การกำกับมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมของไฮเพอร์กราฟเค-เอกรูปบางรูป

นายอัฐวิช นริศยาพร

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557

SUPER EDGE-MAGIC LABELING OF k-UNIFORM HYPERGRAPHS

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A Thesis Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science Program in Mathematics

Department of Mathematics and Computer Science

Faculty of Science

Chulalongkorn University

Academic Year 2014

Thesis Title	SUPER EDGE-MAGIC LABELING OF SOME k-UNIFORM HYPERGRAPHS
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อัฐวิช นริศยาพร : การกำกับมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมของไฮเพอร์กราฟ เค-เอกรูปบางรูป. (SUPER EDGE-MAGIC LABELING OF SOME *k*-UNIFORM HYPERGRAPHS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: ดร.รตินันท์ บุญเคลือบ, อ. ที่ปรึกษาวิทยานิพนธ์ร่วม: ดร.ศิริรัตน์ สิงหันต์, 79 หน้า.

วิทยานิพนธ์ฉบับนี้ ได้วางนัยทั่วไปของมโนทัศน์ของการกำกับมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมของกราฟไปสู่การกำกับมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมของไฮเพอร์กราฟ ไฮเพอร์กราฟ $H=(V,E)$ เป็นไฮเพอร์กราฟมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อม ถ้ามีฟังก์ชันสมนัยหนึ่งต่อหนึ่ง $f:V\cup E\rightarrow\{1,2,3,\ldots,|V|+|E|\}$ ที่สอดคล้องกับเงื่อนไขต่อไปนี้ (i) มีค่าคงตัว Λ ซึ่งทำให้ $\sum_{v\in e}f(v)+f(e)=\Lambda$ สำหรับทุก $e\in E$ และ (ii) $f(V)=\{1,2,3,\ldots,|V|\}$ เงื่อนไขจำเป็นสำหรับไฮเพอร์กราฟจะเป็นไฮเพอร์กราฟมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมได้รับการพิสูจน์ นอกจากนี้ไฮเพอร์กราฟบริบูรณ์เค-เอกรูป $\left(K_n^{(k)}\right)$ เป็นไฮเพอร์กราฟมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมก็ต่อเมื่อ $k\in\{0,1\}$ หรือ $n\in\{k,k+1\}$ ไฮเพอร์พาธเค-เอกรูป $\left({}^{m}P_n^{(k)}\right)$ เป็นไฮเพอร์กราฟมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมเสมอ และไฮเพอร์ไซเคิลเค-เอกรูป $\left({}^{m}C_n^{(k)}\right)$ เป็น ไฮเพอร์กราฟมหัศจรรย์อย่างยิ่งยวดบนเส้นเชื่อมก็ต่อเมื่อ n เป็นจำนวนคี่ หรือ $k\neq 2m$

ภาควิชา	คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต..
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ปีการศึกษา	2557	

5672146823 : MAJOR MATHEMATICS

KEYWORDS : HYPERGRAPHS/ COMPLETE k-UNIFORM HYPERGRAPHS/ HYPERPATHS/ HYPERCYCLES/ SUPER EDGE-MAGIC LABELING

AUTHAWICH NARISSAYAPORN : SUPER EDGE-MAGIC LABELING OF k-UNIFORM HYPERGRAPHS. ADVISOR : RATINAN BOONKLURB, Ph.D., CO-ADVISOR : SIRIRAT SINGHUN, Ph.D., 79 pp.

In this thesis, we generalize the concept of super edge-magic labeling in graph to the super edge-magic labeling in hypergraph. A hypergraph $H = (V, E)$ is super edge-magic if there is a bijection $f : V \cup E \to \{1, 2, 3, \ldots, |V|+|E|\}$ satisfying (i) there exists a constant Λ such that for all $e \in E$, $\sum_{v\in e} f(v)+f(e) = \Lambda$ and (ii) $f(V) = \{1, 2, 3, \ldots, |V|\}$. A necessary condition for a hypergraph being super edge-magic is proved. In particular, the complete k-uniform hypergraph $(K_n^{(k)})$ is super edge-magic if and only if $k \in \{0, 1\}$ or $n \in \{k, k+1\}$, the k-uniform hyperpath $({}^mP_n^{(k)})$ is always super edge-magic, and the k-uniform hypercycle $({}^mC_n^{(k)})$ is super edge-magic if and only if n is odd or $k \neq 2m$.

Department :	..Mathematics and ..	Student's Signature :	....
	...Computer Science...	Advisor's Signature :	....
Field of Study :	....Mathematics		
Academic Year :	...........2014	Co-advisor's Signature :	....

ACKNOWLEDGEMENTS

In the achievement of my Master Thesis, I absolutely thank my thesis advisor and thesis co-advisor, Dr. Ratinan Boonklurb and Dr. Sirirat Singhun, respectively. They not only give some advices but also take care all of my academic life. Moreover, I am indebted to my thesis committee: Assistant Professor Dr. Tuangrat Chaichana, Dr. Teeraphong Phongpattanacharoen and Assistant Professor Dr. Pattira Ruengsinsub. I am sincere to their advices and comments.

Furthermore, I would like to express my special thanks to all of my teachers who have coached me for my knowledge, skills and abilities. Also, I wish to express my best regrad to my family and my friends who encourage throughout my study.

Finally, I would like to thank Chulalongkorn University where gives me all of invaluable experiences.

CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Graph labeling is an assignment of integers to the vertices or edges or both of the graph which satisfies certain conditions. There are some varieties of labeling such as graceful labeling and magic labeling. In this thesis, we investigate the labeling called *super edge-magic labeling* which is a combination between the edge-magic labeling and one extra property, i.e., for a graph G, the bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ is called super edge-magic labeling if it satisfies (1) there is a constant λ such that for every edge $xy \in E(G)$, $f(x) + f(y) + f(xy) = \lambda$, and (2) $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. Note that, a graph admitting this labeling is said to be *super edge-magic*.

There are many researches about the super edge-magic labeling. According to [3], several classes of graphs were studied whether they are super edge-magic or not. For example in [1], they showed that all odd cycles C_n are super edge-magic, and all wheel graphs W_n are not super edge-magic. Moreover, they also showed the important necessary condition for a graph being super edge-magic, i.e., if a graph G is super edge-magic, then $|E(G)| \leq 2|V(G)| - 3$.

However, there are few articles about hypergraph labeling. Therefore, we decided to investigate this type of labeling on hypergraphs, especially the *k-uniform hypergraphs.* In Chapter 2, we give a definition of hypergraphs and introduce some classes of hypergraphs, namely *complete k-uniform hypergraph*, *m-node k-uniform hyperpaths*, and *m-node k-uniform hypercycles.* Then, by extending the one defined in graph, we state the generalized version of the super edge-magic labeling for a hypergraph. Similarly, we can establish the necessary condition for hypergraph being super edge-magic. By using this necessary condition, we can show in Chapter 3 that a complete k-uniform hypergraph with n vertices is super edge-magic if and only if $k \in \{0, 1\}$ or $n \in \{k, k+1\}$.

Furthermore, by assign a super edge-magic labeling to the small hypergraph directly, and extend them to the larger one. We explore that some classes of hypergraphs which we defined in Chapter 2 are super edge-magic. Our result are shown in Chapter 4 to Chapter 5.

In Chapter 4 and Chapter 5, we show our results on m-node k-uniform hyperpaths and m-node k-uniform hypercycles according to the small-to-large idea. We find that all k-uniform hyperpaths are super edge-magic. However, some classes of k-uniform hypercycles are super edge-magic and some are not. This agrees with the known result on super edge-magic of cycle in graph theory. Finally, conclusion and some discussion are given in Chapter 6.

CHAPTER II

PRELIMINARIES

In this chapter, we list some notions that are used throughout this thesis. First, Section 2.1 consists of several definitions, properties and classes of hypergraphs. Then, the basic concepts of the super edge-magic labeling and its properties are listed in Section 2.2 and Section 2.3. The last section shows the important condition of a hypergraph admitting this type of labeling.

