Chapter IV

The results

In this chapter, we show the results of the shortest arc of the scaled considered set. The results are obtained from programming by using Mathematica.

4.1 Equilateral Triangle

The aim is to show that an equilateral triangle of unit side can cover every arc of length $\ell_1 = \sqrt{\frac{27}{28}} \approx 0.981981$. To show this we suppose that an arc γ cannot be covered. Then we will show that the length of γ is greater than ℓ_1. Thus, γ must not be covered by the triangle in the standing position and its reflection.

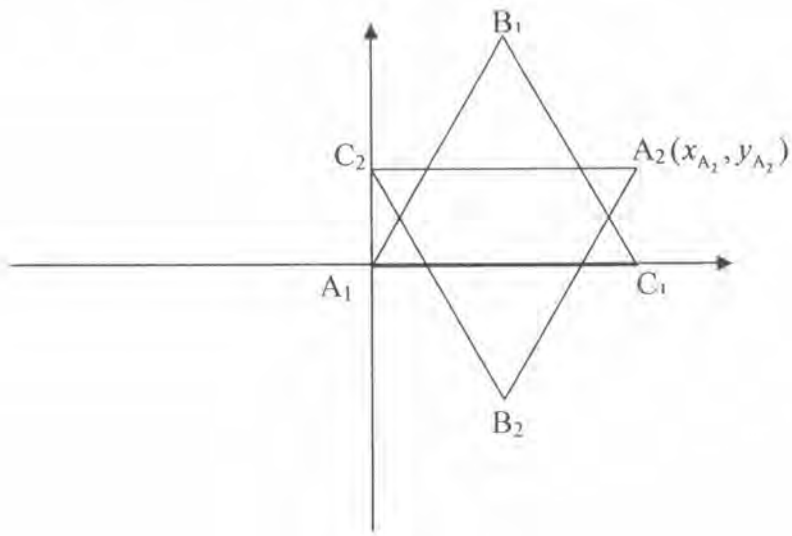


Figure 4.1 : An equilateral triangle $A_1B_1C_1$ in standing position and is its reflection $A_2B_2C_2$

Let $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_4(x_4,y_4)$, and $P_5(x_5,y_5)$ be points on the side $\overline{A_1C_1}$, $\overline{A_1B_1}$, $\overline{A_2C_2}$ and $\overline{A_2B_2}$ of the triangle that touches γ, respectively.

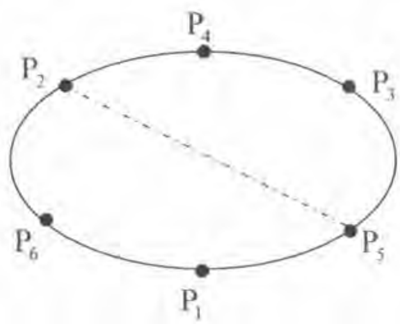
Let $P_3(x_3,y_3)$ and $P_6(x_6,y_6)$ be points which are not in the triangles or on the side $\overline{B_1C_1}$ and $\overline{C_2B_2}$, respectively.

As γ touches all the 6 points, then γ is not shorter than the polysegment connecting those points in the order on γ. There are $\frac{6!}{2} = 360$ ways to make it differently as a path and its reverse are the same. This means 360 cases have to be checked but we can reduce the cases by using the following lemma.

It is clear that there exists a worm with equal or greater convex hall that does not intersect itself. Moreover, the considered set is symmetric. Its reflection does not have to be checked. Thus, the cases can be reduced to the following.

Define $\Pi = P_1\ P_2\ P_3\ P_4 \dots P_n$ is a polysegment that is formed by joining $P_1\ P_2\ P_3\ P_4\ \dots\ P_n$ respectively.

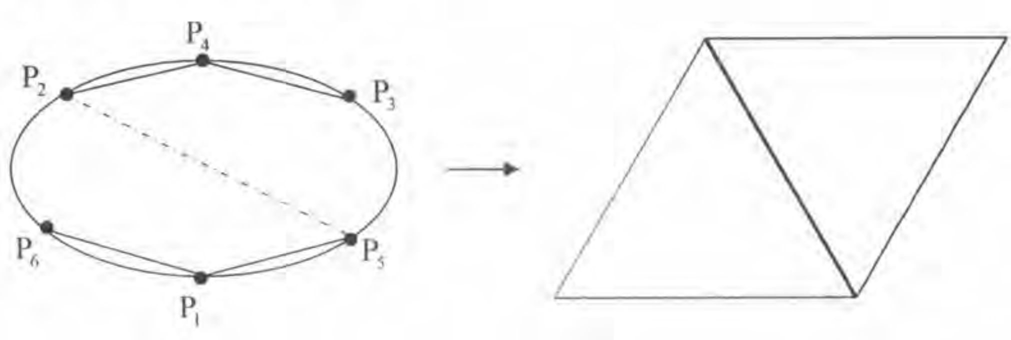
Case 1 P_2 and P_5 are connected.



There will be the following cases

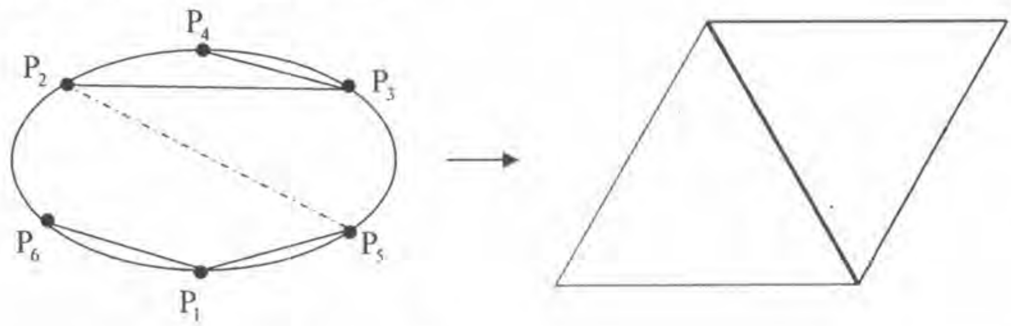
Subcase 1.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



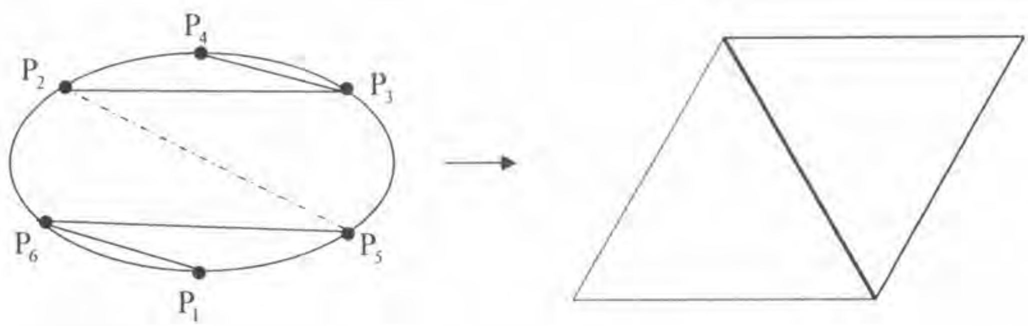
Subcase 1.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$.

By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



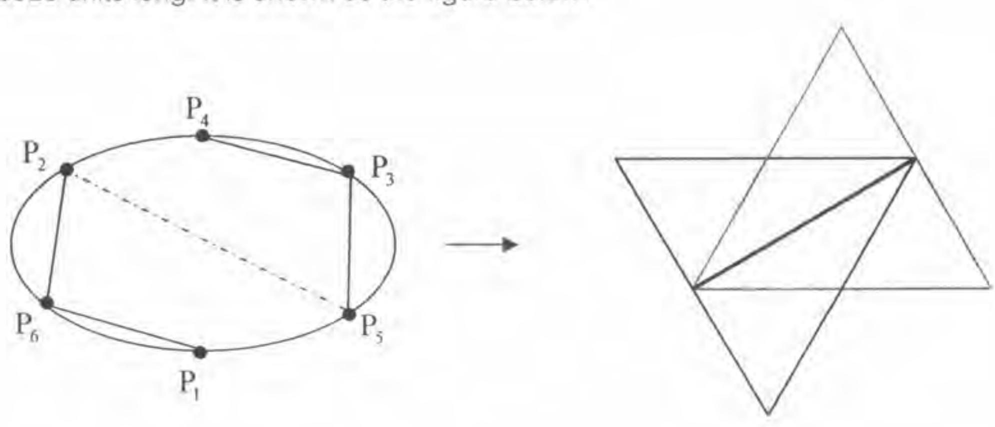
Subcase 1.3 $\Pi = P_1\ P_6\ P_5\ P_2\ P_3\ P_4$

By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



Subcase 1.4 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



Obviously, the given polysegment can be covered by the equilateral triangle though it satisfies all given conditions. So, more conditions must be added in order to find the shortest polysegment that can't be covered. We found that the thickness is the sufficient condition that we need.

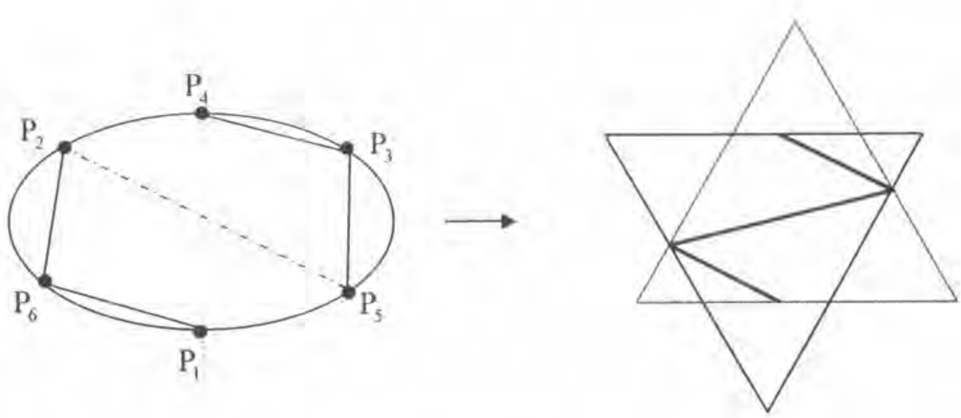
Definition The thickness of an arc is the least width of a strip cover.

We can suppose that we lay down the polysegment on the cover by letting its thickness to the vertical side. This means it is set as flat as possible. The thickness can be form by 2 following conditions.

Conditon T1 The points P_1 and P_4 must be on the same vertical line. This means $x_1 = x_4$.

Condition T2 Without loss of generality, we can say that there is another point P_{1b} which lays on the same horizontal level as P_1 such that $y_1 < y_4 < y_{1b}$.

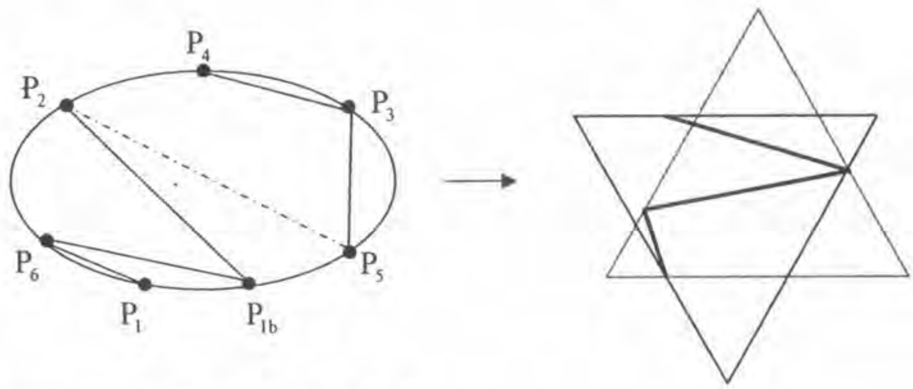
Subcase 1.4.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$ and the **condition T1** is satisfied.
By using the numerical minimization, the shortest polysegment is approximately 1.50213 units long. It is shown as the figure below.



Subcase 1.4.2 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$ and the **condition T2** is satisfied.
There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

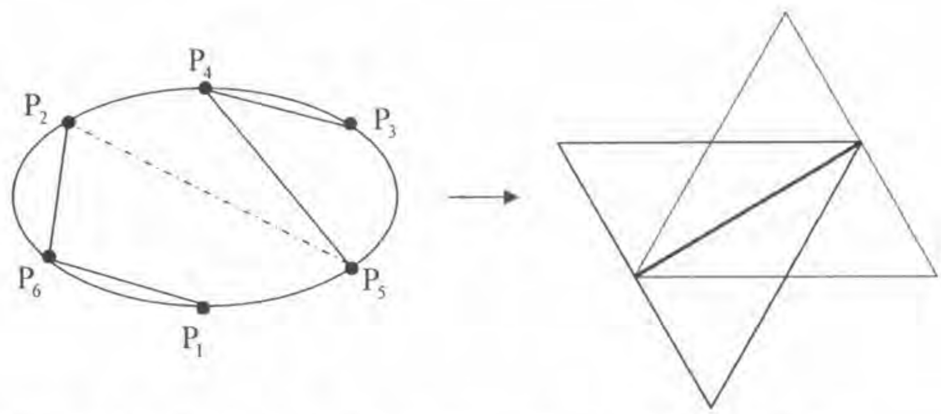
Subcase 1.4.2.1 $\Pi = P_1\ P_6\ P_{1b}\ P_2\ P_5\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 1.541 units long. It is shown as the figure below.



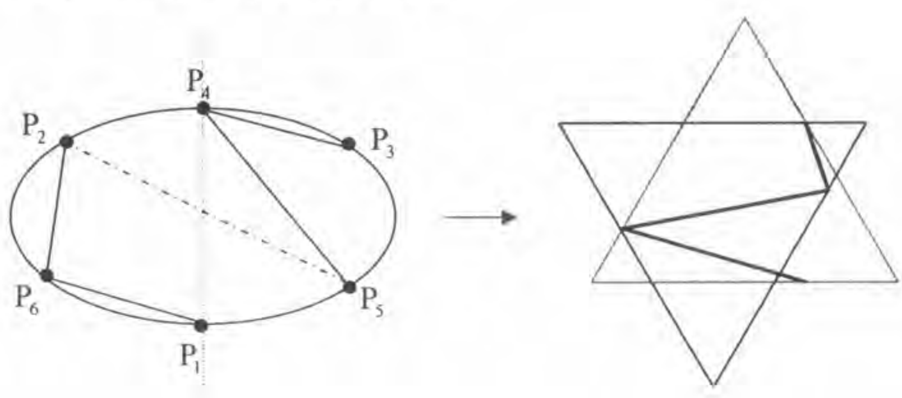
Subcase 1.5 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.866025 unit long. It is shown as the figure below.



Subcase 1.5.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.541 units long. It is shown as the figure below.

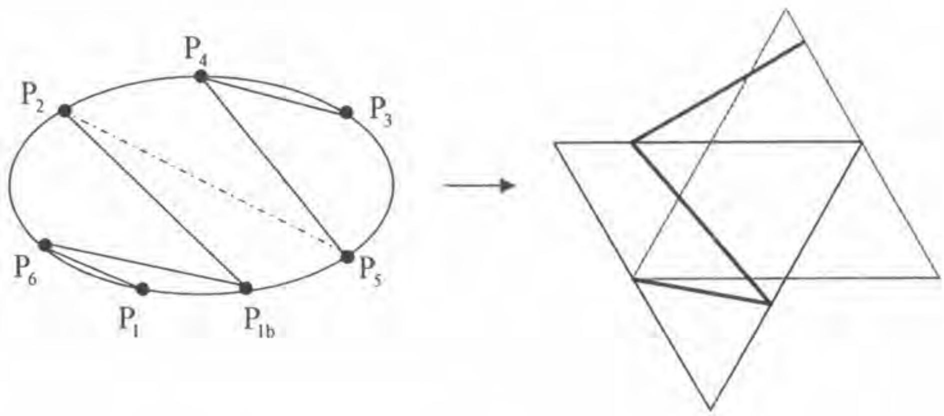


Subcase 1.5.2 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

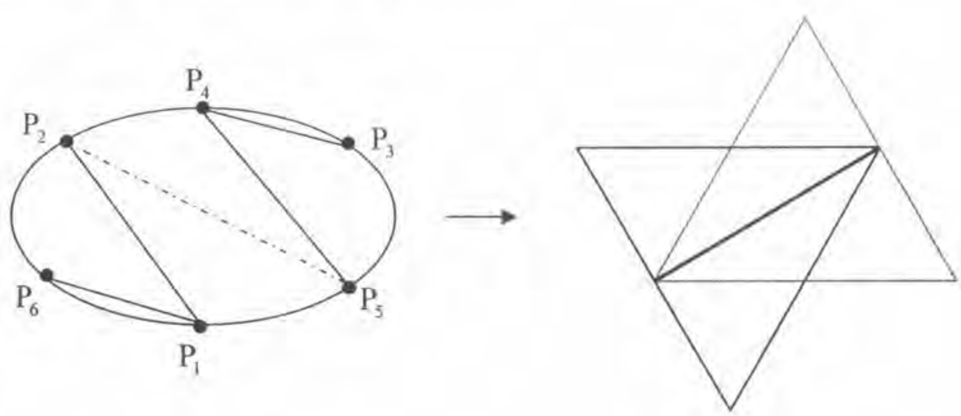
Subcase 1.5.2.1 $\Pi = P_1\ P_6\ P_{1b}\ P_2\ P_5\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1.79561 units long. It is shown as the figure below.



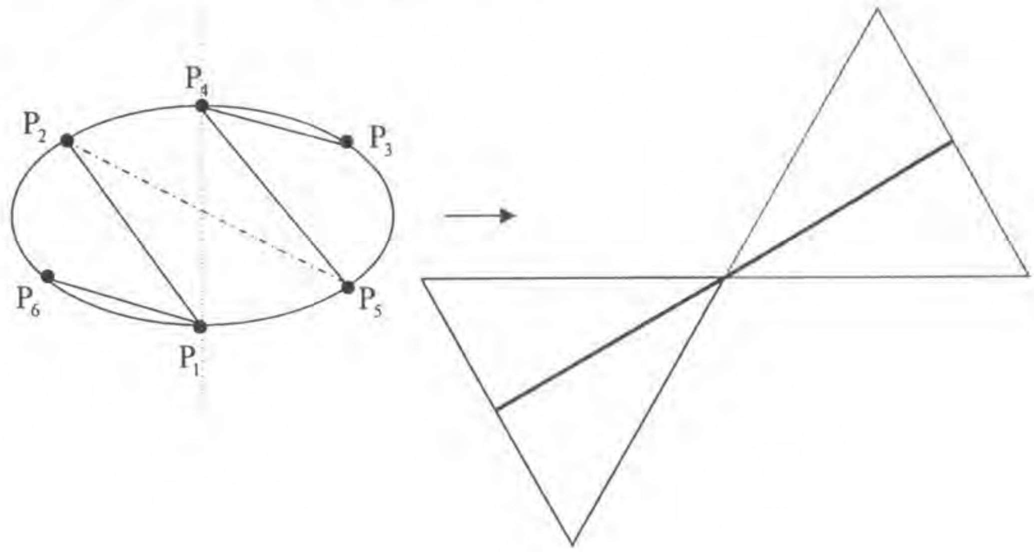
Subcase 1.6 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.866025 unit long. It is shown as the figure below.



Subcase 1.6.1 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.73205 units long. It is shown as the figure below.

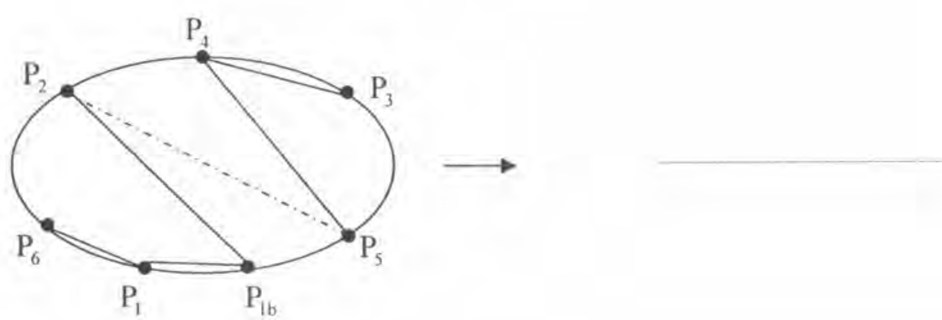


Subcase 1.6.2 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$ and the condition T2 is satisfied.

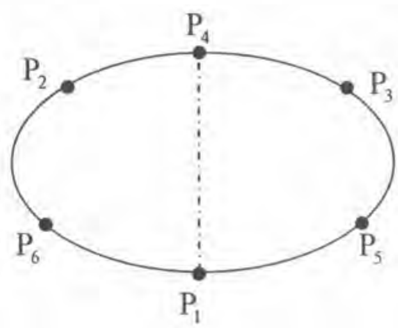
There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

Subcase 1.6.2.1 $\Pi = P_6\ P_1\ P_{1b} P_2\ P_5\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



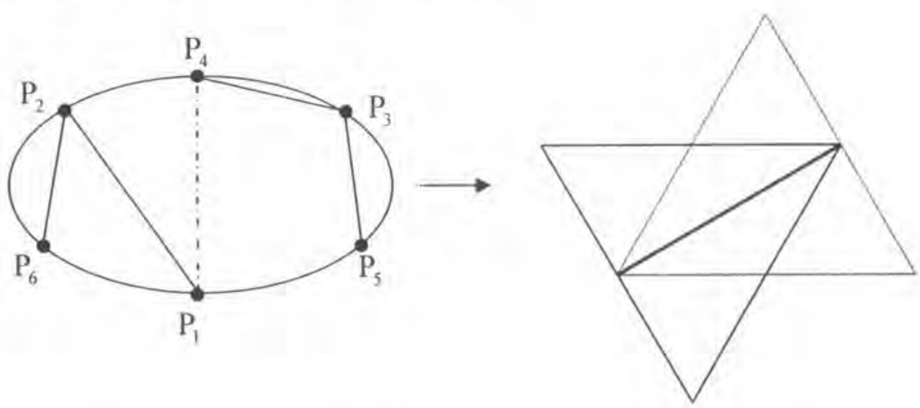
Case 2 P_1 and P_4 are connected.



There will be the following cases

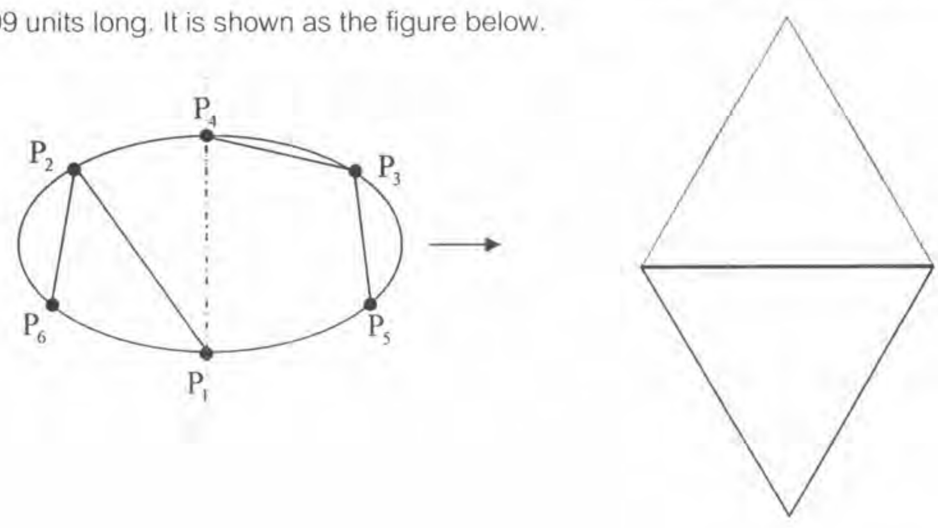
Subcase 2.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$.

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



Subcase 2.1.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.999999 units long. It is shown as the figure below.

