shuffle of M in section 4.1.

Theorem 47. *Let $C \in \mathscr{C}_d$ and Q be a d-fold stochastic matrix. If C is absolutely continuous, then so is the Q-shuffle of C.*

Proof. Assume that a d-copula C is absolutely continuous, that is for all $E \in \mathcal{B}(\mathbb{I}^d)$, $\lambda^d(E) = 0$ implies that $\mu_C(E) = 0$. Let $E \in \mathcal{B}(\mathbb{I}^d)$ be such that $\lambda^d(E) = 0$. Since ϕ is a measure preserving transformation,

$$\lambda^d(\phi^{-1}(E)) = \lambda^d(E) = 0.$$

By the absolutely continuity of C,

$$\mu_{C_Q}(E) = \mu_C(\phi^{-1}(E)) = 0.$$

Thus, Q-shuffle of C is absolutely continuous. □

Examples 48-53 contains selected examples of bivariate shuffles of copulas. An example of a shuffle of 3-copula is given in Example 54. Detailed derivations of C_Q and verifications of the equation $C * M_Q = C_Q$ are given in the APPENDIX.

Example 48. *Let $C \in \mathscr{C}_2$ and Q be the 3×3 doubly stochastic matrix associated with M; that is $Q = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$. Then*

$$C_Q(x_1, x_2) = C(x_1, x_2).$$

Thus, $M_Q(x_1, x_2) = M(x_1, x_2)$. It can be shown that

$$(C * M_Q)(x, y) = C_Q(x, y).$$

Example 49. *Let $C \in \mathscr{C}_2$ and Q be the 2×2 doubly stochastic matrix associated with W; that is $Q = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Then*

$$C_Q(x_1, x_2) = C(x_1, \min\{x_2 + \frac{1}{2}, 1\} - C(x_1, \frac{1}{2}) + C(x_1, \max\{x_2 - \frac{1}{2}, 0\}).$$

Thus, $M(x_1, \min\{x_2+\frac{1}{2}, 1\} - M(x_1, \frac{1}{2}) + M(x_1, \max\{x_2-\frac{1}{2}, 0\})$. *It can be shown that*

$$(C * M_Q)(x, y) = C_Q(x, y).$$

Example 50. *Let* $C \in \mathscr{C}_2$ *and* Q *be the* 2×2 *doubly stochastic matrix associated with* Π*; that is* $Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$. *Then*

$$C_Q(x_1, x_2) = \begin{cases} C(x_1, x_2) & ; x_2 \in \left(0, \frac{1}{4}\right), \\ C(x_1, \frac{1}{4}) + C(x_1, x_2 + \frac{1}{4}) - C(x_1, \frac{1}{2}) & ; x_2 \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ C(x_1, x_2 - \frac{1}{4}) + C(x_1, \frac{3}{4}) - C(x_1, \frac{1}{2}) & ; x_2 \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ C(x_1, x_2) & ; x_2 \in \left(\frac{3}{4}, 1\right). \end{cases}$$

Thus,

$$M_Q(x_1, x_2) = \begin{cases} M(x_1, x_2) & ; x_2 \in \left(0, \frac{1}{4}\right), \\ M(x_1, \frac{1}{4}) + M(x_1, x_2 + \frac{1}{4}) - M(x_1, \frac{1}{2}) & ; x_2 \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ M(x_1, x_2 - \frac{1}{4}) + M(x_1, \frac{3}{4}) - M(x_1, \frac{1}{2}) & ; x_2 \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ M(x_1, x_2) & ; x_2 \in \left(\frac{3}{4}, 1\right). \end{cases}$$

It can be shown that

$$(C * M_Q)(x, y) = C_Q(x, y).$$

Example 51. *Let* $C \in \mathscr{C}_2$ *and* Q *be the* 2×2 *doubly stochastic matrix associated with* 2*-copula* A *such that* $Q = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$. *Then*

$$C_Q(x_1, x_2) = \begin{cases} C(x_1, x_2) & ; x_2 \in \left(0, \frac{1}{3}\right), \\ C(x_1, \frac{1}{3}) + C(x_1, x_2 + \frac{1}{6}) - C(x_1, \frac{1}{2}) & ; x_2 \in \left(\frac{1}{3}, \frac{1}{2}\right), \\ C(x_1, x_2 - \frac{1}{6}) + C(x_1, \frac{2}{3}) - C(x_1, \frac{1}{2}) & ; x_2 \in \left(\frac{1}{2}, \frac{2}{3}\right), \\ C(x_1, x_2) & ; x_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Thus,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\ M(x_1,\frac{1}{3}) + M(x_1,x_2+\frac{1}{6}) - M(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{3},\frac{1}{2}\right), \\ M(x_1,x_2-\frac{1}{6}) + M(x_1,\frac{2}{3}) - M(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{2},\frac{2}{3}\right), \\ M(x_1,x_2) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

It can be shown that

$$(C * M_Q)(x,y) = C_Q(x,y).$$

Example 52. *Let $C \in \mathscr{C}_2$ and Q be the 3×3 doubly stochastic matrix associated with 2-copula A such that* $Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$. *Then*

$$C_Q(x_1,x_2) = \begin{cases} C(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\ C(x_1,\frac{1}{3}) + C(x_1,x_2+\frac{1}{3}) - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{3},\frac{2}{3}\right), \\ C(x_1,x_2-\frac{1}{3}) + x_1 - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

Thus,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\ M(x_1,\frac{1}{3}) + M(x_1,x_2+\frac{1}{3}) - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{3},\frac{2}{3}\right), \\ M(x_1,x_2-\frac{1}{3}) + x_1 - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

It can be shown that

$$(C * M_Q)(x,y) = C_Q(x,y).$$

Example 53. *Let $C \in \mathscr{C}_2$ and Q be the 3×3 doubly stochastic matrix associated with*

2-copula A such that $Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{12} \\ \frac{1}{12} & 0 & \frac{1}{4} \end{bmatrix}$. *Then*

$$C_Q(x_1,x_2) = \begin{cases} C(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{12}\right), \\ C(x_1,\frac{1}{12}) + C(x_1,x_2+\frac{7}{12}) - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{12},\frac{1}{3}\right), \\ C(x_1,x_2-\frac{1}{4}) + C(x_1,\frac{11}{12}) - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{3},\frac{7}{12}\right), \\ C(x_1,\frac{1}{3}) + C(x_1,x_2+\frac{1}{3}) - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{7}{12},\frac{2}{3}\right), \\ C(x_1,x_2-\frac{1}{3}) + x_1 - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

Thus,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{12}\right), \\ M(x_1,\frac{1}{12}) + M(x_1,x_2+\frac{7}{12}) - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{12},\frac{1}{3}\right), \\ M(x_1,x_2-\frac{1}{4}) + M(x_1,\frac{11}{12}) - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{3},\frac{7}{12}\right), \\ M(x_1,\frac{1}{3}) + M(x_1,x_2+\frac{1}{3}) - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{7}{12},\frac{2}{3}\right), \\ M(x_1,x_2-\frac{1}{3}) + x_1 - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

It can be shown that

$$(C * M_Q)(x,y) = C_Q(x,y).$$

Example 54. *Let $C \in \mathscr{C}_3$ and Q be the $2\times2\times2$ three-fold stochastic matrix associated with the 3-copula $A(x,y,z) = W(x, M(y,z))$, $x,y,z \in \mathbb{I}$. Then for any $x_1,x_2,x_3 \in \mathbb{I}$,*

$$\begin{aligned} C_Q(x_1,x_2,x_3) = {} & C(x_1,\min\{x_2+\frac{1}{2},1\},\min\{x_3+\frac{1}{2},1\}) - C(x_1,\min\{x_2+\frac{1}{2},1\},\frac{1}{2}) \\ & + C(x_1,\min\{x_2+\frac{1}{2},1\},\max\{x_3-\frac{1}{2},0\}) - C(x_1,\frac{1}{2},\min\{x_3+\frac{1}{2},1\}) \\ & + C(x_1,\frac{1}{2},\frac{1}{2}) - C(x_1,\frac{1}{2},\max\{x_3-\frac{1}{2},0\}) \\ & + C(x_1,\max\{x_2-\frac{1}{2},0\},\min\{x_3+\frac{1}{2},1\}) - C(x_1,\max\{x_2-\frac{1}{2},0\},\frac{1}{2}) \\ & + C(x_1,\max\{x_2-\frac{1}{2},0\},\max\{x_3-\frac{1}{2},0\}). \end{aligned}$$

4.3 $C * M_Q = C_Q$ generalized

In the previous section, it was shown that, in 2-dimension, the $*$-product of C and Q-shuffle of M is a Q-shuffle of C, where $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^2}$ is a doubly stochastic matrix. So in this section, we shall obtain its generalization for the analogous number of partitions m.

Theorem 55. *Let $C \in \mathscr{C}_2$ and $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^2}$ a doubly stochastic matrix. Then the $*$-product of C and Q-shuffle of M is a Q-shuffle of C; that is,*

$$C * M_Q = C_Q$$

Proof. By Definition 44, for any Borel set $E \subseteq \mathbb{I}^2$, $\mu_{M_Q}(E) = \mu_M(\phi^{-1}(E))$ and $\mu_{C_Q}(E) = \mu_C(\phi^{-1}(E))$. We claim that $C * M_Q = C_Q$. It enough to show that

$$\mu_{C*M_Q} = \mu_{C_Q}.$$

Thus, by theorem 32 and Proposition 31,

$$\begin{aligned}
\mu_{C*M_Q}(I \times J) &= (\mu_C * \mu_{M_Q})(I \times J) \\
&= \int_0^1 \frac{d}{dt}\mu_C(I \times [0,t])\frac{d}{dt}\mu_{M_Q}([0,t] \times J)dt \\
&= \int_0^1 \frac{d}{dt}\mu_C(I \times [0,t])\frac{d}{dt}\mu_M(\phi^{-1}([0,t] \times J))dt \\
&= \int_0^1 \frac{d}{dt}\mu_C(I \times [0,t])\frac{d}{dt}\mu_M([0,t] \times \phi_2^{-1}(J))dt \\
&= (\mu_C * \mu_M)(I \times \phi_2^{-1}(J)) \\
&= (\mu_C * \mu_M)(\phi^{-1}(I \times J)) \\
&= (\mu_{C*M})(\phi^{-1}(I \times J)) \\
&= \mu_C(\phi^{-1}(I \times J)) \\
&= \mu_{C_Q}(I \times J),
\end{aligned}$$

which is the desired assertion. □

Remark 56. *Let* $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i} \in \mathcal{I}_m^2}$ *is a doubly stochastic matrix with respect to a* 2*-copula* A*. Then* $C_Q = C * M_Q \to C * A$ *in uniform norm as* $m \to \infty$*.*

CHAPTER V

MULTIVARIATE PRODUCT

Darsow Nguyen and Olsen introduced $*$-product for bivariate copulas in [1]. In this chapter, we will define a new product operation in d dimension.

5.1 Univariate shuffling of d-copulas

First, we define a product operation between 2-copulas and d-copulas, which is important in this thesis. And we have several theorems and properties.

Definition 57. *Let $k = 1, \dots, d$, $A \in \mathscr{C}_2$ and $B \in \mathscr{C}_d$. The $\heartsuit_k$-**product** of A and B is the function $A \heartsuit_k B \colon \mathbb{I}^d \to \mathbb{I}$ defined by*

$$A \heartsuit_k B(x_1, \dots, x_d) = \int_0^1 \partial_1 A(t, x_k) \partial_k B(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_d)\, dt,$$

wherever the above integral exists.

Theorem 58. *Let $k = 1, \dots, d$, $A \in \mathscr{C}_2$ and $B \in \mathscr{C}_d$. Then $A \heartsuit_k B$ is a d-copula.*

Proof. **(C1)** It is obvious that $A \heartsuit_k B(x_1, \dots, x_d) = 0$ whenever $x_l = 0$ for some l.

(C2) Let $x_1, \dots, x_d \in \mathbb{I}$.

Case 1. Every component is 1, except for x_k. Then

$$\begin{aligned} A \heartsuit_k B(\overbrace{1, \dots, 1}^{k-1 \text{ terms}}, x_k, \overbrace{1, \dots, 1}^{d-k \text{ terms}}) &= \int_0^1 \partial_1 A(t, x_k) \partial_k B(\overbrace{1, \dots, 1}^{k-1 \text{ terms}}, t, \overbrace{1, \dots, 1}^{d-k \text{ terms}})\, dt \\ &= \int_0^1 \partial_1 A(t, x_k)(1)\, dt \\ &= A(1, x_k) - A(0, x_k) \\ &= x_k. \end{aligned}$$

Case 2. Every component is 1, except for x_j, $j \neq k$. WLOG, let $j < k$. Then

$$\begin{aligned}
&A \heartsuit_k B(\overbrace{1,\dots,1}^{j-1 \text{ terms}}, x_j, \overbrace{1,\dots,1}^{d-j \text{ terms}}) \\
&= \int_0^1 \partial_1 A(t,1)\partial_k B(\overbrace{1,\dots,1}^{j-1 \text{ terms}}, x_j, \overbrace{1,\dots,1}^{k-j-1 \text{ terms}}, t, \overbrace{1,\dots,1}^{d-k \text{ terms}})\,dt \\
&= \int_0^1 (1)\partial_k B(\overbrace{1,\dots,1}^{j-1 \text{ terms}}, x_j, \overbrace{1,\dots,1}^{k-j-1 \text{ terms}}, t, \overbrace{1,\dots,1}^{d-k \text{ terms}})\,dt \\
&= B(\overbrace{1,\dots,1}^{j-1 \text{ terms}}, x_j, \overbrace{1,\dots,1}^{k-j-1 \text{ terms}}, 1, \overbrace{1,\dots,1}^{d-k \text{ terms}}) \\
&\quad - B(\overbrace{1,\dots,1}^{j-1 \text{ terms}}, x_j, \overbrace{1,\dots,1}^{k-j-1 \text{ terms}}, 0, \overbrace{1,\dots,1}^{d-k \text{ terms}}) \\
&= x_j.
\end{aligned}$$

(C3) For any $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{I}^d$,

$$\begin{aligned}
&V_{A \heartsuit_k B}([\boldsymbol{a}, \boldsymbol{b}]) \\
&= \triangle^d_{(a_d,b_d)} \cdots \triangle^1_{(a_1,b_1)} A \heartsuit_k B(x_1,\dots,x_d) \\
&= \triangle^d_{(a_d,b_d)} \cdots \triangle^1_{(a_1,b_1)} \int_0^1 \partial_1 A(t,x_k)\partial_k B(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_d)\,dt \\
&= \int_0^1 \partial_1 [A(t,b_k) - A(t,a_k)] \\
&\qquad \times \partial_k [\triangle^d_{(a_d,b_d)} \cdots \triangle^{k+1}_{(a_{k+1},b_{k+1})} \triangle^{k-1}_{(a_{k-1},b_{k-1})} \cdots \triangle^1_{(a_1,b_1)} B(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_d)]\,dt \\
&\geq 0, \text{ by Remark 22.}
\end{aligned}$$

Hence, $A \heartsuit_k B$ is a d-copula. □

Proposition 59. *For each $k = 1,\dots,d$, and $C \in \mathscr{C}_d$,*

$$\begin{aligned}
M \heartsuit_k C(x_1,\dots,x_d) &= C(x_1,\dots,x_d), \\
W \heartsuit_k C(x_1,\dots,x_d) &= C(x_1,\dots,x_{k-1},1,x_{k+1},\dots,x_d) \\
&\quad - C(x_1,\dots,x_{k-1},1-x_k,x_{k+1},\dots,x_d), \\
\Pi \heartsuit_k C(x_1,\dots,x_d) &= x_k C(x_1,\dots,x_{k-1},1,x_{k+1},\dots,x_d),
\end{aligned}$$

for all $(x_1,\dots,x_d) \in \mathbb{I}^d$.

Proof. For $(x_1, \ldots, x_d) \in \mathbb{I}^d$,

$$\begin{aligned} M \heartsuit_k C(x_1, \ldots, x_d) &= \int_0^1 \partial_1 M(t, x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= \int_0^{x_k} (1) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= C(x_1, \ldots, x_d), \end{aligned}$$

$$\begin{aligned} W \heartsuit_k C(x_1, \ldots, x_d) &= \int_0^1 \partial_1 W(t, x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= \int_0^1 \frac{\partial}{\partial t} \max\{t + x_k - 1, 0\} \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= \int_{1-x_k}^1 (1) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= C(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d) \\ &\quad - C(x_1, \ldots, x_{k-1}, 1 - x_k, x_{k+1}, \ldots, x_d) \end{aligned}$$

and

$$\begin{aligned} \Pi \heartsuit_k C(x_1, \ldots, x_d) &= \int_0^1 \partial_1 \Pi(t, x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= \int_0^1 x_k \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ &= x_k C(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d). \end{aligned}$$

□

Proposition 60. *For each $k = 1, \ldots, d$, and $C \in \mathscr{C}_2$,*

$$\begin{aligned} C \heartsuit_k M^d(x_1, \ldots, x_d) &= C(M^{d-1}(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d), x_k), \\ C \heartsuit_k \Pi^d(x_1, \ldots, x_d) &= \Pi^d(x_1, \ldots, x_d), \end{aligned}$$

for $(x_1, \ldots, x_d) \in \mathbb{I}^d$.

Proof. For $(x_1, \ldots, x_d) \in \mathbb{I}^d$,

$$C \heartsuit_k M^d(x_1, \ldots, x_d) = \int_0^1 \partial_1 C(t, x_k) \partial_k M^d(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt$$

$$
\begin{aligned}
&= \int_0^{\min\{x_1,\dots,x_{k-1},x_{k+1},\dots,x_d\}} \partial_1 C(t, x_k)(1)\,dt \\
&= C(\min\{x_1,\dots,x_{k-1},x_{k+1},\dots,x_d\}, x_k) \\
&= C(M^{d-1}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_d), x_k)
\end{aligned}
$$

and

$$
\begin{aligned}
C \heartsuit_k \Pi^d(x_1,\dots,x_d) &= \int_0^1 \partial_1 C(t, x_k)\partial_k \Pi^d(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_d)\,dt \\
&= \int_0^1 \partial_1 C(t, x_k)(x_1\cdots x_{k-1}x_{k+1}\cdots x_d)\,dt \\
&= x_1\cdots x_{k-1}x_{k+1}\cdots x_d C(1, x_k) \\
&= x_1\cdots x_{k-1}x_{k+1}\cdots x_d(x_k) \\
&= \Pi^d(x_1,\dots,x_d).
\end{aligned}
$$

$\square$

Next, we will define the $\heartsuit_k$-product between a doubly stochastic measure and a d-fold stochastic measure. The resulting $\heartsuit_k$-product is a k-axis shuffle of the d-fold stochastic measure.