2.1 Hypergraphs

A hypergraph is the generalization of graphs, since every edge of a hypergraph may be incident to no or many vertices. We give the definition of a hypergraph as follows.

Definition 2.1. [6] A *hypergraph* H is the pair $(V(H), E(H))$ where $V(H)$ is a finite set and $E(H) \subseteq P(V(H))$. The sets $V(H)$ and $E(H)$ are called *vertex set* and *hyperedge set*, respectively.

If there is no ambiguity, we may denote $V(H)$ as V and $E(H)$ as E. For more convenient, we let $|V| = p$ and $|E| = q$. Notice that, by definition, the vertex set of a hypergraph can be empty. However, in this thesis, our hypergraphs consist at least one vertex.

Example 2.1. A hypergraph H, shown in Figure 2.1, has the vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and the hyperedge set $E(H) = \{e_1, e_2, e_3, e_4, e_5\}$.

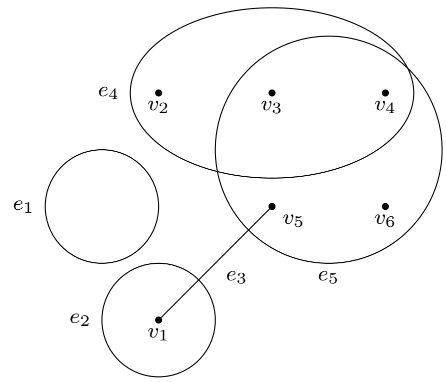

Figure 2.1: A hypergraph H

Notice that, $|V(H)| = p = 6$ and $|E(H)| = q = 5$. Moreover, we may write all hyperedges as $e_1 = \varnothing, e_2 = \{v_1\}, e_3 = \{v_1, v_5\}, e_4 = \{v_2, v_3, v_4\}$ and $e_5 = \{v_3, v_4, v_5, v_6\}$.

In graph, we have the concept of a degree of each vertex. This concept can be generalized in hypergraph. Moreover, in hypergraph, we can have both degrees for each vertex and each hyperedge.

Definition 2.2. [6] Let v be a vertex of a hypergraph H. The *degree* of v is the cardinality of $\{e \in E|\ v \in e\}$. A vertex v is said to be *pendant*, if its degree is one.

Example 2.2. According to the hypergraph H in Example 2.1, only v_2 and v_6 are pendant. The other vertices have degree 2.

Definition 2.3. [6] Assume that e is a hyperedge of a hypergraph H. The *degree* of e is the cardinality of $\{v \in V|\ v \in e\}$. Moreover, H is called k-*uniform*, if every hyperedge $e \in E$ has degree k.

Note that, we use the notation $H^{(k)}$ to denote a k-uniform hypergraph H.

Example 2.3. According to the hypergraph H in Example 2.1, the degree of hyperedge e_i is $i-1$ for $i \in \{1,2,3,4,5\}$. Thus, H is not a uniform hypergraph. However, we show some examples of 3-uniform hypergraphs in Figure 2.2.

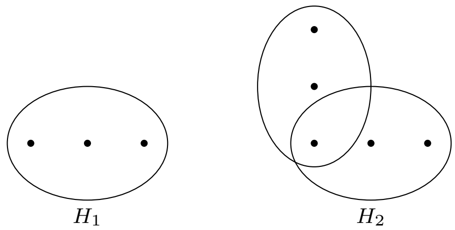

Figure 2.2: 3-uniform hypergraphs

Notice that, a 2-uniform hypergraph is also a graph and vice versa. In the usual graph theory, we used to have the definitions for a complete graph, K_n; a path graph, P_n; and a cycle, C_n. Now, we can define a complete hypergraph, a hyperpath, and a hypercycle in hypergraph in a similar manner. However, for simplicity, here we define only a uniform type of complete hypergraphs, hyperpaths and hypercycles.

Definition 2.4. [6] Let n and k be integers such that $0 \leq k \leq n$. A *complete k-uniform hypergraph*, $K_n^{(k)}$, is a hypergraph that consists of $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and E is the family of all k-subset of V.

Notice that, $K_n^{(k)}$ has n vertices and $\binom{n}{k}$ hyperedges.

Example 2.4. Consider a complete 3-uniform hypergraph $K_4^{(3)}$ with 4 vertices and $\binom{4}{3} = 4$ hyperedges. If the vertex set $V(K_4^{(3)})$ is $\{v_1, v_2, v_3, v_4\}$, then the hyperedge set $E(K_4^{(3)})$ of $K_4^{(3)}$ is the set that contains all 3-subset of $V(K_4^{(3)})$, i.e., $E(K_4^{(3)}) = \{e_1, e_2, e_3, e_4\}$ where $e_1 = \{v_1, v_2, v_3\}, e_2 = \{v_1, v_2, v_4\}, e_3 = \{v_1, v_3, v_4\}$, and $e_4 = \{v_2, v_3, v_4\}$.

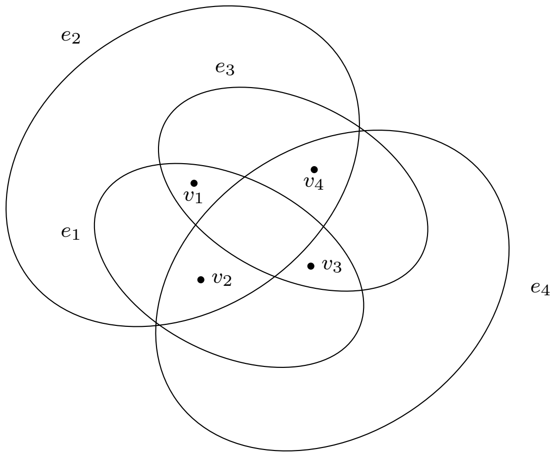

Figure 2.3: A complete 3-uniform hypergraph, $K_4^{(3)}$

Definition 2.5. Let m, n, and k be integers such that $m \geq 1, n \geq 2$, and $k \geq 2m$. An *m-node k-uniform hyperpath*, ${}^mP_n^{(k)}$, is a hypergraph consists of hyperedge set $E = \{e_1, e_2, e_3, \ldots, e_n\}$ and vertex set $V = \bigcup_{i=1}^n e_i$ where

$$e_i = \begin{cases} \bigcup\limits_{j=1}^{m} \{w_{i,j}, w_{i+1,j}\} & \text{if } k = 2m, \\ \bigcup\limits_{j=1}^{m} \{w_{i,j}, w_{i+1,j}\} \cup \{v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,k-2m}\} & \text{if } k > 2m, \end{cases}$$

for $i \in \{1, 2, 3, \ldots, n\}$.

Notice that, ${}^mP_n^{(k)}$ has n hyperedges, $(n-1)m$ vertices of degree 2 and $(n-1)(k-2m)+k$ pendant vertices. Therefore, ${}^mP_n^{(k)}$ has $(n-1)(k-m)+k$ vertices.

Example 2.5. Consider a 1-node 4-uniform hyperpath, ${}^1P_4^{(4)}$. According to Definition 2.5, all hyperedge of ${}^1P_4^{(4)}$ are

$$e_1 = \{w_{1,1}, w_{2,1}, v_{1,1}, v_{1,2}\}, e_2 = \{w_{2,1}, w_{3,1}, v_{2,1}, v_{2,2}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, v_{3,1}, v_{3,2}\}, e_4 = \{w_{4,1}, w_{5,1}, v_{4,1}, v_{4,2}\}.$$

We can draw ${}^1P_4^{(4)}$ as in Figure 2.4.