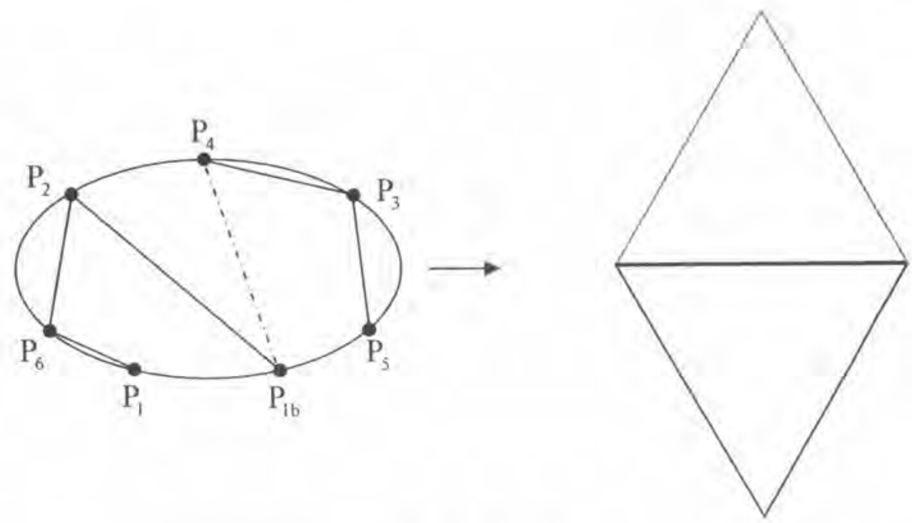


Subcase 2.1.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

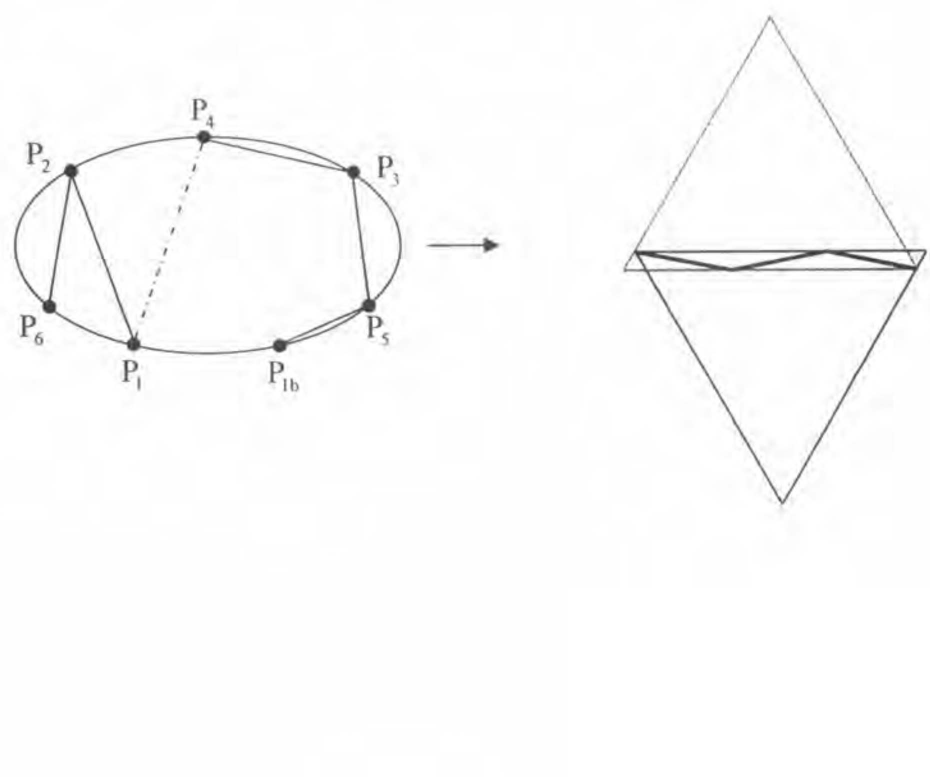
Subcase 2.1.2.1 $\Pi = P_1\ P_6\ P_2\ P_{1b}\ P_4\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



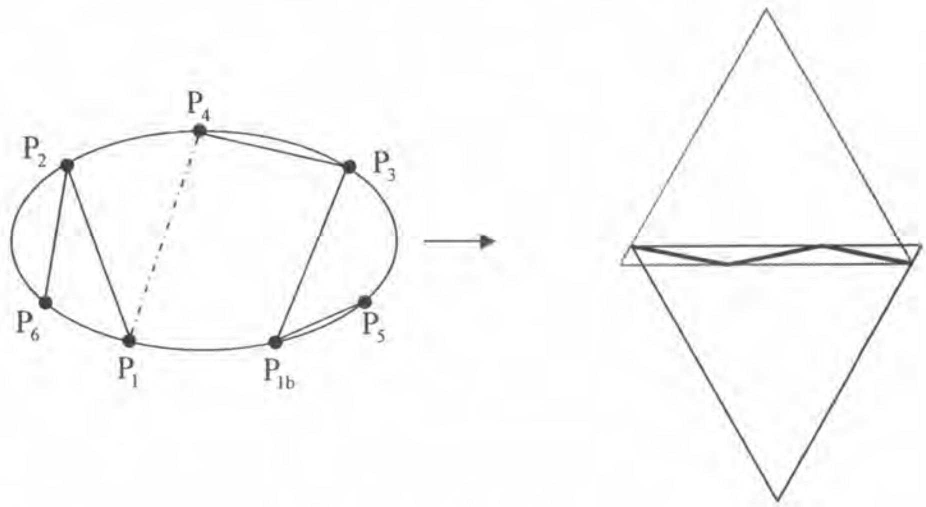
Subcase 2.1.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



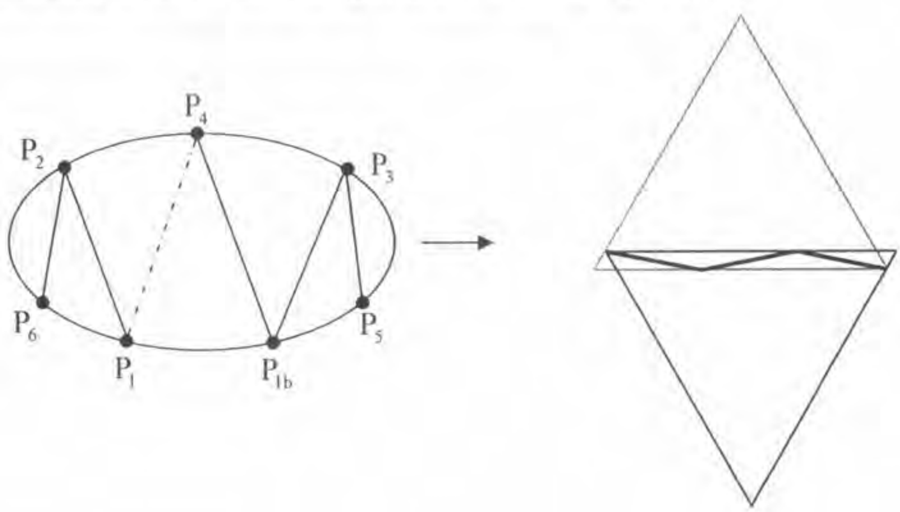
Subcase 2.1.2.3 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



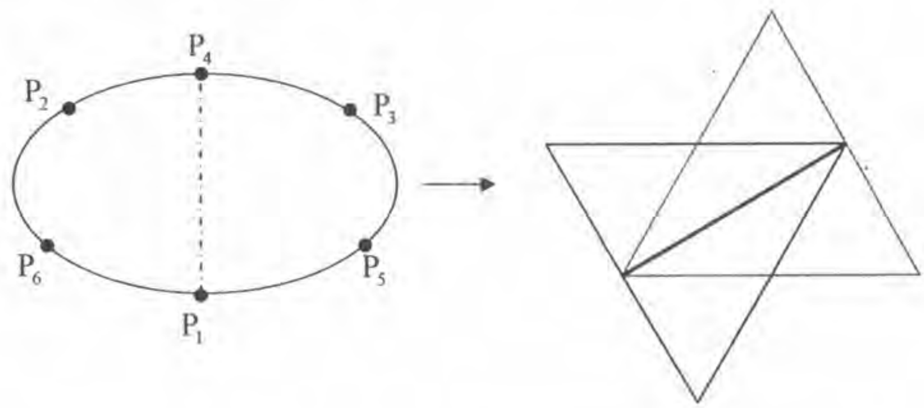
Subcase 2.1.2.4 $\Pi = P_6\ P_2\ P_1\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



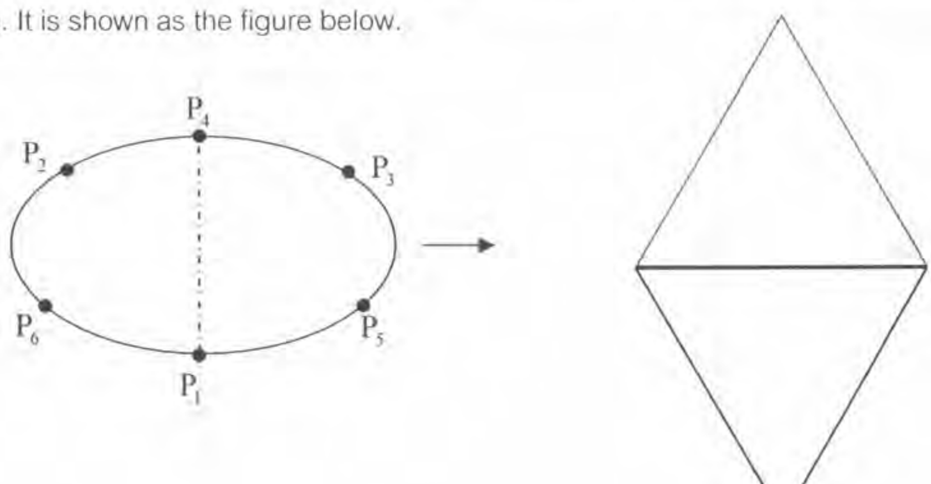
Subcase 2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below



Subcase 2.2.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

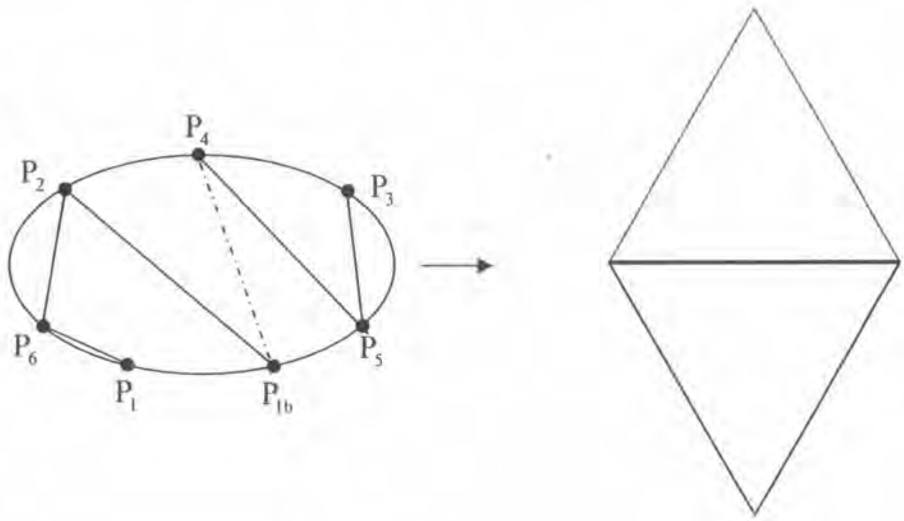


Subcase 2.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

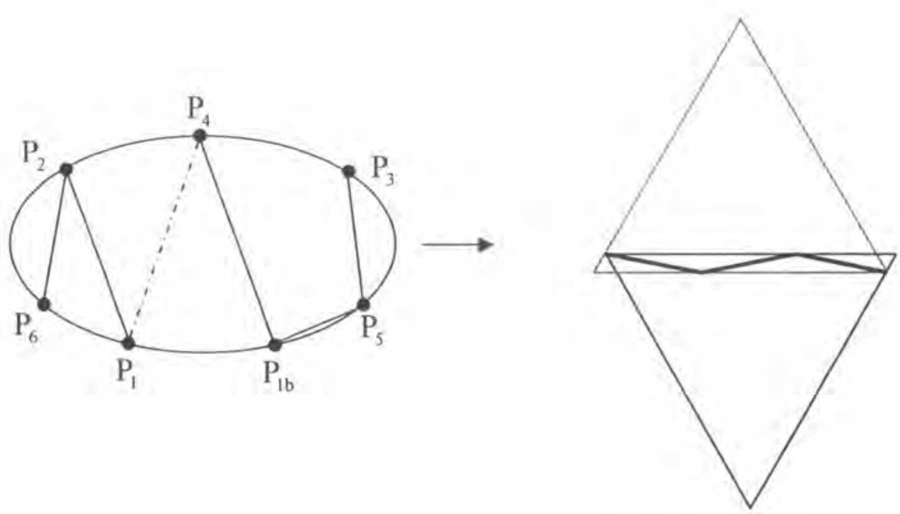
Subcase 2.2.2.1 $\Pi = P_1\ P_6\ P_2\ P_{1b}\ P_4\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



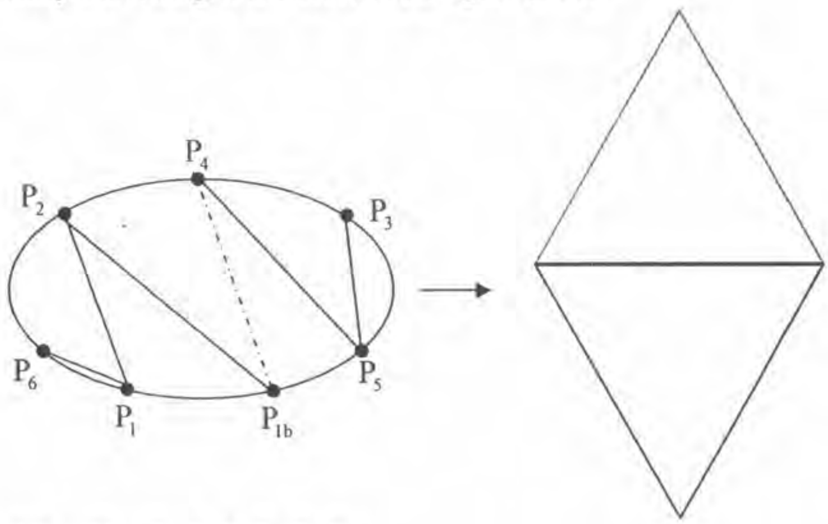
Subcase 2.2.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_{1b}\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



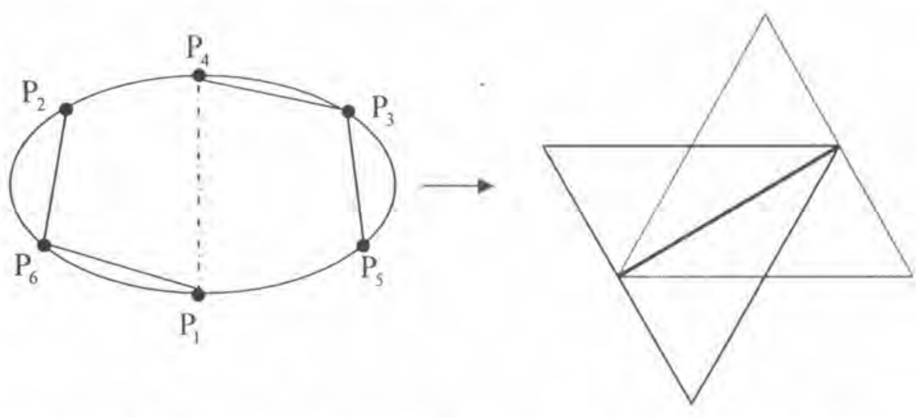
Subcase 2.2.2.3 $\Pi = P_6\ P_1\ P_2\ P_{1b}\ P_4\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



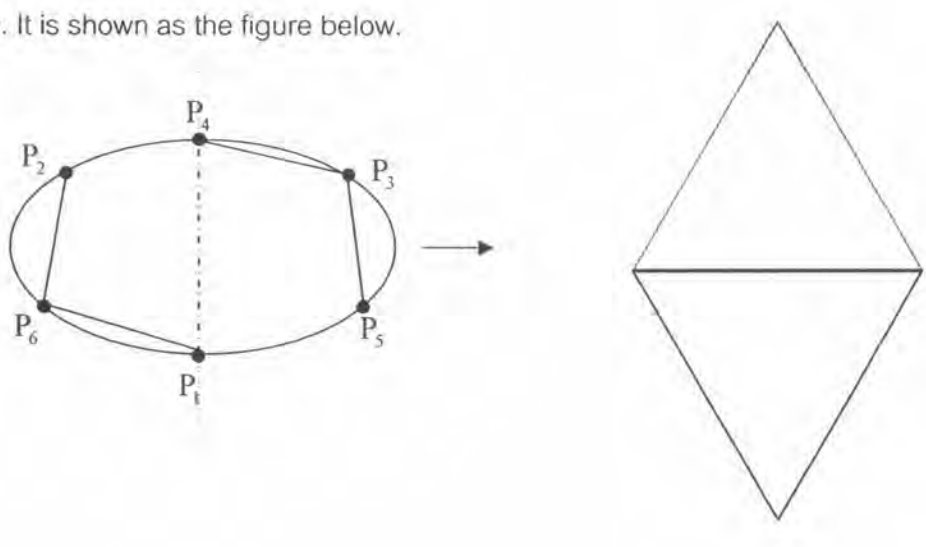
Subcase 2.3 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5$

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below



Subcase 2.3.1 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

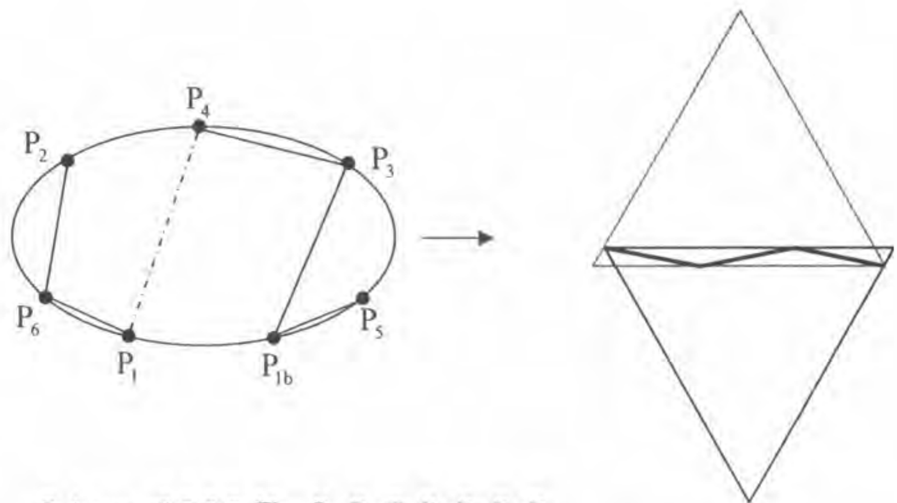


Subcase 2.3.2 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

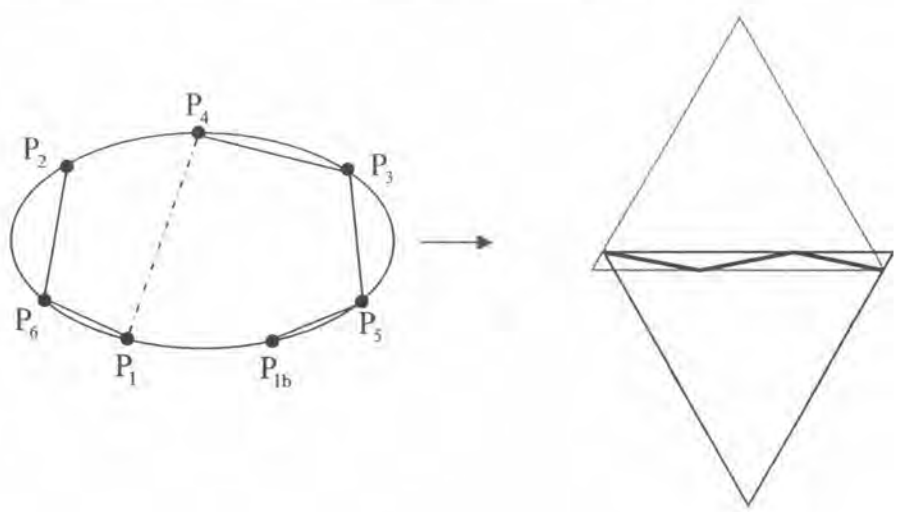
Subcase 2.3.2.1 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



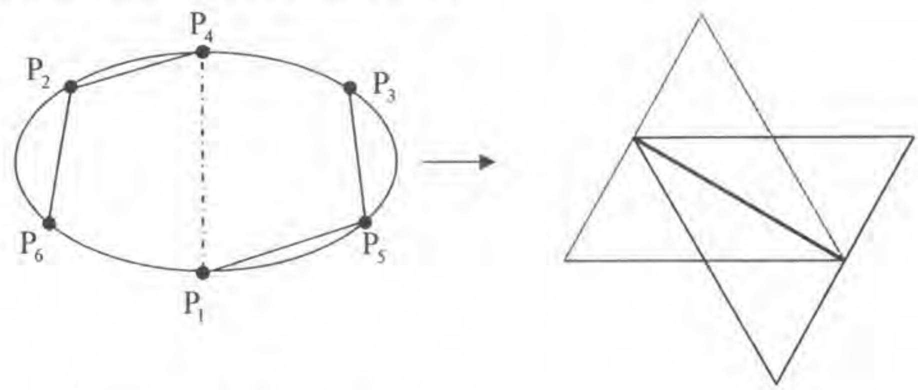
Subcase 2.3.2.2 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



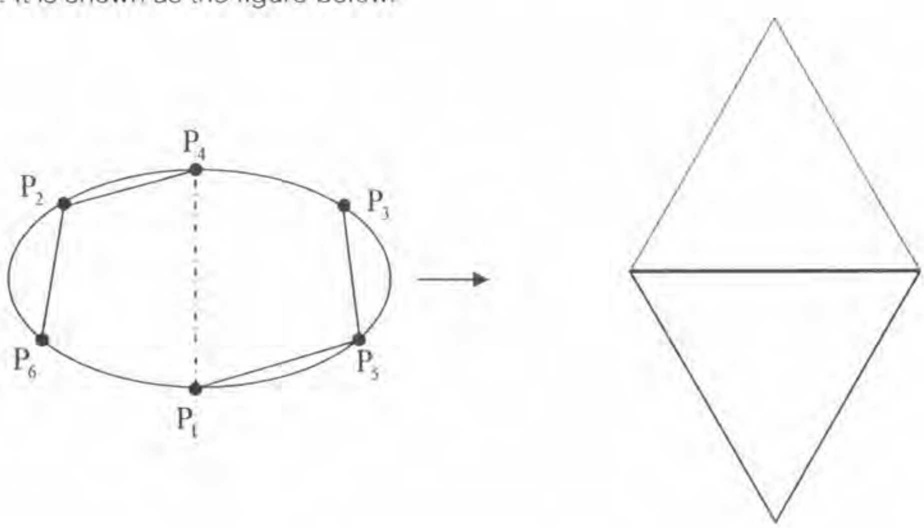
Subcase 2.4 $\Pi = P_6\ P_2\ P_4\ P_1\ P_5\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below



Subcase 2.4.1 $\Pi = P_6\ P_2\ P_4\ P_1\ P_5\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

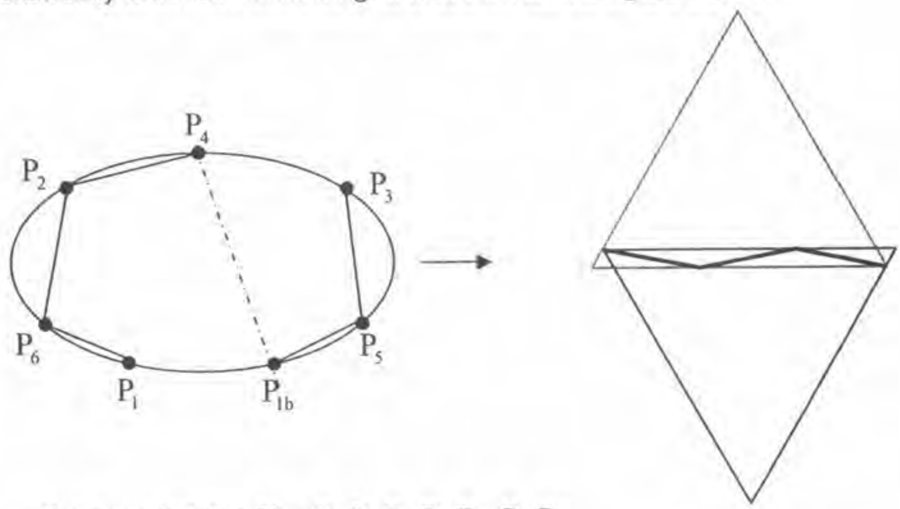


Subcase 2.4.2 $\Pi = P_6\ P_2\ P_4\ P_1\ P_5\ P_3$ and the **condition** T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

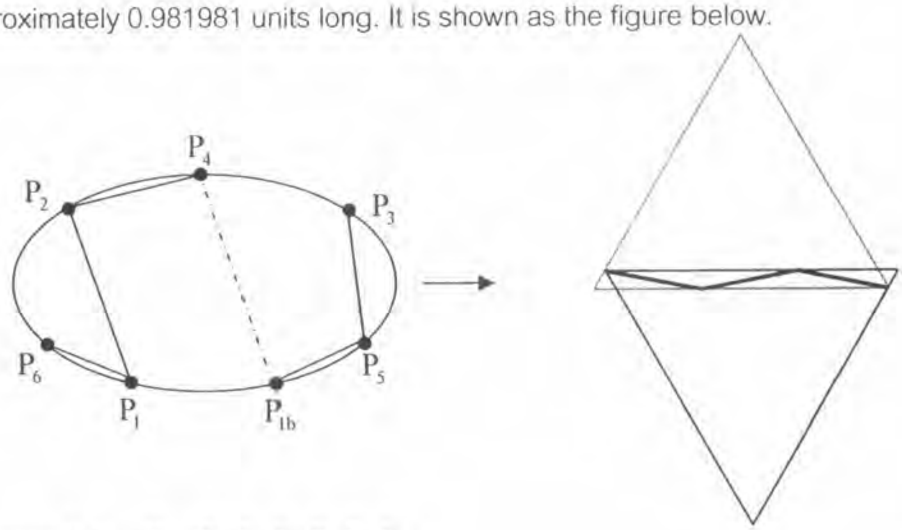
Subcase 2.4.2.1 $\Pi = P_1\ P_6\ P_2\ P_4\ P_{1b}\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



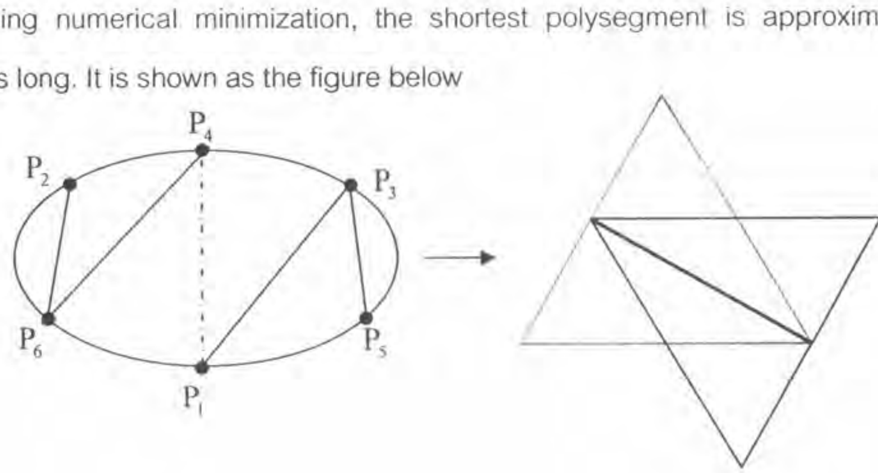
Subcase 2.4.2.2 $\Pi = P_6\ P_1\ P_2\ P_4\ P_{1b}\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



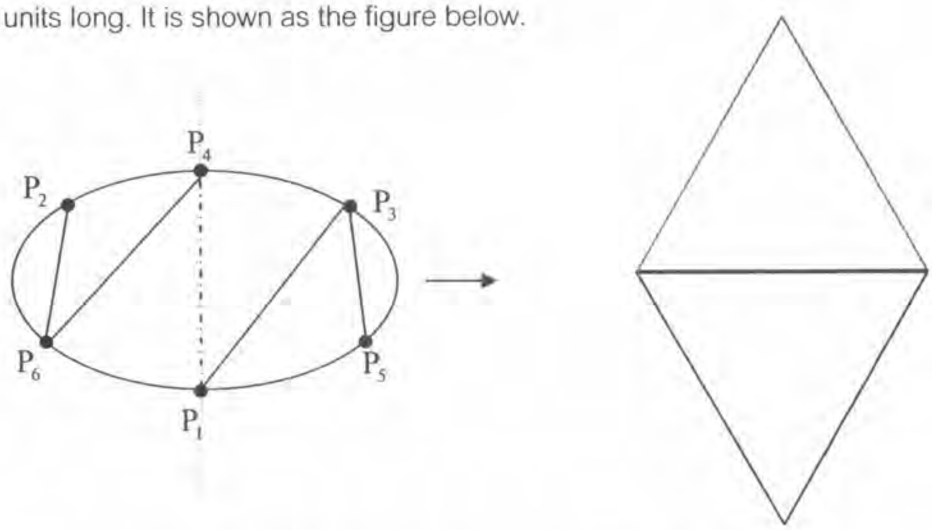
Subcase 2.5 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below



Subcase 2.5.1 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$ and the **condition T1** is satisfied.

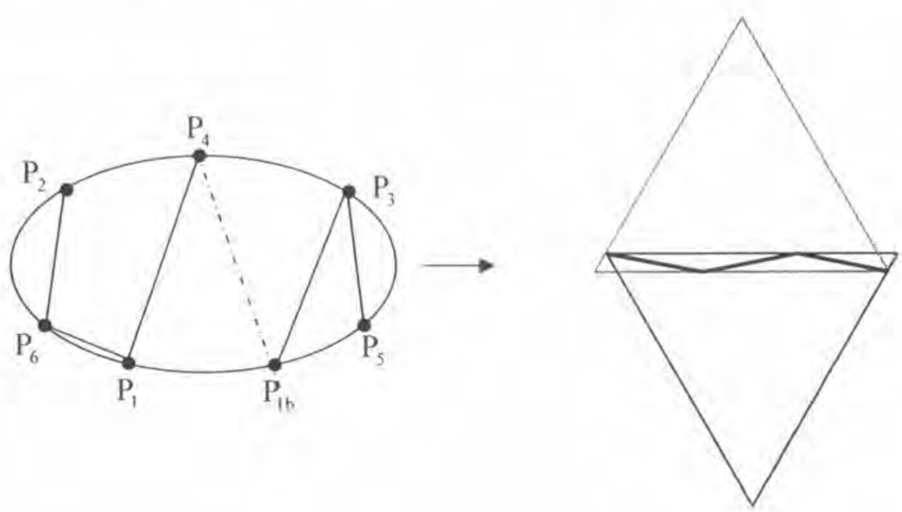
By using the numerical minimization, the shortest polysegment is approximately 1.00054 units long. It is shown as the figure below.