Theorem 61. *If μ is a doubly stochastic measure on $\mathbb{I}^2$ and ν is a d-fold stochastic measure on $\mathbb{I}^d$, then $\mu \heartsuit_k \nu$ defined for Borel sets $J_1,\dots,J_d \subseteq \mathbb{I}$ by*

$$
\mu \heartsuit_k \nu(J_1\times\cdots\times J_d) = \int_0^1 \frac{d}{dt}\mu([0,t]\times J_k)\frac{d}{dt}\nu(J_1\times\cdots\times J_{k-1}\times[0,t]\times J_{k+1}\times\cdots\times J_d)\,dt
$$

gives rise to a d-fold stochastic measure on $\mathbb{I}^d$.
Furthermore, if A is the 2-copula associated with doubly stochastic measure μ_A and B is the d-copula associated with d-fold stochastic measure μ_B, then

$$
\mu_{A\heartsuit_k B} = \mu_A \heartsuit_k \mu_B.
$$

Proof. Since μ and ν are increasing in t, $\frac{d}{dt}\mu([0,t]\times J_k)$ and $\frac{d}{dt}\nu(J_1\times\cdots\times J_{k-1}\times[0,t]\times J_{k+1}\times\cdots\times J_d)$ exist almost surely. First, we will show that $\mu \heartsuit_k \nu$ is a d-fold stochastic measure on $\mathbb{I}^d$. Assume that μ is doubly stochastic and ν is d-fold stochastic. Let $J_1, \ldots, J_d \subseteq \mathbb{I}$. Claim that

$$\mu \heartsuit_k \nu(\mathbb{I}^{l-1} \times J_l \times \mathbb{I}^{d-l}) = \lambda(J_l).$$

Case 1. If $k = l$, then

$$\begin{aligned}
\mu \heartsuit_k \nu(\mathbb{I}^{l-1} \times J_l \times \mathbb{I}^{d-l}) &= \mu \heartsuit_k \nu(\mathbb{I}^{k-1} \times J_k \times \mathbb{I}^{d-k}) \\
&= \int_0^1 \frac{d}{dt}\mu([0,t] \times J_k)\frac{d}{dt}\nu(\mathbb{I}^{k-1} \times [0,t] \times \mathbb{I}^{d-k})\, dt \\
&= \int_0^1 \frac{d}{dt}\mu([0,t] \times J_k)\frac{d}{dt}\lambda([0,t])\, dt \\
&= \int_0^1 \frac{d}{dt}\mu([0,t] \times J_k)(1)\, dt \\
&= \mu([0,1] \times J_k) = \lambda(J_k) = \lambda(J_l).
\end{aligned}$$

Case 2. If $k \neq l$. WLOG, let $k < l$,

$$\begin{aligned}
&\mu \heartsuit_k \nu(\mathbb{I}^{l-1} \times J_l \times \mathbb{I}^{d-l}) \\
&= \int_0^1 \frac{d}{dt}\mu([0,t] \times J_k)\frac{d}{dt}\nu(\mathbb{I}^{k-1} \times [0,t] \times \mathbb{I}^{l-k-1} \times J_l \times \mathbb{I}^{d-l})\, dt \\
&= \int_0^1 \frac{d}{dt}\mu([0,t] \times [0,1])\frac{d}{dt}\nu(\mathbb{I}^{k-1} \times [0,t] \times \mathbb{I}^{l-k-1} \times J_l \times \mathbb{I}^{d-l})\, dt \\
&= \int_0^1 \frac{d}{dt}\lambda([0,t])\frac{d}{dt}\nu(\mathbb{I}^{k-1} \times [0,t] \times \mathbb{I}^{l-k-1} \times J_l \times \mathbb{I}^{d-l})\, dt \\
&= \int_0^1 (1)\frac{d}{dt}\nu(\mathbb{I}^{k-1} \times [0,t] \times \mathbb{I}^{l-k-1} \times J_l \times \mathbb{I}^{d-l})\, dt \\
&= \nu(\mathbb{I}^{l-1} \times J_l \times \mathbb{I}^{d-l}) = \lambda(J_l).
\end{aligned}$$

Next, we will show that $\mu_{A \heartsuit_k B} = \mu_A \heartsuit_k \mu_B$. Let $A \in \mathscr{C}_2$ and $B \in \mathscr{C}_d$. For any $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{I}^d$,

$$\begin{aligned}
&\mu_{A \heartsuit_k B}([\boldsymbol{a}, \boldsymbol{b}]) \\
&= \triangle_{(a_d,b_d)}^d \cdots \triangle_{(a_1,b_1)}^1 \, A \heartsuit_k B(x_1, \ldots, x_d) \\
&= \triangle_{(a_d,b_d)}^d \cdots \triangle_{(a_1,b_1)}^1 \int_0^1 \partial_1 A(t, x_k)\partial_k B(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&= \int_0^1 \frac{d}{dt}\left(A(t, b_k) - A(t, a_k)\right)
\end{aligned}$$

$$\times \frac{d}{dt} \triangle^{d}_{(a_d,b_d)} \cdots \triangle^{k+1}_{(a_{k+1},b_{k+1})} \triangle^{k-1}_{(a_{k-1},b_{k-1})} \cdots \triangle^{1}_{(a_1,b_1)} B(x_1,\dots,x_{k-1},t,x_{k+1},\dots,x_d)\, dt$$

$$= \int_0^1 \frac{d}{dt}\mu_A([0,t]\times[a_k,b_k])$$

$$\times \frac{d}{dt}\mu_B([a_1,b_1]\times\cdots\times[a_{k-1},b_{k-1}]\times[0,t]\times[a_{k+1},b_{k+1}]\times\cdots\times[a_d,b_d])\, dt$$

$$= \mu_A \heartsuit_k \mu_B([\boldsymbol{a},\boldsymbol{b}]).$$

□

Remark 62. *Let $B \in \mathscr{C}_2$ and $C \in \mathscr{C}_d$. For any $k = 1,\dots,d$,*

$$\mu_M \heartsuit_k \mu_C = \mu_{M\heartsuit_k C} = \mu_C,$$

$$\mu_B \heartsuit_k \mu_{\Pi^d} = \mu_{B\heartsuit_k \Pi^d} = \mu_{\Pi^d},$$

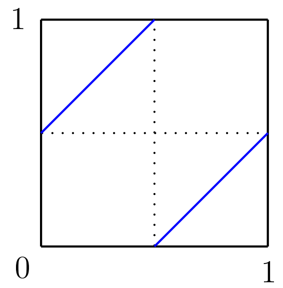

Figure 5.1: the support of S in Example 63.

Example 63. *Let $C \in \mathscr{C}_3$ and $x,y,z \in \mathbb{I}$. Let S be a bivariate shuffle of Min supported on the union of lines $l_1 : y = x + \frac{1}{2}$, $0 \le x \le \frac{1}{2}$ and $l_2 : y = 1 - x$, $\frac{1}{2} < x \le 1$, that is*

$$S(x,y) = M\left(\min\left\{x+\frac{1}{2},1\right\},y\right) - M\left(\frac{1}{2},y\right) + M\left(\max\left\{x-\frac{1}{2},0\right\},y\right).$$

See Figure 5.1. Then, for any $x,y,z \in \mathbb{I}$,

$$S\heartsuit_1 C(x,y,z) = C\left(\min\left\{x+\frac{1}{2},1\right\},y,z\right) - C\left(\frac{1}{2},y,z\right) + C\left(\max\left\{x-\frac{1}{2},0\right\},y,z\right),$$

$$S\heartsuit_2 C(x,y,z) = C\left(x,\min\left\{y+\frac{1}{2},1\right\},z\right) - C\left(x,\frac{1}{2},z\right) + C\left(x,\max\left\{y-\frac{1}{2},0\right\},z\right),$$

$$S\heartsuit_3 C(x,y,z) = C\left(x,y,\min\left\{z+\frac{1}{2},1\right\}\right) - C\left(x,y,\frac{1}{2}\right) + C\left(x,y,\max\left\{z-\frac{1}{2},0\right\}\right).$$

Solution. We will show only the equation for $S\heartsuit_1 C$ because the other equations are analogous.

Case 1. For $x \in \left[0, \frac{1}{2}\right]$,

$$
\begin{aligned}
S\heartsuit_1 C(x,y,z) &= \int_0^1 \partial_1 S(t,x)\partial_1 C(t,y,z)\,dt \\
&= \int_0^1 \frac{\partial}{\partial t} M\left(\min\left\{t+\frac{1}{2},1\right\},x\right)\partial_1 C(t,y,z)\,dt \\
&\quad - \int_0^1 \frac{\partial}{\partial t} M\left(\frac{1}{2},x\right)\partial_1 C(t,y,z)\,dt \\
&\quad + \int_0^1 \frac{\partial}{\partial t} M\left(\max\left\{t-\frac{1}{2},0\right\},x\right)\partial_1 C(t,y,z)\,dt \\
&= \int_0^1 \frac{\partial}{\partial t}(x)\partial_1 C(t,y,z)\,dt - \int_0^1 \frac{\partial}{\partial t}(x)\partial_1 C(t,y,z)\,dt \\
&\quad + \int_0^{x+\frac{1}{2}} \frac{\partial}{\partial t}\left(\max\left\{t-\frac{1}{2},0\right\}\right)\partial_1 C(t,y,z)\,dt \\
&= 0 - 0 + \int_{\frac{1}{2}}^{x+\frac{1}{2}} (1)\partial_1 C(t,y,z)\,dt \\
&= C\left(x+\frac{1}{2},y,z\right) - C\left(\frac{1}{2},y,z\right).
\end{aligned}
$$

Case 2. For $x \in \left[\frac{1}{2}, 1\right]$,

$$
\begin{aligned}
S\heartsuit_1 C(x,y,z) &= \int_0^1 \partial_1 S(t,x)\partial_1 C(t,y,z)\,dt \\
&= \int_0^1 \frac{\partial}{\partial t} M\left(\min\left\{t+\frac{1}{2},1\right\},x\right)\partial_1 C(t,y,z)\,dt \\
&\quad - \int_0^1 \frac{\partial}{\partial t} M\left(\frac{1}{2},x\right)\partial_1 C(t,y,z)\,dt \\
&\quad + \int_0^1 \frac{\partial}{\partial t} M\left(\max\left\{t-\frac{1}{2},0\right\},x\right)\partial_1 C(t,y,z)\,dt \\
&= \int_0^{x-\frac{1}{2}} \frac{d}{dt}\left(\min\left\{t+\frac{1}{2},1\right\}\right)\partial_1 C(t,y,z)\,dt \\
&\quad - \int_0^1 \frac{d}{dt}\left(\frac{1}{2}\right)\partial_1 C(t,y,z)\,dt \\
&\quad + \int_0^1 \frac{d}{dt}\left(\max\left\{t-\frac{1}{2},0\right\}\right)\partial_1 C(t,y,z)\,dt \\
&= \int_0^{x-\frac{1}{2}} (1)\partial_1 C(t,y,z)\,dt - 0 + \int_{\frac{1}{2}}^1 (1)\partial_1 C(t,y,z)\,dt \\
&= C(x-\frac{1}{2},y,z) + C(1,y,z) - C(\frac{1}{2},y,z).
\end{aligned}
$$

Hence, for all $x, y, z \in \mathbb{I}$,

$$S \heartsuit_1 C(x,y,z) = \begin{cases} C\left(x + \frac{1}{2}, y, z\right) - C\left(\frac{1}{2}, y, z\right) & , \text{ if } x \in \left[0, \frac{1}{2}\right], \\ C\left(x - \frac{1}{2}, y, z\right) + C(1, y, z) - C\left(\frac{1}{2}, y, z\right) & , \text{ if } x \in \left[\frac{1}{2}, 1\right], \end{cases}$$

$$= C\left(\min\left\{x + \frac{1}{2}, 1\right\}, y, z\right) - C\left(\frac{1}{2}, y, z\right) + C\left(\max\left\{x - \frac{1}{2}, 0\right\}, y, z\right).$$

□

Example 64. *Let $C \in \mathscr{C}_d$ and $x_1, \ldots, x_d \in \mathbb{I}$. Let S be the bivariate shuffle of Min in Example 63. Then, for all $k = 1, \ldots, d$,*

$$\begin{aligned} S \heartsuit_k C(x_1, \ldots, x_d) = {} & C\left(x_1, \ldots, x_{k-1}, \min\left\{x_k + \frac{1}{2}, 1\right\}, x_{k+1}, \ldots, x_d\right) \\ & - C\left(x_1, \ldots, x_{k-1}, \frac{1}{2}, x_{k+1}, \ldots, x_d\right) \\ & + C\left(x_1, \ldots, x_{k-1}, \max\left\{x_k - \frac{1}{2}, 0\right\}, x_{k+1}, \ldots, x_d\right). \end{aligned}$$

Solution. **Case 1.** If $x_k \in \left[0, \frac{1}{2}\right]$, then

$$\begin{aligned} & S \heartsuit_k C(x_1, \ldots, x_d) \\ & = \int_0^1 \partial_1 S(t, x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & = \int_0^1 \frac{\partial}{\partial t} M\left(\min\left\{t + \frac{1}{2}, 1\right\}, x_k\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & \quad - \int_0^1 \frac{\partial}{\partial t} M\left(\frac{1}{2}, x_k\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & \quad + \int_0^1 \frac{\partial}{\partial t} M\left(\max\left\{t - \frac{1}{2}, 0\right\}, x_k\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & = \int_0^1 \frac{\partial}{\partial t}(x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & \quad - \int_0^1 \frac{\partial}{\partial t}(x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & \quad + \int_0^{x_k + \frac{1}{2}} \frac{\partial}{\partial t}\left(\max\left\{t - \frac{1}{2}, 0\right\}\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & = 0 - 0 + \int_{\frac{1}{2}}^{x_k + \frac{1}{2}} (1) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\ & = C\left(x_1, \ldots, x_{k-1}, x_k + \frac{1}{2}, x_{k+1}, \ldots, x_d\right) - C\left(x_1, \ldots, x_{k-1}, \frac{1}{2}, x_{k+1}, \ldots, x_d\right). \end{aligned}$$

Case 2. If $x_k \in \left[\frac{1}{2}, 1\right]$, then

$$
\begin{aligned}
&S\heartsuit_k C(x_1, \ldots, x_d) \\
&\qquad = \int_0^1 \partial_1 S(t, x_k) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad = \int_0^1 \frac{\partial}{\partial t} M\left(\min\left\{t + \frac{1}{2}, 1\right\}, x_k\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad\quad - \int_0^1 \frac{\partial}{\partial t} M\left(\frac{1}{2}, x_k\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad\quad + \int_0^1 \frac{\partial}{\partial t} M\left(\max\left\{t - \frac{1}{2}, 0\right\}, x_k\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad = \int_0^{x_k - \frac{1}{2}} \frac{d}{dt}\left(\min\left\{t + \frac{1}{2}, 1\right\}\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad\quad - \int_0^1 \frac{d}{dt}\left(\frac{1}{2}\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad\quad + \int_0^1 \frac{d}{dt}\left(\max\left\{t - \frac{1}{2}, 0\right\}\right) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad = \int_0^{x_k - \frac{1}{2}} (1) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt - 0 \\
&\qquad\quad + \int_{\frac{1}{2}}^1 (1) \partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\, dt \\
&\qquad = C\left(x_1, \ldots, x_{k-1}, x_k - \frac{1}{2}, x_{k+1}, \ldots, x_d\right) + C(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d) \\
&\qquad\quad - C\left(x_1, \ldots, x_{k-1}, \frac{1}{2}, x_{k+1}, \ldots, x_d\right).
\end{aligned}
$$

Hence, for all $x_1, \ldots, x_d \in \mathbb{I}$,

$$
S\heartsuit_k C(x_1, \ldots, x_d) =
\begin{cases}
C\left(x_1, \ldots, x_{k-1}, x_k + \frac{1}{2}, x_{k+1}, \ldots, x_d\right) \\
-C\left(x_1, \ldots, x_{k-1}, \frac{1}{2}, x_{k+1}, \ldots, x_d\right) & , \text{ if } x_k \in \left[0, \frac{1}{2}\right], \\
\\
C\left(x_1, \ldots, x_{k-1}, x_k - \frac{1}{2}, x_{k+1}, \ldots, x_d\right) \\
+C(x_1, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_d) & , \text{ if } x_k \in \left[\frac{1}{2}, 1\right], \\
-C\left(x_1, \ldots, x_{k-1}, \frac{1}{2}, x_{k+1}, \ldots, x_d\right)
\end{cases}
$$

$$= C\left(x_1, \ldots, x_{k-1}, \min\left\{x_k + \frac{1}{2}, 1\right\}, x_{k+1}, \ldots, x_d\right)$$
$$- C\left(x_1, \ldots, x_{k-1}, \frac{1}{2}, x_{k+1}, \ldots, x_d\right)$$
$$+ C\left(x_1, \ldots, x_{k-1}, \max\left\{x_k - \frac{1}{2}, 0\right\}, x_{k+1}, \ldots, x_d\right).$$

□

Theorem 65. *Let $X_1, \ldots, X_d$ be continuous random variables with d-copula $C = C_{X_1,\ldots,X_d}$ and marginal distributions $F_1, \ldots, F_d$, respectively. If $F_1, \ldots, F_d$ are strictly increasing, then*

$$E[I_{X_1\le x_1,\ldots,X_{k-1}\le x_{k-1},X_{k+1}\le x_{k+1},\ldots,X_d\le x_d}|X_k = t] = \partial_k C(F_{X_1}(x_1), \ldots, F_{X_k}(t), \ldots, F_{X_d}(x_d)).$$

Proof.

$$E[I_{X_1\le x_1,\ldots,X_{k-1}\le x_{k-1},X_{k+1}\le x_{k+1},\ldots,X_d\le x_d}|X_k]$$
$$= P(X_1 \le x_1, \ldots, X_{k-1} \le x_{k-1}, X_{k+1} \le x_{k+1}, \ldots, X_d \le x_d|X_k)$$

Conditioning on $X_k = t$ is conditioning on an event with probability zero. This is not define, so we make sense by a limiting procedure:

$$P(X_1 \le x_1, \ldots, X_{k-1} \le x_{k-1}, X_{k+1} \le x_{k+1}, \ldots, X_d \le x_d|X_k = t)$$
$$= \lim_{h\to 0} P(X_1 \le x_1, \ldots, X_{k-1} \le x_{k-1}, X_{k+1} \le x_{k+1}, \ldots, X_d \le x_d|t \le X_k \le t+h)$$
$$= \lim_{h\to 0} \frac{P(X_1 \le x_1, \ldots, t \le X_k \le t+h, \ldots, X_d \le x_d)}{P(t \le X_k \le t+h)}$$
$$= \lim_{h\to 0} \frac{P(X_1 \le x_1, \ldots, X_k \le t+h, \ldots, X_d \le x_d) - P(X_1 \le x_1, \ldots, X_k \le t, \ldots, X_d \le x_d)}{P(X_k \le t+h) - P(X_k \le t)}$$
$$= \lim_{h\to 0} \frac{F_{X_1,\ldots,X_d}(x_1, \ldots, x_{k-1}, t+h, x_{k+1}, \ldots, x_d) - F_{X_1,\ldots,X_d}(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)}{F_{X_k}(t+h) - F_{X_k}(t)}$$
$$= \lim_{h\to 0} \frac{C(F_{X_1}(x_1), \ldots, F_{X_k}(t+h), \ldots, F_{X_d}(x_d)) - C(F_{X_1}(x_1), \ldots, F_{X_k}(t), \ldots, F_{X_d}(x_d))}{F_{X_k}(t+h) - F_{X_k}(t)}$$
$$= \partial_k C(F_{X_1}(x_1), \ldots, F_{X_k}(t), \ldots, F_{X_d}(x_d)).$$

□

Theorem 66. *Let $Y, X_1, \ldots, X_d$ be continuous random variables. If $X_1, \ldots, X_d$ are conditionally independent given Y, then*

$$C_{Y,X_k} \heartsuit_k C_{X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_d} = C_{X_1,\ldots,X_d},$$

for all $k = 1, \ldots, d$.