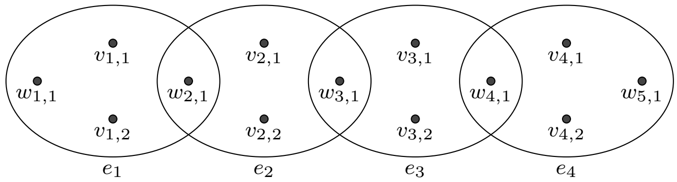

Figure 2.4: A 1-node 4-uniform hyperpath

Definition 2.6. Let m, n, and k be integers such that $m \geq 1, n \geq 3$, and $k \geq 2m$. An *m-node k-uniform hypercycle*, ${}^{m}C_n^{(k)}$, is a hypergraph consists of hyperedge set $E = \{e_1, e_2, e_3, \ldots, e_n\}$ and vertex set $V = \bigcup_{i=1}^{n} e_i$ where

$$e_i = \begin{cases} \bigcup\limits_{j=1}^{m} \{w_{i,j}, w_{i+1,j}\} & \text{if } k = 2m, \\ \bigcup\limits_{j=1}^{m} \{w_{i,j}, w_{i+1,j}\} \cup \{v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,k-2m}\} & \text{if } k > 2m, \end{cases}$$

for $i \in \{1, 2, 3, \ldots, n\}$ and $w_{n+1,j} = w_{1,j}$ for $j \in \{1, 2, 3, \ldots, m\}$.

Notice that, ${}^{m}C_n^{(k)}$ has n hyperedges, nm vertices of degree 2 and $n(k-2m)$ pendant vertices. Therefore, ${}^{m}C_n^{(k)}$ has $n(k-m)$ vertices.

Example 2.6. Consider a 1-node 3-uniform hypercycle, ${}^{1}C_4^{(3)}$. According to Definition 2.6, all hyperedges of ${}^{1}C_4^{(3)}$ are

$$e_1 = \{w_{1,1}, w_{2,1}, v_{1,1}\}, e_2 = \{w_{2,1}, w_{3,1}, v_{2,1}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, v_{3,1}\}, e_4 = \{w_{4,1}, w_{1,1}, v_{4,1}\}.$$

We can draw ${}^{1}C_4^{(3)}$ as in Figure 2.5.

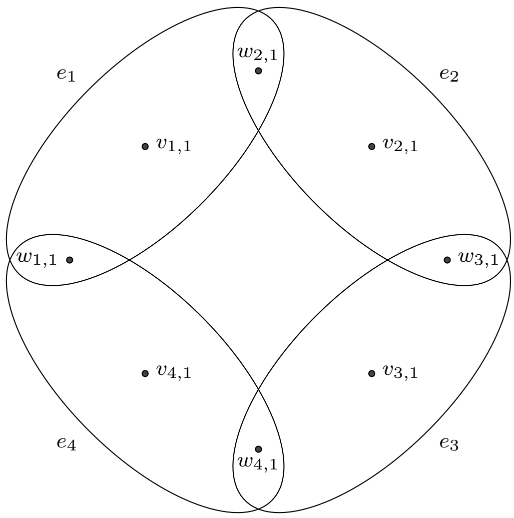

Figure 2.5: A 1-node 3-uniform hypercycle

Note that, hypergraphs $K_n^{(2)}$, ${}^1P_n^{(2)}$, and ${}^1C_n^{(2)}$ are the complete graph K_n, the path graph P_n, and the cycle graph C_n in ordinary graph, respectively.

2.2 The Super Edge-Magic Labeling

According to [1], the *super edge-magic labeling* was first defined for a graph G. It is a bijective function $f : V(G) \cup E(G) \to \{1, 2, 3, \ldots, |V(G)| + |E(G)|\}$ satisfying (i) there exists a constant λ such that for all $xy \in E(G)$, $f(x) + f(y) + f(xy) = \lambda$ and (ii) $f(V(G)) = \{1, 2, 3, \ldots, |V(G)|\}$. Note that, a graph G admits a super edge-magic labeling is said to be *super edge-magic*. In [2], there is an equivalent form of super edge-magic labeling, i.e., a bijective function $f : V(G) \to \{1, 2, 3, \ldots, |V(G)|\}$ such that the set $\{f(x)+f(y) \mid xy \in E(G)\}$ consists of $|E(G)|$ consecutive integers.

In this thesis, we extend the notion of super edge-magic labeling for a hypergraph and its equivalent form, stated as the Definition 2.7 and Theorem 2.8, respectively.

Definition 2.7. For a hypergraph H, the *super edge-magic labeling* is a bijective function $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ satisfying (i) there exists a constant Λ such that for all $e \in E, \sum_{v \in e} f(v) + f(e) = \Lambda$ and (ii) $f(V) = \{1, 2, 3, \ldots, p\}$.

Notice that, in the case of an empty edge, $e = \varnothing$, we let $\sum_{v \in \varnothing} f(v) = 0$. Moreover, a hypergraph admits a super edge-magic labeling is called *super edge-magic* and Λ is called the *magic constant*. Remark that if H is a 2-uniform hypergraph, then Definition 2.7 agrees with the definition of super edge-magic labeling in graph.

Example 2.7. Consider a hyperpath ${}^1P_4^{(2)}$ that has $p = 5$ vertices and $q = 4$ hyperedges. We can give a bijection function $f : V \cup E \to \{1, 2, 3, \ldots, 9\}$ by

$$f(w_{1,1}) = 1, f(w_{2,1}) = 4, f(w_{3,1}) = 2, f(w_{4,1}) = 5, f(w_{5,1}) = 3,$$
$$f(e_1) = 9, f(e_2) = 8, f(e_3) = 7, f(e_4) = 6$$

(shown in Figure 2.6).

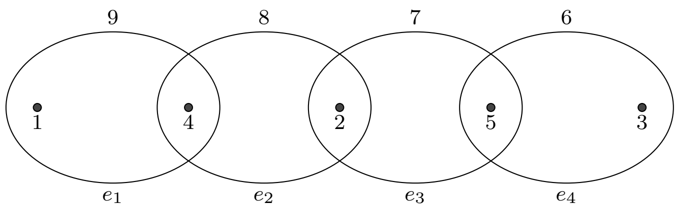

Figure 2.6: The super edge-magic labeling of ${}^1P_4^{(2)}$

Then, there exists a constant $\Lambda = 14$ such that

$$\sum_{v \in e_1} f(v) + f(e_1) = f(w_{1,1}) + f(w_{2,1}) + f(e_1) = 1 + 4 + 9 = 14 = \Lambda,$$

$$\sum_{v \in e_2} f(v) + f(e_2) = f(w_{2,1}) + f(w_{3,1}) + f(e_2) = 4 + 2 + 8 = 14 = \Lambda,$$

$$\sum_{v \in e_3} f(v) + f(e_3) = f(w_{3,1}) + f(w_{4,1}) + f(e_3) = 2 + 5 + 7 = 14 = \Lambda,$$

$$\sum_{v\in e_4} f(v) + f(e_4) = f(w_{4,1}) + f(w_{5,1}) + f(e_4) = 5 + 3 + 6 = 14 = \Lambda.$$

Since $f(V) = f(\{w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1}\}) = \{1, 2, 3, 4, 5\}$, f is a super edge-magic labeling. Hence, ${}^1P_4^{(2)}$ is super edge-magic.

Observe that the labels on each hyperedge of super edge-magic hypergraphs are always consecutive integers. Since the labels on each hyperedge and its vertices add up to constant, the sums of labels of vertices in each hyperedge are also consecutive. Therefore, by this observation, we have the equivalent form of super edge-magic labeling such as Theorem 2.8.

Theorem 2.8. *Let H be a hypergraph. Then, H is super edge-magic if and only if there exists a bijection $f : V \to \{1, 2, 3, \ldots, p\}$ such that $\{\sum_{v\in e} f(v)|\ e \in E\}$ is a set of q consecutive integers. Moreover, the magic constant Λ is $p + q + \min\{\sum_{v\in e} f(v)|\ e \in E\}$.*

Proof. If $E = \varnothing$, then H is trivially super edge-magic and $\{\sum_{v\in e} f(v)|\ e \in E\}$ is an empty set which in this context it can be regarded as a set of 0 consecutive integer. Hence, without loss of generality, we may suppose that $E \neq \varnothing$.