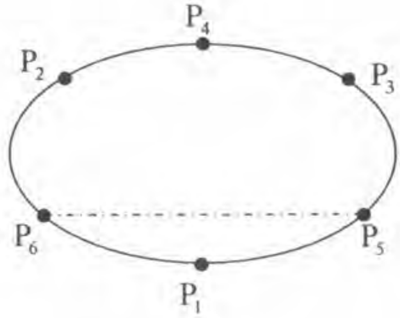
Subcase 2.5.1 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



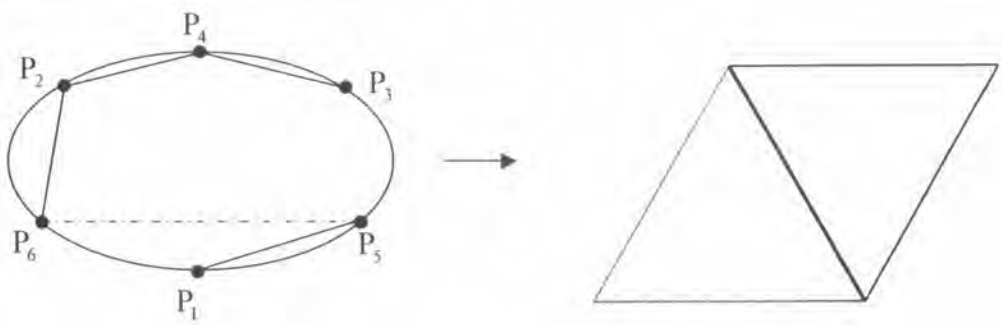
Case 3 P_5 and P_6 are connected.



There will be the following cases

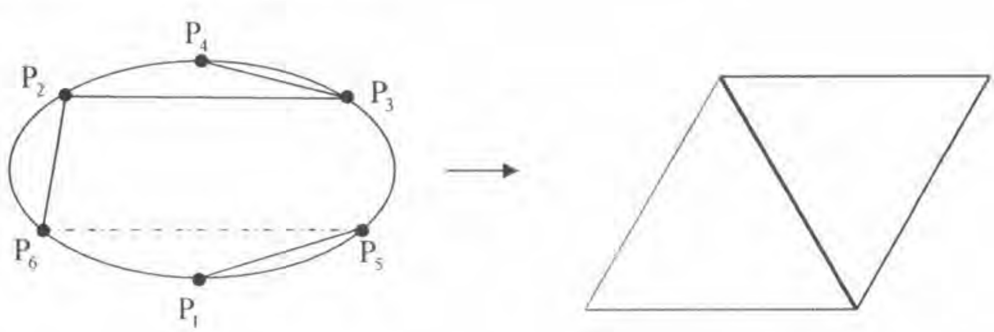
Subcase 3.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

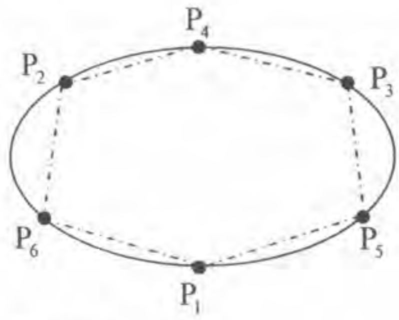


Subcase 3.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$.

By using numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



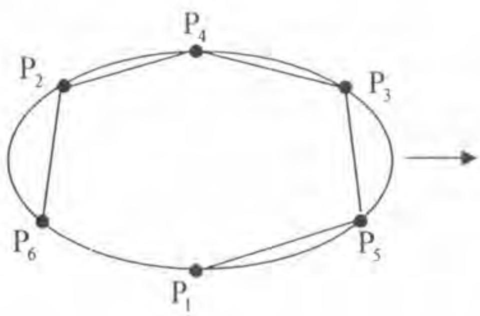
Case 4 Convex arc.

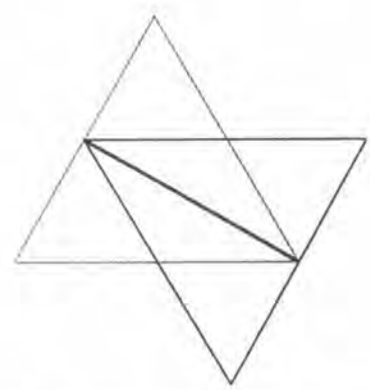


There will be the following cases

Subcase 4.1 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$

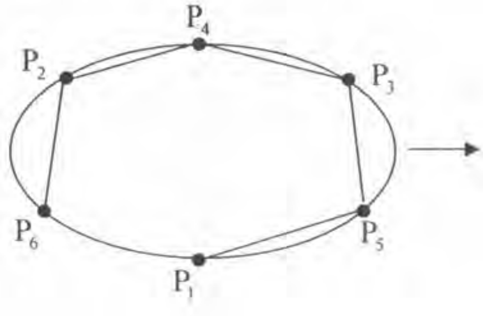
By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.

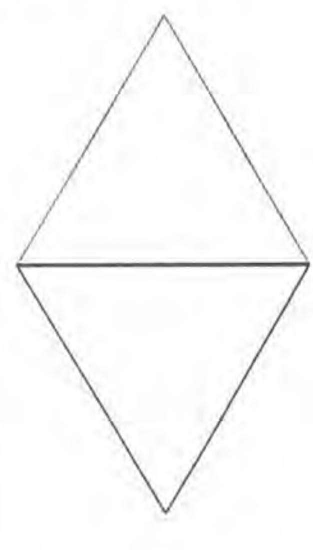




Subcase 4.1.1 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



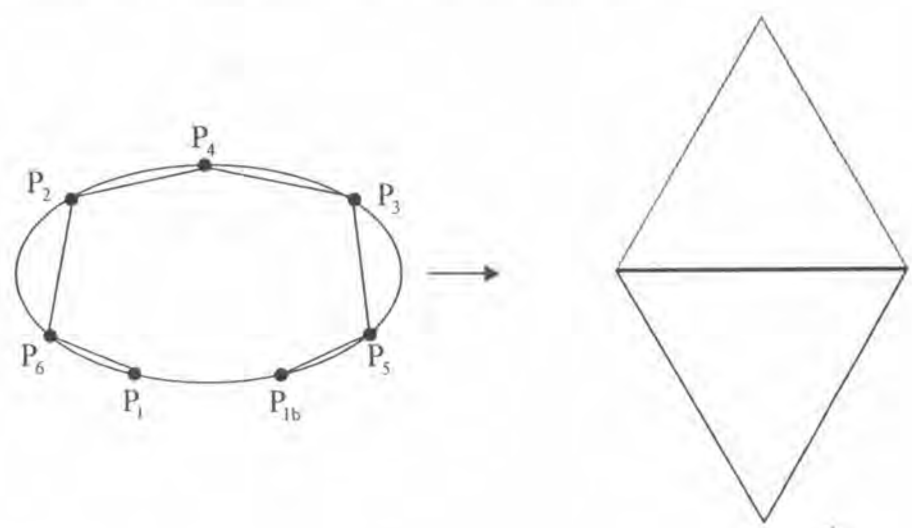


Subcase 4.1.2 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$ and the **condition** T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

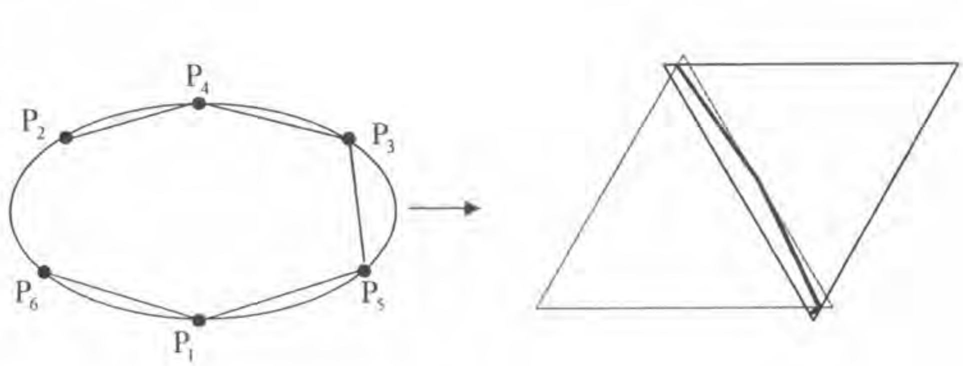
Subcase 4.1.2.1 $\Pi = P_1\ P_6\ P_2\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



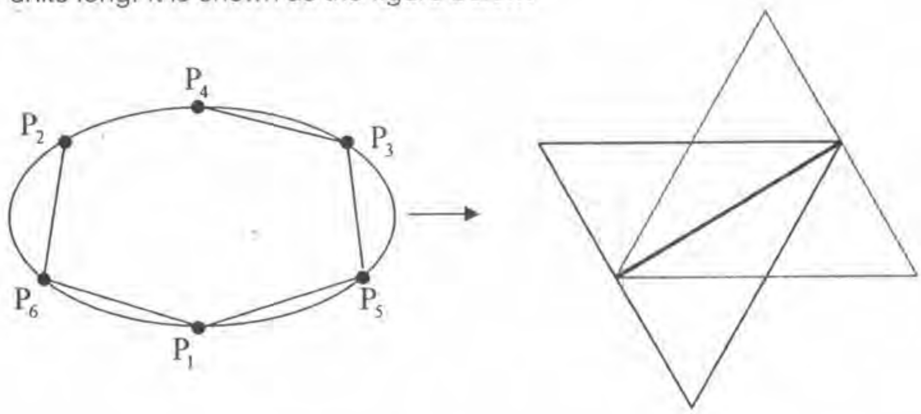
Subcase 4.2 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$

By using numerical minimization, the shortest polysegment is approximately 0.997131 units long. It is shown as the figure below.



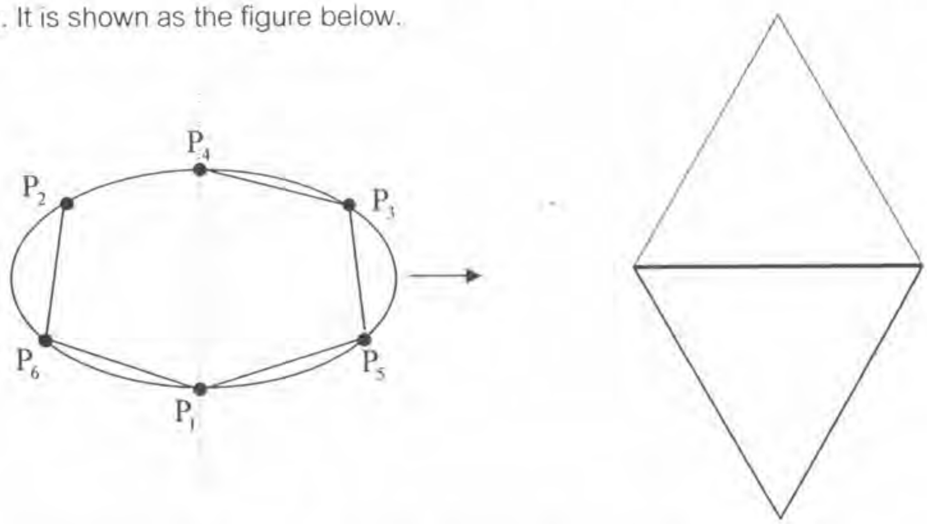
Subcase 4.3 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



Subcase 4.3.1 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

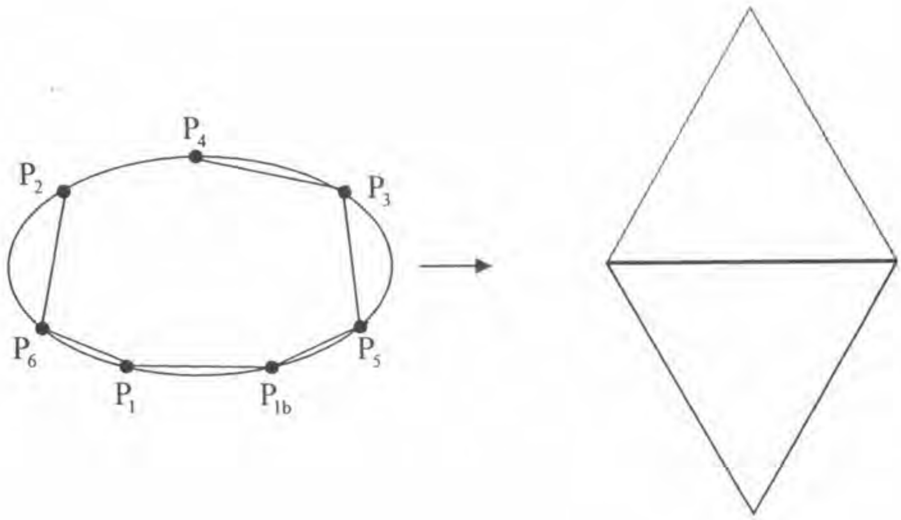


Subcase 4.3.2 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1.Thus, this subcase can be split into the following subcases.

Subcase 4.3.2.1 $\Pi = P_4\ P_3\ P_5\ P_{1b}\ P_1\ P_6\ P_2$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



4.2 isosceles Right angled Triangle

The aim is to show that an isosceles right angled triangle with 1 unit hypotenuse can cover every arc of length $\ell_1 = 0.948683$. To show this we suppose that an arc γ cannot be covered. Then we will show that the length of γ is greater than ℓ_1. Thus, γ must not be covered by the triangle in the standing position and its reflection.

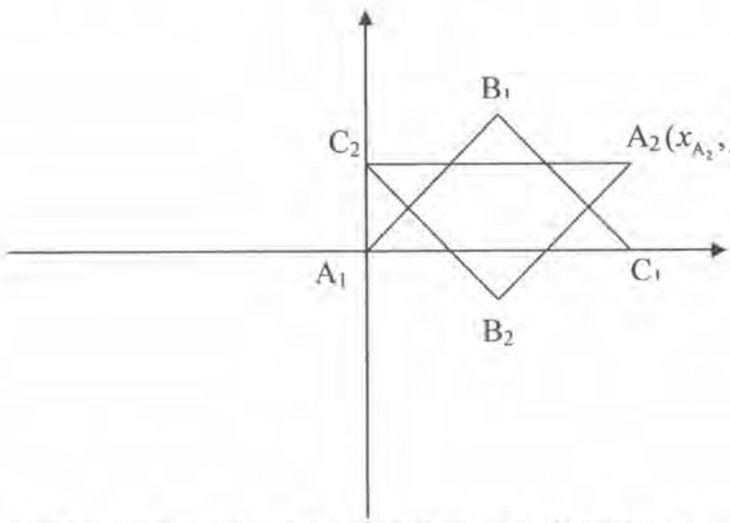
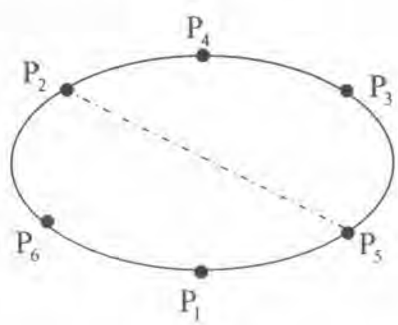


Figure 4.2 : An isosceles right-angled triangle $A_1B_1C_1$ in standing position and its reflection $A_2B_2C_2$

Let $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_4(x_4,y_4)$,and $P_5(x_5,y_5)$ be points on the side $\overline{A_1C_1}$, $\overline{A_1B_1}$ $\overline{A_2C_2}$ and $\overline{A_2B_2}$ of the triangle that touches γ, respectively.

Let $P_3(x_3,y_3)$ and $P_6(x_6,y_6)$ be points which are not in the triangles or on the side $\overline{B_1C_1}$ and $\overline{C_2B_2}$, respectively.

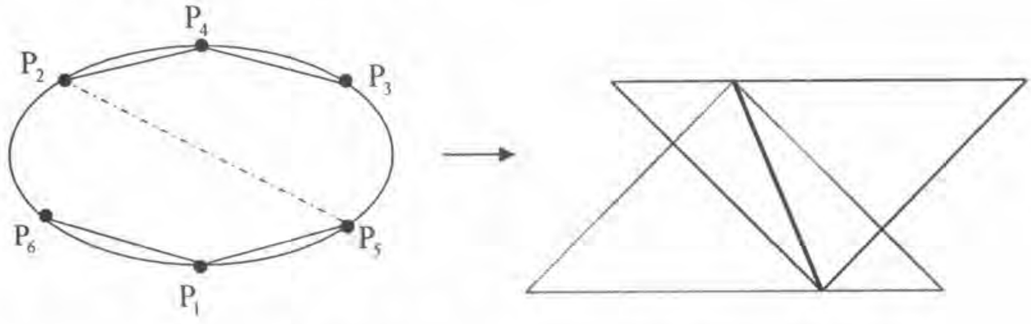
Case 1 P_2 and P_5 are connected.



There will be the following cases

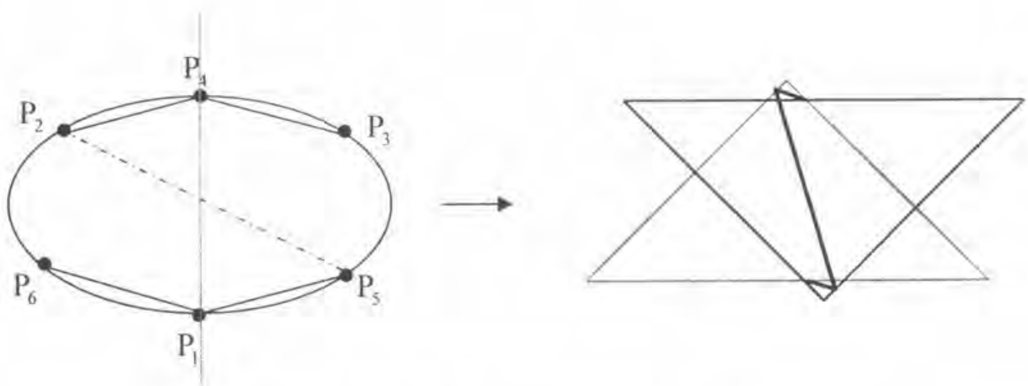
Subcase 1.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.



Subcase 1.1.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.

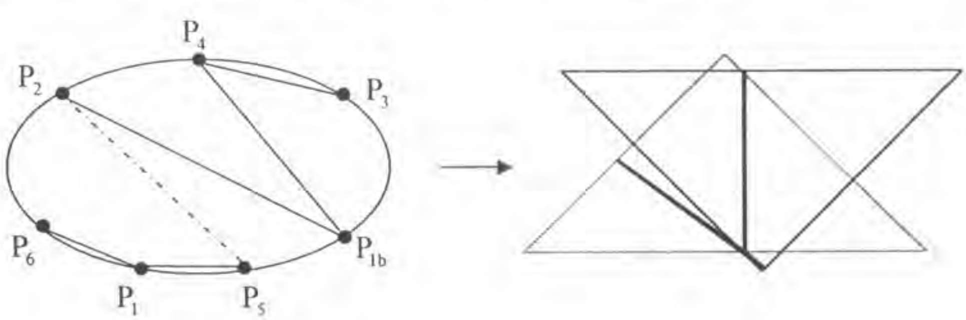


Subcase 1.1.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

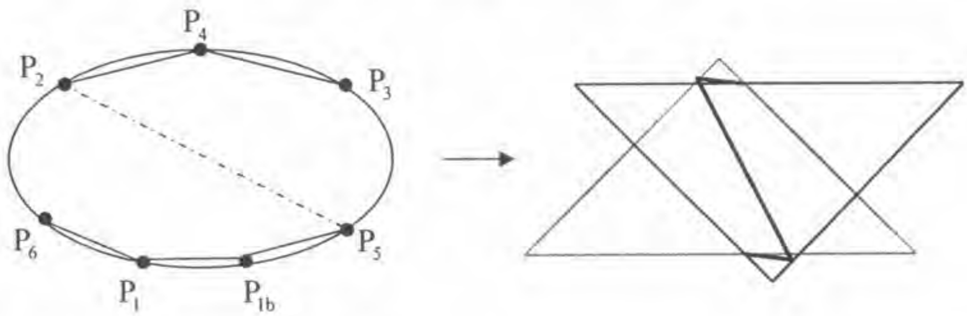
Subcase 1.1.2.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_{1b}\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1.36705 units long. It is shown as the figure below.



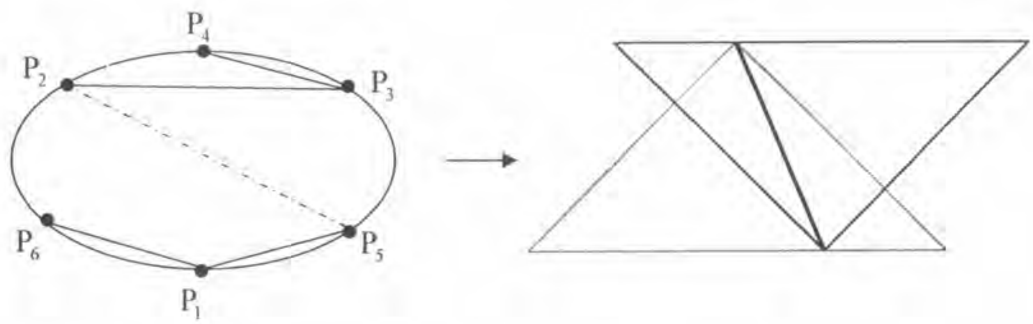
Subcase 1.1.2.2 $\Pi = P_6\ P_1\ P_{1b}\ P_5\ P_2\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.754716 units long. It is shown as the figure below.



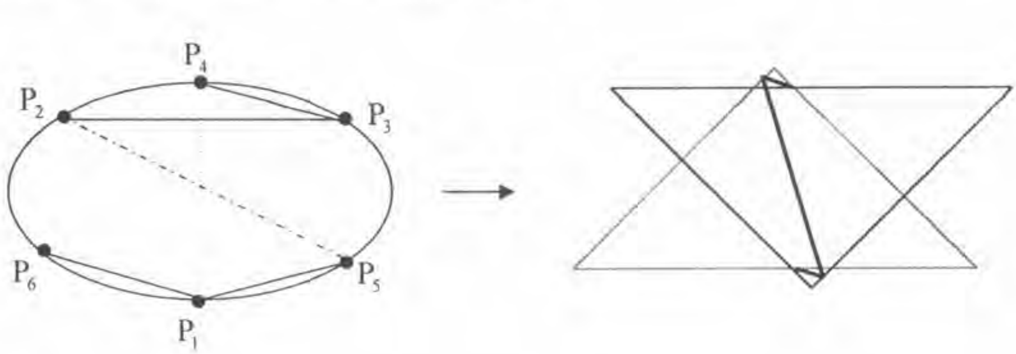
Subcase 1.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$.

By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.



Subcase 1.2.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.

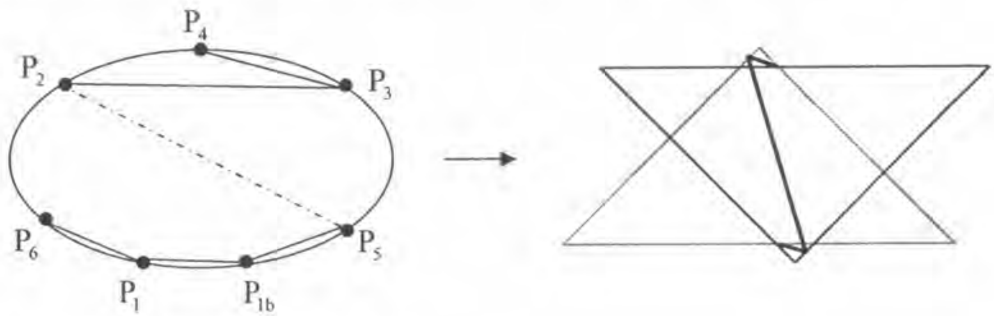


Subcase 1.2.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$ and the **condition** T2 is satisfied.

There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

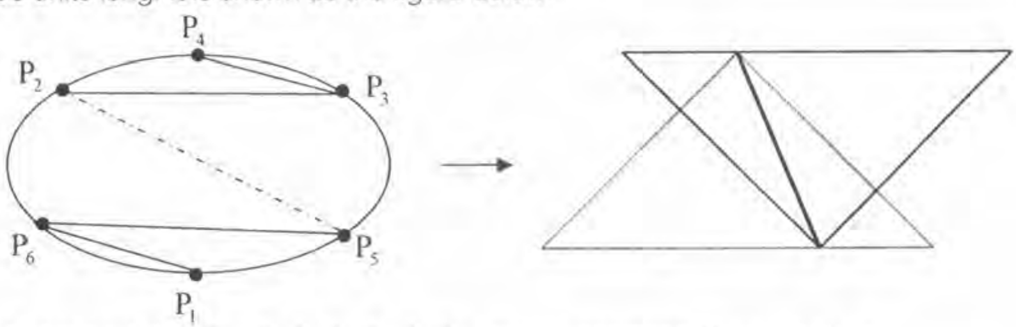
Subcase 1.2.2.1 $\Pi = P_6\ P_1\ P_{1b}\ P_5\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.



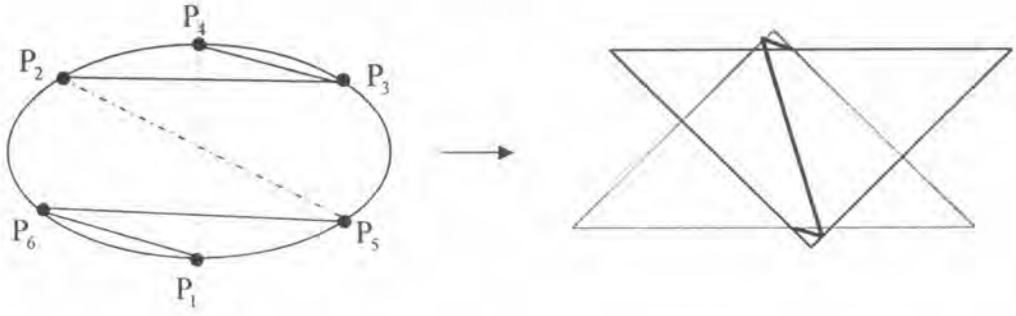
Subcase 1.3 $\Pi = P_1\ P_6\ P_5\ P_2\ P_3\ P_4$

By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.



Subcase 1.3.1 $\Pi = P_1\ P_6\ P_5\ P_2\ P_3\ P_4$ and the **condition** T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.

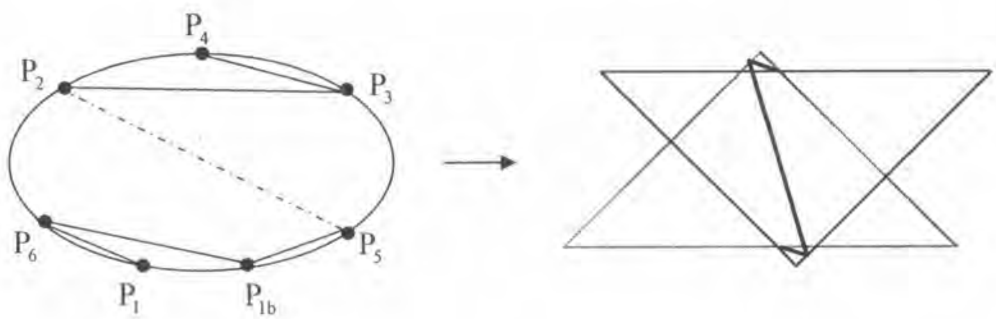


Subcase 1.3.2 $\Pi = P_1\ P_6\ P_5\ P_2\ P_3\ P_4$ and the **condition** T2 is satisfied.

