# CHAPTER II

# TRANSFORMATION SEMIGROUPS

This chapter is our main results. The standard transformation semigroups $G(X)$, $T(X)$, $P(X)$ and $I(X)$ are considered. We characterize when they admit a right nearring structure and a left nearring structure.

Recall that for a nonempty set $X$,

$G(X) =$ the symmetric semigroup on $X$,

$T(X) =$ the full transformation semigroup on $X$,

$P(X) =$ the partial transformation semigroup on $X$,

$I(X) =$ the symmetric inverse transformation semigroup on $X$

(the 1-1 partial transformation semigroup on $X$).

Then $G(X) \subseteq T(X) \subseteq P(X)$ and $G(X) \subseteq I(X) \subseteq P(X)$. Also, $P(X)$ and $I(X)$ have a zero 0 (the empty transformation) but $G(X)$ and $T(X)$ have no zero if $|X| > 1$. For convenience, for $\varnothing \neq A \subseteq X$ and $x \in X$, let $A_x$ be the constant mapping whose domain and range are $A$ and $\{x\}$, respectively and the identity map on $A$ may be denoted by $1_A$. For distinct $a, b \in X$, let $(a\ b) \in G(X)$ be defined by $(a\ b)(a) = b$, $(a\ b)(b) = a$ and $(a\ b)(x) = x$ for all $x \in X \setminus \{a, b\}$. Elements of $P(X)$ may be written by bracket notation. For examples,

$\begin{pmatrix} a \\ b \end{pmatrix} =$ the mapping in $I(X)$ whose domain and

range are $\{a\}$ and $\{b\}$, respectively,

$$\begin{pmatrix} a & x \\ b & x \end{pmatrix}_{x \in X \smallsetminus \{a\}} = \text{ the mapping } f \in T(X) \text{ defined by}$$

$$f(x) = \begin{cases} b & \text{if } x = a, \\ x & \text{ortherwise.} \end{cases}$$

Observe that

$$A_x = \begin{pmatrix} a \\ x \end{pmatrix}_{a \in A} \text{ and } (a\ b) = \begin{pmatrix} a & b & x \\ b & a & x \end{pmatrix}_{x \in X \smallsetminus \{a,b\}}.$$

The following facts will be quoted.

**Theorem 2.1 ([14]).** *Let $X$ be a nonempty set. Then the following statements hold.*

(i) *$G(X)$ admits a ring structure if and only if $|X| \leq 2$.*

(ii) *If $S(X)$ is $T(X)$, $P(X)$ or $I(X)$, then $S(X)$ admits a ring structure if and only if $|X| = 1$.*

Two-sided distribution was used in the proof of Theorem 2.1 in [14]. These results motivate us to characterize when these standard transformation semigroups admit a right nearring structure and a left nearring structure. The following important facts are helpful for our work.

**Theorem 2.2 ([14]).** *For any nonempty set $X$, there is an operation $+$ on $X$ such that $(X, +)$ is an abelian group.*

It can be seen from the proof of Theorem 2.2 in [14] that the identity of the group $(X, +)$ can be specified. Hence we have

**Theorem 2.3.** *If $X$ is a nonempty set and $a \in X$, then there is an operation $+$ on $X$ such that $(X, +)$ is an abelian group with identity $a$.*

**Theorem 2.4 ([3], page 41).** *Let $X$ be a nonempty set and $\theta$ a symbol not representing any element of $X$. For $f \in P(X)$, define $f^* \in T(X \cup \{\theta\})$ by*

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom} f, \\ \theta & \text{if } x \in (X \cup \{\theta\}) \smallsetminus \operatorname{dom} f. \end{cases}$$

*Then $f \mapsto f^*$ is an isomorphism from $P(X)$ onto the subsemigroup $\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}$ of $T(X \cup \{\theta\})$.*

**Theorem 2.5.** *For a nonempty set $X$,*
(i) $G(X) \in \mathcal{RNR}$ *if and only if* $|X| \leq 2$ *and*
(ii) $G(X) \in \mathcal{LNR}$ *if and only if* $|X| \leq 2$.