Assume that H is super edge-magic. Then, there exists a super edge-magic labeling $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ and a constant Λ. Thus, $\Lambda = \sum_{v\in e} f(v) + f(e)$ for all $e \in E$, which implies $\sum_{v\in e} f(v) = \Lambda - f(e)$ for all $e \in E$. Since f is a super edge-magic labeling, we have $f(V) = \{1, 2, 3, \ldots, p\}$. Hence, $f(E) = \{p + 1, p + 2, p + 3, \ldots, p + q\}$. Therefore,

$$\begin{aligned}\left\{ \sum_{v\in e} f(v) \,\middle|\, e \in E \right\} &= \{\Lambda - f(e)|\ e \in E\} \\ &= \{\Lambda - (p + 1), \Lambda - (p + 2), \Lambda - (p + 3), \ldots, \Lambda - (p + q)\}\end{aligned}$$

is a set of q consecutive integers.

On the other hand, assume that the necessary condition holds. Let α be an integer such that $\{\sum_{v\in e} f(v) |\ e \in E\} = \{\alpha+1, \alpha+2, \alpha+3, \ldots, \alpha+q\}$. Then, we define $g : E \to \{p+1, p+2, p+3, \ldots, p+q\}$ by

$$g(e) = p + q + (\alpha + 1) - \sum_{v\in e} f(v)$$

for $e \in E$. Thus, g is bijective. Hence, $f \cup g : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ defined by

$$(f \cup g)(x) = \begin{cases} f(x) & \text{if } x \in V, \\ g(x) & \text{if } x \in E, \end{cases}$$

is a bijection.

To show that $f \cup g$ is a super edge-magic labeling. Let $e \in E$. Then,

$$\begin{aligned} \sum_{v\in e}(f \cup g)(v) + (f \cup g)(e) &= \sum_{v\in e} f(v) + g(e) \\ &= \sum_{v\in e} f(v) + \left(p + q + (\alpha + 1) - \sum_{v\in e} f(v)\right) \\ &= p + q + \alpha + 1. \end{aligned}$$

Therefore, $f \cup g : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ is a super edge-magic labeling of H with $\Lambda = p + q + \alpha + 1 = p + q + \min\{\sum_{v\in e} f(v) |\ e \in E\}$ as desired. □

Corollary 2.9. *A hypergraph H having at most one hyperedge is super edge-magic.*

Proof. Since $|V| = p$, there exists a bijection $f : V \to \{1, 2, 3, \ldots, p\}$.

In the case of $q = 0$, then $\{\sum_{v\in e} f(v) |\ e \in E\} = \varnothing$ and H is trivially super edge-magic.

In the case of $q = 1$, let $E = \{e_1\}$, then $\{\sum_{v\in e} f(v) |\ e \in E\} = \{\sum_{v\in e_1} f(v)\}$ is a singleton which in this context it can be regarded as a set of one consecutive integer. Thus, by Theorem 2.8, H is super edge-magic. □

2.3 The Necessary Condition

The important necessary condition for a graph G being super edge-magic was showed in [1], i.e., if G is super edge-magic such that $|E(G)| \geq 1$, then by investigating the extremal value of λ, the inequality $|E(G)| \leq 2|V(G)| - 3$ is obtained. In Theorem 2.10, we show the necessary condition for a k-uniform hypergraph H being super edge-magic in a similar way.

Theorem 2.10. *Let $H^{(k)}$ be a k-uniform hypergraph such that $q \geq 1$. If $H^{(k)}$ is super edge-magic, then $q \leq kp - k^2 + 1$.*

Proof. Assume that $H^{(k)}$ is super edge-magic. Then, there is a super edge-magic labeling f of $H^{(k)}$. Note that, $f : V \cup E \to \{1, 2, 3, \ldots, p+q\}$ is a bijection such that $\sum_{v \in e} f(v) + f(e) = \Lambda$ for all $e \in E$ and $f(V) = \{1, 2, 3, \ldots, p\}$.

In case of $k = 0$, the hypergraph $H^{(k)}$ has $q = 1$ which satisfies the above inequality. Hence, we assume that $k > 0$.

Since $q \geq 1$, let $e_1, e_2 \in E$ be such that $f(e_1) = p + q$ and $f(e_2) = p + 1$. Then, we obtain inequalities,

$$\Lambda = \sum_{v \in e_1} f(v) + f(e_1) \geq 1 + 2 + 3 + \cdots + k + (p + q)$$

and

$$\Lambda = \sum_{v \in e_2} f(v) + f(e_2) \leq p + (p-1) + (p-2) + \cdots + (p - k + 1) + (p + 1).$$

Hence,

$$1 + 2 + 3 + \cdots + k + (p + q) \leq p + (p-1) + (p-2) + \cdots + (p - k + 1) + (p + 1)$$

which implies $q \leq kp - k^2 + 1$. $\square$

Note that the condition $q \geq 1$ as shown in Theorem 2.10 cannot be omitted. For example, let H be a hypergraph such that $q = 0$. Then, by Corollary 2.9, H is super edge-magic. Since H can be regarded as a $(p+1)$-uniform hypergraph, we have $kp - k^2 + 1 = (p+1)p - (p+1)^2 + 1 = -p < 0 = q$ which contradicts the necessary condition.

CHAPTER III

COMPLETE UNIFORM HYPERGRAPHS

First, we recall that a complete k-uniform hypergraph, $K_n^{(k)}$, has $\binom{n}{k}$ hyperedges. Then, by using Corollary 2.9, we obtain the following theorem.

Theorem 3.1. *$K_n^{(0)}$ and $K_n^{(n)}$ are super edge-magic.*

Proof. Since $K_n^{(0)}$ and $K_n^{(n)}$ have $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ hyperedge, respectively. Thus, $K_n^{(0)}$ and $K_n^{(n)}$ are super edge-magic by Corollary 2.9. $\square$

To give a super edge-magic labeling on a hypergraph H, we then construct only a bijective function between V and $\{1, 2, 3, \ldots, p\}$ satisfying condition in Theorem 2.8. The following theorem is an observation concerning a complete 1-uniform hypergraph and a complete $(n-1)$-uniform hypergraph of n vertices.

Theorem 3.2. *$K_n^{(1)}$ and $K_n^{(n-1)}$ are super edge-magic.*

Proof. In the case of $n = 1$, the complete hypergraphs $K_1^{(1)}$ and $K_1^{(0)}$ are super edge-magic by Theorem 3.1. Hence, we assume that $n > 1$.

Let $V = V(K_n^{(1)}) = V(K_n^{(n-1)}) = \{v_1, v_2, v_3, \ldots, v_n\}$. Note that $E(K_n^{(1)}) = \{\{v_i\}|\ i \in \{1, 2, 3, \ldots, n\}\}$ and $E(K_n^{(n-1)}) = \{V - \{v_i\}|\ i \in \{1, 2, 3, \ldots, n\}\}$ have cardinality n. Then, we define the bijection $f : V \to \{1, 2, 3, \ldots, n\}$ by $f(v_i) = i$. Hence,

$$\left\{ \sum_{v \in e} f(v) \,\middle|\, e \in E(K_n^{(1)}) \right\} = \{i|\ i \in \{1, 2, 3, \ldots, n\}\}$$

and

$$\left\{ \sum_{v\in e} f(v) \;\middle|\; e \in E(K_n^{(n-1)}) \right\} = \{(1+2+3+\cdots+n) - i|\; i \in \{1,2,3,\ldots,n\}\}$$
$$= \left\{ \frac{n(n+1)}{2} - i \;\middle|\; i \in \{1,2,3,\ldots,n\} \right\}$$

which are the sets of n consecutive integers. By Theorem 2.8, $K_n^{(1)}$ and $K_n^{(n-1)}$ are super edge-magic with $\Lambda = 2n+1$ and $\Lambda = \frac{n(n+3)}{2}$, respectively. □

Example 3.1. Consider a complete 1-uniform hypergraph, $K_4^{(1)}$, of which $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$. We define a bijection f from V to $\{1,2,3,4\}$ by $f(v_i) = i$, then $\{\sum_{v\in e} f(v)|\; e \in E\} = \{1,2,3,4\}$ is the set of four consecutive integers. Therefore, by Theorem 2.8, f is a super edge-magic labeling of $K_4^{(1)}$.