There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

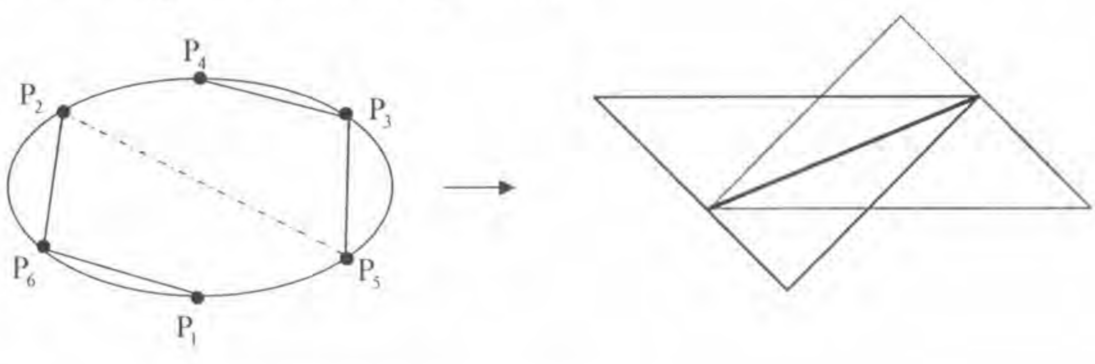
Subcase 1.3.2.1 $\Pi = P_1\ P_6\ P_{1b}\ P_5\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.667673 units long. It is shown as the figure below.



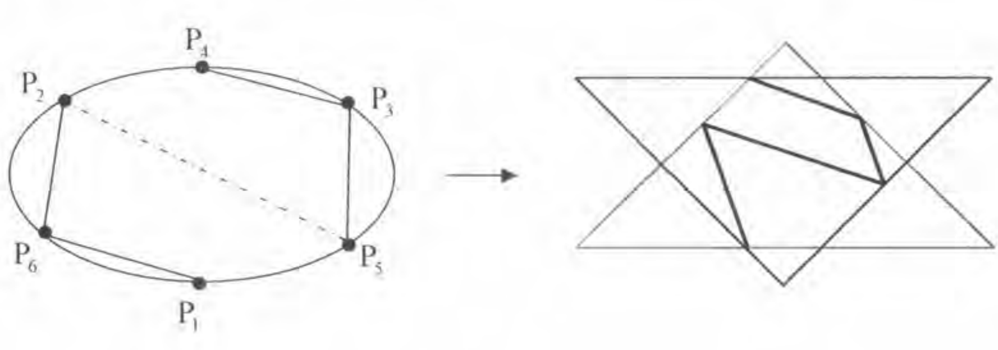
Subcase 1.4 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$

By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.



Subcase 1.4.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$ and the **condition** T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.24259 units long. It is shown as the figure below.

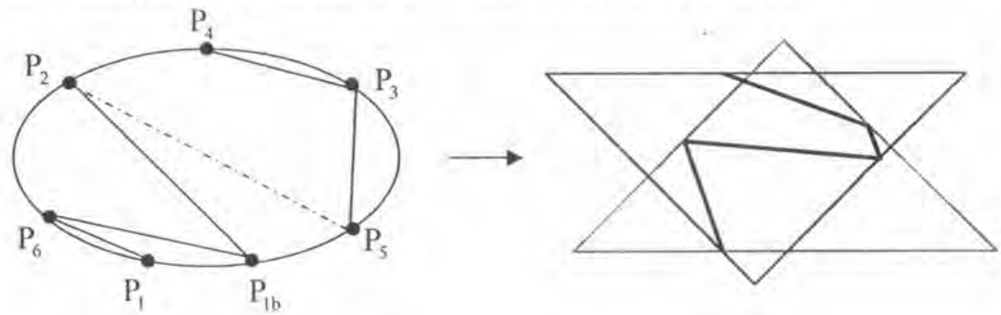


Subcase 1.4.2 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$ and the condition T2 is satisfied.

There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

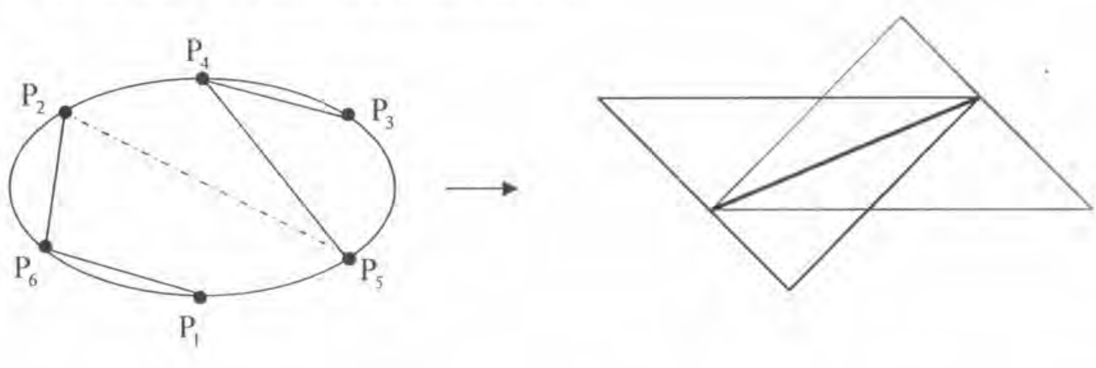
Subcase 1.4.2.1 $\Pi = P_1\ P_6\ P_{1b}\ P_2\ P_5\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 1.19617 units long. It is shown as the figure below.



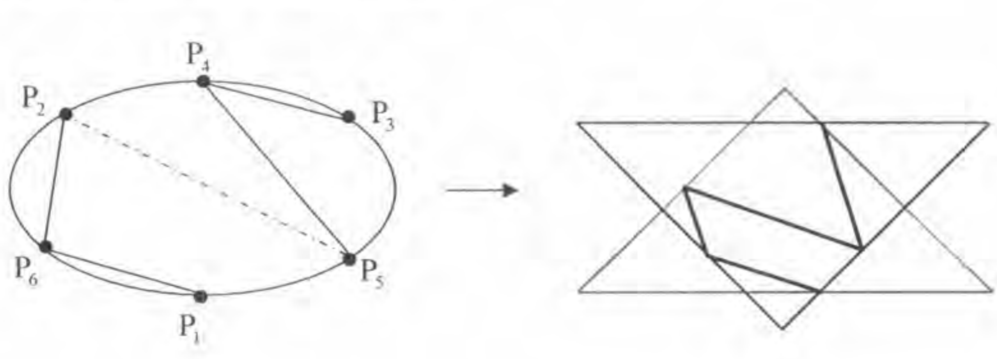
Subcase 1.5 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.



Subcase 1.5.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.24259 units long. It is shown as the figure below.

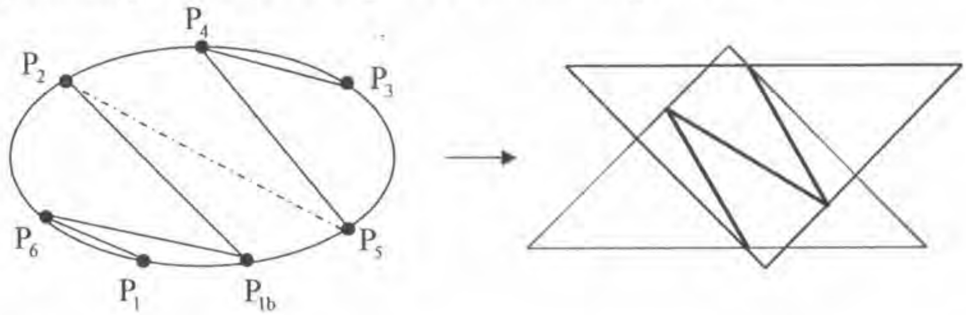


Subcase 1.5.2 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

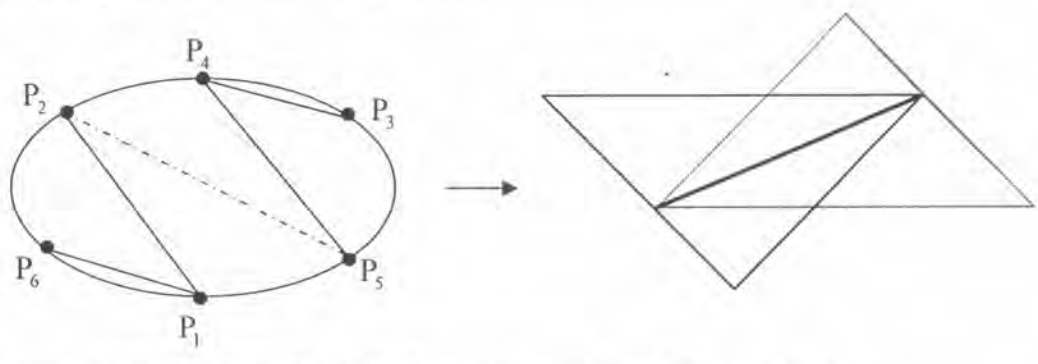
Subcase 1.5.2.1 $\Pi = P_1\ P_6\ P_{1b}\ P_2\ P_5\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1.26769 units long. It is shown as the figure below.



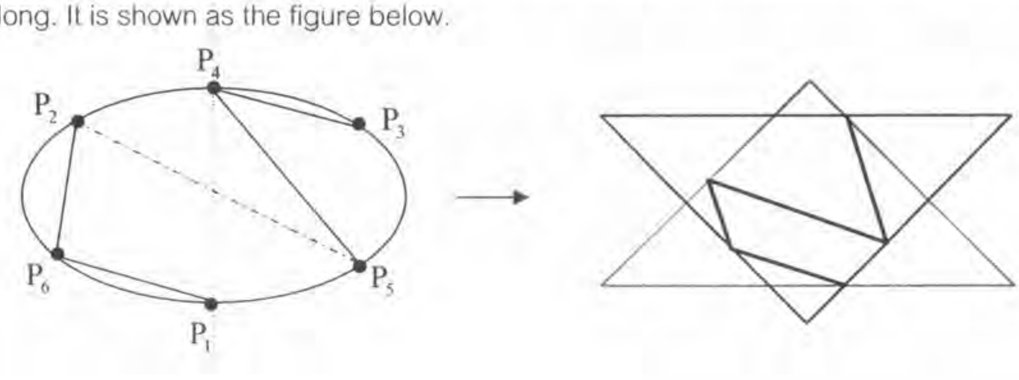
Subcase 1.6 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.



Subcase 1.6.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.24259 units long. It is shown as the figure below.

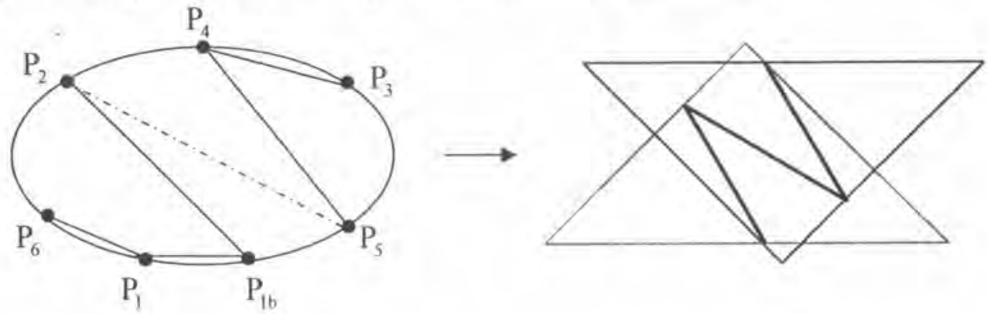


Subcase 1.6.2 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$ and the condition T2 is satisfied.

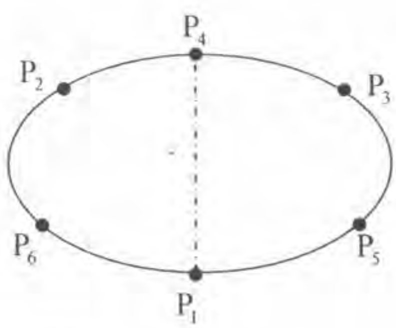
There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

Subcase 1.6.2.1 $\Pi = P_6\ P_1\ P_{1b}\ P_2\ P_5\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1.26770 units long. It is shown as the figure below.



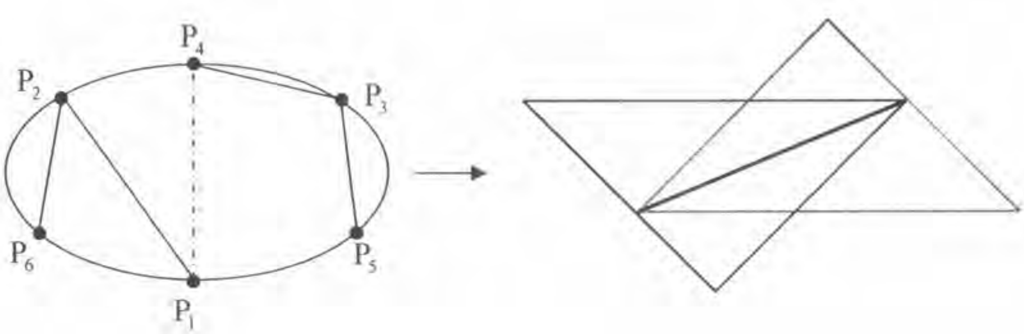
Case 2 P_1 and P_4 are connected.



There will be the following cases

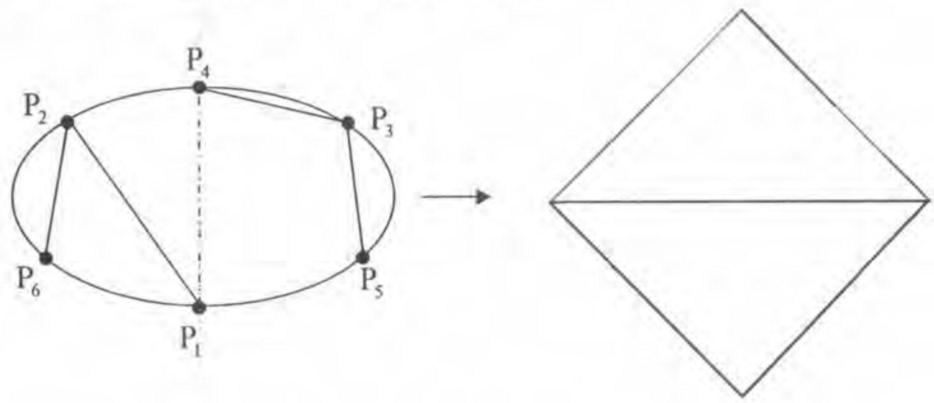
Subcase 2.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$.

By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.



Subcase 2.1.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.999999 units long. It is shown as the figure below.

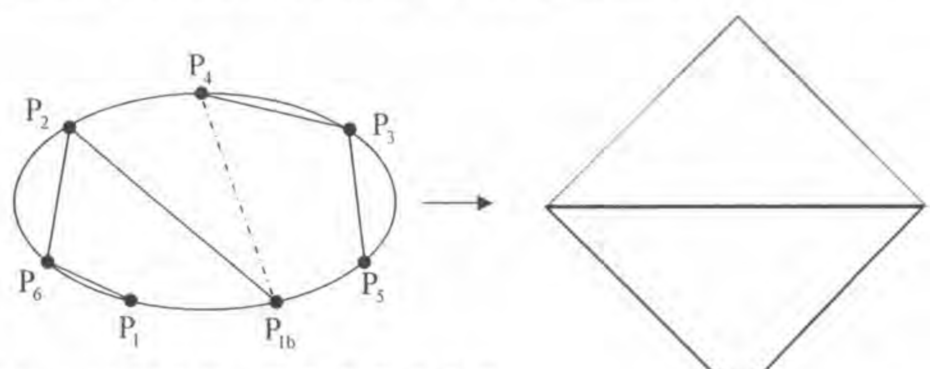


Subcase 2.1.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$ and the **condition** T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

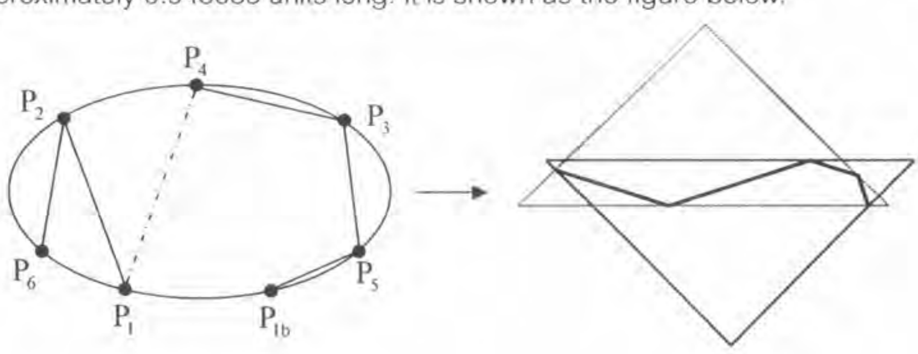
Subcase 3.1.2.1 $\Pi = P_1\ P_6\ P_2\ P_{1b}\ P_4\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



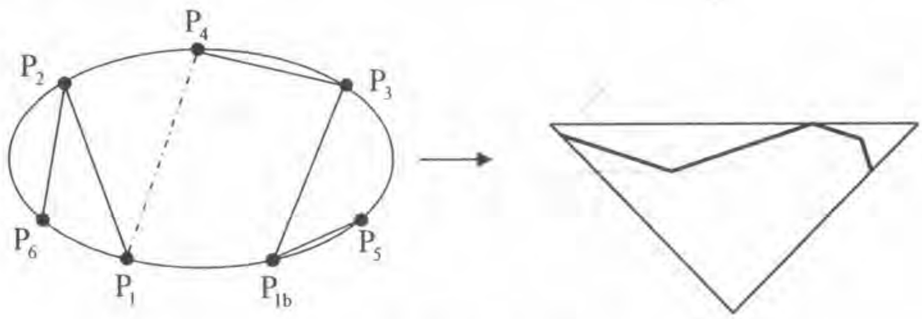
Subcase 2.1.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.



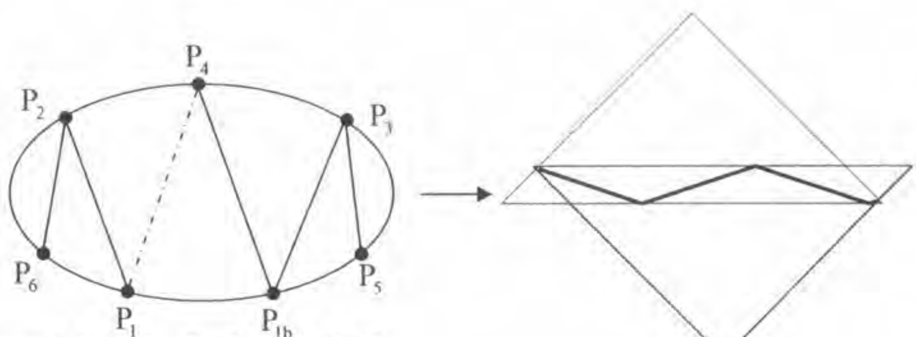
Subcase 2.1.2.3 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.



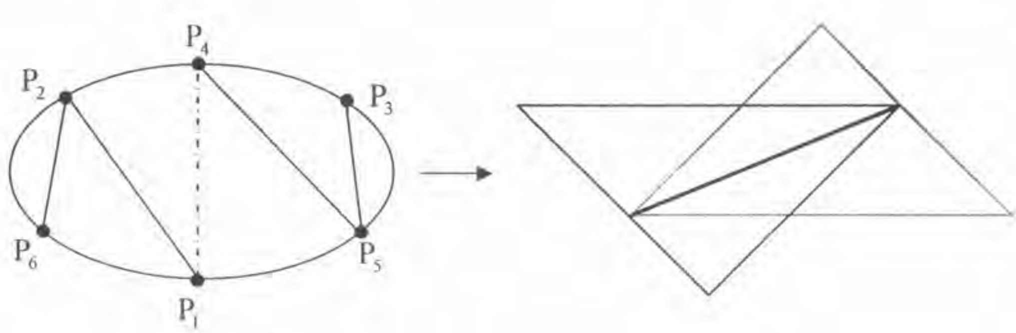
Subcase 2.1.2.4 $\Pi = P_6\ P_2\ P_1\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0. 948683 units long. It is shown as the figure below.



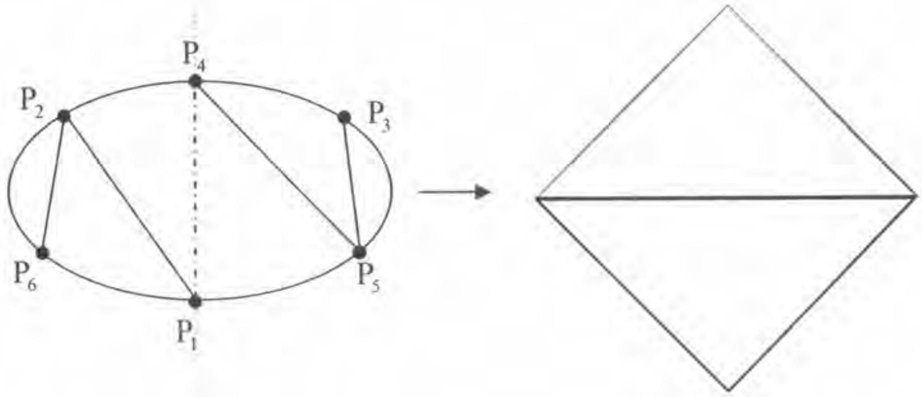
Subcase 2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.765367units long. It is shown as the figure below



Subcase 2.2.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$ and the **condition** T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

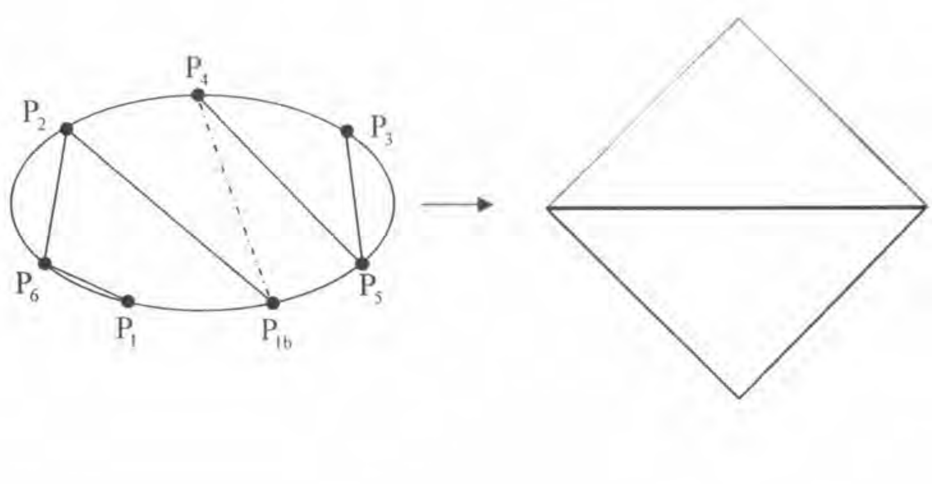


Subcase 2.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$ and the **condition** T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

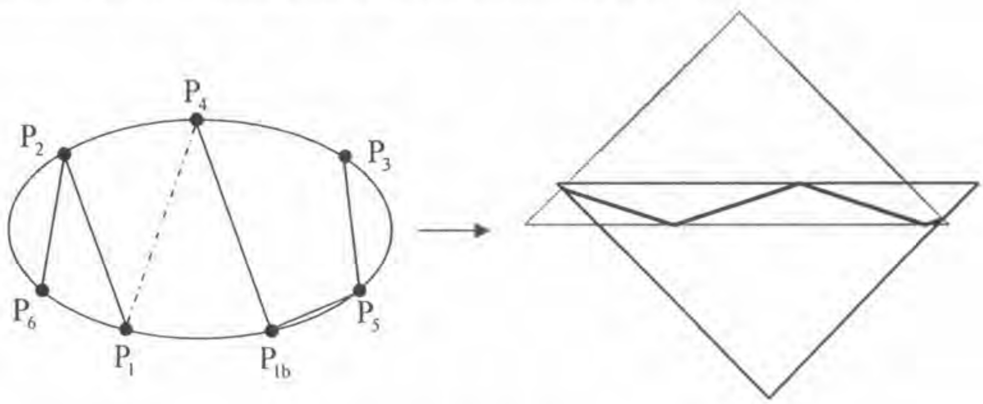
Subcase 2.2.2.1 $\Pi = P_1\ P_6\ P_2\ P_{1b}\ P_4\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



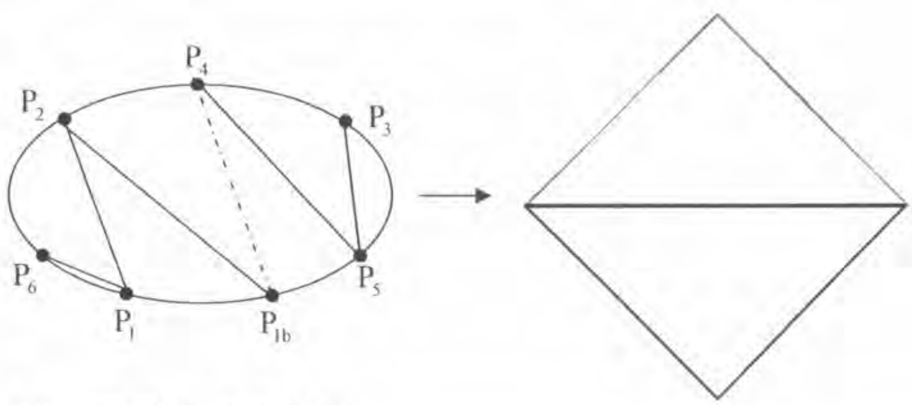
Subcase 2.2.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_{1b}\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0. 948683 units long. It is shown as the figure below.



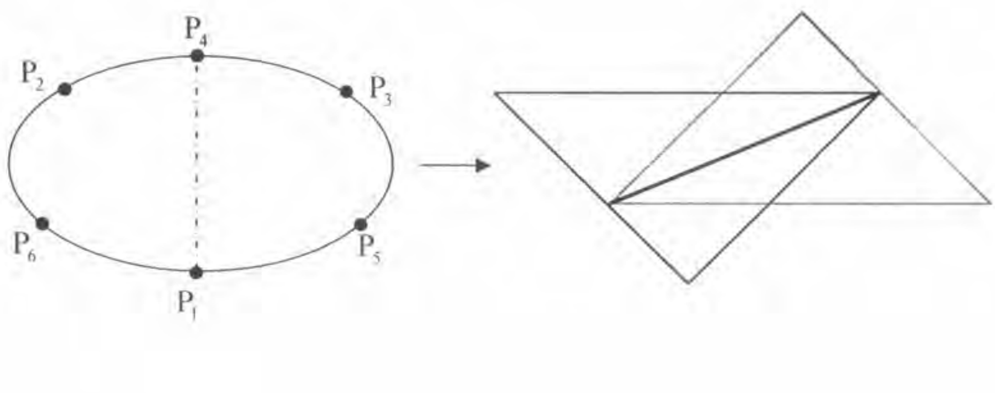
Subcase 2.2.2.3 $\Pi = P_6\ P_1\ P_2\ P_{1b}\ P_4\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



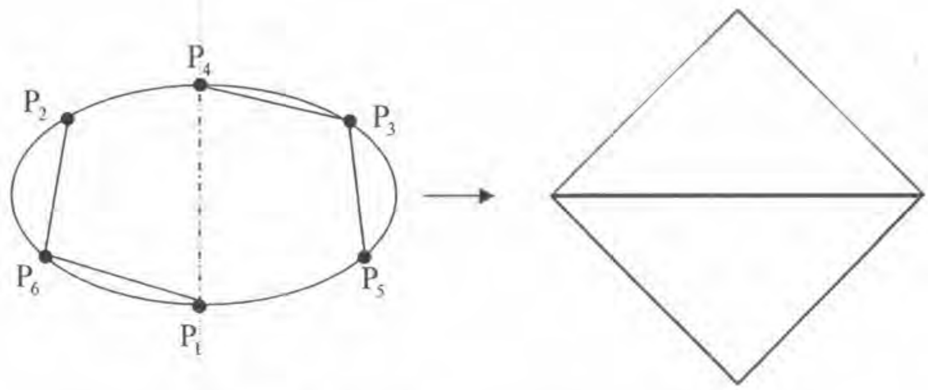
Subcase 2.3 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5$.