*Proof.* If $|X| \leq 2$, then by Theorem 2.1(i), $G(X) \in \mathcal{R}$. Hence the converses of (i) and (ii) hold since $\mathcal{R} \subseteq \mathcal{RNR}$ and $\mathcal{R} \subseteq \mathcal{LNR}$.

Assume that $|X| > 2$. Let $a, b, c$ be distinct elements of $X$. Since $G(X)$ is a group and $1_X$ is the only idempotent of $G(X)$, it follows that $G(X)$ has neither a right zero nor a left zero. First, suppose that $G(X) \in \mathcal{RNR}$. Then there is an operation $+$ on $G^0(X)$ such that $(G^0(X), +, \circ)$ is a right nearring. Thus $1_X + (a\ b) = f$ for some $f \in G^0(X)$, so

$$f = 1_X + (a\ b) = ((a\ b) + 1_X)(a\ b) = (1_X + (a\ b))(a\ b) = f(a\ b).$$

**Case 1** : $f \neq 0$. Then $(a\ b) = 1_X$, a contradiction.

**Case 2** : $f = 0$. Then $1_X + (a\ b) = 0$. Since $(a\ b) \neq (a\ c)$, $1_X + (a\ c) \neq 0$. Let $1_X + (a\ c) = g \in G(X)$. Then

$$g = 1_X + (a\ c) = ((a\ c) + 1_X)(a\ c) = g(a\ c),$$

so $(a\ c) = 1_X$, a contradiction.

This proves that $G(X) \notin \mathcal{RNR}$.

Next, to show that $G(X) \notin \mathcal{LNR}$, suppose on the contrary that $G(X) \in \mathcal{LNR}$. Then there is an operation $+$ on $G^0(X)$ such that $(G^0(X), +, \circ)$ is a left nearring. Then $1_X + (a\ b) = f$ for some $f \in G^0(X)$. Then

$$f = 1_X + (a\ b) = (a\ b)((a\ b) + 1_X) = (a\ b)f.$$

**Case 1** : $f \neq 0$. Then $(a\ b) = 1_X$, a contradiction.

**Case 2** : $f = 0$. Then $1_X + (a\ c) = g$ for some $g \in G(X)$. Since

$$g = 1_X + (a\ c) = (a\ c)((a\ c) + 1_X) = (a\ c)g,$$

we have $(a\ c) = 1_X$, a contradiction.

Hence the proof is complete. □

**Theorem 2.6.** (i) *For any nonempty set $X$, $T(X) \in \mathcal{RNR}$.*
(ii) *For any nonempty set $X$, $T(X) \in \mathcal{LNR}$ if and only if $|X| = 1$.*

*Proof.* (i) Let $X$ be a nonempty set. By Theorem 2.2, there is an operation $+$ on $X$ such that $(X, +)$ is an abelian group. For $f, g \in T(X)$, define

$$(f + g)(x) = f(x) + g(x) \text{ for all } x \in X.$$

By Proposition 1.3, $(T(X), +, \circ)$ is a right nearring, so $T(X) \in \mathcal{RNR}$.

(ii) Suppose that $|X| > 1$. Let $a, b \in X$ be distinct. Since $X_a f = X_a$ and $X_b f = X_b$ for all $f \in T(X)$, it follows that $T(X)$ has no right zero. Suppose that there is an operation $+$ on $T^0(X)$ such that $(T^0(X), +, \circ)$ is a left nearring. Then $X_a + X_b = f$ for some $f \in T^0(X)$.

**Case 1** : $f \neq 0$. Then

$$X_a + X_a = X_a(X_a + X_b) = X_a f = X_a$$

which implies that $X_a = 0$, a contradiction.

**Case 2** : $f = 0$. Then $X_a + X_b = 0$ and

$$X_a + X_a = X_a(X_a + X_b) = X_a 0 = 0.$$

It follows that $X_b = X_a$, a contradiction.

The converse is obtained from Theorem 2.1(ii) since $\mathcal{R} \subseteq \mathcal{LNR}$. $\square$

**Theorem 2.7.** (i) *For any nonempty set* $X$, $P(X) \in \mathcal{RNR}$.
(ii) *For a nonempty set* $X$, $P(X) \in \mathcal{LNR}$ *if and only if* $|X| = 1$.