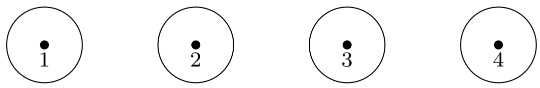

Figure 3.1: The vertex-labeling of a complete 1-uniform hypergraph, $K_4^{(1)}$

Example 3.2. By using the same bijection as in Example 3.1, we then obtain $\{\sum_{v\in e} f(v)|\; e \in E(K_4^{(3)})\} = \{6,7,8,9\}$ which is the set of four consecutive integers. Hence, by Theorem 2.8, f is a super edge-magic labeling of $K_4^{(3)}$.

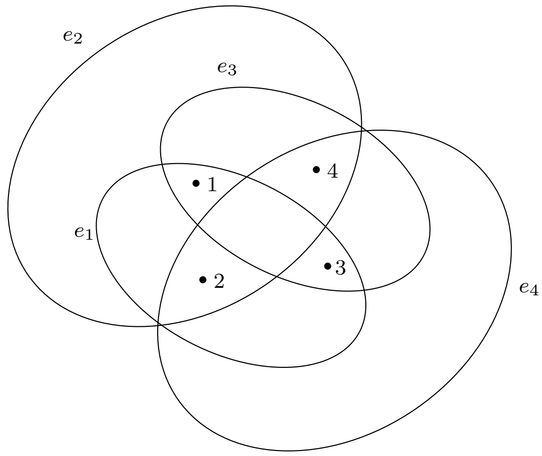

Figure 3.2: The vertex-labeling of a complete 3-uniform hypergraph, $K_4^{(3)}$

If we consider Theorem 3.1 and Theorem 3.2, one may ask a natural question. Are there any other super edge-magic complete uniform hypergraphs? The following lemma will give the answer.

Lemma 3.3. *Let n and k be integers such that $0 \leq k \leq n$. Then, $\binom{n}{k} \leq kn - k^2 + 1$ if and only if $k \in \{0, 1\}$ or $n \in \{k, k+1\}$.*

Proof. It is easy to see that $k \in \{0, 1\}$ or $n \in \{k, k+1\}$ satisfies the inequality. Assume that $k > 1$. We will show that if $n \geq k+2$, then $\binom{n}{k} > kn - k^2 + 1$, by using the mathematical induction on n.

First, let $n = k+2$, thus,

$$\binom{n}{k} = \binom{k+2}{k} = \frac{k^2+3k}{2} + 1 > \frac{k+3k}{2} + 1 = 2k + 1 = kn - k^2 + 1,$$

since $k^2 > k > 1$.

Next, assume $n \geq k+2$ be such that $\binom{n}{k} > kn - k^2 + 1$. Since $n - k \geq 2 > 0$ and $k > 1$, we obtain

$$k(n-k) > n - k$$

$$\begin{aligned}
kn - k^2 + 1 &> n + 1 - k \\
\frac{kn - k^2 + 1}{n + 1 - k} &> 1.
\end{aligned}$$

Observe that,

$$\begin{aligned}
\binom{n+1}{k} &= \frac{n+1}{n+1-k} \cdot \binom{n}{k} \\
&> \left(1 + \frac{k}{n+1-k}\right)(kn - k^2 + 1) \\
&= kn - k^2 + 1 + k\left(\frac{kn - k^2 + 1}{n+1-k}\right) \\
&> kn - k^2 + 1 + k \\
&= k(n+1) - k^2 + 1.
\end{aligned}$$

Therefore, $\binom{n}{k} > kn - k^2 + 1$ for every $n \geq k + 2$. Consequently, since $0 \leq k \leq n$, $\binom{n}{k} \leq kn - k^2 + 1$ if and only if $k \in \{0, 1\}$ or $n \in \{k, k+1\}$. $\square$

Note that n and $\binom{n}{k}$ in Lemma 3.3 are the number of vertices and hyperedges of $K_n^{(k)}$, respectively. By applying Theorem 2.10, we obtain the main result as Theorem 3.4.

Theorem 3.4. *A complete k-uniform hypergraph, $K_n^{(k)}$, is super edge-magic if and only if $k \in \{0, 1\}$ or $n \in \{k, k+1\}$.*

Proof. Assume that $K_n^{(k)}$ is super edge-magic. By Theorem 2.10, $\binom{n}{k} \leq kn - k^2 + 1$. Then, by Lemma 3.3, $k \in \{0, 1\}$ or $n \in \{k, k+1\}$.

On the other hand, assume that $k \in \{0, 1\}$ or $n \in \{k, k+1\}$.
If $k = 0$, then by Theorem 3.1, $K_n^{(k)}$ is super edge-magic.
If $k = 1$, then by Theorem 3.2, $K_n^{(k)}$ is super edge-magic.
If $n = k$, then by Theorem 3.1, $K_n^{(k)}$ is super edge-magic.
If $n = k + 1$, then by Theorem 3.2, $K_n^{(k)}$ is super edge-magic. $\square$

In ordinary graph, according to [1], the super edge-magic complete graphs are only K_1, K_2 and K_3. If we set $k = 2$ in Theorem 3.4, then the only 2-uniform super edge-magic hypergraphs are $K_2^{(2)}$ and $K_3^{(2)}$. Thus, our main result agrees with that in graph. Notice that, by Defintion 2.4, K_1 is not a 2-uniform complete hypergraph. However, K_1 is super edge-magic by Corollary 2.9 since it has no hyperedge.

CHAPTER IV

m-NODE k-UNIFORM HYPERPATHS

In this chapter, we will show that every m-node k-uniform hyperpath ${}^mP_n^{(k)}$, defined in Chapter 2, is super edge-magic. For simplicity, throughout this chapter, we use a hyperpath ${}^mP_n^{(k)}$ instead of an m-node k-uniform hyperpath. First, we begin with the known result in graph, i.e., ${}^1P_n^{(2)}$. For ease of reference, let we give a proof of this result here.

Theorem 4.1. *A hyperpath ${}^1P_n^{(2)}$ is super edge-magic.*

Proof. Note that, ${}^1P_n^{(2)}$ has $p = n+1$ and $q = n$.

Case (i): n is odd. Define $f : V \to \{1, 2, 3, \ldots, n+1\}$ by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1, 3, 5, \ldots, n\}, \\ \dfrac{n+1+i}{2} & \text{if } i \in \{2, 4, 6, \ldots, n+1\}. \end{cases}$$

To show that f is bijective, since $|V| = |\{1, 2, 3, \ldots, n+1\}|$, it suffices to show that f is surjective. Let $a \in \{1, 2, 3, \ldots, n+1\}$.

If $a \in \{1, 2, 3, \ldots, \frac{n+1}{2}\}$, then $2a - 1 \in \{1, 3, 5, \ldots, n\}$. Thus, $f(w_{2a-1,1}) = \frac{1+(2a-1)}{2} = a$.