Proof. Since

$$E[I_{X_1 \le x_1,\ldots,X_d \le x_d} | Y = t] = E[I_{X_1 \le x_1,\ldots,X_{k-1} \le x_{k-1},X_{k+1} \le x_{k+1},\ldots,X_d \le x_d} | Y = t] E[I_{X_k \le x_k} | Y = t],$$

by Theorem 65, we have

$$\begin{aligned}
&\partial_1 C_{Y,X_1,\ldots,X_d}(F_Y(t), F_{X_1}(x_1), \ldots, F_{X_d}(x_d)) \\
&= \partial_k C_{X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_d}(F_{X_1}(x_1), \ldots, F_{X_{k-1}}(x_{k-1}), F_Y(t), F_{X_{k+1}}(x_{k+1}), \ldots, F_{X_d}(x_d)) \\
&\quad \times \partial_1 C_{Y,X_k}(F_Y(t), F_{X_k}(x_k)).
\end{aligned}$$

Since $F_Y, F_{X_1}, \ldots, F_{X_d}$ are continuous on $\mathbb{I}$, for all $u_1, \ldots, u_d \in \mathbb{I}$,

$$\begin{aligned}
&\partial_1 C_{Y,X_1,\ldots,X_d}(t, u_1, \ldots, u_d) \\
&\quad = \partial_k C_{X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_d}(u_1, \ldots, u_{k-1}, t, u_{k+1}, \ldots, u_d) \partial_1 C_{Y,X_k}(t, u_k).
\end{aligned}$$

Consider, for any $u_1, \ldots, u_d \in \mathbb{I}$,

$$\begin{aligned}
&C_{Y,X_k} \heartsuit_k C_{X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_d}(u_1, \ldots, u_d) \\
&= \int_0^1 \partial_1 C_{Y,X_k}(t, x_k) \partial_k C_{X_1,\ldots,X_{k-1},Y,X_{k+1},\ldots,X_d}(u_1, \ldots, u_{k-1}, t, u_{k+1}, \ldots, u_d)\, dt \\
&= \int_0^1 \partial_1 C_{Y,X_1,\ldots,X_d}(t, u_1, \ldots, u_d)\, dt \\
&= C_{X_1,\ldots,X_d}(u_1, \ldots, u_d).
\end{aligned}$$

□

5.2 Multivariate shuffling of d-copulas

Notice that the $\heartsuit_k$-product of a bivariate shuffle of Min and a d-copula C amount to 'k-axis' shuffling of C. So shuffling C in all axes could be defined via $\heartsuit_k$-multiplying by shuffles of Min for all k.

Recall that the $*$-product of a 2-copula C and a Q-shuffle of M is a Q-shuffle of C, where $Q := [Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^2}$ is a doubly stochastic matrix. In this section, we shall state and prove an analogous fact in d dimension. Now, for each $k = 2, 3, \ldots, d$, set

$$A^{(k)}(x_1, x_k) = A(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}),$$

where A is a d-copula and $Q = [Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^d}$ is a d-fold stochastic matrix.

Theorem 67. *Let $C \in \mathscr{C}_d$ and $Q = [Q_{\boldsymbol{i}}]_{\boldsymbol{i}\in\mathcal{I}_m^d}$ be a d-fold stochastic matrix. Then*

$$M_Q^{(d)} \heartsuit_d (M_Q^{(d-1)} \heartsuit_{d-1} \cdots (M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C)) \cdots) = C_Q \tag{2}$$

Proof. Since $M_Q^{(k)}(x_1, x_k) = M_Q^d(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}})$,

$$\begin{aligned}
\mu_{M_Q^{(k)}}([0, x_1] \times [0, x_k]) &= \mu_{M_Q^d}([0, x_1] \times \mathbb{I}^{k-2} \times [0, x_k] \times \mathbb{I}^{d-k}) \\
&= \mu_{M^d}(\phi^{-1}([0, x_1] \times \mathbb{I}^{k-2} \times [0, x_k] \times \mathbb{I}^{d-k})) \\
&= \mu_{M^d}([0, x_1] \times \mathbb{I}^{k-2} \times \phi_k^{-1}[0, x_k] \times \mathbb{I}^{d-k}) \\
&= \mu_M([0, x_1] \times \phi_k^{-1}[0, x_k]),
\end{aligned}$$

since $M^d(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}) = M(x_1, x_k)$. To prove the equation (2), it suffices to show that

$$\mu_{M_Q^{(d)} \heartsuit_d (M_Q^{(d-1)} \heartsuit_{d-1} \cdots (M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C)) \cdots)} = \mu_{C_Q}.$$

By Theorem 61 and Remark 62,

$$\begin{aligned}
&\mu_{M_Q^{(d)} \heartsuit_d (M_Q^{(d-1)} \heartsuit_{d-1} \cdots (M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C)) \cdots)}([0, x_1] \times \cdots \times [0, x_d]) \\
&= \mu_{M_Q^{(d)}} \heartsuit_d \mu_{M_Q^{(d-1)} \heartsuit_{d-1} \cdots (M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C))}([0, x_1] \times \cdots \times [0, x_d]) \\
&= \int_0^1 \frac{d}{dt} \mu_{M_Q^{(d)}}([0, t] \times [0, x_d]) \\
&\qquad \times \frac{d}{dt} \mu_{M_Q^{(d-1)} \heartsuit_{d-1} \cdots (M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C))}([0, x_1] \times \cdots \times [0, x_{d-1}] \times [0, t])\, dt
\end{aligned}$$

$$= \int_0^1 \frac{d}{dt}\mu_M([0,t] \times \phi_d^{-1}[0,x_d])$$
$$\times \frac{d}{dt}\mu_{M_Q^{(d-1)}\heartsuit_{d-1}\cdots(M_Q^{(3)}\heartsuit_3(M_Q^{(2)}\heartsuit_2 C))}([0,x_1] \times \cdots \times [0,x_{d-1}] \times [0,t])\, dt$$
$$= \mu_M \heartsuit_d \mu_{M_Q^{(d-1)}\heartsuit_{d-1}\cdots(M_Q^{(3)}\heartsuit_3(M_Q^{(2)}\heartsuit_2 C))}([0,x_1] \times \cdots \times [0,x_{d-1}] \times \phi_d^{-1}[0,x_d])$$
$$= \mu_{M_Q^{(d-1)}\heartsuit_{d-1}\cdots(M_Q^{(3)}\heartsuit_3(M_Q^{(2)}\heartsuit_2 C))}([0,x_1] \times \cdots \times [0,x_{d-1}] \times \phi_d^{-1}[0,x_d])$$

(by induction)

$$= \mu_C([0,x_1] \times \phi_2^{-1}[0,x_2] \times \cdots \times \phi_d^{-1}[0,x_d])$$
$$= \mu_C(\phi^{-1}([0,x_1] \times \cdots \times [0,x_d]))$$
$$= \mu_{C_Q}([0,x_1] \times \cdots \times [0,x_d]),$$

which is the desired assertion. □

Theorem 68. *For any $A, B \in \mathscr{C}_2$, $C \in \mathscr{C}_d$ and integers $k \neq l$ in $\{1, \ldots, d\}$,*

$$A\heartsuit_k(B\heartsuit_l C) = B\heartsuit_l(A\heartsuit_k C).$$

Proof. WLOG, assume that $k < l$. For any $(x_1, \ldots, x_d) \in \mathbb{I}^d$,

$$A\heartsuit_k(B\heartsuit_l C)(x_1, \ldots, x_d)$$
$$= \int_0^1 \partial_1 A(t, x_k)\partial_k \left[B\heartsuit_l C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_d)\right] dt$$
$$= \int_0^1 \partial_1 A(t, x_k)\partial_k \left[\int_0^1 \partial_1 B(s, x_l)\partial_l C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l-1}, s, x_{l+1}, \ldots, x_d) ds\right] dt$$
$$= \int_0^1 \partial_1 A(t, x_k) \left[\int_0^1 \partial_1 B(s, x_l)\partial^2_{k,l} C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l-1}, s, x_{l+1}, \ldots, x_d) ds\right] dt$$
$$= \int_0^1 \partial_1 B(s, x_l) \left[\int_0^1 \partial_1 A(t, x_k)\partial^2_{l,k} C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l-1}, s, x_{l+1}, \ldots, x_d)\, dt\right] ds$$
$$= \int_0^1 \partial_1 B(s, x_l)\partial_l \left[\int_0^1 \partial_1 A(t, x_k)\partial_k C(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_{l-1}, s, x_{l+1}, \ldots, x_d)\, dt\right] ds$$
$$= \int_0^1 \partial_1 B(s, x_l)\partial_l \left[A\heartsuit_k C(x_1, \ldots, x_{l-1}, s, x_{l+1}, \ldots, x_d)\right] ds$$
$$= B\heartsuit_l(A\heartsuit_k C)(x_1, \ldots, x_d).$$

□

Example 69. *Let $C \in \mathscr{C}_3$ and Q be the $2\times2\times2$ three-fold stochastic matrix associated with a 3-copula A, where $A(x,y,z) = W(x, M(y,z))$ for all $(x,y,z) \in \mathbb{I}^3$. By Example 54, we have*

$$\begin{aligned}
&C_Q(x,y,z)\\
&\quad = C\left(x, \min\left\{y+\frac{1}{2},1\right\}, \min\left\{z+\frac{1}{2},1\right\}\right) - C\left(x, \min\left\{y+\frac{1}{2},1\right\}, \frac{1}{2}\right)\\
&\qquad + C\left(x, \min\left\{y+\frac{1}{2},1\right\}, \max\left\{z-\frac{1}{2},0\right\}\right) - C\left(x, \frac{1}{2}, \min\left\{z+\frac{1}{2},1\right\}\right)\\
&\qquad + C\left(x, \frac{1}{2}, \frac{1}{2}\right) - C\left(x, \frac{1}{2}, \max\left\{z-\frac{1}{2},0\right\}\right)\\
&\qquad + C\left(x, \max\left\{y-\frac{1}{2},0\right\}, \min\left\{z+\frac{1}{2},1\right\}\right) - C\left(x, \max\left\{y-\frac{1}{2},0\right\}, \frac{1}{2}\right)\\
&\qquad + C\left(x, \max\left\{y-\frac{1}{2},0\right\}, \max\left\{z-\frac{1}{2},0\right\}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&M_Q(x,y,z)\\
&\quad = M^3\left(x, \min\left\{y+\frac{1}{2},1\right\}, \min\left\{z+\frac{1}{2},1\right\}\right) - M^3\left(x, \min\left\{y+\frac{1}{2},1\right\}, \frac{1}{2}\right)\\
&\qquad + M^3\left(x, \min\left\{y+\frac{1}{2},1\right\}, \max\left\{z-\frac{1}{2},0\right\}\right) - M^3\left(x, \frac{1}{2}, \min\left\{z+\frac{1}{2},1\right\}\right)\\
&\qquad + M^3\left(x, \frac{1}{2}, \frac{1}{2}\right) - M^3\left(x, \frac{1}{2}, \max\left\{z-\frac{1}{2},0\right\}\right)\\
&\qquad + M^3\left(x, \max\left\{y-\frac{1}{2},0\right\}, \min\left\{z+\frac{1}{2},1\right\}\right) - M^3\left(x, \max\left\{y-\frac{1}{2},0\right\}, \frac{1}{2}\right)\\
&\qquad + M^3\left(x, \max\left\{y-\frac{1}{2},0\right\}, \max\left\{z-\frac{1}{2},0\right\}\right).
\end{aligned}$$

Therefore, by property of M^3,

$$\begin{aligned}
M_Q^{(2)}(x,y) &= M_Q^3(x,y,1)\\
&= M\left(x, \min\left\{y+\frac{1}{2},1\right\}\right) - M\left(x,\frac{1}{2}\right) + M\left(x, \max\left\{y-\frac{1}{2},0\right\}\right),\\
M_Q^{(3)}(x,z) &= M_Q^3(x,1,z)\\
&= M\left(x, \min\left\{z+\frac{1}{2},1\right\}\right) - M\left(x,\frac{1}{2}\right) + M\left(x, \max\left\{z-\frac{1}{2},0\right\}\right).
\end{aligned}$$

Consider, for each $x, y, z \in \mathbb{I}$,

$$\begin{aligned}
&M_Q^{(2)} \heartsuit_2 C(x,y,z)\\
&\quad= \int_0^1 \partial_1 M_Q^{(2)}(t,y)\partial_2 C(x,t,z)\,dt\\
&\quad= \int_0^1 \partial_1 M\left(t, \min\left\{y+\frac{1}{2},1\right\}\right)\partial_2 C(x,t,z)\,dt - \int_0^1 \partial_1 M\left(t,\frac{1}{2}\right)\partial_2 C(x,t,z)\,dt\\
&\qquad+ \int_0^1 \partial_1 M\left(t, \max\left\{y-\frac{1}{2},0\right\}\right)\partial_2 C(x,t,z)\,dt\\
&\quad= \int_0^{\min\{y+\frac{1}{2},1\}} (1)\partial_2 C(x,t,z)\,dt - \int_0^{\frac{1}{2}} (1)\partial_2 C(x,t,z)\,dt\\
&\qquad+ \int_0^{\max\{y-\frac{1}{2},0\}} (1)\partial_2 C(x,t,z)\,dt\\
&\quad= C\left(x, \min\left\{y+\frac{1}{2},1\right\},z\right) - C\left(x,\frac{1}{2},z\right) + C\left(x,\max\left\{y-\frac{1}{2},0\right\},z\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C)(x,y,z)\\
&\quad= \int_0^1 \partial_1 M_Q^{(3)}(t,z)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt\\
&\quad= \int_0^1 \partial_1 M\left(t, \min\left\{z+\frac{1}{2},1\right\}\right)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt\\
&\qquad- \int_0^1 \partial_1 M\left(t,\frac{1}{2}\right)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt\\
&\qquad+ \int_0^1 \partial_1 M\left(t, \max\left\{z-\frac{1}{2},0\right\}\right)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt\\
&\quad= \int_0^{\min\{z+\frac{1}{2},1\}} (1)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt - \int_0^{\frac{1}{2}} (1)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt\\
&\qquad+ \int_0^{\max\{z-\frac{1}{2},0\}} (1)\partial_3[M_Q^{(2)} \heartsuit_2 C(x,y,t)]\,dt\\
&\quad= M_Q^{(2)} \heartsuit_2 C\left(x,y,\min\left\{z+\frac{1}{2},1\right\}\right) - M_Q^{(2)} \heartsuit_2 C\left(x,y,\frac{1}{2}\right)\\
&\qquad+ M_Q^{(2)} \heartsuit_2 C\left(x,y,\max\left\{z-\frac{1}{2},0\right\}\right)
\end{aligned}$$

$$
\begin{aligned}
&= C\left(x, \min\left\{y+\frac{1}{2}, 1\right\}, \min\left\{z+\frac{1}{2}, 1\right\}\right) - C\left(x, \min\left\{y+\frac{1}{2}, 1\right\}, \frac{1}{2}\right) \\
&\quad + C\left(x, \min\left\{y+\frac{1}{2}, 1\right\}, \max\left\{z-\frac{1}{2}, 0\right\}\right) - C\left(x, \frac{1}{2}, \min\left\{z+\frac{1}{2}, 1\right\}\right) \\
&\quad + C\left(x, \frac{1}{2}, \frac{1}{2}\right) - C\left(x, \frac{1}{2}, \max\left\{z-\frac{1}{2}, 0\right\}\right) \\
&\quad + C\left(x, \max\left\{y-\frac{1}{2}, 0\right\}, \min\left\{z+\frac{1}{2}, 1\right\}\right) - C\left(x, \max\left\{y-\frac{1}{2}, 0\right\}, \frac{1}{2}\right) \\
&\quad + C\left(x, \max\left\{y-\frac{1}{2}, 0\right\}, \max\left\{z-\frac{1}{2}, 0\right\}\right) \\
&= C_Q(x, y, z).
\end{aligned}
$$

Then, for all $x, y, z \in \mathbb{I}$,

$$M_Q^{(3)} \heartsuit_3 (M_Q^{(2)} \heartsuit_2 C) = C_Q,$$

5.3 A product of d-copulas

In light of Theorem 67 and 68, we shall define a product operation between d-copulas as follows.

Definition 70. *Let $A, B \in \mathscr{C}_d$. The $\heartsuit$-**product** of A and B is the function $A \heartsuit B \colon \mathbb{I}^d \to \mathbb{I}$ defined as*

$$
\begin{aligned}
&A \heartsuit B(x_1, \ldots, x_d) \\
&\quad = A^{(d)} \heartsuit_d (A^{(d-1)} \heartsuit_{d-1} \cdots (A^{(3)} \heartsuit_3 (A^{(2)} \heartsuit_2 B)) \cdots)(x_1, \ldots, x_d) \\
&\quad = \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial_{2,\ldots,d}^{d-1} B(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d.
\end{aligned}
$$

Remark 71. *By Theorem 67, $M_Q^d \heartsuit C = C_Q$, where C is a d-copula.*

Theorem 72. *Let $A, B \in \mathscr{C}_d$. Then $A \heartsuit B$ is a d-copula.*

Proof. Let $A, B \in \mathscr{C}_d$.

(C1) Obviously, $A \heartsuit B(x_1, \ldots, x_d) = 0$ whenever $x_l = 0$ for some l.

(C2) Let $x_1, \dots, x_d \in \mathbb{I}$.

$$
\begin{aligned}
&A \heartsuit B(\overbrace{1, \dots, 1}^{l-1 \text{ terms}}, x_l, \overbrace{1, \dots, 1}^{d-l \text{ terms}}) \\
&= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, 1) \cdots \partial_1 A^{(l-1)}(t_{l-1}, 1) \partial_1 A^{(l)}(t_l, x_l) \partial_1 A^{(l+1)}(t_{l+1}, 1) \\
&\qquad\qquad \times \cdots \times \partial_1 A^{(2)}(t_2, 1) \partial^{d-1}_{2,\dots,d} B(1, t_2, \dots, t_d)\, dt_2 \cdots dt_d \\
&= \int_0^1 \cdots \int_0^1 \overbrace{(1) \cdots (1)}^{l-1 \text{ terms}} \partial_1 A^{(l)}(t_l, x_l) \overbrace{(1) \cdots (1)}^{d-l \text{ terms}} \partial^{d-1}_{2,\dots,d} B(1, t_2, \dots, t_d)\, dt_2 \cdots dt_d \\
&= \int_0^1 \partial_1 A^{(l)}(t_l, x_l) \left[\int_0^1 \cdots \int_0^1 \partial^{d-1}_{2,\dots,d} B(1, t_2, \dots, t_d)\, dt_2 \cdots dt_{l-1}\, dt_{l+1} \cdots dt_d \right] dt_l \\
&= \int_0^1 \partial_1 A^{(l)}(t_l, x_l) \partial_l B(\overbrace{1, \dots, 1}^{l-1 \text{ terms}}, t_l, \overbrace{1, \dots, 1}^{d-l \text{ terms}})\, dt_l \\
&= \int_0^1 \partial_1 A^{(l)}(t_l, x_l)(1)\, dt_l \\
&= A^{(l)}(1, x_l) = x_l.
\end{aligned}
$$

(C3) Let $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{I}^d$.

$$
\begin{aligned}
&V_{A \heartsuit B}([\boldsymbol{a}, \boldsymbol{b}]) \\
&= \triangle^d_{(a_d, b_d)} \cdots \triangle^1_{(a_1, b_1)} A \heartsuit B(x_1, \dots, x_d) \\
&= \triangle^d_{(a_d, b_d)} \cdots \triangle^1_{(a_1, b_1)} \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \\
&\qquad\qquad \times \partial^{d-1}_{2,\dots,d} B(x_1, t_2, \dots, t_d)\, dt_2 \cdots dt_d \\
&= \int_0^1 \cdots \int_0^1 \frac{d}{dt}[A^{(d)}(t_d, b_d) - A^{(d)}(t_d, a_d)] \cdots \frac{d}{dt}[A^{(2)}(t_2, b_2) - A^{(2)}(t_2, a_2)] \\
&\qquad\qquad \times \partial^{d-1}_{2,\dots,d} [B(b_1, t_2, \dots, t_d) - B(a_1, t_2, \dots, t_d)]\, dt_2 \cdots dt_d \\
&\geq 0, \text{ by Remark 22.}
\end{aligned}
$$

□

Remark 73. *For any $A, B \in \mathscr{C}_2$, $A \heartsuit B = B * A$.*

Proposition 74. *Let $C \in \mathscr{C}_d$. Then*

$$M^d \heartsuit C = C,$$

$$\Pi^d \heartsuit C = \Pi^d = C \heartsuit \Pi^d.$$

Proof. 1. Claim that $M^d \heartsuit C = C$. By properties of M^d, we have

$$M^{(k)}(x_1, x_k) = M^d(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}) = M(x_1, x_k)$$