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below



Subcase 2.3.1 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

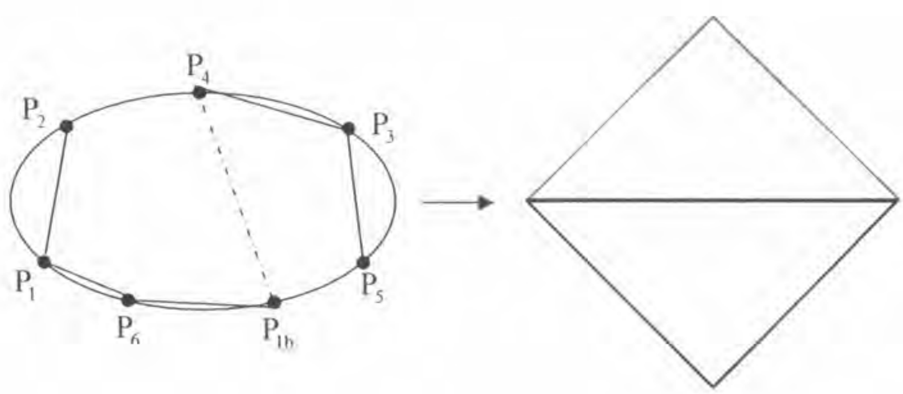


Subcase 2.3.2 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

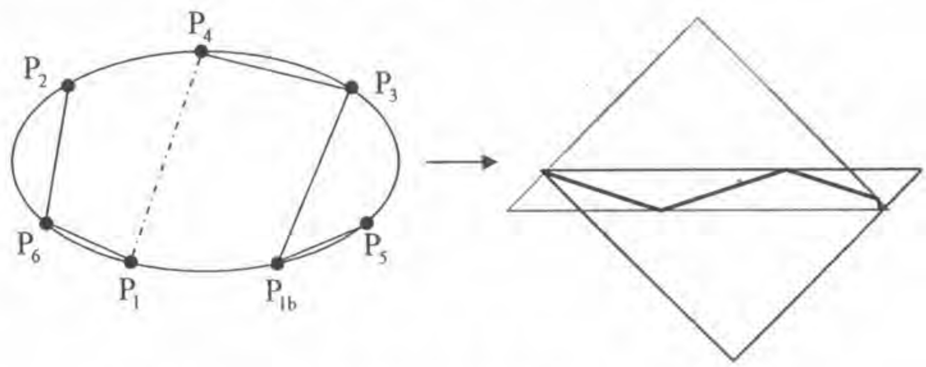
Subcase 2.3.2.1 $\Pi = P_2\ P_1\ P_6\ P_{1b}\ P_4\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



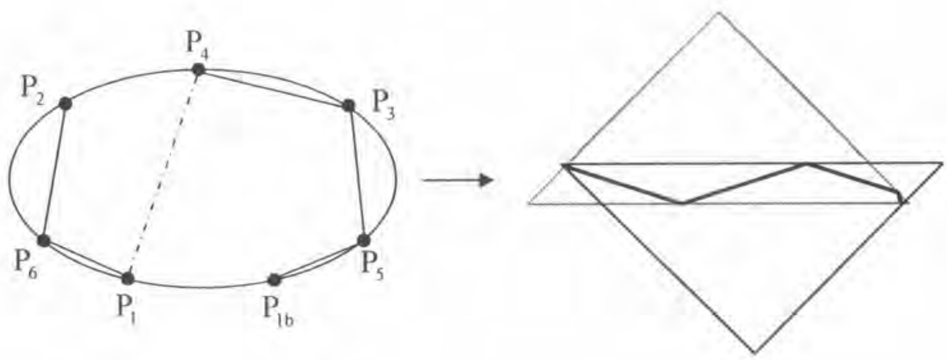
Subcase 2.3.2.2 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.



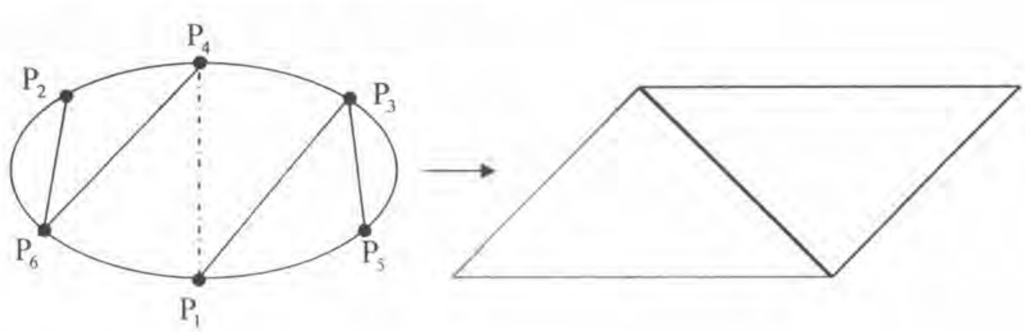
Subcase 2.3.2.3 $\Pi = P_2\ P_6\ P_1\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.



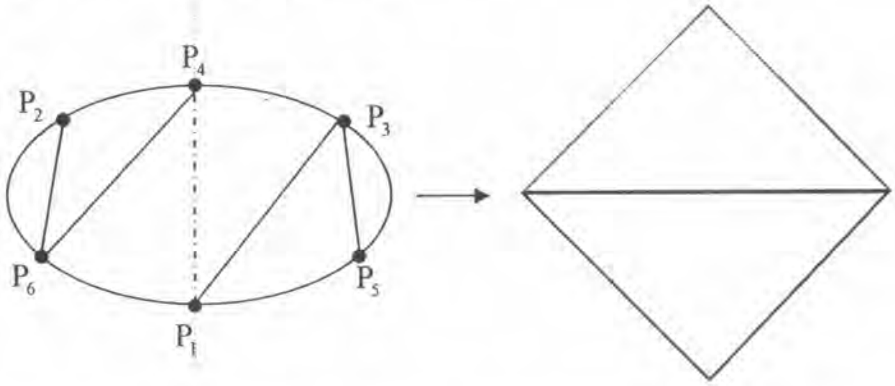
Subcase 2.4 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$

By using numerical minimization, the shortest polysegment is approximately 0.707107 units long. It is shown as the figure below



Subcase 3.9.1 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

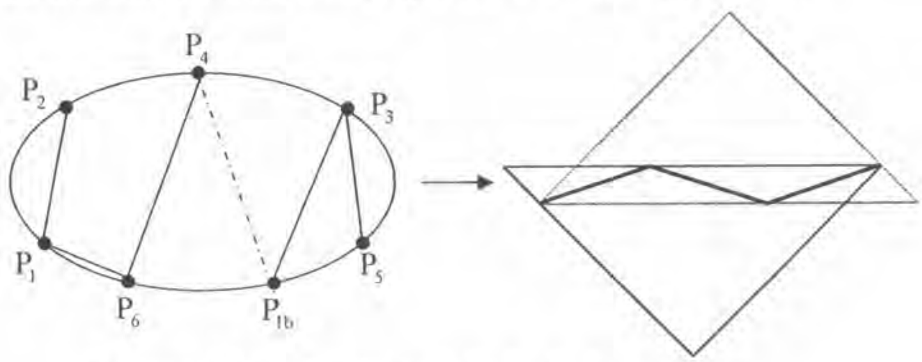


Subcase 2.4.2 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$ and the condition T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

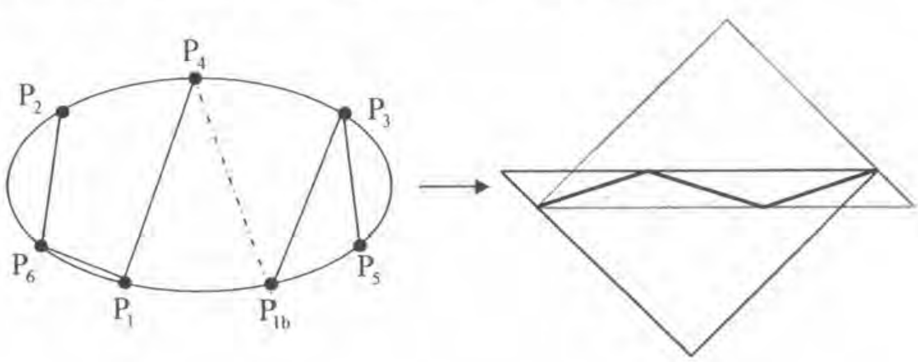
Subcase 2.4.2.1 $\Pi = P_2\ P_1\ P_6\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.948683 units long. It is shown as the figure below.

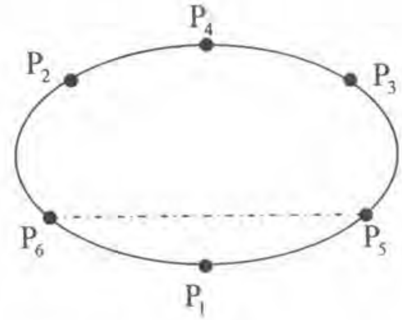


Subcase 2.4.2.2 $\Pi = P_2\ P_6\ P_1\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.981981 units long. It is shown as the figure below.



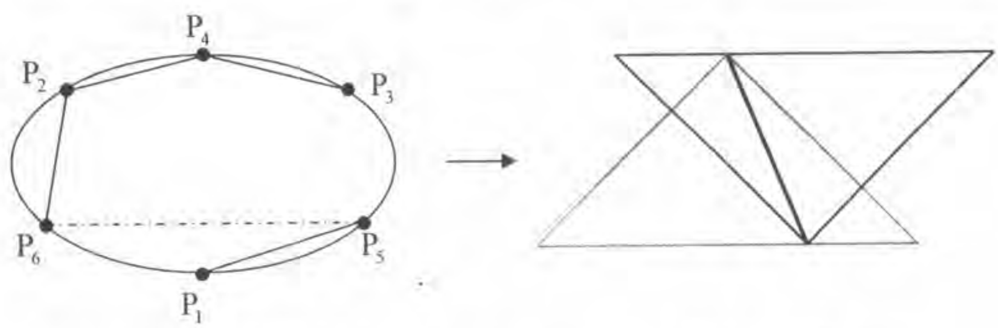
Case 3 P_5 and P_6 are connected.



There will be the following cases

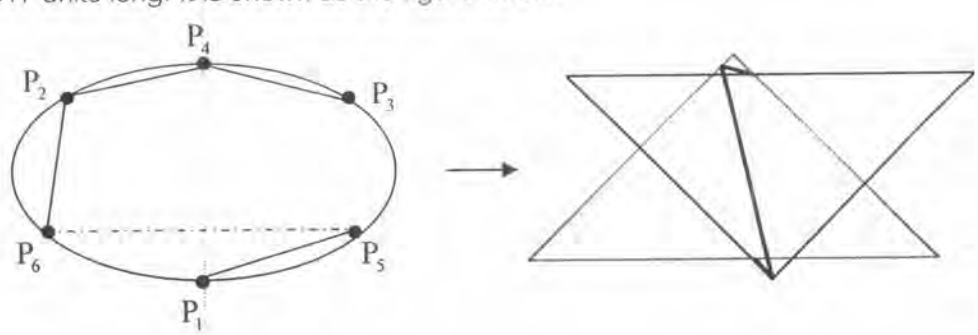
Subcase 3.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.



Subcase 3.1.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.676617 units long. It is shown as the figure below.

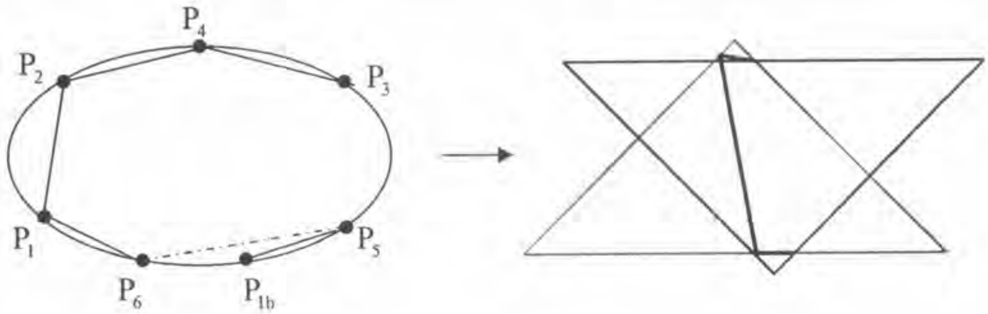


Subcase 3.1.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

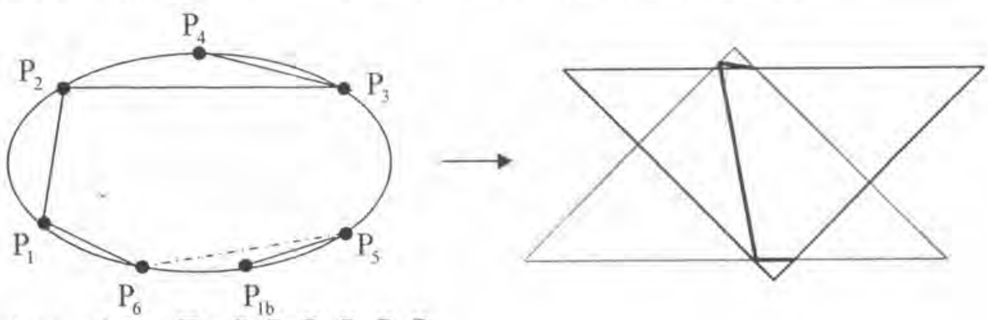
Subcase 3.1.2.1 $\Pi = P_{1b}\ P_5\ P_6\ P_1\ P_2\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.648746 units long. It is shown as the figure below.



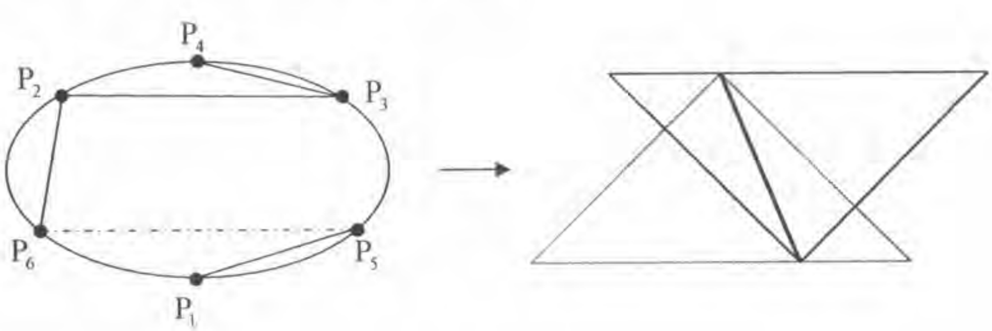
Subcase 3.1.2.2 $\Pi = P_{1b}\ P_5\ P_6\ P_1\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.648746 units long. It is shown as the figure below.



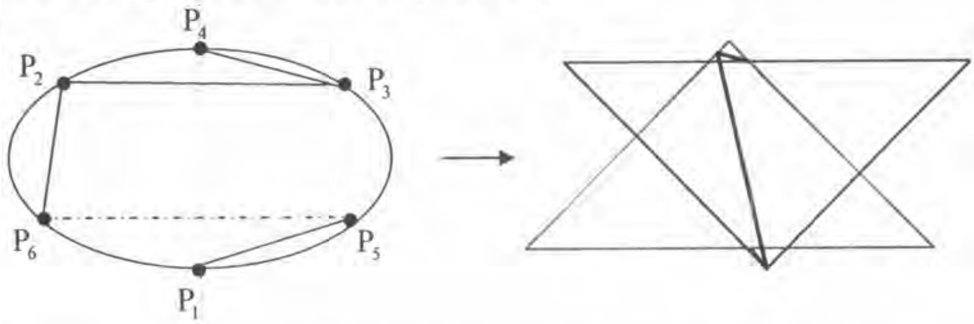
Subcase 3.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$.

By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.



Subcase 3.2.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.676617 units long. It is shown as the figure below.

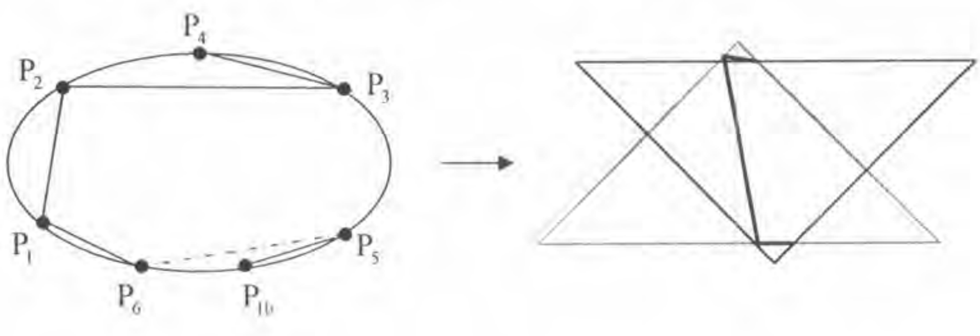


Subcase 3.2.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$ and the **condition T2** is satisfied.

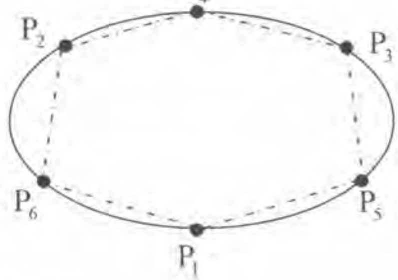
There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

Subcase 3.2.2.1 $\Pi = P_{1b}\ P_5\ P_6\ P_1\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.648747 units long. It is shown as the figure below.



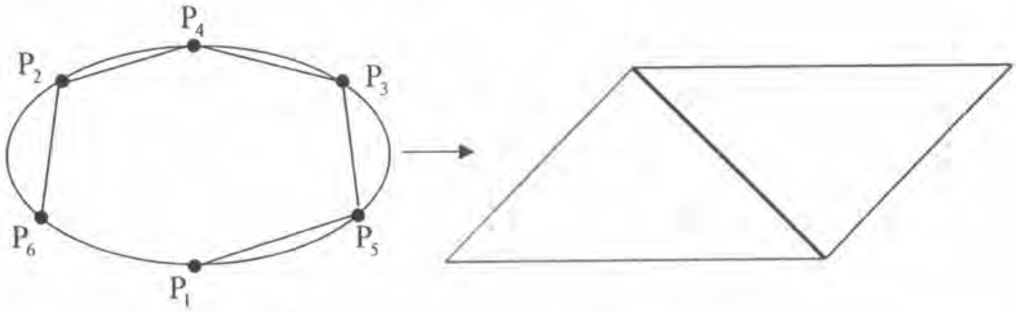
Case 4 Convex arc.



There will be the following cases

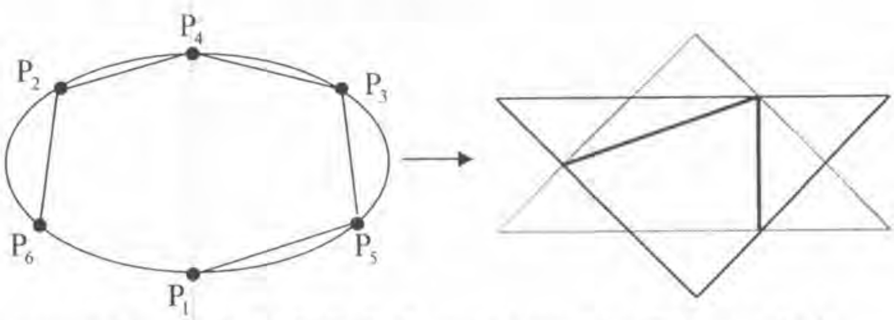
Subcase 4.1 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$.

By using numerical minimization, the shortest polysegment is approximately 0.707107 units long. It is shown as the figure below.



Subcase 4.1.1 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.859389 units long. It is shown as the figure below.

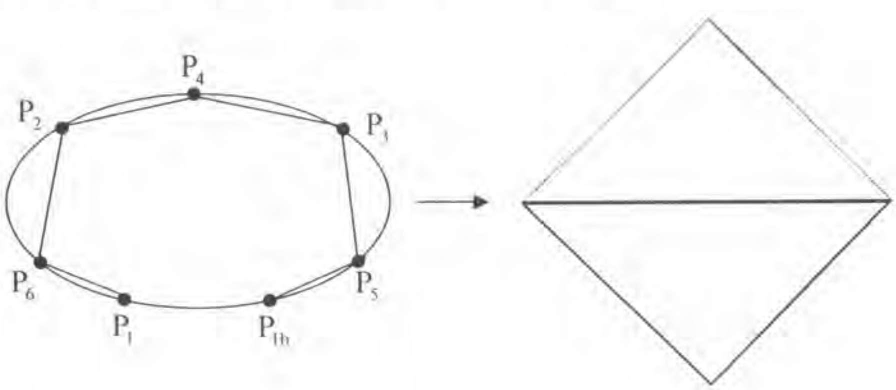


Subcase 4.1.2 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$ and the condition T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

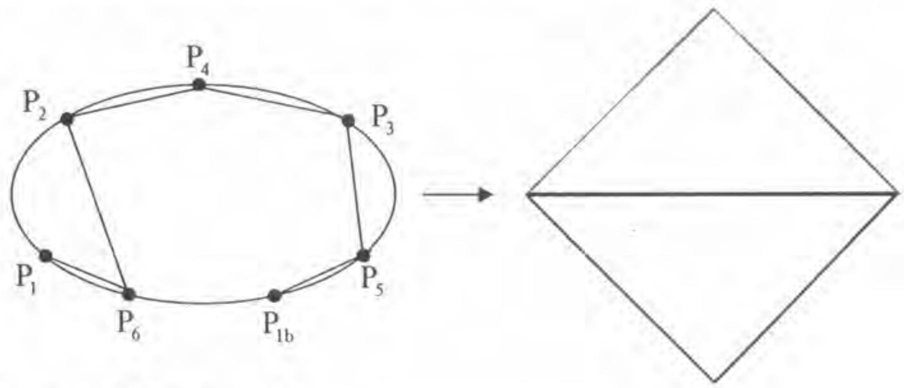
Subcase 4.1.2.1 $\Pi = P_1\ P_6\ P_2\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



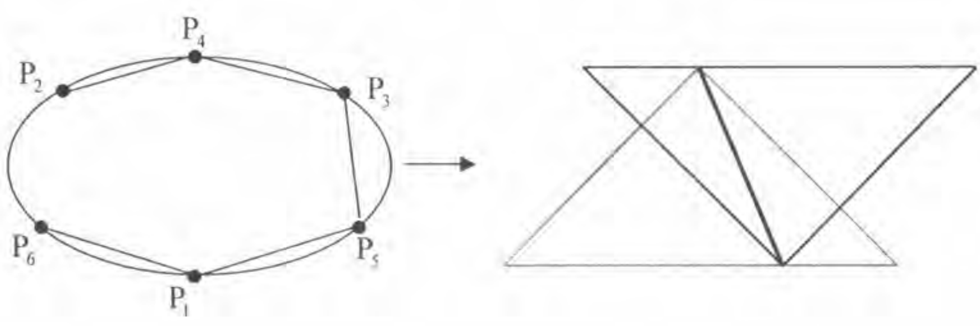
Subcase 4.1.2.2 $\Pi = P_1\ P_6\ P_2\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



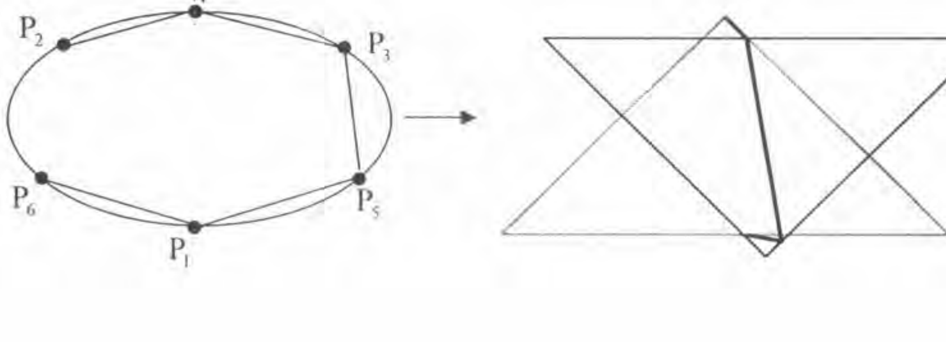
Subcase 4.2 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$

By using numerical minimization, the shortest polysegment is approximately 0.541196 units long. It is shown as the figure below.



Subcase 4.2.1 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.621262 units long. It is shown as the figure below.

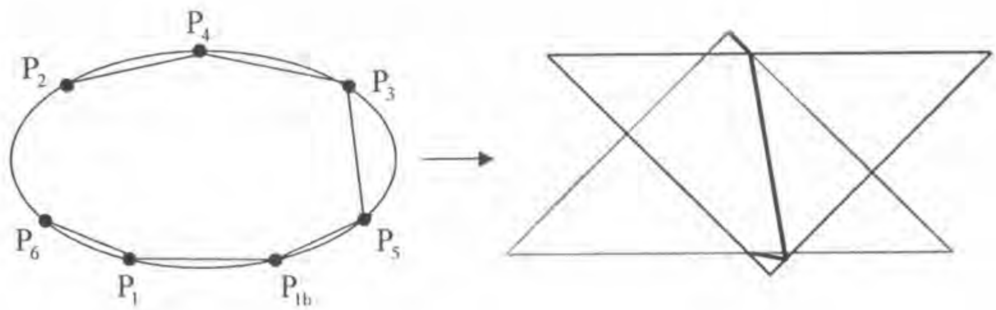


Subcase 4.2.2 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$ and the condition T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

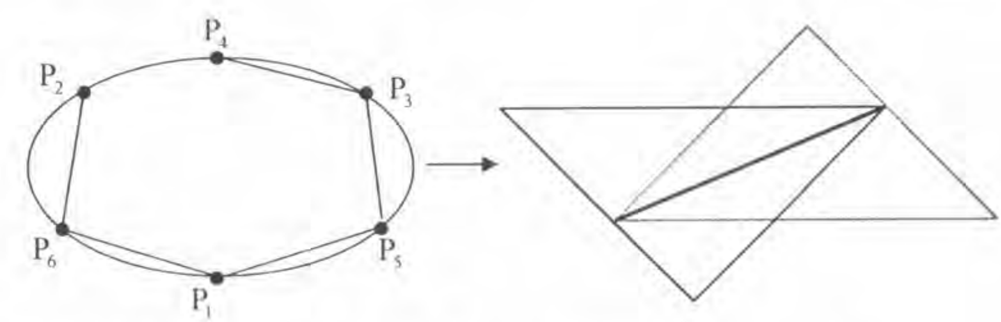
Subcase 4.2.2.1 $\Pi = P_2\ P_4\ P_3\ P_5\ P_{1b}\ \ P_1\ P_6$

By using the numerical minimization, the shortest polysegment is approximately 0.621262 unit long. It is shown as the figure below.