*Proof.* (i) Let $X$ be a nonempty set and $\theta$ a symbol not representing any element of $X$. For $f \in P(X)$, define $f^* \in T(X \cup \{\theta\})$ as in Theorem 2.4. By Theorem 2.4, the mapping $f \mapsto f^*$ is an isomorphism from $P(X)$ onto the subsemigroup $\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}$ of $T(X \cup \{\theta\})$. From Theorem 2.3, there is an operation $+$ on $X \cup \{\theta\}$ such that $(X \cup \{\theta\}, +)$ is an abelian group with identity $\theta$. Then $(T(X \cup \{\theta\}), +, \circ)$ is a right nearring by Proposition 1.3 where

$$(f+g)(x) = f(x) + g(x) \text{ for all } f, g \in T(X \cup \{\theta\}) \text{ and } x \in X.$$

If $g, h \in T(X \cup \{\theta\})$ are such that $g(\theta) = \theta = h(\theta)$, then

$$(g+h)(\theta) = g(\theta) + h(\theta) = \theta + \theta = \theta,\ (gh)(\theta) = g(h(\theta)) = g(\theta) = \theta,$$
$$(-g)(\theta) = -g(\theta) = -\theta = \theta.$$

Therefore $\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}$ is a subnearring of the right nearring $(T(X \cup \{\theta\}), +, \circ)$. But $P(X)$ is isomorphic to $(\{g \in T(X \cup \{\theta\}) \mid g(\theta) = \theta\}, \circ)$, so $P(X) \in \mathcal{RNR}$.

(ii) Assume that $|X| > 1$. Let $a$ and $b$ be distinct elements of $X$. Suppose that there is an operation $+$ on $P(X)$ such that $(P(X), +, \circ)$ is a left nearring. Then $\begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = f$ for some $f \in P(X)$. Then

$$\begin{pmatrix} a \\ a \end{pmatrix} f = \begin{pmatrix} a \\ a \end{pmatrix} \left( \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) = \begin{pmatrix} a \\ a \end{pmatrix} + 0 = \begin{pmatrix} a \\ a \end{pmatrix},$$

$$\binom{b}{b}f = \binom{b}{b}\left(\binom{a}{a} + \binom{a}{b}\right) = 0 + \binom{a}{b} = \binom{a}{b}.$$

It follows that

$$a = \binom{a}{a}(a) = \left(\binom{a}{a}f\right)(a) = \binom{a}{a}f(a) \text{ and}$$

$$b = \binom{a}{b}(a) = \left(\binom{b}{b}f\right)(a) = \binom{b}{b}f(a)$$

which imply respectively that $f(a) = a$ and $f(a) = b$. This is a contradiction.

The converse holds by Theorem 2.1(ii). □

**Theorem 2.8.** *Let $X$ be a nonempty set.*

(i) $I(X) \in \mathcal{RNR}$ *if and only if* $|X| = 1$.

(ii) $I(X) \in \mathcal{LNR}$ *if and only if* $|X| = 1$.

*Proof.* (i) Assume that $|X| > 1$ and let $a, b \in X$ be distinct. Suppose that there is an operation $+$ on $I(X)$ such that $(I(X), +, \circ)$ is a right nearring. Then $\binom{a}{a} + \binom{b}{a} = f$ for some $f \in I(X)$, so

$$f\binom{a}{a} = \left(\binom{a}{a} + \binom{b}{a}\right)\binom{a}{a} = \binom{a}{a} + 0 = \binom{a}{a},$$

$$f\binom{b}{b} = \left(\binom{a}{a} + \binom{b}{a}\right)\binom{b}{b} = 0 + \binom{b}{a} = \binom{b}{a}.$$

These imply that

$$f(a) = \left(f\binom{a}{a}\right)(a) = \binom{a}{a}(a) = a \text{ and}$$

$$f(b) = \left(f\binom{b}{b}\right)(b) = \binom{b}{a}(b) = a.$$

It is a contradiction since $f$ is 1-1.

The converse is obtained from Theorem 2.1(ii).

(ii) The proof can be given the same as that of Theorem 2.7(ii). □