Consider, for any $(x_1, \ldots, x_d) \in \mathbb{I}^d$,

$$\begin{aligned}
&M^d \heartsuit C(x_1, \ldots, x_d) \\
&\quad = \int_0^1 \cdots \int_0^1 \partial_1 M^{(d)}(t_d, x_d) \cdots \partial_1 M^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad = \int_0^1 \cdots \int_0^1 \partial_1 M(t_d, x_d) \cdots \partial_1 M(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad = \int_0^{x_d} \cdots \int_0^{x_2} \overbrace{(1) \cdots (1)}^{d-1 \text{ terms}} \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad = C(x_1, \ldots, x_d).
\end{aligned}$$

2. Claim that $\Pi^d \heartsuit C = \Pi^d$. By properties of Π^d, we have

$$\Pi^{(k)}(x_1, x_k) = \Pi^d(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}) = x_1 x_k$$

Consider, for any $(x_1, \ldots, x_d) \in \mathbb{I}^d$,

$$\begin{aligned}
&\Pi^d \heartsuit C(x_1, \ldots, x_d) \\
&\quad = \int_0^1 \cdots \int_0^1 \partial_1 \Pi^{(d)}(t_d, x_d) \cdots \partial_1 \Pi^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad = \int_0^1 \cdots \int_0^1 \frac{\partial}{\partial t_d}(t_d x_d) \cdots \frac{\partial}{\partial t_2}(t_2 x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad = \int_0^1 \cdots \int_0^1 (x_d) \cdots (x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&\quad = (x_d \cdots x_2) C(x_1, 1, \ldots, 1) = \Pi^d(x_1, \ldots, x_d).
\end{aligned}$$

3. Claim that $C\heartsuit\Pi^d = \Pi^d$.

$$\begin{aligned}
&C\heartsuit\Pi^d(x_1,\dots,x_d)\\
&= \int_0^1 \cdots \int_0^1 \partial_1 C^{(d)}(t_d,x_d)\cdots\partial_1 C^{(2)}(t_2,x_2)\partial^{d-1}_{2,\dots,d}\Pi(x_1,t_2,\dots,t_d)\,dt_2\cdots dt_d\\
&= \int_0^1 \cdots \int_0^1 \partial_1 C^{(d)}(t_d,x_d)\cdots\partial_1 C^{(2)}(t_2,x_2)\frac{\partial^{d-1}}{\partial t_d\cdots\partial t_2}(x_1t_2\cdots t_d)\,dt_2\cdots dt_d\\
&= \int_0^1 \cdots \int_0^1 \partial_1 C^{(d)}(t_d,x_d)\cdots\partial_1 C^{(2)}(t_2,x_2)(x_1)\,dt_2\cdots dt_d\\
&= (x_1)C^{(2)}(1,x_2)\cdots C^{(d)}(1,x_d) = \Pi^d(x_1,\dots,x_d).
\end{aligned}$$

□

Next, we have 2 examples which shows that $C\heartsuit M^d$ may or may not equal to C, where C is a d-copula, respectively. In particular, $\heartsuit$ is not commutative.

Example 75. *Let $C(x,y,z) = M(W(x,y),z)$ for all $x,y,z \in \mathbb{I}$. Then*

$$C^{(2)}(x,y) = C(x,y,1) = M(W(x,y),1) = W(x,y),$$

$$C^{(3)}(x,z) = C(x,1,z) = M(W(x,1),z) = M(x,z).$$

For any $(x,y,z)\in\mathbb{I}^3$,

$$\begin{aligned}
C\heartsuit M(x,y,z) &= \int_0^1\int_0^1 \partial_1 C^{(3)}(t_3,z)\partial_1 C^{(2)}(t_2,y)\partial^2_{2,3}M^3(x,t_2,t_3)\,dt_2\,dt_3\\
&= \int_0^1\int_0^1 \partial_1 M(t_3,z)\partial_1 W(t_2,y)\partial^2_{2,3}M^3(x,t_2,t_3)\,dt_2\,dt_3\\
&= \int_0^1 \partial_1 W(t_2,y)\frac{\partial}{\partial t_2}\left[\int_0^1 \partial_1 M(t_3,z)\partial_3 M^3(x,t_2,t_3)\,dt_3\right]dt_2\\
&= \int_0^1 \partial_1 W(t_2,y)\frac{\partial}{\partial t_2}\left[\int_0^z (1)\partial_3 M^3(x,t_2,t_3)\,dt_3\right]dt_2\\
&= \int_0^1 \partial_1 W(t_2,y)\partial_2 M^3(x,t_2,z)\,dt_2\\
&= \int_{1-y}^1 (1)\partial_2 M^3(x,t_2,z)\,dt_2\\
&= M^3(x,1,z) - M^3(x,1-y,z).
\end{aligned}$$

Consider at $(x,y,z)=\left(\frac{1}{2},\frac{3}{4},\frac{1}{4}\right)$,

$$C\heartsuit M^3\left(\frac{1}{2},\frac{3}{4},\frac{1}{4}\right)=M^3\left(\frac{1}{2},1,\frac{1}{4}\right)-M^3\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right)=\frac{1}{4}-\frac{1}{4}=0$$

but

$$C\left(\frac{1}{2},\frac{3}{4},\frac{1}{4}\right)=M\left(W\left(\frac{1}{2},\frac{3}{4}\right),\frac{1}{4}\right)=M\left(\frac{1}{4},\frac{1}{4}\right)=\frac{1}{4}.$$

Hence, $C\heartsuit M^3\neq C$.

Example 76. *Let* $C(x,y,z)=\Pi(W(x,y),z)$ *for all* $x,y,z\in\mathbb{I}$. *Then*

$$C^{(2)}(x,y)=C(x,y,1)=\Pi(W(x,y),1)=W(x,y),$$

$$C^{(3)}(x,z)=C(x,1,z)=\Pi(W(x,1),z)=\Pi(x,z).$$

For any $(x,y,z)\in\mathbb{I}^3$,

$$\begin{aligned}
C\heartsuit M^3(x,y,z)&=\int_0^1\int_0^1\partial_1C^{(3)}(t_3,z)\partial_1C^{(2)}(t_2,y)\partial^2_{2,3}M^3(x,t_2,t_3)\,dt_2\,dt_3\\
&=\int_0^1\int_0^1\partial_1\Pi(t_3,z)\partial_1W(t_2,y)\partial^2_{2,3}M^3(x,t_2,t_3)\,dt_2\,dt_3\\
&=\int_0^1\partial_1W(t_2,y)\frac{\partial}{\partial t_2}\left[\int_0^1\partial_1\Pi(t_3,z)\partial_3M^3(x,t_2,t_3)\,dt_3\right]dt_2\\
&=\int_0^1\partial_1W(t_2,y)\frac{\partial}{\partial t_2}\left[\int_0^1(z)\partial_3M^3(x,t_2,t_3)\,dt_3\right]dt_2\\
&=\int_0^1\partial_1W(t_2,y)\partial_2zM^3(x,t_2,1)\,dt_2\\
&=\int_{1-y}^1(1)\partial_2zM^3(x,t_2,1)\,dt_2\\
&=z(M^3(x,1,1)-M^3(x,1-y,1))\\
&=z(x-M(x,1-y))\\
&=\begin{cases}z(x+y-1) & \text{if } x\geq 1-y\\ z(x-x)=z(0) & \text{if } x\leq 1-y\end{cases}\\
&=zW(x,y)=\Pi(W(x,y),z).
\end{aligned}$$

Hence, $C\heartsuit M^3=C$.

Theorem 77. *The ♡-product is associative; that is*

$$A\heartsuit(B\heartsuit C) = (A\heartsuit B)\heartsuit C,$$

where A, B and C are d-copulas.

Proof. Let $(x_1, \ldots, x_d) \in \mathbb{I}^d$. Set $T^{(k)} = (A\heartsuit B)^{(k)}$. Since

$$\begin{aligned}
&A\heartsuit B(x_1, \ldots, x_d) \\
&\quad = \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} B(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d,
\end{aligned}$$

for each k,

$$\begin{aligned}
T^{(k)}(x_1, x_k) &= A\heartsuit B(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ terms}}, x_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}}) \\
&= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, 1) \cdots \partial_1 A^{(k+1)}(t_{k+1}, 1) \partial_1 A^{(k)}(t_k, x_k) \partial_1 A^{(k-1)}(t_{k-1}, 1) \\
&\qquad\qquad \times \cdots \times \partial_1 A^{(2)}(t_2, 1) \partial^{d-1}_{2,\ldots,d} B(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&= \int_0^1 \cdots \int_0^1 \overbrace{(1) \cdots (1)}^{d-k \text{ terms}}\, \partial_1 A^{(k)}(t_k, x_k)\, \overbrace{(1) \cdots (1)}^{k-2 \text{ terms}} \\
&\qquad\qquad \times \partial^{d-1}_{2,\ldots,d} B(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\
&= \int_0^1 \partial_1 A^{(k)}(t_k, x_k) \partial_k B(x_1, \overbrace{1, \ldots, 1}^{k-2 \text{ trems}}, t_k, \overbrace{1, \ldots, 1}^{d-k \text{ terms}})\, dt_k \\
&= \int_0^1 \partial_1 A^{(k)}(t_k, x_k) \partial_2 B^{(k)}(x_1, t_k)\, dt_k,
\end{aligned}$$

so we have

$$\begin{aligned}
\partial_1 T^{(k)}(t'_k, x_k) &= \frac{\partial}{\partial t'_k} \int_0^1 \partial_1 A^{(k)}(t_k, x_k) \partial_2 B^{(k)}(t'_k, t_k)\, dt_k \\
&= \int_0^1 \partial_1 A^{(k)}(t_k, x_k) \partial^2_{1,2} B^{(k)}(t'_k, t_k)\, dt_k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&A\heartsuit(B\heartsuit C)(x_1, \ldots, x_d) \\
&\quad = \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} B\heartsuit C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d
\end{aligned}$$

$$= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2)$$
$$\times \frac{\partial^{d-1}}{\partial t_d \cdots \partial t_2} \left[\int_0^1 \cdots \int_0^1 \partial_1 B^{(d)}(t'_d, t_d) \cdots \partial_1 B^{(2)}(t'_2, t_2) \right.$$
$$\left. \times\, \partial^{d-1}_{2,\ldots,d} C(x_1, t'_2, \ldots, t'_d)\, dt'_2 \cdots dt'_d \right] dt_2 \cdots dt_d$$
$$= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2)$$
$$\times \left[\int_0^1 \cdots \int_0^1 \partial^2_{1,2} B^{(d)}(t'_d, t_d) \cdots \partial^2_{1,2} B^{(2)}(t'_2, t_2) \right.$$
$$\left. \times\, \partial^{d-1}_{2,\ldots,d} C(x_1, t'_2, \ldots, t'_d)\, dt'_2 \cdots dt'_d \right] dt_2 \cdots dt_d$$
$$= \int_0^1 \cdots \int_0^1 \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2)$$
$$\times\, \partial^2_{1,2} B^{(d)}(t'_d, t_d) \cdots \partial^2_{1,2} B^{(2)}(t'_2, t_2)$$
$$\times\, \partial^{d-1}_{2,\ldots,d} C(x_1, t'_2, \ldots, t'_d)\, dt'_2 \cdots dt'_d\, dt_2 \cdots dt_d$$
$$= \int_0^1 \cdots \int_0^1 \left[\int_0^1 \partial_1 A^{(d)}(t_d, x_d) \partial^2_{1,2} B^{(d)}(t'_d, t_d)\, dt_d \right]$$
$$\times \cdots \times \left[\int_0^1 \partial_1 A^{(2)}(t_2, x_2) \partial^2_{1,2} B^{(2)}(t'_2, t_2)\, dt_2 \right]$$
$$\times\, \partial^{d-1}_{2,\ldots,d} C(x_1, t'_2, \ldots, t'_d)\, dt'_2 \cdots dt'_d$$
$$= \int_0^1 \cdots \int_0^1 \partial_1 T^{(d)}(t'_d, x_d) \cdots \partial_1 T^{(2)}(t'_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t'_2, \ldots, t'_d)\, dt'_2 \cdots dt'_d$$
$$= (A \heartsuit B) \heartsuit C(x_1, \ldots, x_d).$$

□

$A \heartsuit B$ can be defined for A, B in the span of $\mathscr{C}_d$. As such, $\heartsuit$ is left distributive.

Theorem 78. *Let* $A, B, C \in \mathscr{C}_d$. *Then*

$$A \heartsuit (B + C) = (A \heartsuit B) + (A \heartsuit C).$$

Proof. For $(x_1, \ldots, x_d) \in \mathbb{I}^d$,

$$A \heartsuit (B + C)(x_1, \ldots, x_d)$$
$$= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} (B + C)(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$

$$= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} B(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$+ \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$= (A \heartsuit B)(x_1, \ldots, x_d) + (A \heartsuit C)(x_1, \ldots, x_d).$$

□

But ♡ is not right distributive.

Theorem 79. *Let $A, B, C \in \mathscr{C}_d$. Then*

$$(A + B) \heartsuit C = \sum_{\substack{X_k \in \{A^{(k)}, B^{(k)}\} \\ 2 \le k \le d}} X_d \heartsuit_d (X_{d-1} \heartsuit_{d-1} \cdots (X_2 \heartsuit_2 C)).$$

Proof. Let $(x_1, \ldots, x_d) \in \mathbb{I}^d$. Set $T^{(k)} = (A + B)^{(k)} = A^{(k)} + B^{(k)}$. Consider,

$$(A + B) \heartsuit C(x_1, \ldots, x_d)$$
$$= \int_0^1 \cdots \int_0^1 \partial_1 T^{(d)}(t_d, x_d) \cdots \partial_1 T^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$= \int_0^1 \cdots \int_0^1 \partial_1 (A^{(d)} + B^{(d)})(t_d, x_d) \cdots \partial_1 (A^{(2)} + B^{(2)})(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$+ \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \cdots \partial_1 A^{(3)}(t_3, x_3) \partial_1 B^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$+ \cdots$$
$$+ \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_d, x_d) \partial_1 B^{(d-1)}(t_{d-1}, x_{d-1}) \cdots \partial_1 B^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$+ \int_0^1 \cdots \int_0^1 \partial_1 B^{(d)}(t_d, x_d) \cdots \partial_1 B^{(2)}(t_2, x_2) \partial^{d-1}_{2,\ldots,d} C(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d$$
$$= [A \heartsuit_d \cdots (A \heartsuit_2 C)(x_1, \ldots, x_d)] + [A \heartsuit_d \cdots (A \heartsuit_3 (B \heartsuit_2 C))(x_1, \ldots, x_d)]$$
$$+ \cdots + [A \heartsuit_d (B \heartsuit_{d-1} \cdots (B \heartsuit_2 C))(x_1, \ldots, x_d)] + [B \heartsuit_d \cdots (B \heartsuit_2 C)(x_1, \ldots, x_d)]$$
$$= \sum_{\substack{X_k \in \{A^{(k)}, B^{(k)}\} \\ 2 \le k \le d}} X_d \heartsuit_d (X_{d-1} \heartsuit_{d-1} \cdots (X_2 \heartsuit_2 C)).$$

□

Definition 80. *Let $A \in \mathscr{C}_d$ and σ be a permutation on $\{1,2,\ldots,d\}$. We define the* ***σ-transpose*** *A_σ of A by*

$$A_\sigma(x_1,\ldots,x_d) = A(x_{\sigma(1)},\ldots,x_{\sigma(d)}).$$

Theorem 81. *Let A, B be d-copula. Then*

$$(A \heartsuit B)_\sigma = A_\sigma \heartsuit B_\sigma.$$

Proof. Let $y, x_1, \ldots, x_d \in \mathbb{I}$. Set

$$A_\sigma^{(l)}(x_1, x_l) = A_\sigma(x_1,\ldots,x_d), \text{ where } x_i = 1 \text{ for all } i \neq 1, l.$$

Let $\sigma(j) = l$,

$$\begin{aligned} A_\sigma^{(l)}(x_1, x_l) &= A_\sigma(x_1,\ldots,x_d) \\ &= A(x_{\sigma(1)},\ldots,x_{\sigma(d)}) \\ &= A(x_1, x_{\sigma(2)},\ldots,x_{\sigma(j-1)}, x_{\sigma(j)}, x_{\sigma(j+1)},\ldots,x_{\sigma(d)}) \\ &= A(x_1, x_{\sigma(2)},\ldots,x_{\sigma(j-1)}, x_l, x_{\sigma(j+1)},\ldots,x_{\sigma(d)}) \\ &= A(x_1, \overbrace{1,\ldots,1}^{j-2 \text{ terms}}, x_l, \overbrace{1,\ldots,1}^{d-j \text{ terms}}) \\ &= A^{(j)}(x_1, x_l). \end{aligned}$$

Thus,

$$A_\sigma^{(\sigma(j))}(x_1, y) = A_\sigma^{(l)}(x_1, y) = A^{(j)}(x_1, y).$$

Finally,

$$\begin{aligned} &A_\sigma \heartsuit B_\sigma(x_1,\ldots,x_d) \\ &\quad = \int_0^1 \cdots \int_0^1 \partial_1 A_\sigma^{(d)}(t_d, x_d) \cdots \partial_1 A_\sigma^{(2)}(t_2, x_2) \partial_{2,\ldots,d}^{d-1} B_\sigma(x_1, t_2, \ldots, t_d)\, dt_2 \cdots dt_d \\ &\quad = \int_0^1 \cdots \int_0^1 \partial_1 A_\sigma^{(\sigma(d))}(t_{\sigma(d)}, x_{\sigma(d)}) \cdots \partial_1 A_\sigma^{(\sigma(2))}(t_{\sigma(2)}, x_{\sigma(2)}) \\ &\qquad\qquad \times \partial_{2,\ldots,d}^{d-1} B(x_1, t_{\sigma(2)}, \ldots, t_{\sigma(d)})\, dt_{\sigma(2)} \cdots dt_{\sigma(d)} \end{aligned}$$

$$= \int_0^1 \cdots \int_0^1 \partial_1 A^{(d)}(t_{\sigma(d)}, x_{\sigma(d)}) \cdots \partial_1 A^{(2)}(t_{\sigma(2)}, x_{\sigma(2)})$$
$$\times\, \partial^{d-1}_{2,\ldots,d} B(x_1, t_{\sigma(2)}, \ldots, t_{\sigma(d)})\, dt_{\sigma(2)} \cdots dt_{\sigma(d)}$$
$$= A \heartsuit B(x_1, x_{\sigma(2)}, \ldots, x_{\sigma(d)})$$
$$= (A \heartsuit B)_\sigma(x_1, \ldots, x_d).$$

□

Theorem 82. *If random variables X, Z are conditionally independent given a random variable Y, then*

$$C_{Y,Z} \heartsuit C_{X,Y} = C_{X,Z}.$$

Proof. By Remark 73, $C_{Y,Z} \heartsuit C_{X,Y} = C_{X,Y} * C_{Y,Z} = C_{X,Z}$. □

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[9]

APPENDIX

In this appendix, we have several examples for Q-shuffles of C.

Example 83 (48)**.** *Let $C \in \mathscr{C}_2$ and Q be the 3×3 doubly stochastic matrix associated with M; that is* $Q = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$. *Then*

$Q_{(0,2)} = 0$	$Q_{(1,2)} = 0$	$Q_{(2,2)} = \frac{1}{3}$
$Q_{(0,1)} = 0$	$Q_{(1,1)} = \frac{1}{3}$	$Q_{(2,1)} = 0$
$Q_{(0,0)} = \frac{1}{3}$	$Q_{(1,0)} = 0$	$Q_{(2,0)} = 0$

1. $q_{\boldsymbol{i}}^k = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}_m^d, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}_m^d$ *and* $k = 1, 2, \ldots, d$.