If $a \in \{\frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \ldots, n+1\}$, then $2a - n - 1 \in \{2, 4, 6, \ldots, n+1\}$. Thus, $f(w_{2a-n-1,1}) = \frac{n+1+(2a-n-1)}{2} = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i =$

$\{w_{i,1}, w_{i+1,1}\} \in E$. Since for $i \in \{1,3,5,\ldots,n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{1+i}{2}\right) + \left(\frac{n+1+(i+1)}{2}\right) \\ &= \frac{n+3}{2} + i, \end{aligned}$$

and for $i \in \{2,4,6,\ldots,n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{n+1+i}{2}\right) + \left(\frac{1+(i+1)}{2}\right) \\ &= \frac{n+3}{2} + i. \end{aligned}$$

Hence, for $i \in \{1,2,3,\ldots,n\}$,

$$\sum_{v \in e_i} f(v) = f(w_{i,1}) + f(w_{i+1,1}) = \frac{n+3}{2} + i.$$

Thus, $\{\sum_{v\in e_i} f(v) | \; i \in \{1,2,3,\ldots,n\}\} = \{\frac{n+3}{2}+1, \frac{n+3}{2}+2, \frac{n+3}{2}+3, \ldots, \frac{n+3}{2}+n\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = \frac{5n+7}{2}$.

Case (ii): n is even. Define $f : V \to \{1,2,3,\ldots,n+1\}$ by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1,3,5,\ldots,n+1\}, \\ \dfrac{n+2+i}{2} & \text{if } i \in \{2,4,6,\ldots,n\}. \end{cases}$$

To show that f is bijective, since $|V| = |\{1,2,3,\ldots,n+1\}|$, it suffices to show that f is surjective. Let $a \in \{1,2,3,\ldots,n+1\}$.

If $a \in \{1,2,3,\ldots,\frac{2}{2}\}$, then $2a-1 \in \{1,3,5,\ldots,n+1\}$. Thus, $f(w_{2a-1,1}) = \frac{1+(2a-1)}{2} = a$.

If $a \in \{\frac{n+4}{2}, \frac{n+6}{2}, \frac{n+8}{2}, \ldots, n+1\}$, then $2a-n-2 \in \{2,4,6,\ldots,n\}$. Thus, $f(w_{2a-n-2,1}) = \frac{n+2+(2a-n-2)}{2} = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}\} \in E$. Since for $i \in \{1, 3, 5, \ldots, n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{1+i}{2}\right) + \left(\frac{n+2+(i+1)}{2}\right) \\ &= \frac{n+4}{2} + i, \end{aligned}$$

and for $i \in \{2, 4, 6, \ldots, n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{n+2+i}{2}\right) + \left(\frac{1+(i+1)}{2}\right) \\ &= \frac{n+4}{2} + i. \end{aligned}$$

Hence, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\sum_{v \in e_i} f(v) = f(w_{i,1}) + f(w_{i+1,1}) = \frac{n+4}{2} + i.$$

Thus, $\{\sum_{v \in e_i} f(v) |\ i \in \{1, 2, 3, \ldots, n\}\} = \{\frac{n+4}{2} + 1, \frac{n+4}{2} + 2, \frac{n+4}{2} + 3, \ldots, \frac{n+4}{2} + n\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = \frac{5n+8}{2}$. Consequently, the hyperpath ${}^1P_n^{(2)}$ is super edge-magic.

□

Example 4.1. For the hypergraph ${}^1P_5^{(2)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1}, w_{6,1},$$

and hyperedges

$$\begin{aligned} &e_1 = \{w_{1,1}, w_{2,1}\}, e_2 = \{w_{2,1}, w_{3,1}\}, e_3 = \{w_{3,1}, w_{4,1}\}, \\ &e_4 = \{w_{4,1}, w_{5,1}\}, e_5 = \{w_{5,1}, w_{6,1}\}, \end{aligned}$$

by Theorem 4.1, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 4, f(w_{3,1}) = 2,$$

$$f(w_{4,1}) = 5, f(w_{5,1}) = 3, f(w_{6,1}) = 6.$$

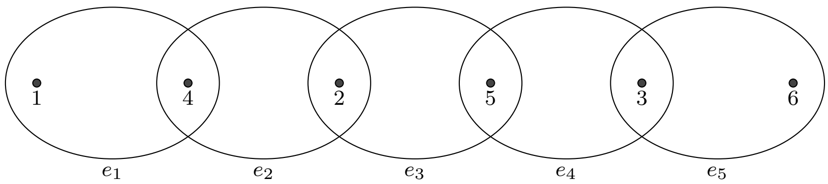

Figure 4.1: The vertex-labeling of ${}^1P_5^{(2)}$

The vertex-labeling of ${}^1P_5^{(2)}$ is shown in Figure 4.1 and we see that

$$\sum_{v \in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) = 1 + 4 = 5,$$

$$\sum_{v \in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) = 4 + 2 = 6,$$

$$\sum_{v \in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) = 2 + 5 = 7,$$

$$\sum_{v \in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) = 5 + 3 = 8,$$

$$\sum_{v \in e_5} f(v) = f(w_{5,1}) + f(w_{6,1}) = 3 + 6 = 9.$$

Then, $\{\sum_{v \in e_i} f(v) | \ i \in \{1, 2, 3, 4, 5\}\} = \{5, 6, 7, 8, 9\}$ is a set of five consecutive integers. Hence, by Theorem 2.8, ${}^1P_5^{(2)}$ is super edge-magic.

Example 4.2. For the hypergraph ${}^1P_4^{(2)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}\}, e_2 = \{w_{2,1}, w_{3,1}\}, e_3 = \{w_{3,1}, w_{4,1}\}, e_4 = \{w_{4,1}, w_{5,1}\},$$

by Theorem 4.1, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 4, f(w_{3,1}) = 2, f(w_{4,1}) = 5, f(w_{5,1}) = 3.$$

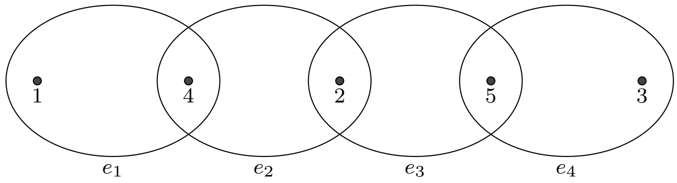

Figure 4.2: The vertex-labeling of ${}^1P_4^{(2)}$

The vertex-labeling of ${}^1P_4^{(2)}$ is shown in Figure 4.2 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) = 1 + 4 = 5,$$

$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) = 4 + 2 = 6,$$

$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) = 2 + 5 = 7,$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) = 5 + 3 = 8.$$

Then, $\{\sum_{v\in e_i} f(v)|\ i \in \{1,2,3,4\}\} = \{5,6,7,8\}$ is a set of four consecutive integers. Hence, by Theorem 2.8, ${}^1P_4^{(2)}$ is super edge-magic.

Next, we construct super edge-magic labelings for small hyperpaths ${}^1P_n^{(3)}$ (in Theorem 4.2), ${}^2P_n^{(4)}$ (in Theorem 4.3), and ${}^2P_n^{(5)}$ (in Theorem 4.4).

Theorem 4.2. *A hyperpath* ${}^1P_n^{(3)}$ *is super edge-magic.*

Proof. Note that, ${}^1P_n^{(3)}$ has $p = 2n+1$ and $q = n$.

Define $f : V \to \{1,2,3,\dots,2n+1\}$ by

$$\begin{aligned} f(w_{i,1}) &= i && \text{if } i \in \{1,2,3,\dots,n+1\},\\ f(v_{i,1}) &= 2n+2-i && \text{if } i \in \{1,2,3,\dots,n\}. \end{aligned}$$

To show that f is bijective, since $|V| = |\{1,2,3,\dots,2n+1\}|$, it suffices to show that f is surjective. Let $a \in \{1,2,3,\dots,2n+1\}$.