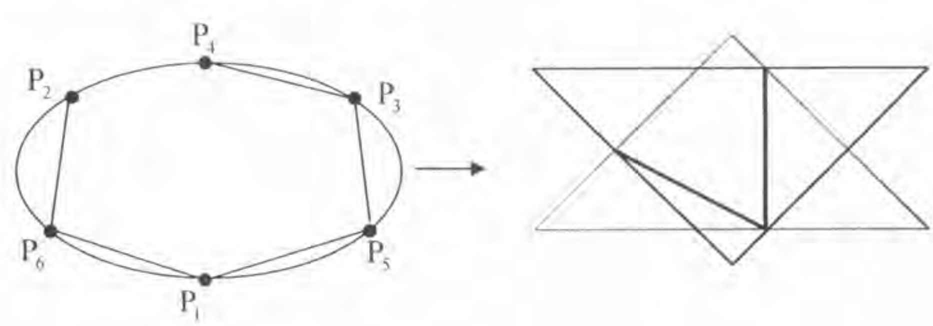
Subcase 4.3 $\quad \Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$

By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.



Subcase 4.3.1 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.845299 units long. It is shown as the figure below.

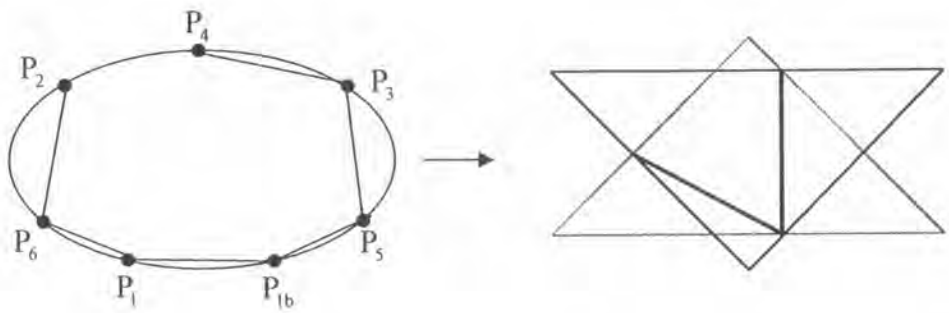


Subcase 4.3.2 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$ and the **condition** T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

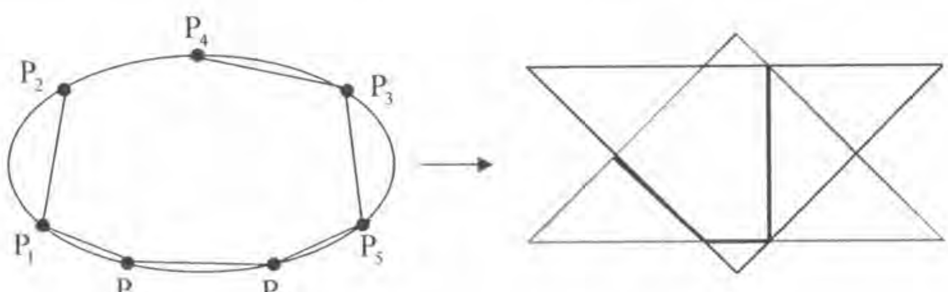
Subcase 4.3.2.1 $\Pi = P_4\ P_3\ P_5\ P_{1b}\ P_1\ P_6\ P_2$

By using the numerical minimization, the shortest polysegment is approximately 0.845299 units long. It is shown as the figure below.



Subcase 4.3.2.2 $\Pi = P_4\ P_3\ P_5\ P_{1b}\ P_6\ P_1\ P_2$

By using the numerical minimization, the shortest polysegment is approximately 0.876003 units long. It is shown as the figure below.



3.3 A 30°- 60°-90° triangle

The objective is to show that a 30°- 60°-90° triangle with 1 unit hypotenuse can cover every arc of length $\ell_1 = \dfrac{9}{(3+4\sqrt{3})} \simeq 0.906508$. To show this we suppose that an arc γ cannot be covered. Then we will show that the length of γ is greater than ℓ_1. Thus, γ must not be covered by the triangle in the standing position and its reflection.

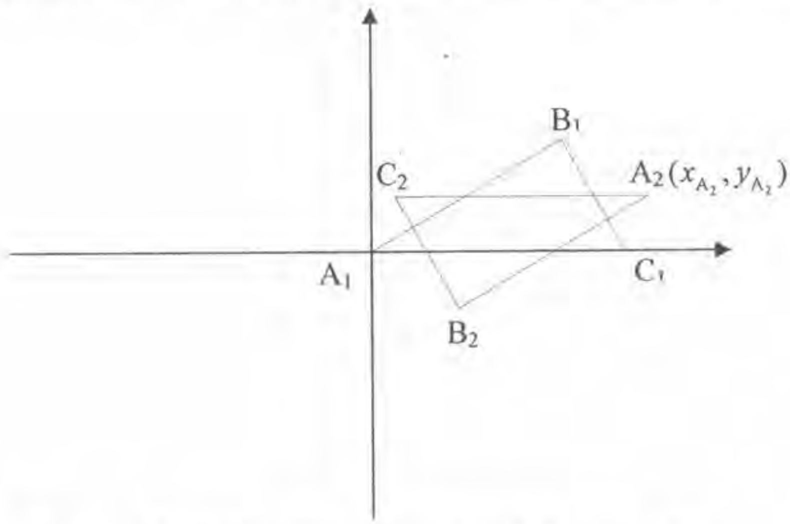


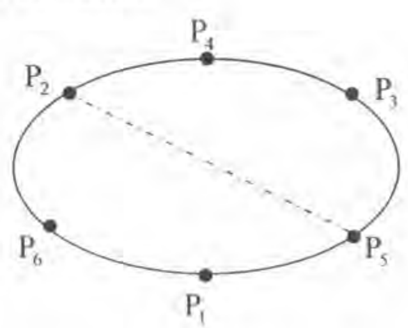
Figure 4.3 : A 30°- 60°-90° triangle $A_1B_1C_1$ in standing position and its reflection $A_2B_2C_2$

Let $P_1(x_1,y_1)$, $P_2(x_2,y_2)$, $P_4(x_4,y_4)$, and $P_5(x_5,y_5)$ be points on the side $\overline{A_1C_1}$, $\overline{A_1B_1}$, $\overline{A_2C_2}$ and $\overline{A_2B_2}$ of the triangle that touches γ, respectively.

Let $P_3(x_3,y_3)$ and $P_6(x_6,y_6)$ be points which are not in the triangles or on the side $\overline{B_1C_1}$ and $\overline{C_2B_2}$, respectively.

Unfortunately, the set is not symmetric. Thus each arc and its reflection have to be checked.

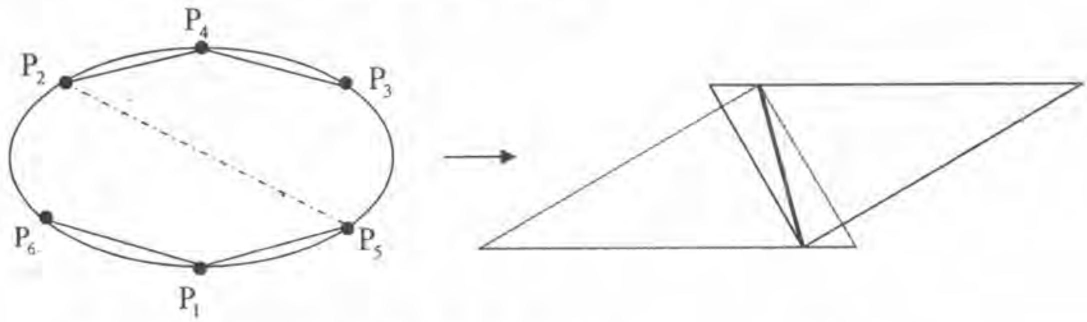
Case 1 P_2 and P_5 are connected.



There will be the following cases

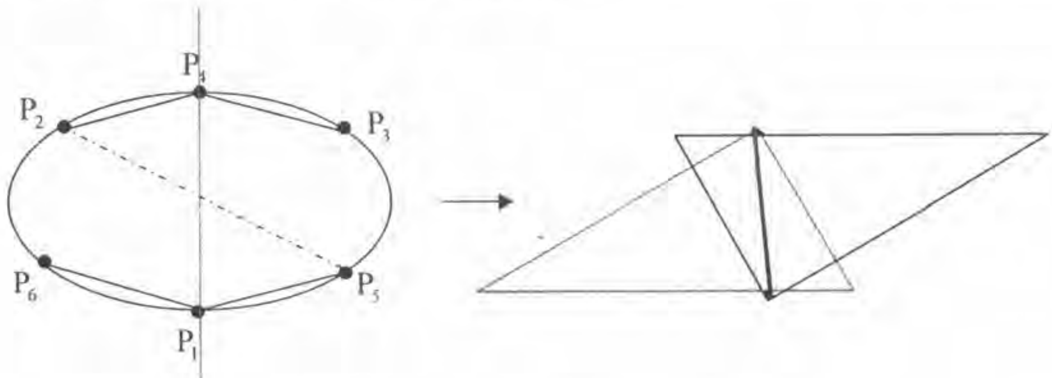
Subcase 1.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.



Subcase 1.1.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.

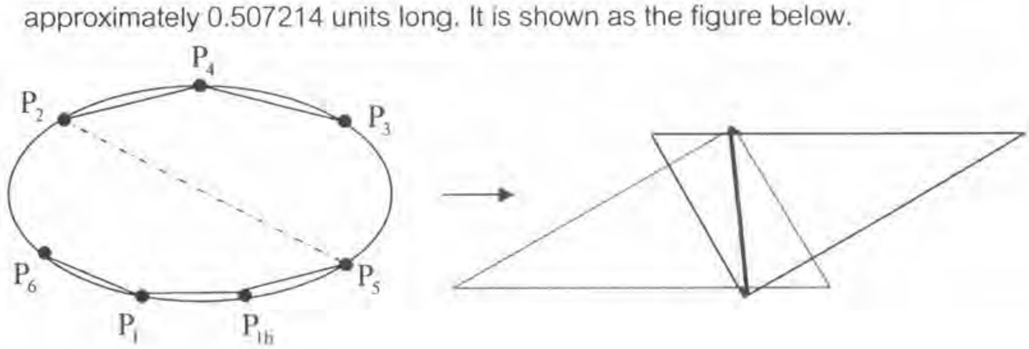


Subcase 1.1.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_4\ P_3$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

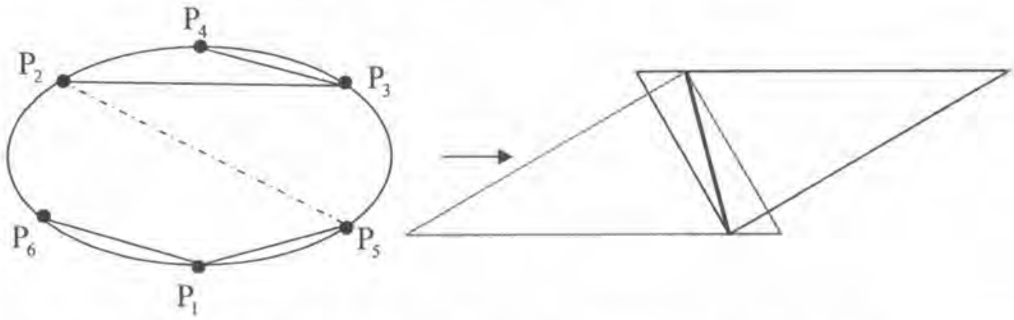
Subcase 1.1.2.4 $\Pi = P_6\ P_1\ P_{1b}\ P_5\ P_2\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.



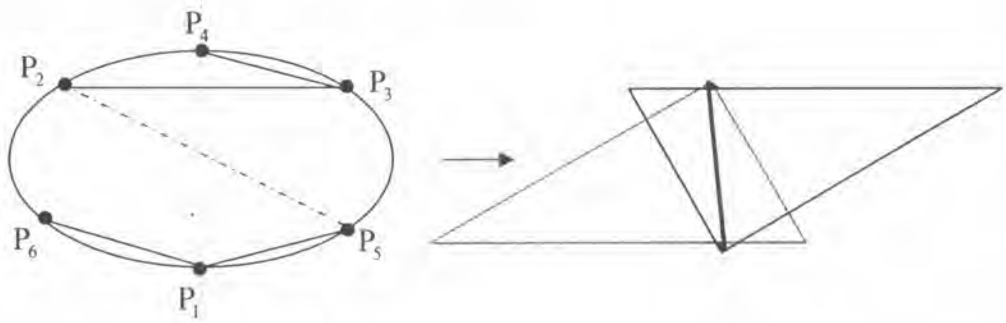
Subcase 1.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$.

By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.



Subcase 1.2.1 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.

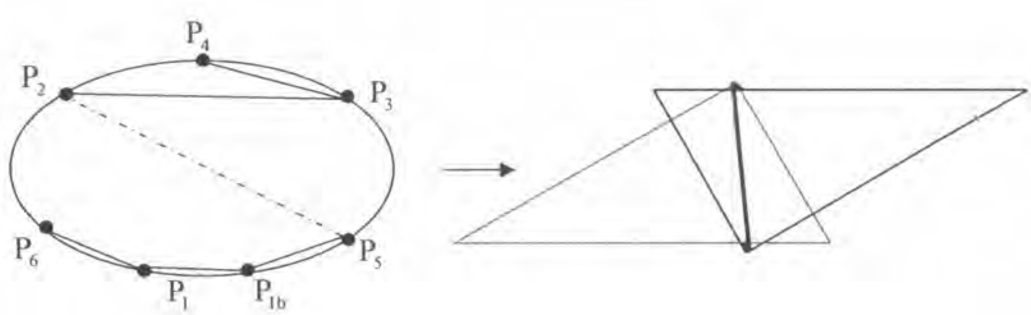


Subcase 1.2.2 $\Pi = P_6\ P_1\ P_5\ P_2\ P_3\ P_4$ and the **condition** T2 is satisfied.

There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

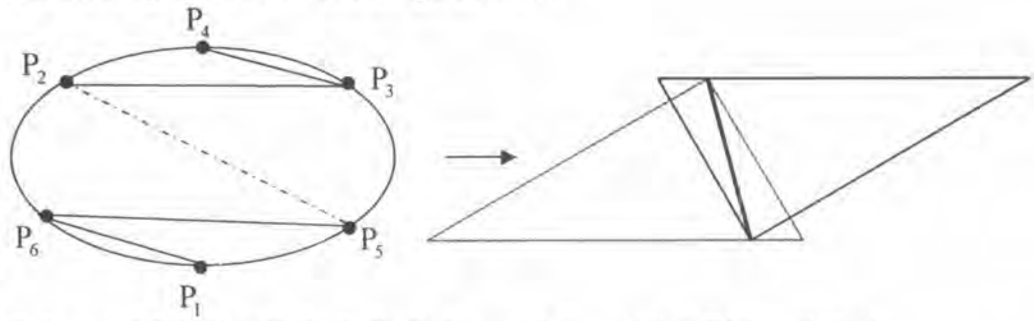
Subcase 1.2.2.1 $\Pi = P_6\ P_1\ P_{1b}\ P_5\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.



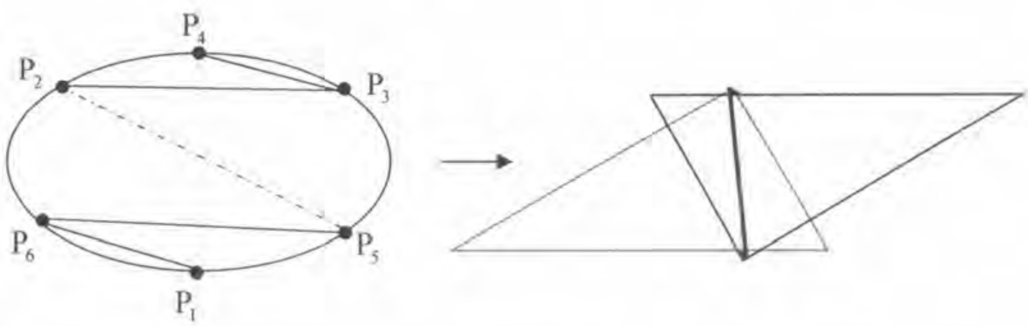
Subcase 1.3 $\Pi = P_1 \ P_6 \ P_5 \ P_2 \ P_3 \ P_4$

By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.



Subcase 1.3.1 $\Pi = P_1 \ P_6 \ P_5 \ P_2 \ P_3 \ P_4$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.

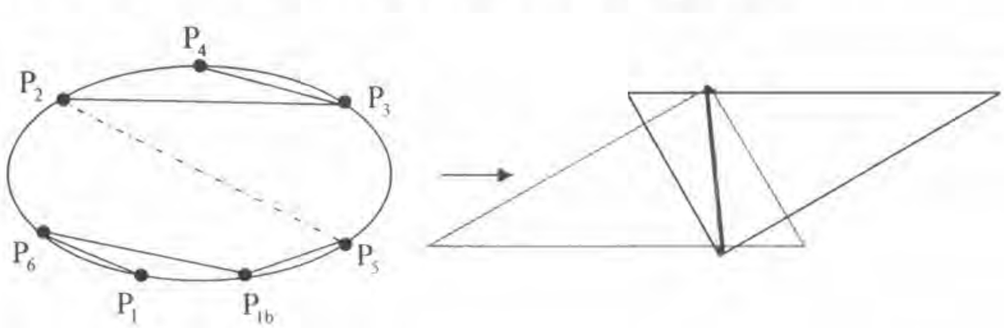


Subcase 1.3.2 $\Pi = P_1 \ P_6 \ P_5 \ P_2 \ P_3 \ P_4$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1.Thus, this subcase can be split into the following subcases.

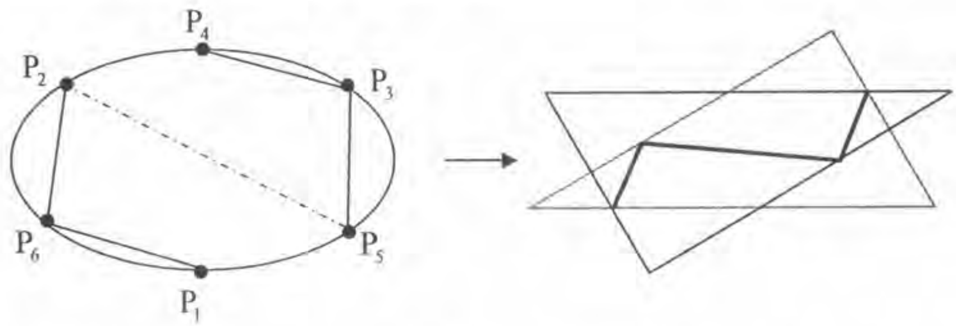
Subcase 1.3.2.1 $\Pi = P_1 \ P_6 \ P_{1b} \ P_5 \ P_2 \ P_3 \ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.507214 units long. It is shown as the figure below.



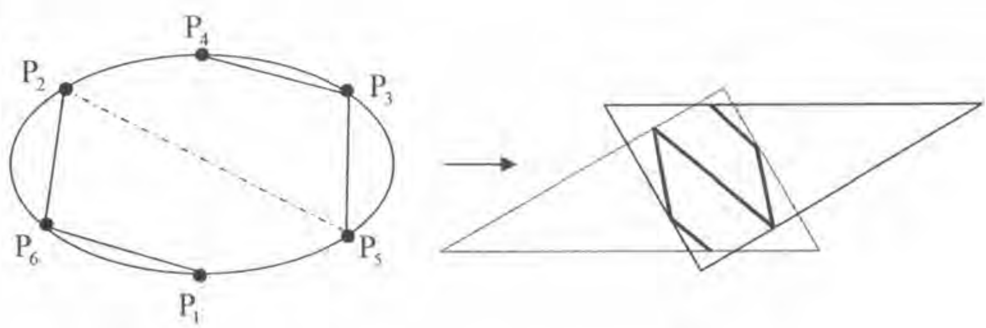
Subcase 1.4 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$

By using numerical minimization, the shortest polysegment is approximately 0.841949 units long. It is shown as the figure below.



Subcase 1.4.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.17298 units long. It is shown as the figure below.

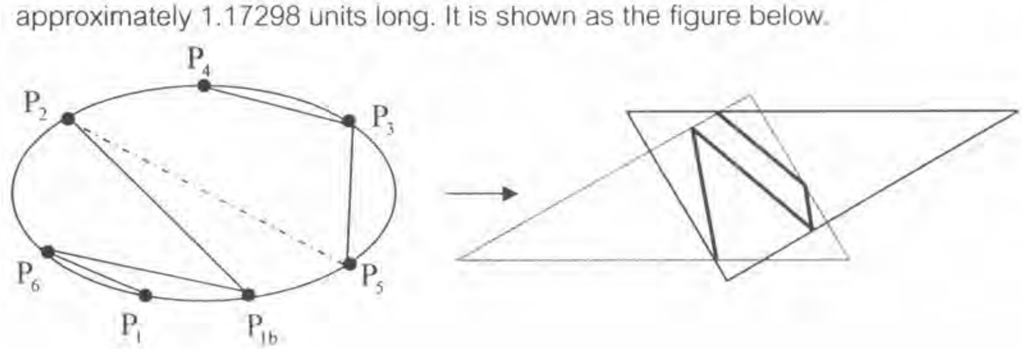


Subcase 1.4.2 $\Pi = P_1\ P_6\ P_2\ P_5\ P_3\ P_4$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

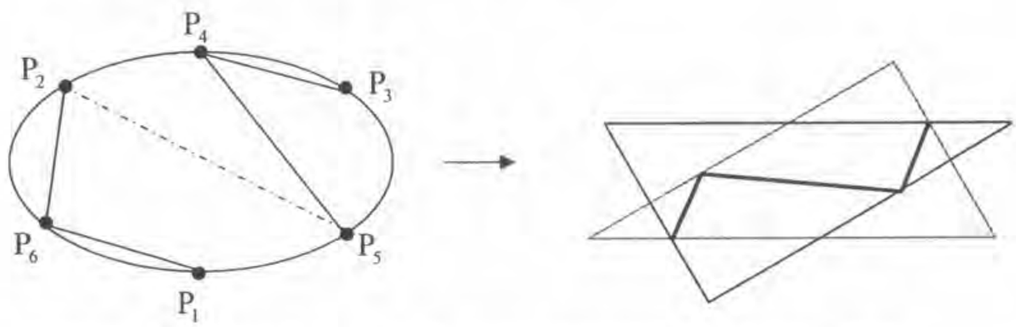
Subcase 1.4.2.1 $\Pi = P_1\ P_6\ P_{1b}\ P_2\ P_5\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 1.17298 units long. It is shown as the figure below.



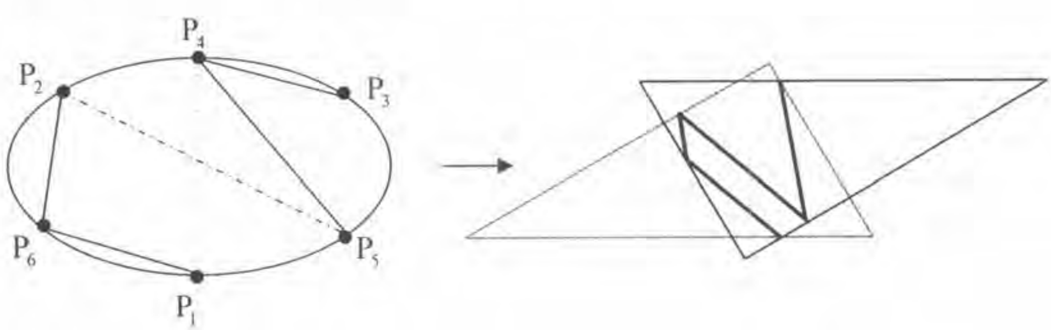
Subcase 1.5 $\Pi = P_1 P_6 P_2 P_5 P_4 P_3$

By using numerical minimization, the shortest polysegment is approximately 0.841949 units long. It is shown as the figure below.



Subcase 1.5.1 $\Pi = P_1 P_6 P_2 P_5 P_4 P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.17298 units long. It is shown as the figure below.

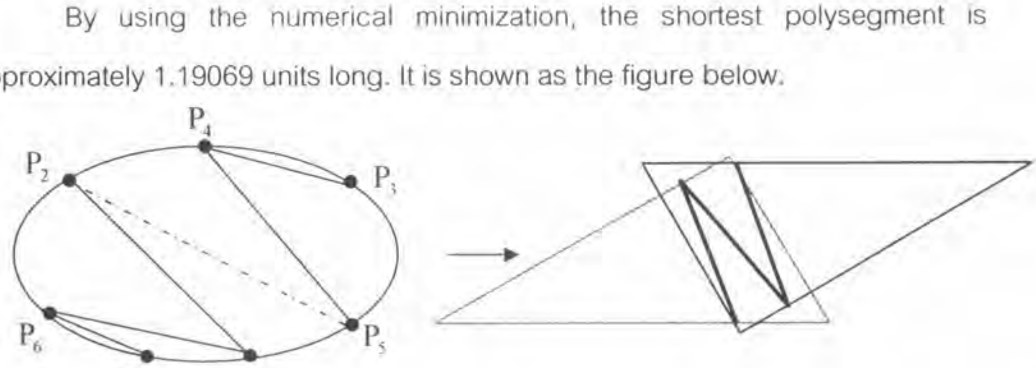


Subcase 1.5.2 $\Pi = P_1 P_6 P_2 P_5 P_4 P_3$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

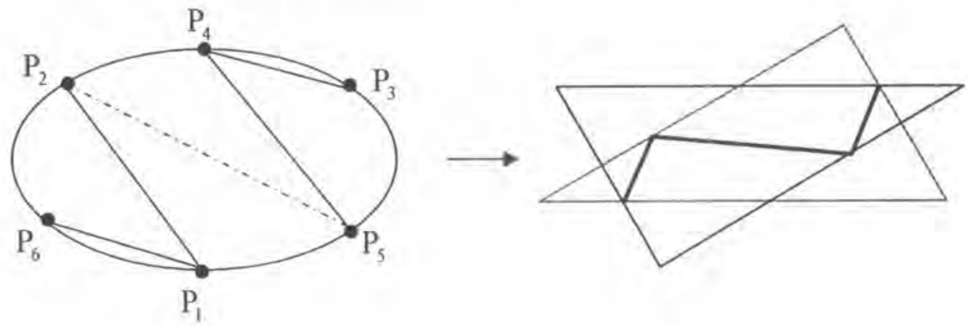
Subcase 1.5.2.1 $\Pi = P_1 P_6 P_{1b} P_2 P_5 P_4 P_3$

By using the numerical minimization, the shortest polysegment is approximately 1.19069 units long. It is shown as the figure below.



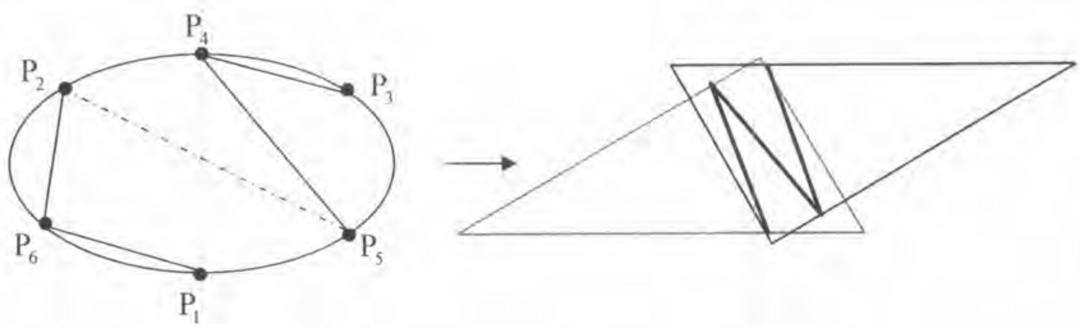
Subcase 1.6 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.841948 units long. It is shown as the figure below.