$q^1_{(0,2)} = \frac{1}{3}$	$q^1_{(1,2)} = \frac{1}{3}$	$q^1_{(2,2)} = 0$
$q^1_{(0,1)} = \frac{1}{3}$	$q^1_{(1,1)} = 0$	$q^1_{(2,1)} = 0$
$q^1_{(0,0)} = 0$	$q^1_{(1,0)} = 0$	$q^1_{(2,0)} = 0$

$q^2_{(0,2)} = 0$	$q^2_{(1,2)} = 0$	$q^2_{(2,2)} = 0$
$q^2_{(0,1)} = 0$	$q^2_{(1,1)} = 0$	$q^2_{(2,1)} = \frac{1}{3}$
$q^2_{(0,0)} = 0$	$q^2_{(1,0)} = \frac{1}{3}$	$q^2_{(2,0)} = \frac{1}{3}$

2. $J_{\boldsymbol{i}}^k = \left(\dfrac{i_k}{m} + q_{\boldsymbol{i}}^k, \dfrac{i_k}{m} + q_{\boldsymbol{i}}^k + Q_{\boldsymbol{i}}\right)$.

$$J^1_{(0,0)} = \left(0, \frac{1}{3}\right) = J^2_{(0,0)},$$

$$J^1_{(1,1)} = \left(\frac{1}{3}, \frac{2}{3}\right) = J^2_{(1,1)},$$

$$J^1_{(2,2)} = \left(\frac{2}{3}, 1\right) = J^2_{(2,2)}.$$

3. For any $k = 1, \dots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q_{\boldsymbol{i}}^k - Q_{\boldsymbol{i}}^1 \text{ for } t \in J_{\boldsymbol{i}}^1$$

and

$$\phi(t_1, \dots, t_d) = (\phi_1(t_1), \dots, \phi_d(t_d)),$$

for all $t_1, \dots, t_d \in \mathbb{I}$.

If $t \in J^1_{(0,0)} = \left(0, \frac{1}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-0}{3} + 0 - 0 = t$.

If $t \in J^1_{(1,1)} = \left(\frac{1}{3}, \frac{2}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-1}{3} + 0 - 0 = t$.

If $t \in J^1_{(2,2)} = \left(\frac{2}{3}, 1\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{2-2}{3} + 0 - 0 = t$.

Therefore, for all $t_1, t_2 \in \mathbb{I}$,

$$\phi(t_1, t_2) = (t_1, t_2).$$

Hence,

$$\phi^{-1}(t_1, t_2) = (t_1, t_2).$$

4. For any $x_1, x_2 \in \mathbb{I}$,

$$\begin{aligned} C_Q(x_1, x_2) &= \mu_{C_Q}([0, x_1] \times [0, x_2]) \\ &= \mu_C(\phi^{-1}([0, x_1] \times [0, x_2])) \\ &= \mu_C([0, x_1] \times [0, x_2]) \\ &= C(x_1, x_2). \end{aligned}$$

Thus, $M_Q(x_1, x_2) = M(x_1, x_2)$ *for all* $x_1, x_2 \in \mathbb{I}$. *Consider, for each* $x, y \in \mathbb{I}$,

$$\begin{aligned} (C * M_Q)(x, y) &= \int_0^1 \partial_2 C(x, t) \partial_1 M_Q(t, y)\, dt \\ &= \int_0^1 \partial_2 C(x, t) \partial_1 M(t, y)\, dt \\ &= \int_0^y \partial_2 C(x, t)(1)\, dt \\ &= C(x, y) \\ &= C_Q(x, y). \end{aligned}$$

Example 84 (49)**.** *Let $C \in \mathscr{C}_2$ and Q be the 2×2 doubly stochastic matrix associated with W; that is $Q = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Then*

$Q_{(0,1)} = \frac{1}{2}$	$Q_{(1,1)} = 0$
$Q_{(0,0)} = 0$	$Q_{(1,0)} = \frac{1}{2}$

1. $q_{\boldsymbol{i}}^k = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}_m^d, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}_m^d$ *and* $k = 1, 2, \ldots, d$.

$q_{(0,1)}^1 = 0$	$q_{(1,1)}^1 = \frac{1}{2}$
$q_{(0,0)}^1 = 0$	$q_{(1,0)}^1 = 0$

$q_{(0,1)}^2 = 0$	$q_{(1,1)}^2 = \frac{1}{2}$
$q_{(0,0)}^2 = 0$	$q_{(1,0)}^2 = 0$

2. $J_{\boldsymbol{i}}^k = \left(\frac{i_k}{m} + q_{\boldsymbol{i}}^k, \frac{i_k}{m} + q_{\boldsymbol{i}}^k + Q_{\boldsymbol{i}}\right)$.

$$J_{(0,1)}^1 = \left(0, \frac{1}{2}\right) = J_{(1,0)}^2,$$

$$J_{(1,0)}^1 = \left(\frac{1}{2}, 1\right) = J_{(0,1)}^2.$$

3. *For any* $k = 1, \ldots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q_{\boldsymbol{i}}^k - Q_{\boldsymbol{i}}^1 \text{ for } t \in J_{\boldsymbol{i}}^1$$

and

$$\phi(t_1, \ldots, t_d) = (\phi_1(t_1), \ldots, \phi_d(t_d)),$$

for all $t_1, \ldots, t_d \in \mathbb{I}$.

If $t \in J_{(0,1)}^1 = \left(0, \frac{1}{2}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-0}{2} + 0 - 0 = t + \frac{1}{2}$.

If $t \in J_{(1,0)}^1 = \left(\frac{1}{2}, 1\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-1}{2} + 0 - 0 = t - \frac{1}{2}$.

Thus, for any $t_1, t_2 \in \mathbb{I}$,

$$\phi(t_1, t_2) = \begin{cases} \left(t_1, t_2 + \frac{1}{2}\right) & ; t_2 \in \left(0, \frac{1}{2}\right), \\ \left(t_1, t_2 - \frac{1}{2}\right) & ; t_2 \in \left(\frac{1}{2}, 1\right). \end{cases}$$

Hence,

$$\phi^{-1}(t_1,t_2)=\begin{cases}\left(t_1,t_2+\dfrac{1}{2}\right) & ;t_2\in\left(0,\frac{1}{2}\right),\\ \left(t_1,t_2-\dfrac{1}{2}\right) & ;t_2\in\left(\frac{1}{2},1\right).\end{cases}$$

4. For any $x_1,x_2\in\mathbb{I}$,

$$C_Q(x_1,x_2)=\mu_{C_Q}([0,x_1]\times[0,x_2])=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2])).$$

Case 1. $x_2\in\left(0,\frac{1}{2}\right)$. *Then*

$$\begin{aligned}C_Q(x_1,x_2)&=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2]))\\&=\mu_C\left([0,x_1]\times\left[\frac{1}{2},x_2+\frac{1}{2}\right]\right)\\&=C\left(x_1,x_2+\frac{1}{2}\right)-C\left(x_1,\frac{1}{2}\right).\end{aligned}$$

Case 2. $x_2\in\left(\frac{1}{2},1\right)$. *Then*

$$\begin{aligned}C_Q(x_1,x_2)&=\mu_C\left(\phi^{-1}\left([0,x_1]\times\left[0,\frac{1}{2}\right]\right)\right)+\mu_C\left(\phi^{-1}\left([0,x_1]\times\left[\frac{1}{2},x_2\right]\right)\right)\\&=\mu_C\left([0,x_1]\times\left[\frac{1}{2},1\right]\right)+\mu_C\left([0,x_1]\times\left[0,x_2-\frac{1}{2}\right]\right)\\&=x_1-C\left(x_1,\frac{1}{2}\right)+C\left(x_1,x_2-\frac{1}{2}\right).\end{aligned}$$

Thus, for any $x_1,x_2\in\mathbb{I}$,

$$\begin{aligned}C_Q(x_1,x_2)&=\begin{cases}C\left(x_1,x_2+\dfrac{1}{2}\right)-C\left(x_1,\dfrac{1}{2}\right) & ;x_2\in\left(0,\frac{1}{2}\right),\\ x_1-C\left(x_1,\dfrac{1}{2}\right)+C\left(x_1,x_2-\dfrac{1}{2}\right) & ;x_2\in\left(\frac{1}{2},1\right)\end{cases}\\&=C\left(x_1,\min\left\{x_2+\frac{1}{2},1\right\}\right)-C\left(x_1,\frac{1}{2}\right)+C\left(x_1,\max\left\{x_2-\frac{1}{2},0\right\}\right).\end{aligned}$$

Therefore, for any $x_1,x_2\in\mathbb{I}$,

$$M_Q(x_1,x_2)=M\left(x_1,\min\left\{x_2+\frac{1}{2},1\right\}\right)-M\left(x_1,\frac{1}{2}\right)+M\left(x_1,\max\left\{x_2-\frac{1}{2},0\right\}\right).$$

Next, we determine $C*M_Q$, *let* $x,y\in\mathbb{I}$.

Case 1. $y \in (0, \frac{1}{2})$. *Then* $M_Q(x, y) = M\left(x, y + \frac{1}{2}\right) - M\left(x, \frac{1}{2}\right)$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x, y) \\
&= \int_0^1 \partial_2 C(x, t) \partial_1 M_Q(t, y)\, dt \\
&= \int_0^1 \partial_2 C(x, t) \partial_1 M\left(t, y + \frac{1}{2}\right) dt - \int_0^1 \partial_2 C(x, t) \partial_1 M\left(t, \frac{1}{2}\right) dt \\
&= \int_0^{y+\frac{1}{2}} \partial_2 C(x, t)(1)\, dt - \int_0^{\frac{1}{2}} \partial_2 C(x, t)(1)\, dt \\
&= C\left(x, y + \frac{1}{2}\right) - C\left(x, \frac{1}{2}\right).
\end{aligned}$$

Case 2. $y \in (\frac{1}{2}, 1)$. *Then* $M_Q(x, y) = x - M\left(x, \frac{1}{2}\right) + M\left(x, y - \frac{1}{2}\right)$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x, y) \\
&= \int_0^1 \partial_2 C(x, t) \partial_1 M_Q(t, y)\, dt \\
&= \int_0^1 \partial_2 C(x, t) \frac{d}{dt}(t)\, dt - \int_0^1 \partial_2 C(x, t) \partial_1 M(t, \frac{1}{2})\, dt \\
&\quad + \int_0^1 \partial_2 C(x, t) \partial_1 M(t, y - \frac{1}{2})\, dt \\
&= \int_0^1 \partial_2 C(x, t)(1)\, dt - \int_0^{\frac{1}{2}} \partial_2 C(x, t)(1)\, dt + \int_0^{y-\frac{1}{2}} \partial_2 C(x, t)(1)\, dt \\
&= x - C\left(x, \frac{1}{2}\right) + C\left(x, y - \frac{1}{2}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
(C * M_Q)(x, y) &= \begin{cases} C\left(x, y + \frac{1}{2}\right) - C\left(x, \frac{1}{2}\right) & ; y \in \left(0, \frac{1}{2}\right), \\ x - C\left(x, \frac{1}{2}\right) + C\left(x, y - \frac{1}{2}\right) & ; y \in \left(\frac{1}{2}, 1\right) \end{cases} \\
&= C\left(x, \min\left\{y + \frac{1}{2}, 1\right\}\right) - C\left(x, \frac{1}{2}\right) + C\left(x, \max\left\{y - \frac{1}{2}, 0\right\}\right) \\
&= C_Q(x, y).
\end{aligned}$$

Example 85 (50). *Let $C \in \mathscr{C}_2$ and Q be the 2×2 doubly stochastic matrix associated with Π; that is* $Q = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$. *Then*

$Q_{(0,1)} = \frac{1}{4}$	$Q_{(1,1)} = \frac{1}{4}$
$Q_{(0,0)} = \frac{1}{4}$	$Q_{(1,0)} = \frac{1}{4}$

1. $q_{\boldsymbol{i}}^k = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}_m^d, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}_m^d$ *and* $k = 1, \dots, d$.

$q_{(0,1)}^1 = \frac{1}{4}$	$q_{(1,1)}^1 = \frac{1}{4}$
$q_{(0,0)}^1 = 0$	$q_{(1,0)}^1 = 0$

$q_{(0,1)}^2 = 0$	$q_{(1,1)}^2 = \frac{1}{4}$
$q_{(0,0)}^2 = 0$	$q_{(1,0)}^2 = \frac{1}{4}$

2. $J_{\boldsymbol{i}}^k = \left(\frac{i_k}{m} + q_{\boldsymbol{i}}^k, \frac{i_k}{m} + q_{\boldsymbol{i}}^k + Q_{\boldsymbol{i}}\right)$.

$$J_{(0,0)}^1 = \left(0, \frac{1}{4}\right) = J_{(0,0)}^2,$$

$$J_{(0,1)}^1 = \left(\frac{1}{4}, \frac{1}{2}\right) = J_{(1,0)}^2,$$

$$J_{(1,0)}^1 = \left(\frac{1}{2}, \frac{3}{4}\right) = J_{(0,1)}^2,$$

$$J_{(1,1)}^1 = \left(\frac{3}{4}, 1\right) = J_{(1,1)}^2.$$

3. For any $k = 1, \dots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q_{\boldsymbol{i}}^k - Q_{\boldsymbol{i}}^1 \text{ for } t \in J_{\boldsymbol{i}}^1$$

and

$$\phi(t_1, \dots, t_d) = (\phi_1(t_1), \dots, \phi_d(t_d)),$$

for all $t_1, \dots, t_d \in \mathbb{I}$.

If $t \in J_{(0,0)}^1 = \left(0, \frac{1}{4}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-0}{2} + 0 - 0 = t$.

If $t \in J^1_{(0,1)} = \left(\frac{1}{4}, \frac{1}{2}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-0}{2} + 0 - \frac{1}{4} = t + \frac{1}{4}$.

If $t \in J^1_{(1,0)} = \left(\frac{1}{2}, \frac{3}{4}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-1}{2} + \frac{1}{4} - 0 = t - \frac{1}{4}$.

If $t \in J^1_{(1,1)} = \left(\frac{3}{4}, 1\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-1}{2} + \frac{1}{4} - \frac{1}{4} = t$.

Thus, for any $t_1, t_2 \in \mathbb{I}$,

$$\phi(t_1, t_2) = \begin{cases} (t_1, t_2) & ; t_2 \in \left(0, \frac{1}{4}\right), \\ \left(t_1, t_2 + \dfrac{1}{4}\right) & ; t_2 \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ \left(t_1, t_2 - \dfrac{1}{4}\right) & ; t_2 \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ (t_1, t_2) & ; t_2 \in \left(\frac{3}{4}, 1\right). \end{cases}$$

Hence,

$$\phi^{-1}(t_1, t_2) = \begin{cases} (t_1, t_2) & ; t_2 \in \left(0, \frac{1}{4}\right), \\ \left(t_1, t_2 + \dfrac{1}{4}\right) & ; t_2 \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ \left(t_1, t_2 - \dfrac{1}{4}\right) & ; t_2 \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ (t_1, t_2) & ; t_2 \in \left(\frac{3}{4}, 1\right). \end{cases}$$

4. *For any* $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1, x_2) = \mu_{C_Q}([0, x_1] \times [0, x_2]) = \mu_C(\phi^{-1}([0, x_1] \times [0, x_2])).$$

Case 1. $x_2 \in \left(0, \frac{1}{4}\right)$. *Then*

$$\begin{aligned} C_Q(x_1, x_2) &= \mu_C(\phi^{-1}([0, x_1] \times [0, x_2])) \\ &= \mu_C([0, x_1] \times [0, x_2]) \\ &= C(x_1, x_2). \end{aligned}$$

Case 2. $x_2 \in \left(\frac{1}{4}, \frac{1}{2}\right)$. *Then*

$$\begin{aligned} C_Q(x_1, x_2) &= \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[0, \frac{1}{4}\right]\right)\right) + \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[\frac{1}{4}, x_2\right]\right)\right) \\ &= \mu_C\left([0, x_1] \times \left[0, \frac{1}{4}\right]\right) + \mu_C\left([0, x_1] \times \left[\frac{1}{2}, x_2 + \frac{1}{4}\right]\right) \end{aligned}$$

$$= C\left(x_1, \frac{1}{4}\right) + C\left(x_1, x_2 + \frac{1}{4}\right) - C\left(x_1, \frac{1}{2}\right).$$

Case 3. $x_2 \in \left(\frac{1}{2}, \frac{3}{4}\right)$. *Then*

$$\begin{aligned}
C_Q(x_1, x_2) &= \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[0, \frac{1}{4}\right]\right)\right) + \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[\frac{1}{4}, \frac{1}{2}\right]\right)\right) \\
&\quad + \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[\frac{1}{2}, x_2\right]\right)\right) \\
&= \mu_C\left([0, x_1] \times \left[0, \frac{1}{4}\right]\right) + \mu_C\left([0, x_1] \times \left[\frac{1}{2}, \frac{3}{4}\right]\right) \\
&\quad + \mu_C\left([0, x_1] \times \left[\frac{1}{4}, x_2 - \frac{1}{4}\right]\right) \\
&= \mu_C\left([0, x_1] \times \left[0, x_2 - \frac{1}{4}\right]\right) + \mu_C\left([0, x_1] \times \left[\frac{1}{2}, \frac{3}{4}\right]\right) \\
&= C\left(x_1, x_2 - \frac{1}{4}\right) + C\left(x_1, \frac{3}{4}\right) - C\left(x_1, \frac{1}{2}\right).
\end{aligned}$$

Case 4. $x_2 \in \left(\frac{3}{4}, 1\right)$. *Then*

$$\begin{aligned}
C_Q(x_1, x_2) &= \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[0, \frac{1}{4}\right]\right)\right) + \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[\frac{1}{4}, \frac{1}{2}\right]\right)\right) \\
&\quad + \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[\frac{1}{2}, \frac{3}{4}\right]\right)\right) + \mu_C\left(\phi^{-1}\left([0, x_1] \times \left[\frac{3}{4}, x_2\right]\right)\right) \\
&= \mu_C\left([0, x_1] \times \left[0, \frac{1}{4}\right]\right) + \mu_C\left([0, x_1] \times \left[\frac{1}{2}, \frac{3}{4}\right]\right) \\
&\quad + \mu_C\left([0, x_1] \times \left[\frac{1}{4}, \frac{1}{2}\right]\right) + \mu_C\left([0, x_1] \times \left[\frac{3}{4}, x_2\right]\right) \\
&= \mu_C([0, x_1] \times [0, x_2]) \\
&= C(x_1, x_2).
\end{aligned}$$

Thus, for any $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1, x_2) = \begin{cases}
C(x_1, x_2) & ; x_2 \in \left(0, \frac{1}{4}\right), \\
C\left(x_1, \frac{1}{4}\right) + C\left(x_1, x_2 + \frac{1}{4}\right) - C\left(x_1, \frac{1}{2}\right) & ; x_2 \in \left(\frac{1}{4}, \frac{1}{2}\right), \\
C\left(x_1, x_2 - \frac{1}{4}\right) + C\left(x_1, \frac{3}{4}\right) - C\left(x_1, \frac{1}{2}\right) & ; x_2 \in \left(\frac{1}{2}, \frac{3}{4}\right), \\
C(x_1, x_2) & ; x_2 \in \left(\frac{3}{4}, 1\right).
\end{cases}$$

Therefore, for any $x_1, x_2 \in \mathbb{I}$,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{4}\right), \\ M(x_1,\frac{1}{4}) + M(x_1,x_2+\frac{1}{4}) - M(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{4},\frac{1}{2}\right), \\ M(x_1,x_2-\frac{1}{4}) + M(x_1,\frac{3}{4}) - M(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{2},\frac{3}{4}\right), \\ M(x_1,x_2) & ; x_2 \in \left(\frac{3}{4},1\right). \end{cases}$$

Next, we determine $C * M_Q$, *let* $x, y \in \mathbb{I}$.