If $a \in \{1, 2, 3, \ldots, n+1\}$, then $f(w_{a,1}) = a$;

If $a \in \{n+2, n+3, n+4, \ldots, 2n+1\}$, then $2n+2-a \in \{1, 2, 3, \ldots, n\}$. Thus, $f(v_{2n+2-a,1}) = 2n+2-(2n+2-a) = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, v_{i,1}\} \in E$. We observe that, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= i + (i+1) \\ &= 1 + 2i. \end{aligned}$$

Hence, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(v_{i,1}) \\ &= (1+2i) + (2n+2-i) \\ &= 2n+3+i. \end{aligned}$$

Thus, $\{\sum_{v \in e_i} f(v) | \, i \in \{1, 2, 3, \ldots, n\}\} = \{2n+4, 2n+5, 2n+6, \ldots, 3n+3\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = 5n+5$. $\square$

Example 4.3. For the hypergraph ${}^1P_4^{(3)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

$$v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, v_{1,1}\}, e_2 = \{w_{2,1}, w_{3,1}, v_{2,1}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, v_{3,1}\}, e_4 = \{w_{4,1}, w_{5,1}, v_{4,1}\},$$

by Theorem 4.2, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 2, f(w_{3,1}) = 3, f(w_{4,1}) = 4, f(w_{5,1}) = 5,$$
$$f(v_{1,1}) = 9, f(v_{2,1}) = 8, f(v_{3,1}) = 7, f(v_{4,1}) = 6.$$

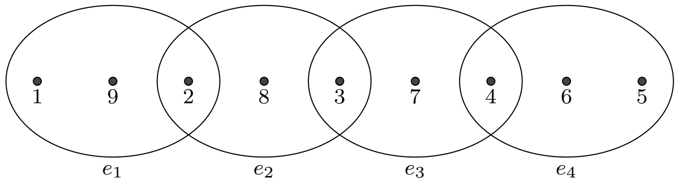

Figure 4.3: The vertex-labeling of ${}^1P_4^{(3)}$

The vertex-labeling of ${}^1P_4^{(3)}$ is shown in Figure 4.3 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(v_{1,1}) = 1 + 2 + 9 = 12,$$
$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(v_{2,1}) = 2 + 3 + 8 = 13,$$
$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(v_{3,1}) = 3 + 4 + 7 = 14,$$
$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(v_{4,1}) = 4 + 5 + 6 = 15.$$

Then, $\{\sum_{v\in e_i} f(v) |\ i \in \{1, 2, 3, 4\}\} = \{12, 13, 14, 15\}$ is a set of four consecutive integers. Hence, by Theorem 2.8, ${}^1P_4^{(3)}$ is super edge-magic.

Theorem 4.3. *A hyperpath* ${}^2P_n^{(4)}$ *is super edge-magic.*

Proof. Note that, ${}^2P_n^{(4)}$ has $p = 2n + 2$ and $q = n$.

Case (i): n is odd. Define $f : V \to \{1, 2, 3, \dots, 2n + 2\}$ by

$$f(w_{i,j}) = \begin{cases} i & \text{if } i \in \{1, 2, 3, \dots, n+1\} \text{ and } j = 1, \\ \dfrac{4n+5-i}{2} & \text{if } i \in \{1, 3, 5, \dots, n\} \text{ and } j = 2, \\ \dfrac{3n+5-i}{2} & \text{if } i \in \{2, 4, 6, \dots, n+1\} \text{ and } j = 2. \end{cases}$$

To show that f is bijective, since $|V| = |\{1, 2, 3, \ldots, 2n+2\}|$, it suffices to show that f is surjective. Let $a \in \{1, 2, 3, \ldots, 2n+2\}$.

If $a \in \{1, 2, 3, \ldots, n+1\}$, then $f(w_{a,1}) = a$.

If $a \in \{n+2, n+3, n+4, \ldots, \frac{3n+3}{2}\}$, then $3n+5-2a \in \{2, 4, 6, \ldots, n+1\}$. Thus, $f(w_{3n+5-2a,2}) = \frac{3n+5-(3n+5-2a)}{2} = a$.

If $a \in \{\frac{3n+5}{2}, \frac{3n+7}{2}, \frac{3n+9}{2}, \ldots, 2n+2\}$, then $4n+5-2a \in \{1, 3, 5, \ldots, n\}$. Thus, $f(w_{4n+5-2a,1}) = \frac{4n+5-(4n+5-2a)}{2} = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, w_{i,2}, w_{i+1,2}\} \in E$. Since for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= i + (i+1) \\ &= 1 + 2i, \end{aligned}$$

for $i \in \{1, 3, 5, \ldots, n\}$,

$$\begin{aligned} f(w_{i,2}) + f(w_{i+1,2}) &= \frac{4n+5-i}{2} + \frac{3n+5-(i+1)}{2} \\ &= \frac{7n+9}{2} - i, \end{aligned}$$

for $i \in \{2, 4, 6, \ldots, n-1\}$,

$$\begin{aligned} f(w_{i,2}) + f(w_{i+1,2}) &= \frac{3n+5-i}{2} + \frac{4n+5-(i+1)}{2} \\ &= \frac{7n+9}{2} - i. \end{aligned}$$

Hence, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (1+2i) + \left(\frac{7n+9}{2} - i\right) \\ &= \frac{7n+11}{2} + i. \end{aligned}$$

Thus, $\{\sum_{v\in e_i} f(v) |\, i \in \{1,2,3,\ldots,n\}\} = \{\frac{7n+11}{2}+1, \frac{7n+11}{2}+2, \frac{7n+11}{2}+3, \ldots, \frac{7n+11}{2}+n\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = \frac{13n+17}{2}$.

Case (ii): n is even. Define $f : V \to \{1,2,3,\ldots,2n+2\}$ by

$$f(w_{i,j}) = \begin{cases} i & \text{if } i \in \{1,2,3,\ldots,n+1\} \text{ and } j=1, \\ \dfrac{3n+5-i}{2} & \text{if } i \in \{1,3,5,\ldots,n+1\} \text{ and } j=2, \\ \dfrac{4n+6-i}{2} & \text{if } i \in \{2,4,6,\ldots,n\} \text{ and } j=2. \end{cases}$$

To show that f is bijective, since $|V| = |\{1,2,3,\ldots,2n+2\}|$, it suffices to show that f is surjective. Let $a \in \{1,2,3,\ldots,2n+2\}$.

If $a \in \{1,2,3,\ldots,n+1\}$, then $f(w_{a,1}) = a$.

If $a \in \{n+2, n+3, n+4, \ldots, \frac{3n+4}{2}\}$, then $3n+5-2a \in \{1,3,5,\ldots,n+1\}$. Thus, $f(w_{3n+5-2a,2}) = \frac{3n+5-(3n+5-2a)}{2} = a$.

If $a \in \{\frac{3n+6}{2}, \frac{3n+8}{2}, \frac{3n+10}{2}, \ldots, 2n+2\}$, then $4n+6-2a \in \{2,4,6,\ldots,n\}$. Thus, $f(w_{4n+6-2a,2}) = \frac{4n+6-(4n+6-2a)}{2} = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1,2,3,\ldots,n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, w_{i,2}, w_{i+1,2}\} \in E$. Since for $i \in \{1,2,3,\ldots,n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= i + (i+1) \\ &= 1 + 2i, \end{aligned}$$

for $i \in \{1,3,5,\ldots,n-1\}$,

$$\begin{aligned} f(w_{i,2}) + f(w_{i+1,2}) &= \frac{3n+5-i}{2} + \frac{4n+6-(i+1)}{2} \\ &= \frac{7n+10}{2} - i, \end{aligned}$$

for $i \in \{2,4,6,\ldots,n\}$,

$$f(w_{i,2}) + f(w_{i+1,2}) = \frac{4n+6-i}{2} + \frac{3n+5-(i+1)}{2}$$

$$= \frac{7n+10}{2} - i.$$

Hence, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (1+2i) + \left(\frac{7n+10}{2} - i\right) \\ &= \frac{7n+12}{2} + i.\end{aligned}$$

Thus, $\{\sum_{v \in e_i} f(v) |\, i \in \{1, 2, 3, \ldots, n\}\} = \{\frac{7n+12}{2}+1, \frac{7n+12}{2}+2, \frac{7n+12}{2}+3, \ldots, \frac{7n+12}{2}+n\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = \frac{13n+18}{2}$. Consequently, the hyperpath ${}^2P_n^{(4)}$ is super edge-magic.