Subcase 1.6.1 $\Pi = P_1\ P_6\ P_2\ P_5\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1.19069 units long. It is shown as the figure below.

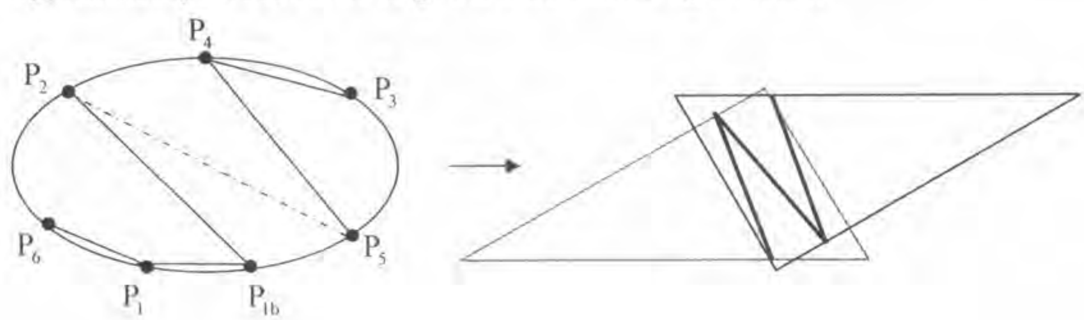


Subcase 1.6.2 $\Pi = P_6\ P_1\ P_2\ P_5\ P_4\ P_3$ and the **condition T2** is satisfied.

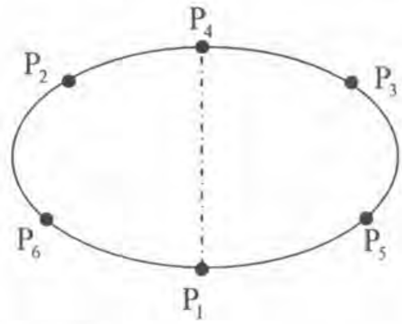
There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

Subcase 1.6.2.1 $\Pi = P_6\ P_1\ P_{1b}\ P_2\ P_5\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 1.19069 units long. It is shown as the figure below.



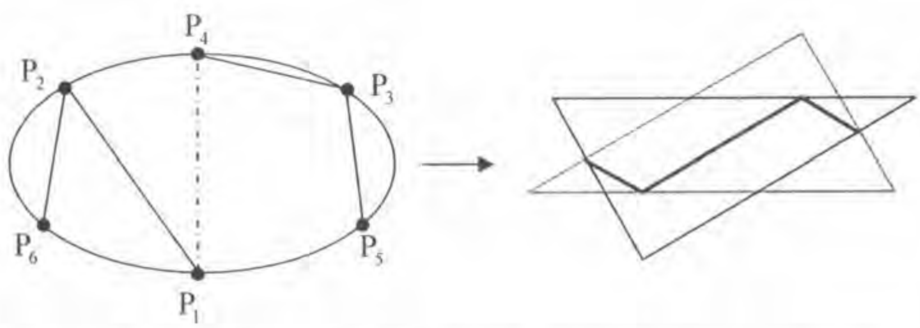
Case 2 P_1 and P_4 are connected.



There will be the following cases

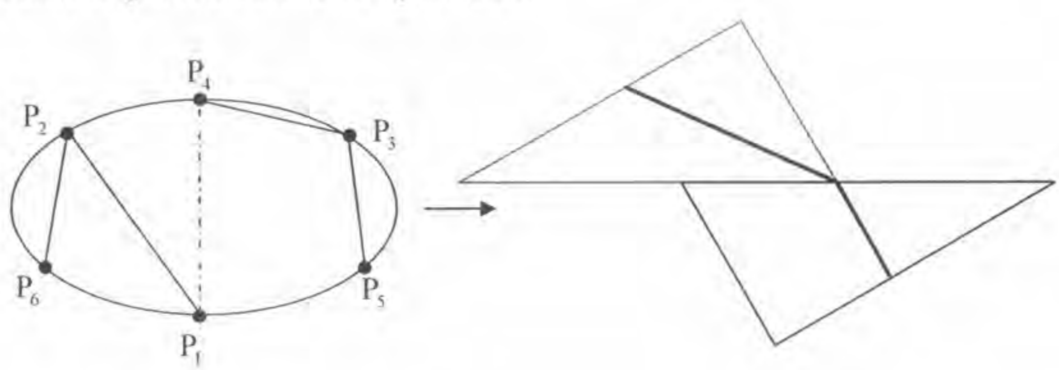
Subcase 2.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$.

By using numerical minimization, the shortest polysegment is approximately 0.765367 units long. It is shown as the figure below.



Subcase 2.1.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.908249 units long. It is shown as the figure below.

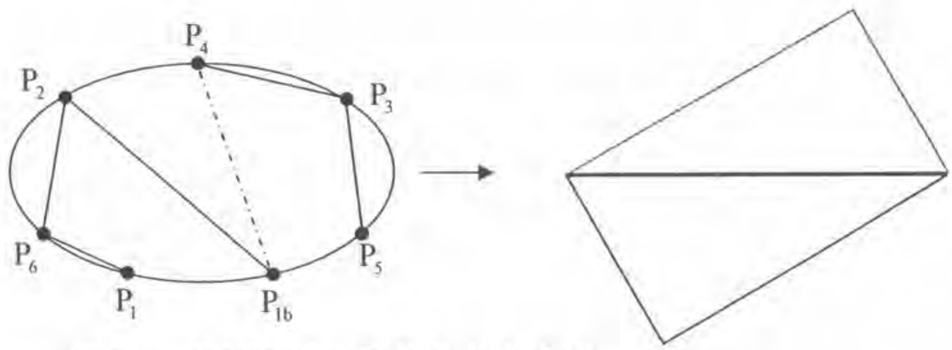


Subcase 2.1.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

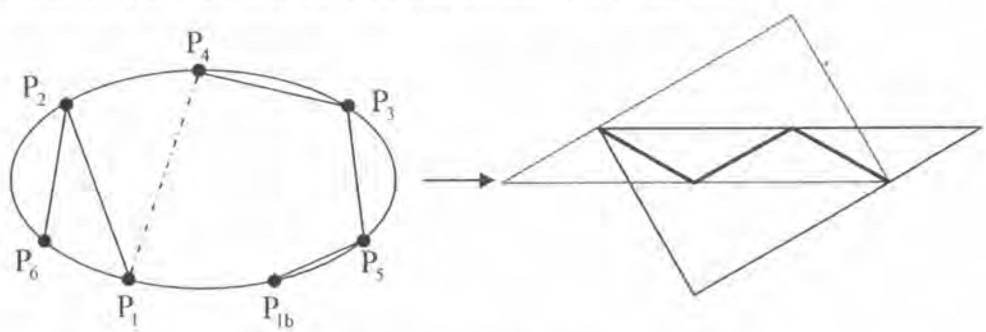
Subcase 2.1.2.1 $\Pi = P_1\ P_6\ P_2\ P_{1b}\ P_4\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



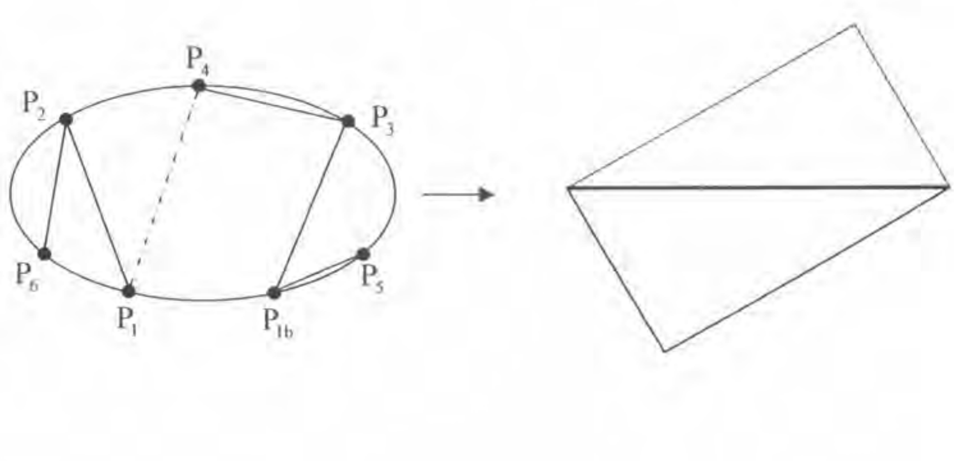
Subcase 2.1.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



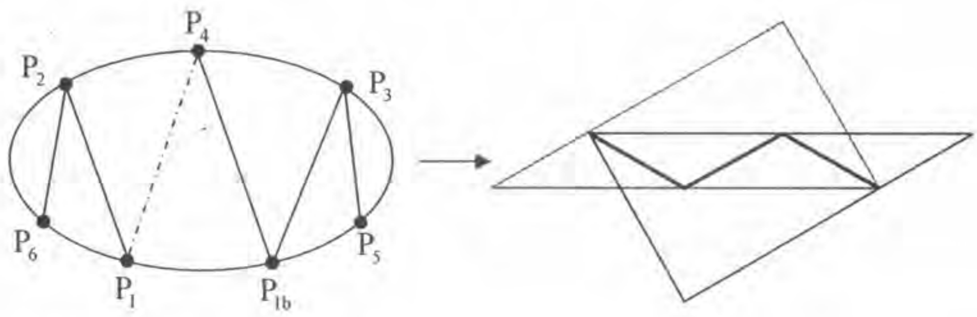
Subcase 2.1.2.3 $\Pi = P_6\ P_2\ P_1\ P_4\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



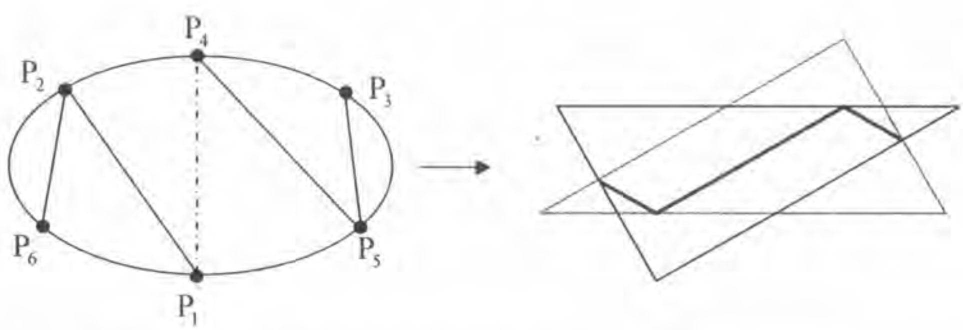
Subcase 2.1.2.4 $\Pi = P_6\ P_2\ P_1\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0. 866025 units long. It is shown as the figure below.



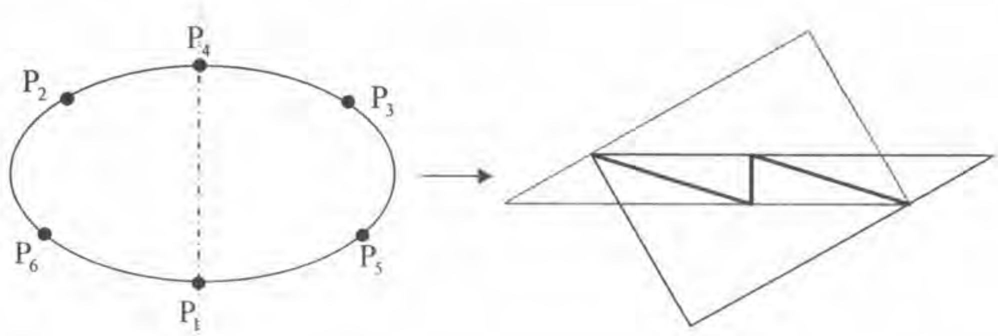
Subcase 2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below



Subcase 2.2.1 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.947294 units long. It is shown as the figure below.

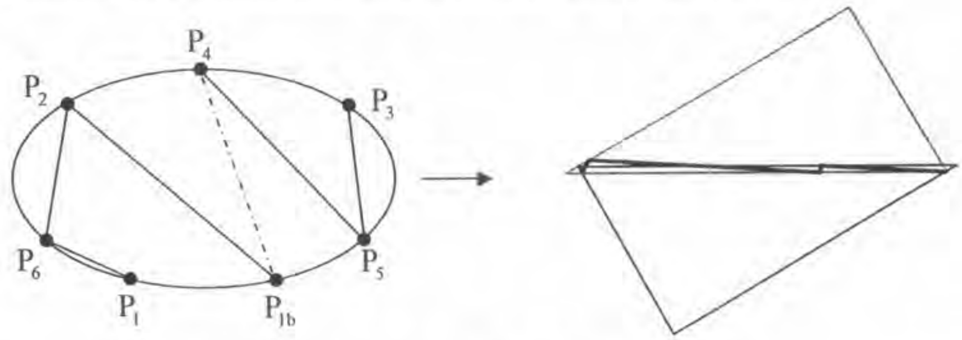


Subcase 2.2.2 $\Pi = P_6\ P_2\ P_1\ P_4\ P_5\ P_3$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

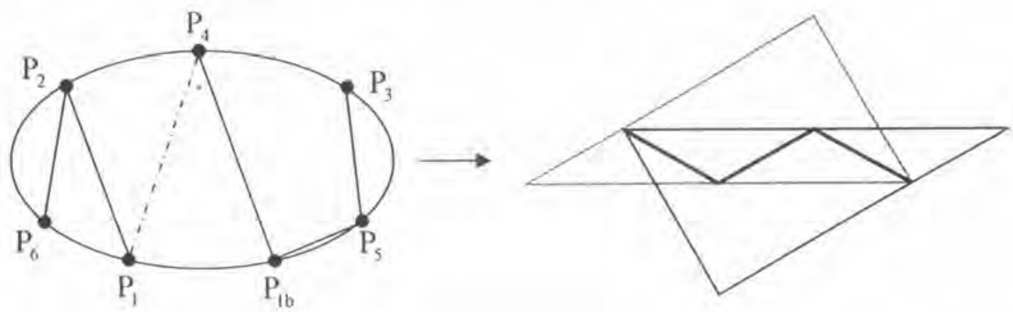
Subcase 2.1.2.1 $\Pi = P_1 P_6 P_2 P_{1b} P_4 P_5 P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.998695 units long. It is shown as the figure below.



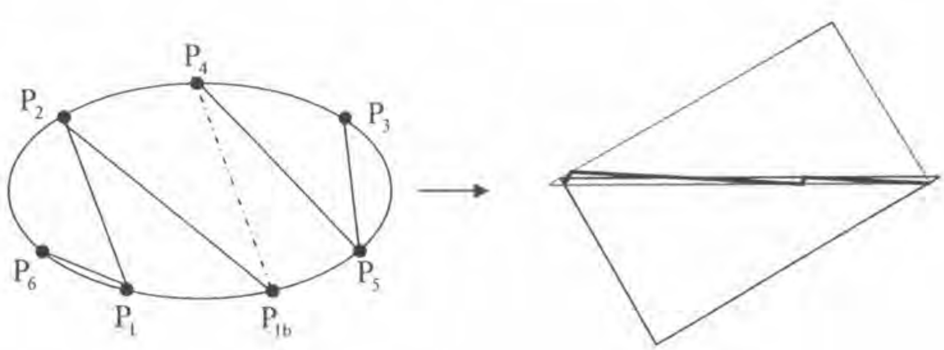
Subcase 2.2.2.2 $\Pi = P_6 P_2 P_1 P_4 P_{1b} P_5 P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



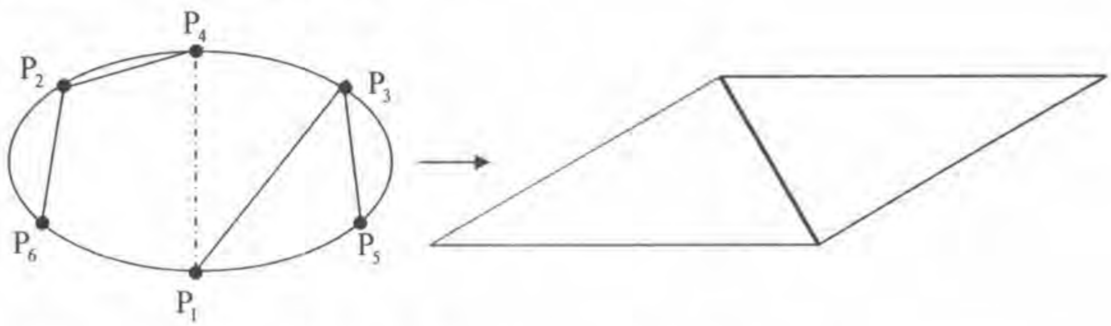
Subcase 3.2.2.3 $\Pi = P_6 P_1 P_2 P_{1b} P_4 P_5 P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.998696 units long. It is shown as the figure below.



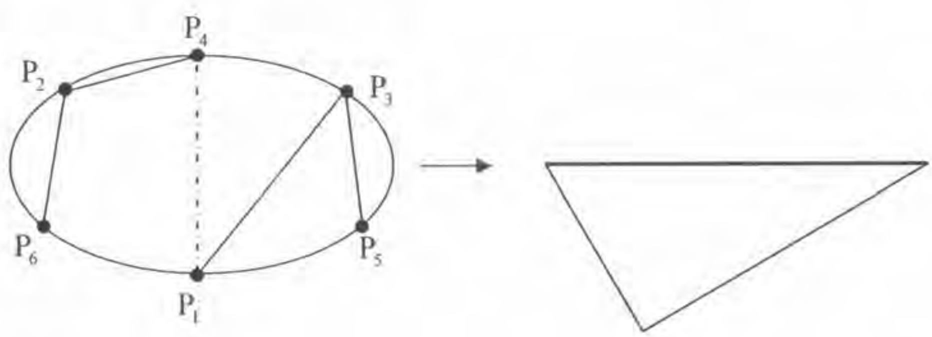
Subcase 2.3 $\Pi = P_6\ P_2\ P_4\ P_1\ P_3\ P_5$

By using numerical minimization, the shortest polysegment is approximately 0.5 units long. It is shown as the figure below



Subcase 2.3.1 $\Pi = P_6\ P_2\ P_4\ P_1\ P_3\ P_5$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

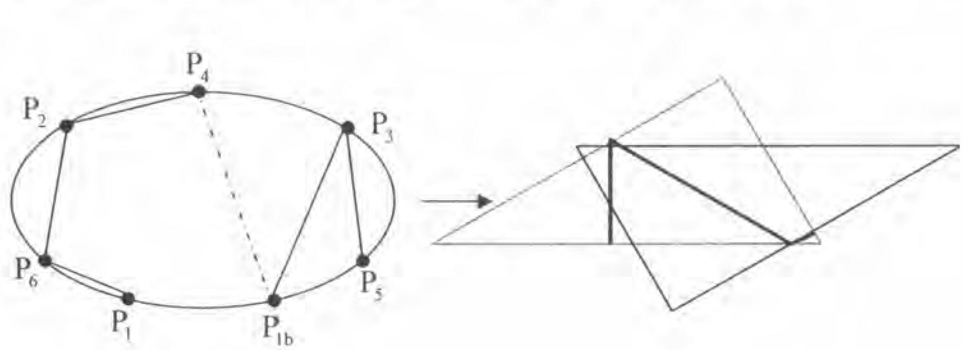


Subcase 2.3.2 $\Pi = P_6\ P_2\ P_4\ P_1\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

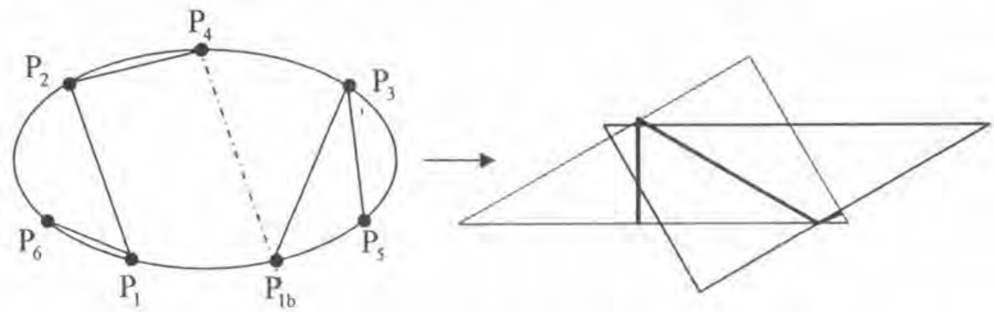
Subcase 2.3.2.1 $\Pi = P_1\ P_6\ P_2\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



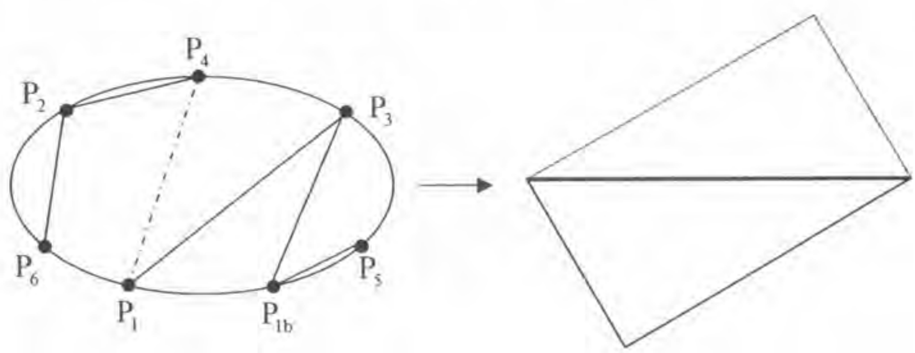
Subcase 2.3.2.2 $\Pi = P_6\ P_1\ P_2\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.866025units long. It is shown as the figure below.



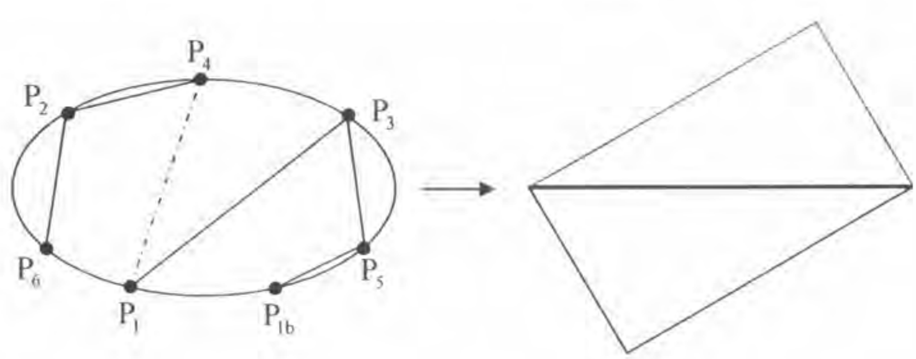
Subcase 2.3.2.3 $\Pi = P_6\ P_2\ P_4\ P_1\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



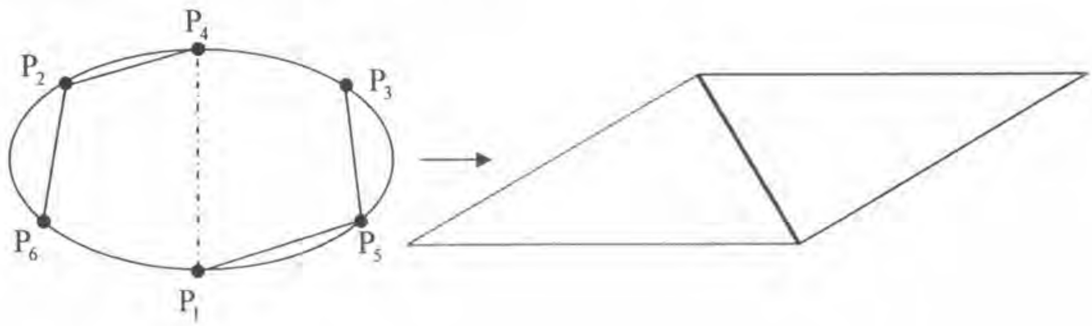
Subcase 2.3.2.4 $\Pi = P_6\ P_2\ P_4\ P_1\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.



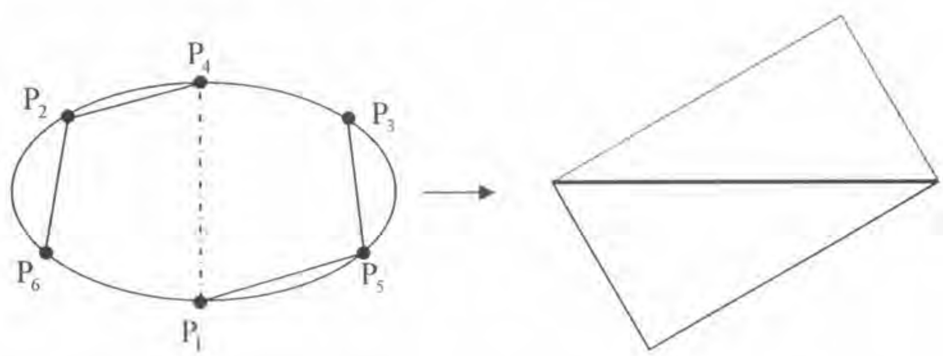
Subcase 2.4 $\Pi = P_6\ P_2\ P_4\ P_1\ P_5\ P_3$

By using numerical minimization, the shortest polysegment is approximately 0.707107 units long. It is shown as the figure below



Subcase 2.4.1 $\Pi = P_6\ P_2\ P_4\ P_1\ P_5\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

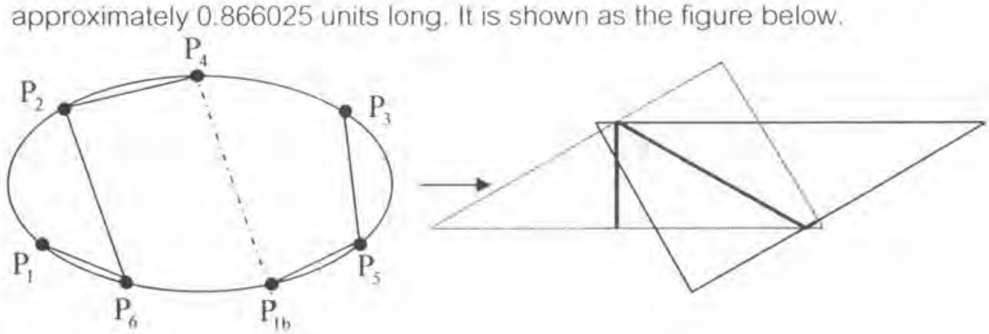


Subcase 2.4.2 $\Pi = P_6\ P_2\ P_4\ P_1\ P_5\ P_3$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

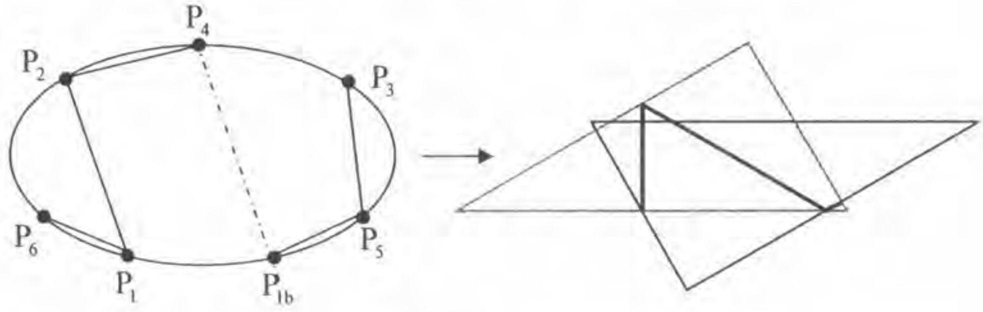
Subcase 2.4.2.1 $\Pi = P_1\ P_6\ P_2\ P_4\ P_{1b}\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



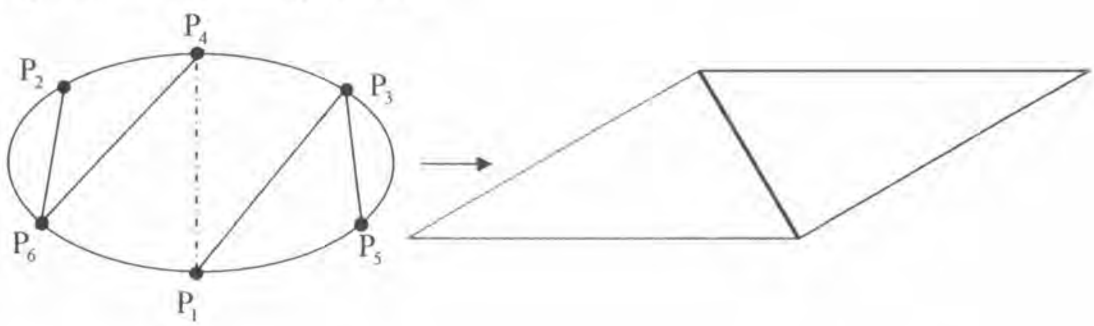
Subcase 2.4.2.2 $\Pi = P_6\ P_1\ P_2\ P_4\ P_{1b}\ P_5\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



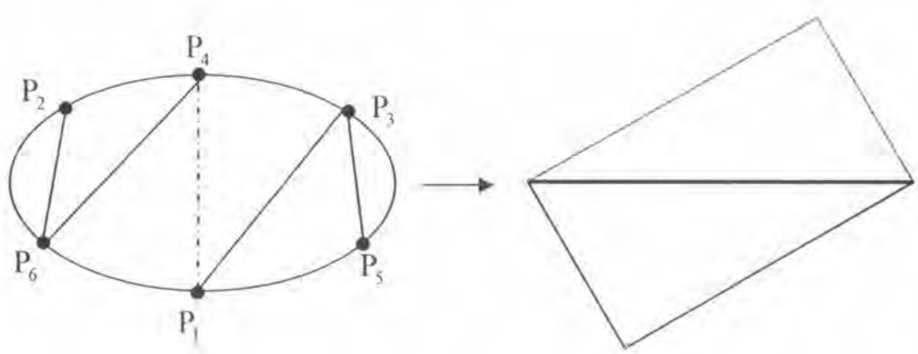
Subcase 2.5 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$

By using numerical minimization, the shortest polysegment is approximately 0.5 units long. It is shown as the figure below



Subcase 2.5.1 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 1 unit long. It is shown as the figure below.