Case 1. $y \in (0,\frac{1}{4})$. *Then* $M_Q(x,y) = M(x,y)$. *Consider,*

$$\begin{aligned} (C * M_Q)(x,y) &= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y)\,dt \\ &= \int_0^y \partial_2 C(x,t)(1)\,dt \\ &= C(x,y). \end{aligned}$$

Case 2. $y \in (\frac{1}{4},\frac{1}{2})$. *Then* $M_Q(x,y) = M(x,\frac{1}{4}) + M(x,y+\frac{1}{4}) - M(x,\frac{1}{2})$. *Consider,*

$$\begin{aligned} &(C * M_Q)(x,y) \\ &\quad= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &\quad= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,\frac{1}{4})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y+\frac{1}{4})\,dt \\ &\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t,\frac{1}{2})\,dt \\ &\quad= \int_0^{\frac{1}{4}} \partial_2 C(x,t)(1)\,dt + \int_0^{y+\frac{1}{4}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{1}{2}} \partial_2 C(x,t)(1)\,dt \\ &\quad= C(x,\frac{1}{4}) + C(x,y+\frac{1}{4}) - C(x,\frac{1}{2}). \end{aligned}$$

Case 3. $y \in (\frac{1}{2},\frac{3}{4})$. *Then* $M_Q(x,y) = M(x,y-\frac{1}{4}) + M(x,\frac{3}{4}) - M(x,\frac{1}{2})$. *Consider,*

$$\begin{aligned} &(C * M_Q)(x,y) \\ &\quad= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \end{aligned}$$

$$= \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y-\frac{1}{4})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{3}{4})\,dt$$
$$- \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{1}{2})\,dt$$
$$= \int_0^{y-\frac{1}{4}} \partial_2 C(x,t)(1)\,dt + \int_0^{\frac{3}{4}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{1}{2}} \partial_2 C(x,t)(1)\,dt$$
$$= C(x, y-\frac{1}{4}) + C(x, \frac{3}{4}) - C(x, \frac{1}{2}).$$

Case 4. $y \in (\frac{3}{4}, 1)$. *Then* $M_Q(x,y) = M(x,y)$. *Consider,*

$$\begin{aligned}(C * M_Q)(x,y) &= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y)\,dt \\ &= \int_0^y \partial_2 C(x,t)(1)\,dt \\ &= C(x,y).\end{aligned}$$

Hence,

$$(C * M_Q)(x,y) = \begin{cases} C(x,y) & ; y \in \left(0, \frac{1}{4}\right), \\ C(x,\frac{1}{4}) + C(x, y+\frac{1}{4}) - C(x,\frac{1}{2}) & ; y \in \left(\frac{1}{4}, \frac{1}{2}\right), \\ C(x, y-\frac{1}{4}) + C(x,\frac{3}{4}) - C(x,\frac{1}{2}) & ; y \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ C(x,y) & ; y \in \left(\frac{3}{4}, 1\right) \end{cases}$$
$$= C_Q(x,y).$$

Example 86 (51)**.** *Let* $C \in \mathscr{C}_2$ *and* Q *be the* 2×2 *doubly stochastic matrix associated with 2-copula* A *such that* $Q = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$. *Then*

$Q_{(0,1)} = \frac{1}{6}$	$Q_{(1,1)} = \frac{1}{3}$
$Q_{(0,0)} = \frac{1}{3}$	$Q_{(1,0)} = \frac{1}{6}$

1. $q^k_{\boldsymbol{i}} = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}^d_m, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}^d_m$ *and* $k = 1, 2$.

$q^1_{(0,1)} = \frac{1}{3}$	$q^1_{(1,1)} = \frac{1}{6}$
$q^1_{(0,0)} = 0$	$q^1_{(1,0)} = 0$

$q^2_{(0,1)} = 0$	$q^2_{(1,1)} = \frac{1}{6}$
$q^2_{(0,0)} = 0$	$q^2_{(1,0)} = \frac{1}{3}$

2. $J^k_{\boldsymbol{i}} = \left(\frac{i_k}{m} + q^k_{\boldsymbol{i}}, \frac{i_k}{m} + q^k_{\boldsymbol{i}} + Q_{\boldsymbol{i}}\right)$.

$$J^1_{(0,0)} = \left(0, \frac{1}{3}\right) = J^2_{(0,0)},$$

$$J^1_{(0,1)} = \left(\frac{1}{3}, \frac{1}{2}\right) = J^2_{(1,0)},$$

$$J^1_{(1,0)} = \left(\frac{1}{2}, \frac{2}{3}\right) = J^2_{(0,1)},$$

$$J^1_{(1,1)} = \left(\frac{2}{3}, 1\right) = J^2_{(1,1)}.$$

3. *For any* $k = 1, \ldots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q^k_{\boldsymbol{i}} - Q^1_{\boldsymbol{i}} \text{ for } t \in J^1_{\boldsymbol{i}}$$

and

$$\phi(t_1, \ldots, t_d) = (\phi_1(t_1), \ldots, \phi_d(t_d)),$$

for all $t_1, \ldots, t_d \in \mathbb{I}$.

If $t \in J^1_{(0,0)} = \left(0, \frac{1}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-0}{2} + 0 - 0 = t$.

If $t \in J^1_{(0,1)} = \left(\frac{1}{3}, \frac{1}{2}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-0}{2} + 0 - \frac{1}{3} = t + \frac{1}{6}$.

If $t \in J^1_{(1,0)} = \left(\frac{1}{2}, \frac{2}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-1}{2} + \frac{1}{3} - 0 = t - \frac{1}{6}$.

If $t \in J^1_{(1,1)} = \left(\frac{2}{3}, 1\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-1}{2} + \frac{1}{6} - \frac{1}{6} = t$.

Thus, for any $t_1, t_2 \in \mathbb{I}$,

$$\phi(t_1,t_2) = \begin{cases} (t_1,t_2) & ; t_2 \in \left(0, \frac{1}{3}\right), \\ (t_1, t_2 + \frac{1}{6}) & ; t_2 \in \left(\frac{1}{3}, \frac{1}{2}\right), \\ (t_1, t_2 - \frac{1}{6}) & ; t_2 \in \left(\frac{1}{2}, \frac{2}{3}\right), \\ (t_1,t_2) & ; t_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Hence,

$$\phi^{-1}(t_1,t_2) = \begin{cases} (t_1,t_2) & ; t_2 \in \left(0, \frac{1}{3}\right), \\ (t_1, t_2 + \frac{1}{6}) & ; t_2 \in \left(\frac{1}{3}, \frac{1}{2}\right), \\ (t_1, t_2 - \frac{1}{6}) & ; t_2 \in \left(\frac{1}{2}, \frac{2}{3}\right), \\ (t_1,t_2) & ; t_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

4. *For any* $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1,x_2) = \mu_{C_Q}([0,x_1] \times [0,x_2]) = \mu_C(\phi^{-1}([0,x_1] \times [0,x_2])).$$

Case 1. $x_2 \in \left(0, \frac{1}{3}\right)$. *Then*

$$\begin{aligned} C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1] \times [0,x_2])) \\ &= \mu_C([0,x_1] \times [0,x_2]) \\ &= C(x_1,x_2). \end{aligned}$$

Case 2. $x_2 \in \left(\frac{1}{3}, \frac{1}{2}\right)$. *Then*

$$\begin{aligned} C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1] \times [0,\frac{1}{3}])) + \mu_C(\phi^{-1}([0,x_1] \times [\frac{1}{3}, x_2])) \\ &= \mu_C([0,x_1] \times [0,\frac{1}{3}]) + \mu_C([0,x_1] \times [\frac{1}{2}, x_2 + \frac{1}{6}]) \\ &= C(x_1, \frac{1}{3}) + C(x_1, x_2 + \frac{1}{6}) - C(x_1, \frac{1}{2}). \end{aligned}$$

Case 3. $x_2 \in \left(\frac{1}{2}, \frac{2}{3}\right)$. *Then*

$$\begin{aligned}
C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{3}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},\frac{1}{2}])) \\
&\quad + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{2},x_2])) \\
&= \mu_C([0,x_1]\times[0,\frac{1}{3}]) + \mu_C([0,x_1]\times[\frac{1}{2},\frac{2}{3}]) \\
&\quad + \mu_C([0,x_1]\times[\frac{1}{3},x_2-\frac{1}{6}]) \\
&= \mu_C([0,x_1]\times[0,x_2-\frac{1}{6}]) + \mu_C([0,x_1]\times[\frac{1}{2},\frac{2}{3}]) \\
&= C(x_1,x_2-\frac{1}{6}) + C(x_1,\frac{2}{3}) - C(x_1,\frac{1}{2}).
\end{aligned}$$

Case 4. $x_2 \in \left(\frac{2}{3}, 1\right)$. *Then*

$$\begin{aligned}
C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{3}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},\frac{1}{2}])) \\
&\quad + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{2},\frac{2}{3}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{2}{3},x_2])) \\
&= \mu_C([0,x_1]\times[0,\frac{1}{3}]) + \mu_C([0,x_1]\times[\frac{1}{2},\frac{2}{3}]) \\
&\quad + \mu_C([0,x_1]\times[\frac{1}{3},\frac{1}{2}]) + \mu_C([0,x_1]\times[\frac{2}{3},x_2]) \\
&= \mu_C([0,x_1]\times[0,x_2]) \\
&= C(x_1,x_2).
\end{aligned}$$

Thus, for any $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1,x_2) = \begin{cases}
C(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\
C(x_1,\frac{1}{3}) + C(x_1,x_2+\frac{1}{6}) - C(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{3},\frac{1}{2}\right), \\
C(x_1,x_2-\frac{1}{6}) + C(x_1,\frac{2}{3}) - C(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{2},\frac{2}{3}\right), \\
C(x_1,x_2) & ; x_2 \in \left(\frac{2}{3},1\right).
\end{cases}$$

Therefore, for any $x_1, x_2 \in \mathbb{I}$,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\ M(x_1,\frac{1}{3}) + M(x_1,x_2+\frac{1}{6}) - M(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{3},\frac{1}{2}\right), \\ M(x_1,x_2-\frac{1}{6}) + M(x_1,\frac{2}{3}) - M(x_1,\frac{1}{2}) & ; x_2 \in \left(\frac{1}{2},\frac{2}{3}\right), \\ M(x_1,x_2) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

Next, we determine $C * M_Q$, let $x, y \in \mathbb{I}$.

Case 1. $y \in (0, \frac{1}{3})$. *Then $M_Q(x,y) = M(x,y)$. Consider,*

$$\begin{aligned} (C * M_Q)(x,y) &= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y)\,dt \\ &= \int_0^y \partial_2 C(x,t)(1)\,dt \\ &= C(x,y). \end{aligned}$$

Case 2. $y \in (\frac{1}{3}, \frac{1}{2})$. *Then $M_Q(x,y) = M(x,\frac{1}{3}) + M(x,y+\frac{1}{6}) - M(x,\frac{1}{2})$. Consider,*

$$\begin{aligned} &(C * M_Q)(x,y) \\ &\quad= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &\quad= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,\frac{1}{3})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y+\frac{1}{6})\,dt \\ &\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t,\frac{1}{2})\,dt \\ &\quad= \int_0^{\frac{1}{3}} \partial_2 C(x,t)(1)\,dt + \int_0^{y+\frac{1}{6}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{1}{2}} \partial_2 C(x,t)(1)\,dt \\ &\quad= C(x,\frac{1}{3}) + C(x,y+\frac{1}{6}) - C(x,\frac{1}{2}). \end{aligned}$$

Case 3. $y \in (\frac{1}{2}, \frac{2}{3})$. *Then $M_Q(x,y) = M(x,y-\frac{1}{6}) + M(x,\frac{2}{3}) - M(x,\frac{1}{2})$. Consider,*

$$\begin{aligned} &(C * M_Q)(x,y) \\ &\quad= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \end{aligned}$$

$$= \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y - \frac{1}{6})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{2}{3})\,dt$$
$$- \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{1}{2})\,dt$$
$$= \int_0^{y-\frac{1}{6}} \partial_2 C(x,t)(1)\,dt + \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{1}{2}} \partial_2 C(x,t)(1)\,dt$$
$$= C(x, y - \frac{1}{6}) + C(x, \frac{2}{3}) - C(x, \frac{1}{2}).$$

Case 4. $y \in (\frac{2}{3}, 1)$. *Then* $M_Q(x,y) = M(x,y)$. *Consider,*

$$\begin{aligned}(C * M_Q)(x,y) &= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y)\,dt \\ &= \int_0^y \partial_2 C(x,t)(1)\,dt \\ &= C(x,y).\end{aligned}$$

Hence,

$$(C * M_Q)(x,y) = \begin{cases} C(x,y) & ; y \in \left(0, \frac{1}{3}\right), \\ C(x, \frac{1}{3}) + C(x, y + \frac{1}{6}) - C(x, \frac{1}{2}) & ; y \in \left(\frac{1}{3}, \frac{1}{2}\right), \\ C(x, y - \frac{1}{6}) + C(x, \frac{2}{3}) - C(x, \frac{1}{2}) & ; y \in \left(\frac{1}{2}, \frac{2}{3}\right), \\ C(x,y) & ; y \in \left(\frac{2}{3}, 1\right) \end{cases}$$
$$= C_Q(x,y).$$

Example 87 (52)**.** *Let* $C \in \mathscr{C}_2$ *and* Q *be the* 3×3 *doubly stochastic matrix associated with 2-copula* A *such that* $Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{bmatrix}$. *Then*

$Q_{(0,2)} = 0$	$Q_{(1,2)} = \frac{1}{3}$	$Q_{(2,2)} = 0$
$Q_{(0,1)} = 0$	$Q_{(1,1)} = 0$	$Q_{(2,1)} = \frac{1}{3}$
$Q_{(0,0)} = \frac{1}{3}$	$Q_{(1,0)} = 0$	$Q_{(2,0)} = 0$

1. $q_{\boldsymbol{i}}^k = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}_m^d, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}_m^d$ *and* $k = 1, 2, \ldots, d$.

$q^1_{(0,2)} = \frac{1}{3}$	$q^1_{(1,2)} = 0$	$q^1_{(2,2)} = \frac{1}{3}$
$q^1_{(0,1)} = \frac{1}{3}$	$q^1_{(1,1)} = 0$	$q^1_{(2,1)} = 0$
$q^1_{(0,0)} = 0$	$q^1_{(1,0)} = 0$	$q^1_{(2,0)} = 0$

$q^2_{(0,2)} = 0$	$q^2_{(1,2)} = 0$	$q^2_{(2,2)} = \frac{1}{3}$
$q^2_{(0,1)} = 0$	$q^2_{(1,1)} = 0$	$q^2_{(2,1)} = 0$
$q^2_{(0,0)} = 0$	$q^2_{(1,0)} = \frac{1}{3}$	$q^2_{(2,0)} = \frac{1}{3}$

2. $J_{\boldsymbol{i}}^k = \left(\dfrac{i_k}{m} + q_{\boldsymbol{i}}^k, \dfrac{i_k}{m} + q_{\boldsymbol{i}}^k + Q_{\boldsymbol{i}}\right)$.

$$J^1_{(0,0)} = \left(0, \frac{1}{3}\right) = J^2_{(0,0)},$$

$$J^1_{(1,2)} = \left(\frac{1}{3}, \frac{2}{3}\right) = J^2_{(2,1)},$$

$$J^1_{(2,1)} = \left(\frac{2}{3}, 1\right) = J^2_{(1,2)}.$$

3. *For any* $k = 1, \ldots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q_{\boldsymbol{i}}^k - Q_{\boldsymbol{i}}^1 \text{ for } t \in J_{\boldsymbol{i}}^1$$

and

$$\phi(t_1, \ldots, t_d) = (\phi_1(t_1), \ldots, \phi_d(t_d)),$$

for all $t_1, \ldots, t_d \in \mathbb{I}$.

If $t \in J^1_{(0,0)} = \left(0, \frac{1}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-0}{3} + 0 - 0 = t$.

If $t \in J^1_{(1,2)} = \left(\frac{1}{3}, \frac{2}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{2-1}{3} + 0 - 0 = t + \frac{1}{3}$.

If $t \in J^1_{(2,1)} = \left(\frac{2}{3}, 1\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-2}{3} + 0 - 0 = t - \frac{1}{3}$.

Thus, for any $t_1, t_2 \in \mathbb{I}$,

$$\phi(t_1, t_2) = \begin{cases} (t_1, t_2) & ; t_2 \in \left(0, \frac{1}{3}\right), \\ (t_1, t_2 + \frac{1}{3}) & ; t_2 \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ (t_1, t_2 - \frac{1}{3}) & ; t_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Hence,

$$\phi^{-1}(t_1,t_2)=\begin{cases}(t_1,t_2) & ;t_2\in\left(0,\frac{1}{3}\right),\\ (t_1,t_2+\frac{1}{3}) & ;t_2\in\left(\frac{1}{3},\frac{2}{3}\right),\\ (t_1,t_2-\frac{1}{3}) & ;t_2\in\left(\frac{2}{3},1\right).\end{cases}$$

4. *For any* $x_1,x_2\in\mathbb{I}$,

$$C_Q(x_1,x_2)=\mu_{C_Q}([0,x_1]\times[0,x_2])=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2])).$$

Case 1. $x_2\in\left(0,\frac{1}{3}\right)$. *Then*

$$\begin{aligned}C_Q(x_1,x_2)&=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2]))\\&=\mu_C([0,x_1]\times[0,x_2])\\&=C(x_1,x_2).\end{aligned}$$

Case 2. $x_2\in\left(\frac{1}{3},\frac{2}{3}\right)$. *Then*

$$\begin{aligned}C_Q(x_1,x_2)&=\mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{3}]))+\mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},x_2]))\\&=\mu_C([0,x_1]\times[0,\frac{1}{3}])+\mu_C([0,x_1]\times[\frac{2}{3},x_2+\frac{1}{3}])\\&=C(x_1,\frac{1}{3})+C(x_1,x_2+\frac{1}{3})-C(x_1,\frac{2}{3}).\end{aligned}$$

Case 3. $x_2\in\left(\frac{2}{3},1\right)$. *Then*

$$\begin{aligned}C_Q(x_1,x_2)&=\mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{3}]))+\mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},\frac{2}{3}]))\\&\quad+\mu_C(\phi^{-1}([0,x_1]\times[\frac{2}{3},x_2]))\\&=\mu_C([0,x_1]\times[0,\frac{1}{3}])+\mu_C([0,x_1]\times[\frac{2}{3},1])\\&\quad+\mu_C([0,x_1]\times[\frac{1}{3},x_2-\frac{1}{3}])\\&=\mu_C([0,x_1]\times[0,x_2-\frac{1}{3}])+\mu_C([0,x_1]\times[\frac{2}{3},1])\\&=C(x_1,x_2-\frac{1}{3})+x_1-C(x_1,\frac{2}{3}).\end{aligned}$$

Thus, for any $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1,x_2) = \begin{cases} C(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\ C(x_1,\frac{1}{3}) + C(x_1,x_2+\frac{1}{3}) - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{3},\frac{2}{3}\right), \\ C(x_1,x_2-\frac{1}{3}) + x_1 - C(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

Therefore, for any $x_1, x_2 \in \mathbb{I}$,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0,\frac{1}{3}\right), \\ M(x_1,\frac{1}{3}) + M(x_1,x_2+\frac{1}{3}) - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{1}{3},\frac{2}{3}\right), \\ M(x_1,x_2-\frac{1}{3}) + x_1 - M(x_1,\frac{2}{3}) & ; x_2 \in \left(\frac{2}{3},1\right). \end{cases}$$

Next, we determine $C * M_Q$, *let* $x, y \in \mathbb{I}$.

Case 1. $y \in (0, \frac{1}{3})$. *Then* $M_Q(x,y) = M(x,y)$. *Consider,*

$$\begin{aligned}(C * M_Q)(x,y) &= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y)\,dt \\ &= \int_0^y \partial_2 C(x,t)(1)\,dt \\ &= C(x,y).\end{aligned}$$

Case 2. $y \in (\frac{1}{3}, \frac{2}{3})$. *Then* $M_Q(x,y) = M(x,\frac{1}{3}) + M(x,y+\frac{1}{3}) - M(x,\frac{2}{3})$. *Consider,*

$$\begin{aligned}&(C * M_Q)(x,y) \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\ &= \int_0^1 \partial_2 C(x,t)\partial_1 M(t,\frac{1}{3})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t,y+\frac{1}{3})\,dt \\ &\quad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t,\frac{2}{3})\,dt \\ &= \int_0^{\frac{1}{3}} \partial_2 C(x,t)(1)\,dt + \int_0^{y+\frac{1}{3}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt \\ &= C(x,\frac{1}{3}) + C(x,y+\frac{1}{3}) - C(x,\frac{2}{3}).\end{aligned}$$

Case 3. $y \in (\frac{2}{3}, 1)$. *Then* $M_Q(x,y) = M(x, y-\frac{1}{3}) + x - M(x, \frac{2}{3})$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x,y) \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y-\frac{1}{3})\,dt + \int_0^1 \partial_2 C(x,t)\frac{\partial}{\partial t} t\,dt \\
&\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{2}{3})\,dt \\
&\quad = \int_0^{y-\frac{1}{3}} \partial_2 C(x,t)(1)\,dt + \int_0^1 \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt \\
&\quad = C(x, y-\frac{1}{3}) + x - C(x, \frac{2}{3}).
\end{aligned}$$

Hence,

$$\begin{aligned}
(C * M_Q)(x,y) &= \begin{cases} C(x,y) & ; y \in \left(0, \frac{1}{3}\right), \\ C(x, \frac{1}{3}) + C(x, y+\frac{1}{3}) - C(x, \frac{2}{3}) & ; y \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ C(x, y-\frac{1}{3}) + x - C(x, \frac{2}{3}) & ; y \in \left(\frac{2}{3}, 1\right) \end{cases} \\
&= C_Q(x,y).
\end{aligned}$$

Example 88 (53)**.** *Let* $C \in \mathscr{C}_2$ *and* Q *be the* 3×3 *doubly stochastic matrix associated with 2-copula* A *such that* $Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{12} \\ \frac{1}{12} & 0 & \frac{1}{4} \end{bmatrix}$. *Then*

$Q_{(0,2)} = 0$	$Q_{(1,2)} = \frac{1}{3}$	$Q_{(2,2)} = 0$
$Q_{(0,1)} = \frac{1}{4}$	$Q_{(1,1)} = 0$	$Q_{(2,1)} = \frac{1}{12}$
$Q_{(0,0)} = \frac{1}{12}$	$Q_{(1,0)} = 0$	$Q_{(2,0)} = \frac{1}{4}$

1. $q_{\boldsymbol{i}}^k = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}_m^d, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}_m^d$ *and* $k = 1, 2, \ldots, d$.

$q^1_{(0,2)} = \frac{1}{3}$	$q^1_{(1,2)} = 0$	$q^1_{(2,2)} = \frac{1}{3}$
$q^1_{(0,1)} = \frac{1}{12}$	$q^1_{(1,1)} = 0$	$q^1_{(2,1)} = \frac{1}{4}$
$q^1_{(0,0)} = 0$	$q^1_{(1,0)} = 0$	$q^1_{(2,0)} = 0$

$q^2_{(0,2)} = 0$	$q^2_{(1,2)} = 0$	$q^2_{(2,2)} = \frac{1}{3}$
$q^2_{(0,1)} = 0$	$q^2_{(1,1)} = \frac{1}{4}$	$q^2_{(2,1)} = \frac{1}{4}$
$q^2_{(0,0)} = 0$	$q^2_{(1,0)} = \frac{1}{12}$	$q^2_{(2,0)} = \frac{1}{12}$

2. $J^k_{\boldsymbol{i}} = \left(\frac{i_k}{m} + q^k_{\boldsymbol{i}}, \frac{i_k}{m} + q^k_{\boldsymbol{i}} + Q_{\boldsymbol{i}}\right).$

$$J^1_{(0,0)} = \left(0, \frac{1}{12}\right), J^2_{(0,0)} = \left(0, \frac{1}{12}\right),$$

$$J^1_{(0,1)} = \left(\frac{1}{12}, \frac{1}{3}\right), J^2_{(0,1)} = \left(\frac{1}{3}, \frac{7}{12}\right),$$

$$J^1_{(1,2)} = \left(\frac{1}{3}, \frac{2}{3}\right), J^2_{(1,2)} = \left(\frac{2}{3}, 1\right),$$

$$J^1_{(2,0)} = \left(\frac{2}{3}, \frac{11}{12}\right), J^2_{(2,0)} = \left(\frac{1}{12}, \frac{1}{3}\right),$$

$$J^1_{(2,1)} = \left(\frac{11}{12}, 1\right), J^2_{(2,1)} = \left(\frac{7}{12}, \frac{2}{3}\right).$$

3. *For any* $k = 1, \ldots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q^k_{\boldsymbol{i}} - Q^1_{\boldsymbol{i}} \text{ for } t \in J^1_{\boldsymbol{i}}$$

and

$$\phi(t_1, \ldots, t_d) = (\phi_1(t_1), \ldots, \phi_d(t_d)),$$

for all $t_1, \ldots, t_d \in \mathbb{I}$.

If $t \in J^1_{(0,0)} = \left(0, \frac{1}{12}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-0}{3} + 0 - 0 = t$.

If $t \in J^1_{(0,1)} = \left(\frac{1}{12}, \frac{1}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-0}{3} + 0 - \frac{1}{12} = t + \frac{1}{4}$.

If $t \in J^1_{(1,2)} = \left(\frac{1}{3}, \frac{2}{3}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{2-1}{3} + 0 - 0 = t + \frac{1}{3}$.

If $t \in J^1_{(2,0)} = \left(\frac{2}{3}, \frac{11}{12}\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{0-2}{3} + \frac{1}{12} - 0 = t - \frac{7}{12}$.

If $t \in J^1_{(2,1)} = \left(\frac{11}{12}, 1\right)$, *then* $\phi_1(t) = t$ *and* $\phi_2(t) = t + \frac{1-2}{3} + \frac{1}{4} - \frac{1}{4} = t - \frac{1}{3}$.

Thus, for any $t_1, t_2 \in \mathbb{I}$,

$$\phi(t_1,t_2) = \begin{cases} (t_1, t_2) & ; t_2 \in \left(0, \frac{1}{12}\right), \\ (t_1, t_2 + \frac{1}{4}) & ; t_2 \in \left(\frac{1}{12}, \frac{1}{3}\right), \\ (t_1, t_2 + \frac{1}{3}) & ; t_2 \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ (t_1, t_2 - \frac{7}{12}) & ; t_2 \in \left(\frac{2}{3}, \frac{11}{12}\right), \\ (t_1, t_2 - \frac{1}{3}) & ; t_2 \in \left(\frac{11}{12}, 1\right). \end{cases}$$

Hence,

$$\phi^{-1}(t_1,t_2) = \begin{cases} (t_1, t_2) & ; t_2 \in \left(0, \frac{1}{12}\right), \\ (t_1, t_2 + \frac{7}{12}) & ; t_2 \in \left(\frac{1}{12}, \frac{1}{3}\right), \\ (t_1, t_2 - \frac{1}{4}) & ; t_2 \in \left(\frac{1}{3}, \frac{7}{12}\right), \\ (t_1, t_2 + \frac{1}{3}) & ; t_2 \in \left(\frac{7}{12}, \frac{2}{3}\right), \\ (t_1, t_2 - \frac{1}{3}) & ; t_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

4. For any $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1, x_2) = \mu_{C_Q}([0, x_1] \times [0, x_2]) = \mu_C(\phi^{-1}([0, x_1] \times [0, x_2])).$$

Case 1. $x_2 \in \left(0, \frac{1}{12}\right)$. *Then*

$$\begin{aligned} C_Q(x_1, x_2) &= \mu_C(\phi^{-1}([0, x_1] \times [0, x_2])) \\ &= \mu_C([0, x_1] \times [0, x_2]) \\ &= C(x_1, x_2). \end{aligned}$$

Case 2. $x_2 \in \left(\frac{1}{12}, \frac{1}{3}\right)$. *Then*

$$\begin{aligned} C_Q(x_1, x_2) &= \mu_C(\phi^{-1}([0, x_1] \times [0, \frac{1}{12}])) + \mu_C(\phi^{-1}([0, x_1] \times [\frac{1}{12}, x_2])) \\ &= \mu_C([0, x_1] \times [0, \frac{1}{12}]) + \mu_C([0, x_1] \times [\frac{2}{3}, x_2 + \frac{7}{12}]) \\ &= C(x_1, \frac{1}{12}) + C(x_1, x_2 + \frac{7}{12}) - C(x_1, \frac{2}{3}). \end{aligned}$$

Case 3. $x_2 \in \left(\frac{1}{3}, \frac{7}{12}\right)$. *Then*

$$\begin{aligned}
C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{12}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{12},\frac{1}{3}])) \\
&\quad + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},x_2])) \\
&= \mu_C([0,x_1]\times[0,\frac{1}{12}]) + \mu_C([0,x_1]\times[\frac{2}{3},\frac{11}{12}]) \\
&\quad + \mu_C([0,x_1]\times[\frac{1}{12},x_2-\frac{1}{4}]) \\
&= \mu_C([0,x_1]\times[0,x_2-\frac{1}{4}]) + \mu_C([0,x_1]\times[\frac{2}{3},\frac{11}{12}]) \\
&= C(x_1,x_2-\frac{1}{4}) + C(x_1,\frac{11}{12}) - C(x_1,\frac{2}{3}).
\end{aligned}$$

Case 4. $x_2 \in \left(\frac{7}{12}, \frac{2}{3}\right)$. *Then*

$$\begin{aligned}
C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{12}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{12},\frac{1}{3}])) \\
&\quad + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},\frac{7}{12}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{7}{12},x_2])) \\
&= \mu_C([0,x_1]\times[0,\frac{1}{12}]) + \mu_C([0,x_1]\times[\frac{2}{3},\frac{11}{12}]) \\
&\quad + \mu_C([0,x_1]\times[\frac{1}{12},\frac{1}{3}]) + \mu_C([0,x_1]\times[\frac{11}{12},x_2+\frac{1}{3}]) \\
&= \mu_C([0,x_1]\times[0,\frac{1}{3}]) + \mu_C([0,x_1]\times[\frac{2}{3},x_2+\frac{1}{3}]) \\
&= C(x_1,\frac{1}{3}) + C(x_1,x_2+\frac{1}{3}) - C(x_1,\frac{2}{3}).
\end{aligned}$$

Case 5. $x_2 \in \left(\frac{2}{3}, 1\right)$. *Then*

$$\begin{aligned}
C_Q(x_1,x_2) &= \mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{12}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{12},\frac{1}{3}])) \\
&\quad + \mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{3},\frac{7}{12}])) + \mu_C(\phi^{-1}([0,x_1]\times[\frac{7}{12},\frac{2}{3}])) \\
&\quad + \mu_C(\phi^{-1}([0,x_1]\times[\frac{2}{3},x_2])) \\
&= \mu_C([0,x_1]\times[0,\frac{1}{12}]) + \mu_C([0,x_1]\times[\frac{2}{3},\frac{11}{12}]) \\
&\quad + \mu_C([0,x_1]\times[\frac{1}{12},\frac{1}{3}]) + \mu_C([0,x_1]\times[\frac{11}{12},1]) \\
&\quad + \mu_C([0,x_1]\times[\frac{1}{3},x_2-\frac{1}{3}]) \\
&= \mu_C([0,x_1]\times[0,x_2-\frac{1}{3}]) + \mu_C([0,x_1]\times[\frac{2}{3},1])
\end{aligned}$$

$$= C(x_1, x_2 - \frac{1}{3}) + x_1 - C(x_1, \frac{2}{3}).$$

Thus, for any $x_1, x_2 \in \mathbb{I}$,

$$C_Q(x_1,x_2) = \begin{cases} C(x_1,x_2) & ; x_2 \in \left(0, \frac{1}{12}\right), \\ C(x_1, \frac{1}{12}) + C(x_1, x_2 + \frac{7}{12}) - C(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{1}{12}, \frac{1}{3}\right), \\ C(x_1, x_2 - \frac{1}{4}) + C(x_1, \frac{11}{12}) - C(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{1}{3}, \frac{7}{12}\right), \\ C(x_1, \frac{1}{3}) + C(x_1, x_2 + \frac{1}{3}) - C(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{7}{12}, \frac{2}{3}\right), \\ C(x_1, x_2 - \frac{1}{3}) + x_1 - C(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Therefore, for any $x_1, x_2 \in \mathbb{I}$,

$$M_Q(x_1,x_2) = \begin{cases} M(x_1,x_2) & ; x_2 \in \left(0, \frac{1}{12}\right), \\ M(x_1, \frac{1}{12}) + M(x_1, x_2 + \frac{7}{12}) - M(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{1}{12}, \frac{1}{3}\right), \\ M(x_1, x_2 - \frac{1}{4}) + M(x_1, \frac{11}{12}) - M(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{1}{3}, \frac{7}{12}\right), \\ M(x_1, \frac{1}{3}) + M(x_1, x_2 + \frac{1}{3}) - M(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{7}{12}, \frac{2}{3}\right), \\ M(x_1, x_2 - \frac{1}{3}) + x_1 - M(x_1, \frac{2}{3}) & ; x_2 \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Next, we determine $C * M_Q$, *let* $x, y \in \mathbb{I}$.

Case 1. $y \in \left(0, \frac{1}{12}\right)$. *Then* $M_Q(x,y) = M(x,y)$. *Consider,*

$$\begin{aligned} (C * M_Q)(x,y) &= \int_0^1 \partial_2 C(x,t) \partial_1 M_Q(t,y)\, dt \\ &= \int_0^1 \partial_2 C(x,t) \partial_1 M(t,y)\, dt \\ &= \int_0^y \partial_2 C(x,t)(1)\, dt \\ &= C(x,y). \end{aligned}$$

Case 2. $y \in \left(\frac{1}{12}, \frac{1}{3}\right)$. *Then* $M_Q(x,y) = M(x, \frac{1}{12}) + M(x, y + \frac{7}{12}) - M(x, \frac{2}{3})$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x,y) \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{1}{12})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y + \frac{7}{12})\,dt \\
&\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{2}{3})\,dt \\
&\quad = \int_0^{\frac{1}{12}} \partial_2 C(x,t)(1)\,dt + \int_0^{y+\frac{7}{12}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt \\
&\quad = C(x, \frac{1}{12}) + C(x, y + \frac{7}{12}) - C(x, \frac{2}{3}).
\end{aligned}$$

Case 3. $y \in \left(\frac{1}{3}, \frac{7}{12}\right)$. *Then* $M_Q(x,y) = M(x, y - \frac{1}{4}) + M(x, \frac{11}{12}) - M(x, \frac{2}{3})$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x,y) \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y - \frac{1}{4})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{11}{12})\,dt \\
&\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{2}{3})\,dt \\
&\quad = \int_0^{y-\frac{1}{4}} \partial_2 C(x,t)(1)\,dt + \int_0^{\frac{11}{12}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt \\
&\quad = C(x, y - \frac{1}{4}) + C(x, \frac{11}{12}) - C(x, \frac{2}{3}).
\end{aligned}$$

Case 4. $y \in \left(\frac{7}{12}, \frac{2}{3}\right)$. *Then* $M_Q(x,y) = M(x, \frac{1}{3}) + M(x, y + \frac{1}{3}) - M(x, \frac{2}{3})$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x,y) \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt \\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{1}{3})\,dt + \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y + \frac{1}{3})\,dt \\
&\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{2}{3})\,dt \\
&\quad = \int_0^{\frac{1}{3}} \partial_2 C(x,t)(1)\,dt + \int_0^{y+\frac{1}{3}} \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt \\
&\quad = C(x, \frac{1}{3}) + C(x, y + \frac{1}{3}) - C(x, \frac{2}{3}).
\end{aligned}$$

Case 5. $y \in \left(\frac{2}{3}, 1\right)$. *Then* $M_Q(x,y) = M(x, y-\frac{1}{3}) + x - M(x, \frac{2}{3})$. *Consider,*

$$\begin{aligned}
&(C * M_Q)(x,y)\\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M_Q(t,y)\,dt\\
&\quad = \int_0^1 \partial_2 C(x,t)\partial_1 M(t, y-\frac{1}{3})\,dt + \int_0^1 \partial_2 C(x,t)\frac{\partial}{\partial t} t\,dt\\
&\qquad - \int_0^1 \partial_2 C(x,t)\partial_1 M(t, \frac{2}{3})\,dt\\
&\quad = \int_0^{y-\frac{1}{3}} \partial_2 C(x,t)(1)\,dt + \int_0^1 \partial_2 C(x,t)(1)\,dt - \int_0^{\frac{2}{3}} \partial_2 C(x,t)(1)\,dt\\
&\quad = C(x, y-\frac{1}{3}) + x - C(x, \frac{2}{3}).
\end{aligned}$$

Hence,

$$\begin{aligned}
(C * M_Q)(x,y) &= \begin{cases}
C(x,y) & ; y \in \left(0, \frac{1}{12}\right),\\
C(x, \frac{1}{12}) + C(x, y+\frac{7}{12}) - C(x, \frac{2}{3}) & ; y \in \left(\frac{1}{12}, \frac{1}{3}\right),\\
C(x, y-\frac{1}{4}) + C(x, \frac{11}{12}) - C(x, \frac{2}{3}) & ; y \in \left(\frac{1}{3}, \frac{7}{12}\right),\\
C(x, \frac{1}{3}) + C(x, y+\frac{1}{3}) - C(x, \frac{2}{3}) & ; y \in \left(\frac{7}{12}, \frac{2}{3}\right),\\
C(x, y-\frac{1}{3}) + x - C(x, \frac{2}{3}) & ; y \in \left(\frac{2}{3}, 1\right)
\end{cases}\\
&= C_Q(x,y).
\end{aligned}$$