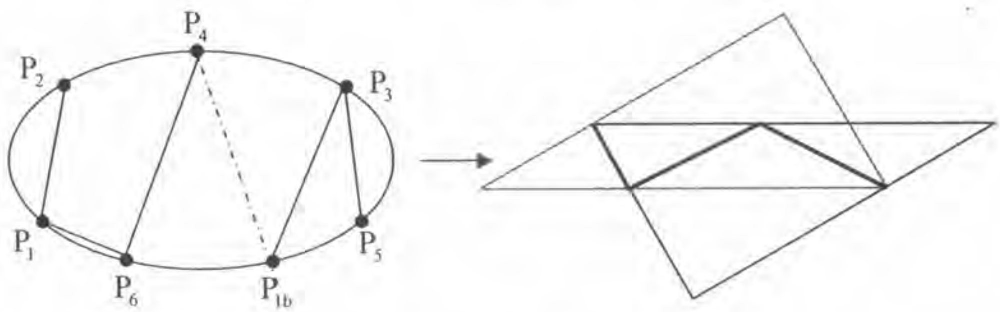


Subcase 2.5.2 $\Pi = P_2\ P_6\ P_4\ P_1\ P_3\ P_5$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

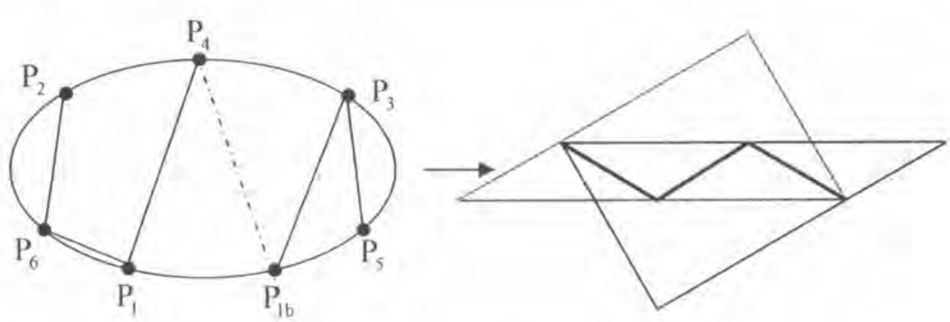
Subcase 2.5.2.1 $\Pi = P_2\ P_1\ P_6\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.891806 units long. It is shown as the figure below.

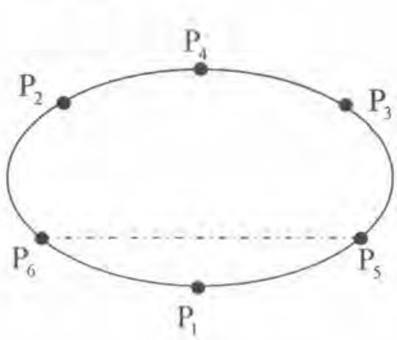


Subcase 2.5.2.2 $\Pi = P_2\ P_6\ P_1\ P_4\ P_{1b}\ P_3\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.866303 units long. It is shown as the figure below.



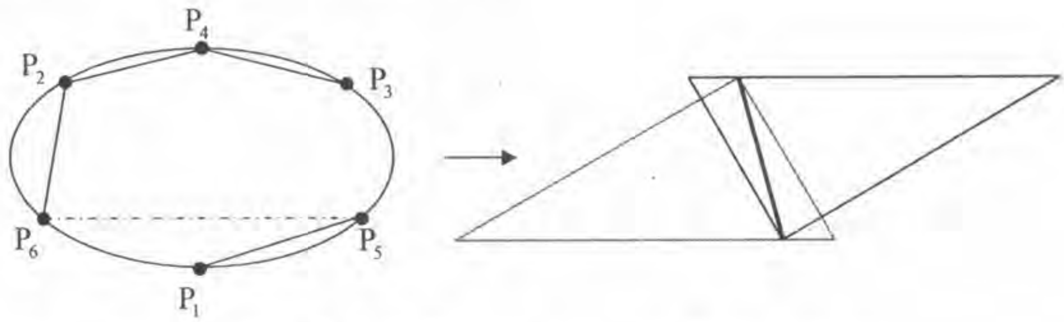
Case 3 P_5 and P_6 are connected.



There will be the following cases

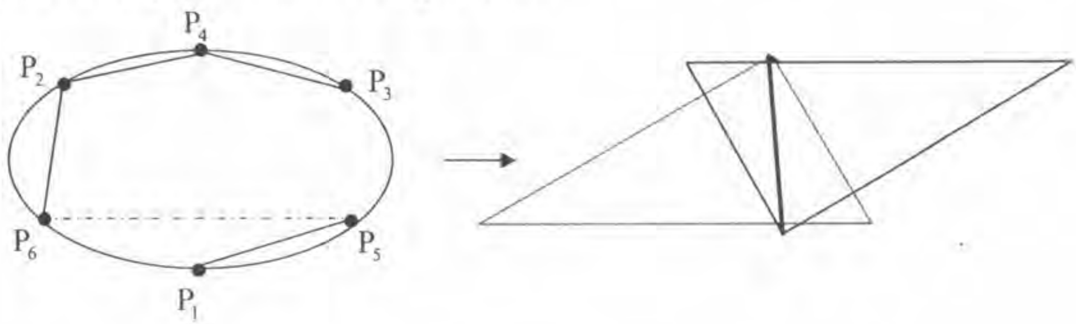
Subcase 3.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$.

By using numerical minimization, the shortest polysegment is approximately 0.451684 units long. It is shown as the figure below.



Subcase 3.1.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.509526 units long. It is shown as the figure below.

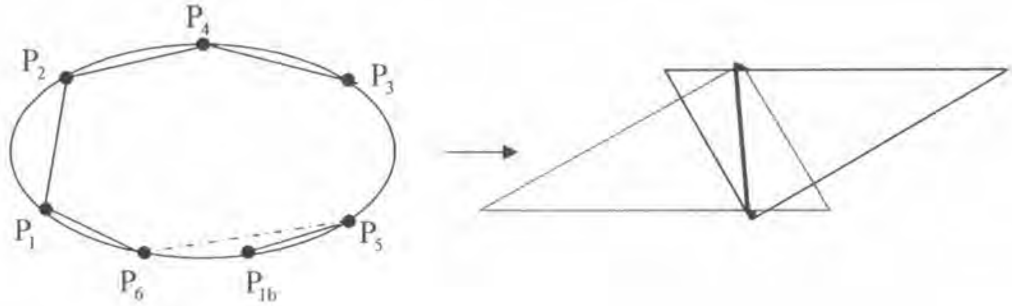


Subcase 3.1.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_4\ P_3$ and the **condition T2** is satisfied.

There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

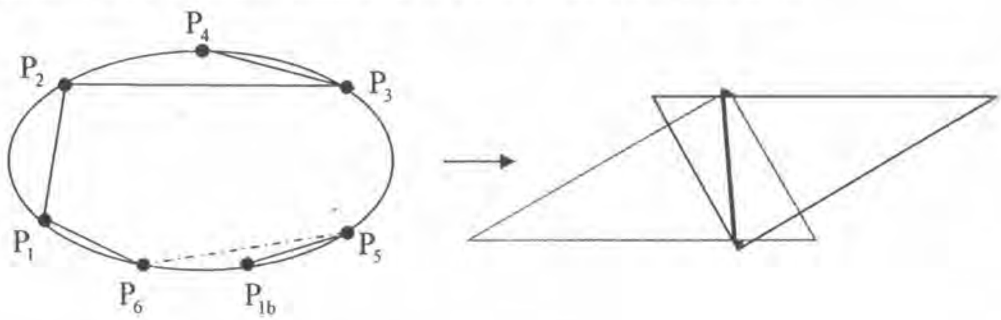
Subcase 3.1.2.1 $\Pi = P_{1b}\ P_5\ P_6\ P_1\ P_2\ P_4\ P_3$

By using the numerical minimization, the shortest polysegment is approximately 0.505962 units long. It is shown as the figure below.



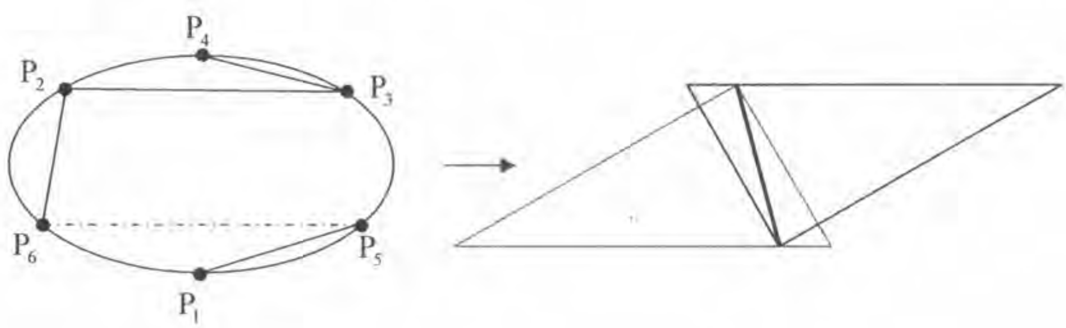
Subcase 3.1.2.2 $\Pi = P_{1b}\ P_5\ P_6\ P_1\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.508501 units long. It is shown as the figure below.



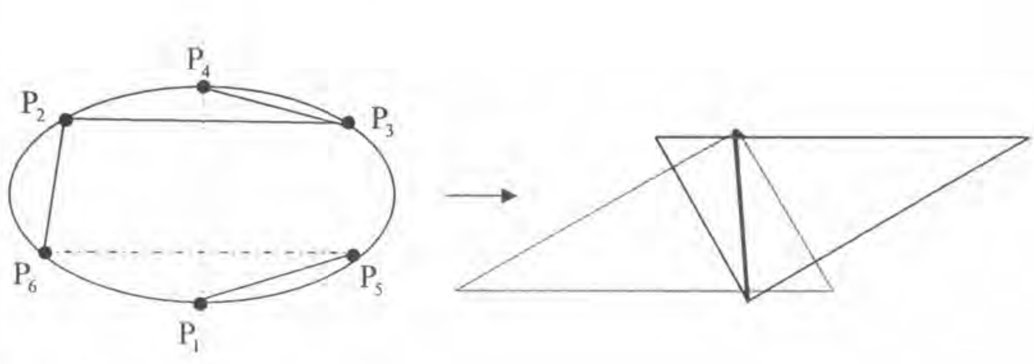
Subcase 3.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$

By using numerical minimization, the shortest polysegment is approximately 0.451684 units long. It is shown as the figure below.



Subcase 3.2.1 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$ and the **condition** T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.509526 units long. It is shown as the figure below.

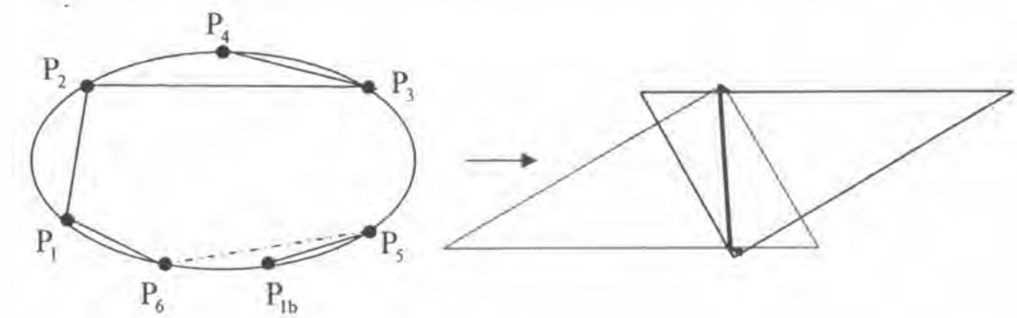


Subcase 3.2.2 $\Pi = P_1\ P_5\ P_6\ P_2\ P_3\ P_4$ and the **condition** T2 is satisfied.

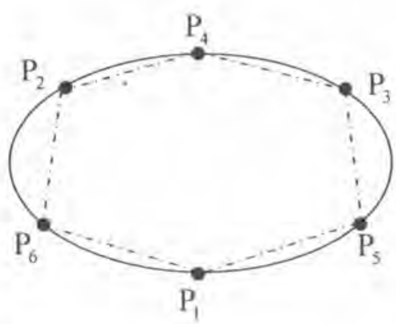
There is P_{1b} which is on the same horizontal of P_1. Thus, this subcase can be split into the following subcases.

Subcase 3.2.2.1 $\Pi = P_{1b}\ P_5\ P_6\ P_1\ P_2\ P_3\ P_4$

By using the numerical minimization, the shortest polysegment is approximately 0.648747 units long. It is shown as the figure below.



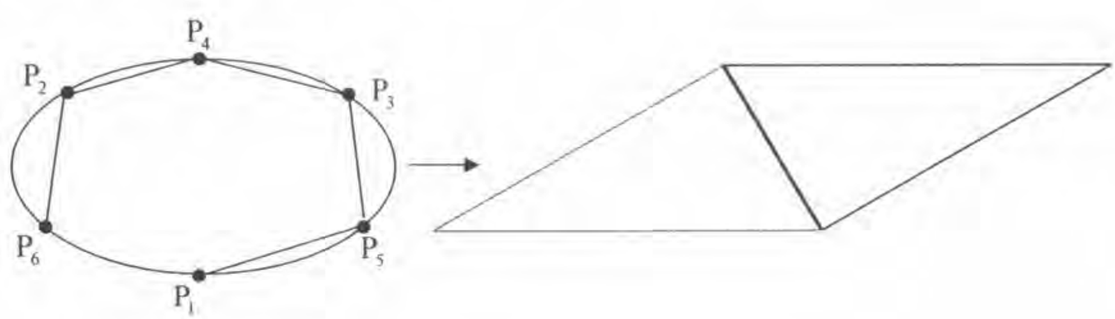
Case 4 Convex arc.



There will be the following cases

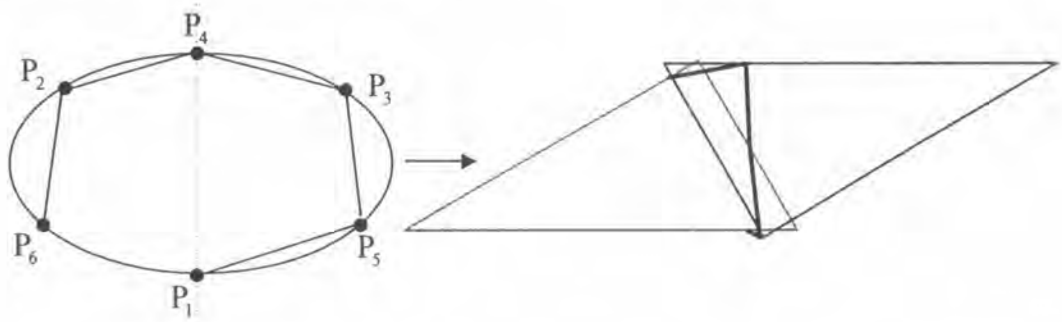
Subcase 4.1 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$.

By using numerical minimization, the shortest polysegment is approximately 0.5 units long. It is shown as the figure below.



Subcase 4.1.1 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$ and the **condition** T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.686052 units long. It is shown as the figure below.

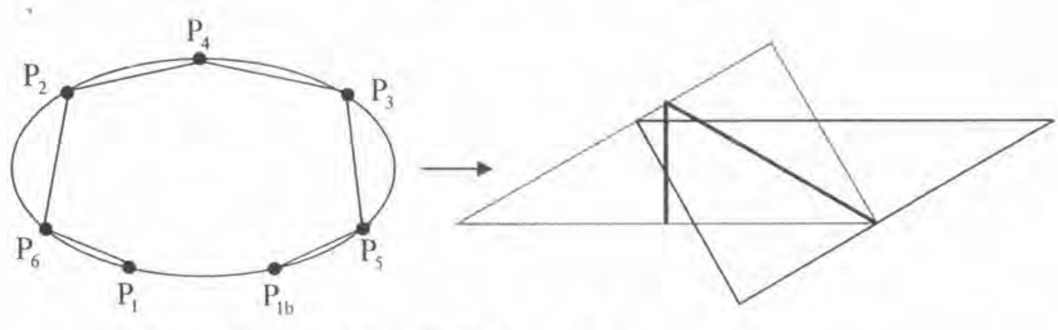


Subcase 4.1.2 $\Pi = P_6\ P_2\ P_4\ P_3\ P_5\ P_1$ and the **condition** T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1.Thus, this subcase can be split into the following subcases.

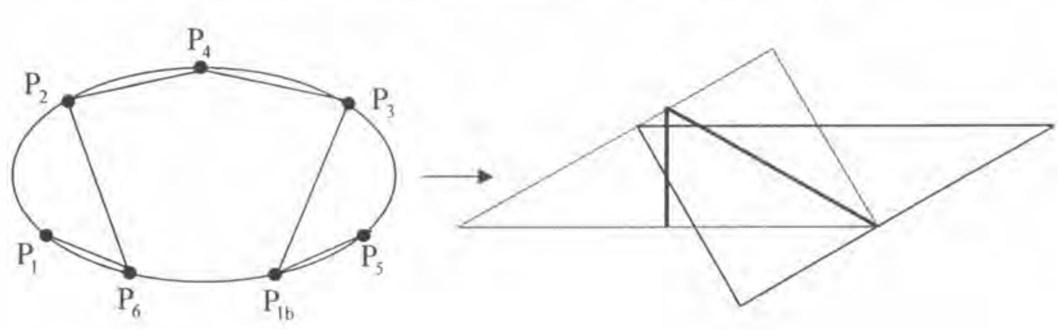
Subcase 4.1.2.1 $\Pi = P_1\ P_6\ P_2\ P_4\ P_3\ P_5\ P_{1b}$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



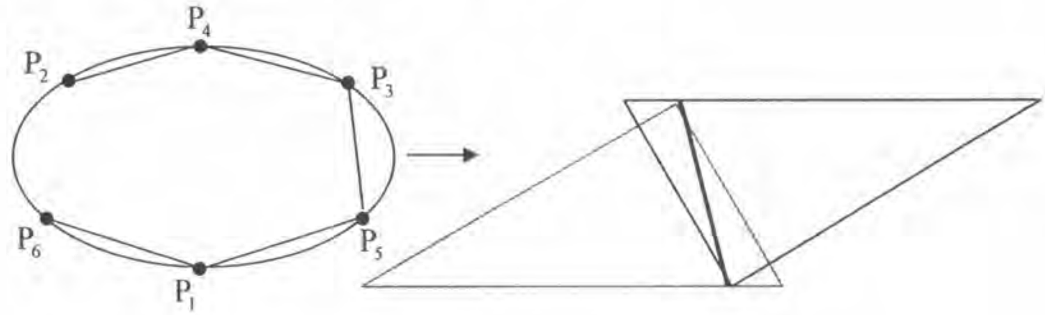
Subcase 4.1.2.2 $\Pi = P_1\ P_6\ P_2\ P_4\ P_3\ P_{1b}\ P_5$

By using the numerical minimization, the shortest polysegment is approximately 0.866025 units long. It is shown as the figure below.



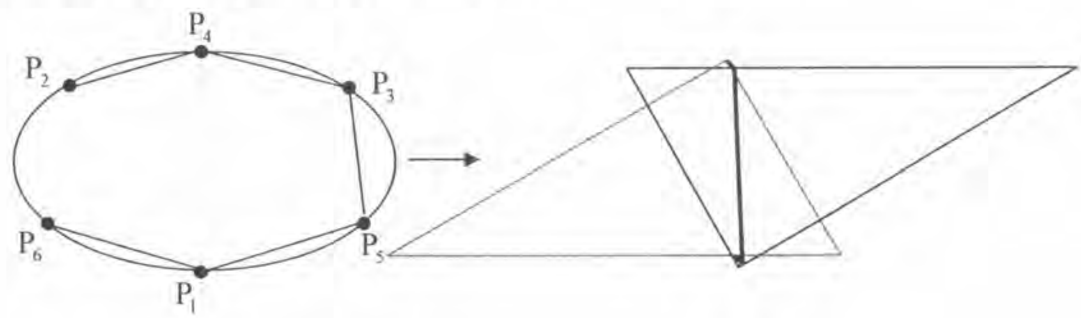
Subcase 4.2 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$

By using numerical minimization, the shortest polysegment is approximately 0.448288 units long. It is shown as the figure below.



Subcase 4.2.1 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$ and the **condition T1** is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.484994 units long. It is shown as the figure below.

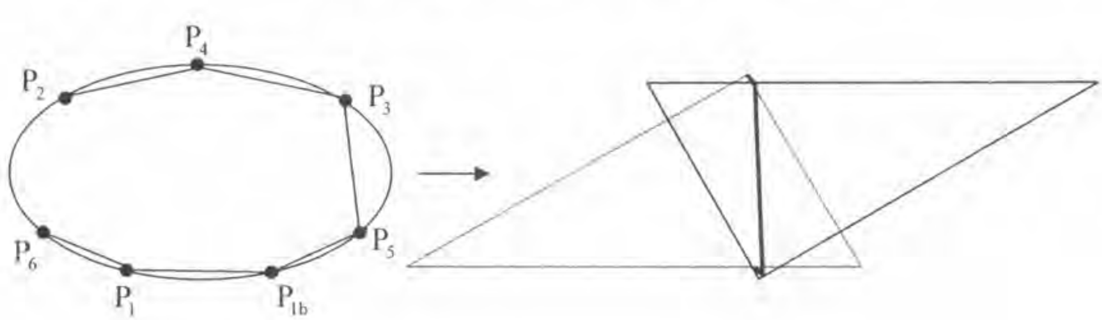


Subcase 4.2.2 $\Pi = P_2\ P_4\ P_3\ P_5\ P_1\ P_6$ and the **condition T2** is satisfied.

There is a P_{1b} which is on the same vertical of P_1. Thus, this subcase can be split into the following subcases.

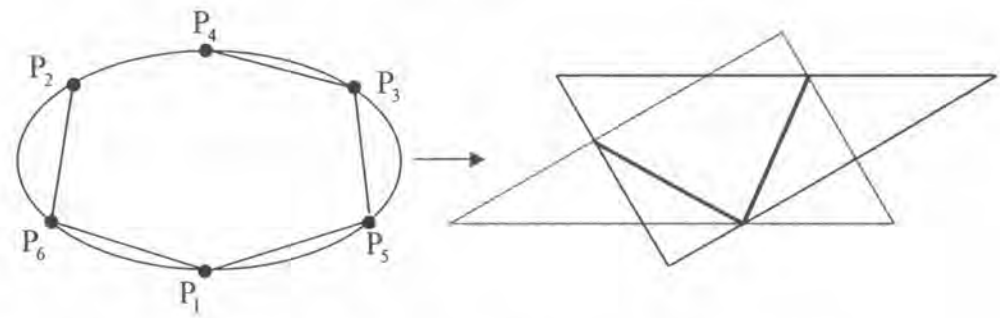
Subcase 4.2.2.1 $\Pi = P_2\ P_4\ P_3\ P_5\ P_{1b} P_1\ P_6$

By using the numerical minimization, the shortest polysegment is approximately 0.48519 units long. It is shown as the figure below.



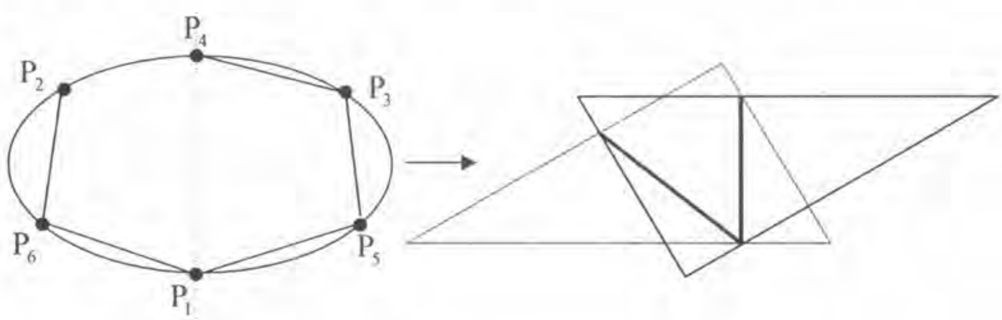
Subcase 4.3 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$

By using numerical minimization, the shortest polysegment is approximately 0.750346 units long. It is shown as the figure below.



Subcase 4.3.1 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$ and the condition T1 is satisfied.

By using the numerical minimization, the shortest polysegment is approximately 0.782044 units long. It is shown as the figure below.



Subcase 4.3.2 $\Pi = P_4\ P_3\ P_5\ P_1\ P_6\ P_2$ and the condition T2 is satisfied.

There is a P_{1b} which is on the same vertical of P_1 .Thus, this subcase can be split into the following subcases.

Subcase 4.3.2.1 $\Pi = P_4\ P_3\ P_5\ P_{1b}\ P_1\ P_6\ P_2$

By using the numerical minimization, the shortest polysegment is approximately 0.782044 units long. It is shown as the figure below.