Example 89 (54)**.** *Let* $C \in \mathscr{C}_3$ *and* Q *be the* $2 \times 2 \times 2$ *three-fold stochastic matrix associated with* 3*-copula* A*, where* $A(x,y,z) = W(x, M(y,z))$ *for all* $x, y, z \in \mathbb{I}$*. Then*

$$\begin{aligned}
Q_{(0,0,0)} &= Q_A([0,\frac{1}{2}] \times [0,\frac{1}{2}] \times [0,\frac{1}{2}]) = A(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 0.\\
Q_{(0,0,1)} &= Q_A([0,\frac{1}{2}] \times [0,\frac{1}{2}] \times [\frac{1}{2},1]) = A(\frac{1}{2},\frac{1}{2},1) - A(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 0 - 0 = 0.\\
Q_{(0,1,0)} &= Q_A([0,\frac{1}{2}] \times [\frac{1}{2},1] \times [0,\frac{1}{2}]) = A(\frac{1}{2},1,\frac{1}{2}) - A(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 0 - 0 = 0.\\
Q_{(0,1,1)} &= Q_A([0,\frac{1}{2}] \times [\frac{1}{2},1] \times [\frac{1}{2},1])\\
&= A(\frac{1}{2},1,1) - A(\frac{1}{2},1,\frac{1}{2}) - A(\frac{1}{2},\frac{1}{2},1) + A(\frac{1}{2},\frac{1}{2},\frac{1}{2})\\
&= \frac{1}{2} - 0 - 0 + 0 = \frac{1}{2}.
\end{aligned}$$

$$
\begin{aligned}
Q_{(1,0,0)} &= Q_A([\frac{1}{2},1]\times[0,\frac{1}{2}]\times[0,\frac{1}{2}]) = A(1,\frac{1}{2},\frac{1}{2}) - A(\frac{1}{2},\frac{1}{2},\frac{1}{2}) = \frac{1}{2} - 0 = \frac{1}{2}.\\
Q_{(1,0,1)} &= Q_A([\frac{1}{2},1]\times[0,\frac{1}{2}]\times[\frac{1}{2},1])\\
&= A(1,\frac{1}{2},1) - A(1,\frac{1}{2},\frac{1}{2}) - A(\frac{1}{2},\frac{1}{2},1) + A(\frac{1}{2},\frac{1}{2},\frac{1}{2})\\
&= \frac{1}{2} - \frac{1}{2} - 0 + 0 = 0.\\
Q_{(1,1,0)} &= Q_A([\frac{1}{2},1]\times[\frac{1}{2},1]\times[0,\frac{1}{2}])\\
&= A(1,1,\frac{1}{2}) - A(1,\frac{1}{2},\frac{1}{2}) - A(\frac{1}{2},1,\frac{1}{2}) + A(\frac{1}{2},\frac{1}{2},\frac{1}{2})\\
&= \frac{1}{2} - \frac{1}{2} - 0 + 0 = 0.\\
Q_{(1,1,1)} &= Q_A([\frac{1}{2},1]\times[\frac{1}{2},1]\times[\frac{1}{2},1])\\
&= A(1,1,1) - A(1,1,\frac{1}{2}) - A(1,\frac{1}{2},1) + A(1,\frac{1}{2},\frac{1}{2})\\
&\quad - A(\frac{1}{2},1,1) + A(\frac{1}{2},1,\frac{1}{2}) + A(\frac{1}{2},\frac{1}{2},1) - A(\frac{1}{2},\frac{1}{2},\frac{1}{2})\\
&= 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + 0 + 0 - 0 = 0.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\sum_{i_1=0} Q_{\boldsymbol{i}} &= Q_{(0,0,0)} + Q_{(0,0,1)} + Q_{(0,1,0)} + Q_{(0,1,1)} = \frac{1}{2}\\
\sum_{i_1=1} Q_{\boldsymbol{i}} &= Q_{(1,0,0)} + Q_{(1,0,1)} + Q_{(1,1,0)} + Q_{(1,1,1)} = \frac{1}{2}\\
\sum_{i_2=0} Q_{\boldsymbol{i}} &= Q_{(0,0,0)} + Q_{(0,0,1)} + Q_{(1,0,0)} + Q_{(1,0,1)} = \frac{1}{2}\\
\sum_{i_2=1} Q_{\boldsymbol{i}} &= Q_{(0,1,0)} + Q_{(0,1,1)} + Q_{(1,1,0)} + Q_{(1,1,1)} = \frac{1}{2}\\
\sum_{i_3=0} Q_{\boldsymbol{i}} &= Q_{(0,0,0)} + Q_{(0,1,0)} + Q_{(1,0,0)} + Q_{(1,1,0)} = \frac{1}{2}\\
\sum_{i_3=1} Q_{\boldsymbol{i}} &= Q_{(0,0,1)} + Q_{(0,1,1)} + Q_{(1,0,1)} + Q_{(1,1,1)} = \frac{1}{2}
\end{aligned}
$$

Thus, Q is a 3-fold stochastic matrix.

1. $q^k_{\boldsymbol{i}} = \sum\{Q_{\boldsymbol{j}} : \boldsymbol{j} \in \mathcal{I}^d_m, \boldsymbol{j} < \boldsymbol{i}$ *and* $j_k = i_k\}$ *where* $\boldsymbol{i} \in \mathcal{I}^d_m$ *and* $k = 1, 2, \ldots, d.$

$q^1_{(0,0,0)}$		$= 0$
$q^1_{(0,0,1)}$	$= Q_{(0,0,0)}$	$= 0$
$q^1_{(0,1,0)}$	$= Q_{(0,0,0)} + Q_{(0,0,1)} = 0 + 0$	$= 0$
$q^1_{(0,1,1)}$	$= Q_{(0,0,0)} + Q_{(0,0,1)} + Q_{(0,1,0)} = 0 + 0 + 0$	$= 0$
$q^1_{(1,0,0)}$		$= 0$
$q^1_{(1,0,1)}$	$= Q_{(1,0,0)}$	$= \frac{1}{2}$
$q^1_{(1,1,0)}$	$= Q_{(1,0,0)} + Q_{(1,0,1)} = \frac{1}{2} + 0$	$= \frac{1}{2}$
$q^1_{(1,1,1)}$	$= Q_{(1,0,0)} + Q_{(1,0,1)} + Q_{(1,1,0)} = \frac{1}{2} + 0 + 0$	$= \frac{1}{2}$

$q^2_{(0,0,0)}$		$= 0$
$q^2_{(0,0,1)}$	$= Q_{(0,0,0)}$	$= 0$
$q^2_{(0,1,0)}$		$= 0$
$q^2_{(0,1,1)}$	$= Q_{(0,1,0)}$	$= 0$
$q^2_{(1,0,0)}$	$= Q_{(0,0,0)} + Q_{(0,0,1)} = 0 + 0$	$= 0$
$q^2_{(1,0,1)}$	$= Q_{(0,0,0)} + Q_{(0,0,1)} + Q_{(1,0,0)} = 0 + 0 + \frac{1}{2}$	$= \frac{1}{2}$
$q^2_{(1,1,0)}$	$= Q_{(0,1,0)} + Q_{(0,1,1)} = 0 + \frac{1}{2}$	$= \frac{1}{2}$
$q^2_{(1,1,1)}$	$= Q_{(0,1,0)} + Q_{(0,1,1)} + Q_{(1,1,0)} = 0 + \frac{1}{2} + 0$	$= \frac{1}{2}$

$q^3_{(0,0,0)}$		$= 0$
$q^3_{(0,0,1)}$		$= 0$
$q^3_{(0,1,0)}$	$= Q_{(0,0,0)}$	$= 0$
$q^3_{(0,1,1)}$	$= Q_{(0,0,1)}$	$= 0$
$q^3_{(1,0,0)}$	$= Q_{(0,0,0)} + Q_{(0,1,0)} = 0 + 0$	$= 0$
$q^3_{(1,0,1)}$	$= Q_{(0,0,1)} + Q_{(0,1,1)} = 0 + \frac{1}{2}$	$= \frac{1}{2}$
$q^3_{(1,1,0)}$	$= Q_{(0,0,0)} + Q_{(0,1,0)} + Q_{(1,0,0)} = 0 + 0 + \frac{1}{2}$	$= \frac{1}{2}$
$q^3_{(1,1,1)}$	$= Q_{(0,0,1)} + Q_{(0,1,1)} + Q_{(1,0,1)} = 0 + \frac{1}{2} + 0$	$= \frac{1}{2}$

2. $J^k_{\boldsymbol{i}} = \left(\frac{i_k}{m} + q^k_{\boldsymbol{i}}, \frac{i_k}{m} + q^k_{\boldsymbol{i}} + Q_{\boldsymbol{i}}\right)$.

$$J^1_{(0,1,1)} = \left(0, \frac{1}{2}\right) = J^2_{(1,0,0)} = J^3_{(1,0,0)},$$

$$J^1_{(1,0,0)} = \left(\frac{1}{2}, 1\right) = J^2_{(0,1,1)} = J^3_{(0,1,1)}.$$

3. *For any* $k = 1, \dots, d$,

$$\phi_k(t) = t + \frac{i_k - i_1}{m} + q^k_{\boldsymbol{i}} - Q^1_{\boldsymbol{i}} \text{ for } t \in J^1_{\boldsymbol{i}}$$

and

$$\phi(t_1, \dots, t_d) = (\phi_1(t_1), \dots, \phi_d(t_d)),$$

for all $t_1, \dots, t_d \in \mathbb{I}$.

If $t \in J^1_{(0,1,1)} = \left(0, \frac{1}{2}\right)$, *then*

$$\begin{aligned} \phi_1(t) &= t, \\ \phi_2(t) &= t + \frac{1-0}{2} + 0 - 0 = t + \frac{1}{2}, \\ \phi_3(t) &= t + \frac{1-0}{2} + 0 - 0 = t + \frac{1}{2}. \end{aligned}$$

If $t \in J^1_{(1,0,0)} = \left(\frac{1}{2}, 1\right)$, *then*

$$\begin{aligned} \phi_1(t) &= t, \\ \phi_2(t) &= t + \frac{0-1}{2} + 0 - 0 = t - \frac{1}{2}, \\ \phi_3(t) &= t + \frac{0-1}{2} + 0 - 0 = t - \frac{1}{2}. \end{aligned}$$

Thus,

$$\phi(t_1, t_2, t_3) = \begin{cases} (t_1, t_2 + \frac{1}{2}, t_3 + \frac{1}{2}) & ; (t_1, t_2, t_3) \in (0,1) \times \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right), \\ (t_1, t_2 + \frac{1}{2}, t_3 - \frac{1}{2}) & ; (t_1, t_2, t_3) \in (0,1) \times \left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right), \\ (t_1, t_2 - \frac{1}{2}, t_3 + \frac{1}{2}) & ; (t_1, t_2, t_3) \in (0,1) \times \left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right), \\ (t_1, t_2 - \frac{1}{2}, t_3 - \frac{1}{2}) & ; (t_1, t_2, t_3) \in (0,1) \times \left(\frac{1}{2}, 1\right) \times \left(\frac{1}{2}, 1\right). \end{cases}$$

Hence,

$$\phi^{-1}(t_1,t_2,t_3)=\begin{cases}(t_1,t_2+\frac{1}{2},t_3+\frac{1}{2}) & ;(t_1,t_2,t_3)\in(0,1)\times\left(0,\frac{1}{2}\right)\times\left(0,\frac{1}{2}\right),\\ (t_1,t_2+\frac{1}{2},t_3-\frac{1}{2}) & ;(t_1,t_2,t_3)\in(0,1)\times\left(0,\frac{1}{2}\right)\times\left(\frac{1}{2},1\right),\\ (t_1,t_2-\frac{1}{2},t_3+\frac{1}{2}) & ;(t_1,t_2,t_3)\in(0,1)\times\left(\frac{1}{2},1\right)\times\left(0,\frac{1}{2}\right),\\ (t_1,t_2-\frac{1}{2},t_3-\frac{1}{2}) & ;(t_1,t_2,t_3)\in(0,1)\times\left(\frac{1}{2},1\right)\times\left(\frac{1}{2},1\right).\end{cases}$$

4. *For any* $x_1,x_2,x_3\in\mathbb{I}$,

$$C_Q(x_1,x_2,x_3)=\mu_{C_Q}([0,x_1]\times[0,x_2]\times[0,x_3])=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2]\times[0,x_3])).$$

Case 1. $x_2\in\left(0,\frac{1}{2}\right)$ *and* $x_3\in\left(0,\frac{1}{2}\right)$. *Then*

$$\begin{aligned}C_Q(x_1,x_2,x_3)&=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2]\times[0,x_3]))\\&=\mu_C([0,x_1]\times[\frac{1}{2},x_2+\frac{1}{2}]\times[\frac{1}{2},x_3+\frac{1}{2}])\\&=C(x_1,x_2+\frac{1}{2},x_3+\frac{1}{2})-C(x_1,x_2+\frac{1}{2},\frac{1}{2})\\&\quad-C(x_1,\frac{1}{2},x_3+\frac{1}{2})+C(x_1,\frac{1}{2},\frac{1}{2}).\end{aligned}$$

Case 2. $x_2\in\left(0,\frac{1}{2}\right)$ *and* $x_3\in\left(\frac{1}{2},1\right)$. *Then*

$$\begin{aligned}&C_Q(x_1,x_2,x_3)\\&=\mu_C(\phi^{-1}([0,x_1]\times[0,x_2]\times[0,\frac{1}{2}]))+\mu_C(\phi^{-1}([0,x_1]\times[0,x_2]\times[\frac{1}{2},x_3]))\\&=\mu_C([0,x_1]\times[\frac{1}{2},x_2+\frac{1}{2}]\times[\frac{1}{2},1])+\mu_C([0,x_1]\times[\frac{1}{2},x_2+\frac{1}{2}]\times[0,x_3-\frac{1}{2}])\\&=C(x_1,x_2+\frac{1}{2},1)-C(x_1,x_2+\frac{1}{2},\frac{1}{2})-C(x_1,\frac{1}{2},1)+C(x_1,\frac{1}{2},\frac{1}{2})\\&\quad+C(x_1,x_2+\frac{1}{2},x_3-\frac{1}{2})-C(x_1,\frac{1}{2},x_3-\frac{1}{2}).\end{aligned}$$

Case 3. $x_2\in\left(\frac{1}{2},1\right)$ *and* $x_3\in\left(0,\frac{1}{2}\right)$. *Then*

$$\begin{aligned}&C_Q(x_1,x_2,x_3)\\&=\mu_C(\phi^{-1}([0,x_1]\times[0,\frac{1}{2}]\times[0,x_3]))+\mu_C(\phi^{-1}([0,x_1]\times[\frac{1}{2},x_2]\times[0,x_3]))\\&=\mu_C([0,x_1]\times[\frac{1}{2},1]\times[\frac{1}{2},x_3+\frac{1}{2}])+\mu_C([0,x_1]\times[0,x_2-\frac{1}{2}]\times[\frac{1}{2},x_3+\frac{1}{2}])\end{aligned}$$

$$= C(x_1, 1, x_3 + \frac{1}{2}) - C(x_1, 1, \frac{1}{2}) - C(x_1, \frac{1}{2}, x_3 + \frac{1}{2}) + C(x_1, \frac{1}{2}, \frac{1}{2})$$
$$+ C(x_1, x_2 - \frac{1}{2}, x_3 + \frac{1}{2}) - C(x_1, x_2 - \frac{1}{2}, \frac{1}{2}).$$

Case 4. *$x_2 \in \left(\frac{1}{2}, 1\right)$ and $x_3 \in \left(\frac{1}{2}, 1\right)$. Then*

$$C_Q(x_1, x_2, x_3)$$
$$= \mu_C(\phi^{-1}([0, x_1] \times [0, \frac{1}{2}] \times [0, \frac{1}{2}])) + \mu_C(\phi^{-1}([0, x_1] \times [0, \frac{1}{2}] \times [\frac{1}{2}, x_3]))$$
$$+ \mu_C(\phi^{-1}([0, x_1] \times [\frac{1}{2}, x_2] \times [0, \frac{1}{2}])) + \mu_C(\phi^{-1}([0, x_1] \times [\frac{1}{2}, x_2] \times [\frac{1}{2}, x_3]))$$
$$= \mu_C([0, x_1] \times [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]) + \mu_C([0, x_1] \times [\frac{1}{2}, 1] \times [0, x_3 - \frac{1}{2}])$$
$$+ \mu_C([0, x_1] \times [0, x_2 - \frac{1}{2}] \times [\frac{1}{2}, 1]) + \mu_C([0, x_1] \times [0, x_2 - \frac{1}{2}] \times [0, x_3 - \frac{1}{2}])$$
$$= C(x_1, 1, 1) - C(x_1, 1, \frac{1}{2}) - C(x_1, \frac{1}{2}, 1) + C(x_1, \frac{1}{2}, \frac{1}{2})$$
$$+ C(x_1, 1, x_3 - \frac{1}{2}) - C(x_1, \frac{1}{2}, x_3 - \frac{1}{2}) + C(x_1, x_2 - \frac{1}{2}, 1)$$
$$- C(x_1, x_2 - \frac{1}{2}, \frac{1}{2}) + C(x_1, x_2 - \frac{1}{2}, x_3 - \frac{1}{2}).$$

Thus, for $x_1, x_2, x_3 \in \mathbb{I}$,

$$
\begin{aligned}
&C_Q(x_1,x_2,x_3)\\
&=\begin{cases}
C(x_1,x_2+\frac{1}{2},x_3+\frac{1}{2})-C(x_1,x_2+\frac{1}{2},\frac{1}{2}) & \\
-C(x_1,\frac{1}{2},x_3+\frac{1}{2})+C(x_1,\frac{1}{2},\frac{1}{2}) & ;x_2\in\left[0,\frac{1}{2}\right],x_3\in\left[0,\frac{1}{2}\right],\\
\\
C(x_1,x_2+\frac{1}{2},1)-C(x_1,x_2+\frac{1}{2},\frac{1}{2}) & \\
-C(x_1,\frac{1}{2},1)+C(x_1,\frac{1}{2},\frac{1}{2}) & ;x_2\in\left[0,\frac{1}{2}\right],x_3\in\left[\frac{1}{2},1\right],\\
+C(x_1,x_2+\frac{1}{2},x_3-\frac{1}{2})-C(x_1,\frac{1}{2},x_3-\frac{1}{2}) & \\
\\
C(x_1,1,x_3+\frac{1}{2})-C(x_1,1,\frac{1}{2}) & \\
-C(x_1,\frac{1}{2},x_3+\frac{1}{2})+C(x_1,\frac{1}{2},\frac{1}{2}) & ;x_2\in\left[\frac{1}{2},1\right],x_3\in\left[0,\frac{1}{2}\right],\\
+C(x_1,x_2-\frac{1}{2},x_3+\frac{1}{2})-C(x_1,x_2-\frac{1}{2},\frac{1}{2}) & \\
\\
C(x_1,1,1)-C(x_1,1,\frac{1}{2})-C(x_1,\frac{1}{2},1) & \\
+C(x_1,\frac{1}{2},\frac{1}{2})+C(x_1,1,x_3-\frac{1}{2}) & \\
-C(x_1,\frac{1}{2},x_3-\frac{1}{2})+C(x_1,x_2-\frac{1}{2},1) & ;x_2\in\left[\frac{1}{2},1\right],x_3\in\left[\frac{1}{2},1\right]\\
-C(x_1,x_2-\frac{1}{2},\frac{1}{2})+C(x_1,x_2-\frac{1}{2},x_3-\frac{1}{2}) &
\end{cases}
\end{aligned}
$$

$$
= \begin{cases}
C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, x_3 + \frac{1}{2}) - C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, \frac{1}{2}) & \\
-C(x_1, \frac{1}{2}, x_3 + \frac{1}{2}) + C(x_1, \frac{1}{2}, \frac{1}{2}) & ; x_3 \in \left[0, \frac{1}{2}\right], \\
+C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, x_3 + \frac{1}{2}) - C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, \frac{1}{2}) & \\
& \\
C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, 1) - C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, \frac{1}{2}) & \\
+C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, x_3 - \frac{1}{2}) - C(x_1, \frac{1}{2}, 1) + C(x_1, \frac{1}{2}, \frac{1}{2}) & \\
-C(x_1, \frac{1}{2}, x_3 - \frac{1}{2}) + C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, 1) & ; x_3 \in \left[\frac{1}{2}, 1\right] \\
-C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, \frac{1}{2}) + C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, x_3 - \frac{1}{2}) &
\end{cases}
$$

$$
\begin{aligned}
&= C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, \min\{x_3 + \frac{1}{2}, 1\}) - C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, \frac{1}{2}) \\
&\quad + C(x_1, \min\{x_2 + \frac{1}{2}, 1\}, \max\{x_3 - \frac{1}{2}, 0\}) - C(x_1, \frac{1}{2}, \min\{x_3 + \frac{1}{2}, 1\}) \\
&\quad + C(x_1, \frac{1}{2}, \frac{1}{2}) - C(x_1, \frac{1}{2}, \max\{x_3 - \frac{1}{2}, 0\}) \\
&\quad + C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, \min\{x_3 + \frac{1}{2}, 1\}) - C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, \frac{1}{2}) \\
&\quad + C(x_1, \max\{x_2 - \frac{1}{2}, 0\}, \max\{x_3 - \frac{1}{2}, 0\}).
\end{aligned}
$$

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