□

Example 4.4. For the hypergraph ${}^2P_5^{(4)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1}, w_{6,1},$$

$$w_{1,2}, w_{2,2}, w_{3,2}, w_{4,2}, w_{5,2}, w_{6,2},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}\}, e_2 = \{w_{2,1}, w_{3,1}, w_{2,2}, w_{3,2}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, w_{3,2}, w_{4,2}\}, e_4 = \{w_{4,1}, w_{5,1}, w_{4,2}, w_{5,2}\},$$

$$e_5 = \{w_{5,1}, w_{6,1}, w_{5,2}, w_{6,2}\},$$

by Theorem 4.3, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 2, f(w_{3,1}) = 3, f(w_{4,1}) = 4, f(w_{5,1}) = 5, f(w_{6,1}) = 6,$$

$$f(w_{1,2}) = 12, f(w_{2,2}) = 9, f(w_{3,2}) = 11, f(w_{4,2}) = 8, f(w_{5,2}) = 10, f(w_{6,2}) = 7.$$

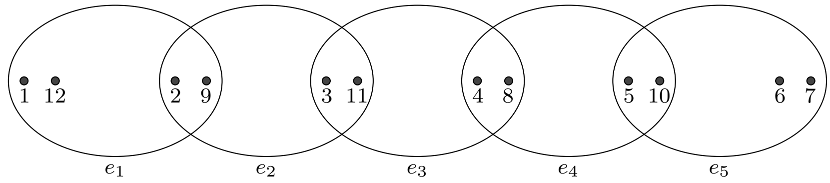

Figure 4.4: The vertex-labeling of ${}^2P_5^{(4)}$

The vertex-labeling of ${}^2P_5^{(4)}$ is shown in Figure 4.4 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(w_{1,2}) + f(w_{2,2}) = 1 + 2 + 12 + 9 = 24,$$

$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(w_{2,2}) + f(w_{3,2}) = 2 + 3 + 9 + 11 = 25,$$

$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(w_{3,2}) + f(w_{4,2}) = 3 + 4 + 11 + 8 = 26,$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(w_{4,2}) + f(w_{5,2}) = 4 + 5 + 8 + 10 = 27,$$

$$\sum_{v\in e_5} f(v) = f(w_{5,1}) + f(w_{6,1}) + f(w_{5,2}) + f(w_{6,2}) = 5 + 6 + 10 + 7 = 28.$$

Then, $\{\sum_{v\in e_i} f(v)|\ i \in \{1, 2, 3, 4, 5\}\} = \{24, 25, 26, 27, 28\}$ is a set of five consecutive integers. Hence, by Theorem 2.8, ${}^2P_5^{(4)}$ is super edge-magic.

Example 4.5. For the hypergraph ${}^2P_4^{(4)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

$$w_{1,2}, w_{2,2}, w_{3,2}, w_{4,2}, w_{5,2}$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}\}, e_2 = \{w_{2,1}, w_{3,1}, w_{2,2}, w_{3,2}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, w_{3,2}, w_{4,2}\}, e_4 = \{w_{4,1}, w_{5,1}, w_{4,2}, w_{5,2}\},$$

by Theorem 4.3, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 2, f(w_{3,1}) = 3, f(w_{4,1}) = 4, f(w_{5,1}) = 5,$$

$$f(w_{1,2}) = 8, f(w_{2,2}) = 10, f(w_{3,2}) = 7, f(w_{4,2}) = 9, f(w_{5,2}) = 6.$$

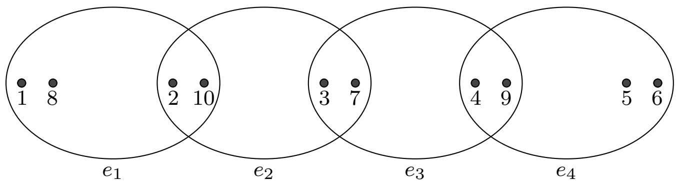

Figure 4.5: The vertex-labeling of ${}^2P_4^{(4)}$

The vertex-labeling of ${}^2P_4^{(4)}$ is shown in Figure 4.5 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(w_{1,2}) + f(w_{2,2}) = 1+2+8+10 = 21,$$

$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(w_{2,2}) + f(w_{3,2}) = 2+3+10+7 = 22,$$

$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(w_{3,2}) + f(w_{4,2}) = 3+4+7+9 = 23,$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(w_{4,2}) + f(w_{5,2}) = 4+5+9+6 = 24.$$

Then, $\{\sum_{v\in e_i} f(v) | \; i \in \{1,2,3,4\}\} = \{21,22,23,24\}$ is a set of four consecutive integers. Hence, by Theorem 2.8, ${}^2P_4^{(4)}$ is super edge-magic.

Theorem 4.4. *A hyperpath ${}^2P_n^{(5)}$ is super edge-magic.*

Proof. Note that, ${}^2P_n^{(5)}$ has $p = 3n+2$ and $q = n$.

Define $f : V \to \{1,2,3,\dots,3n+2\}$ by

$$f(w_{i,j}) = \begin{cases} i & \text{if } i \in \{1,2,3,\dots,n+1\} \text{ and } j = 1, \\ 2n+3-i & \text{if } i \in \{1,2,3,\dots,n+1\} \text{ and } j = 2, \end{cases}$$

$$f(v_{i,1}) = \quad 2n+2+i \quad \text{if } i \in \{1,2,3,\dots,n\}.$$

To show that f is bijective, since $|V| = |\{1,2,3,\dots,3n+2\}|$, it suffices to show that f is surjective. Let $a \in \{1,2,3,\dots,3n+2\}$.

If $a \in \{1, 2, 3, \ldots, n+1\}$, then $f(w_{a,1}) = a$.

If $a \in \{n+2, n+3, n+4, \ldots, 2n+2\}$, then $2n+3-a \in \{1, 2, 3, \ldots, n+1\}$. Thus, $f(w_{2n+3-a,2}) = 2n+3-(2n+3-a) = a$.

If $a \in \{2n+3, 2n+4, 2n+5, \ldots, 3n+2\}$, then $a-2n-2 \in \{1, 2, 3, \ldots, n\}$. Thus, $f(v_{a-2n-2,1}) = 2n+2+(a-2n-2) = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, w_{i,2}, w_{i+1,2}, v_{i,1}\} \in E$. We observe that, for $i \in \{1, 2, 3, \ldots, n+1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= i + (2n+3-i) \\ &= 2n+3. \end{aligned}$$

Hence, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) + f(v_{i,1}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) + f(v_{i,1}) \\ &= (2n+3) + (2n+3) + (2n+2+i) \\ &= 6n+8+i. \end{aligned}$$

Thus, $\{\sum_{v \in e_i} f(v) | \ i \in \{1, 2, 3, \ldots, n\}\} = \{6n+9, 6n+10, 6n+11, \ldots, 7n+8\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = 10n+11$. $\square$

Example 4.6. For the hypergraph ${}^2P_4^{(5)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

$$w_{1,2}, w_{2,2}, w_{3,2}, w_{4,2}, w_{5,2},$$

$$v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}, v_{1,1}\}, e_2 = \{w_{2,1}, w_{3,1}, w_{2,2}, w_{3,2}, v_{2,1}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, w_{3,2}, w_{4,2}, v_{3,1}\}, e_4 = \{w_{4,1}, w_{5,1}, w_{4,2}, w_{5,2}, v_{4,1}\},$$

by Theorem 4.4, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 2, f(w_{3,1}) = 3, f(w_{4,1}) = 4, f(w_{5,1}) = 5,$$

$$f(w_{1,2}) = 10, f(w_{2,2}) = 9, f(w_{3,2}) = 8, f(w_{4,2}) = 7, f(w_{5,2}) = 6,$$

$$f(v_{1,1}) = 11, f(v_{2,1}) = 12, f(v_{3,1}) = 13, f(v_{4,1}) = 14.$$

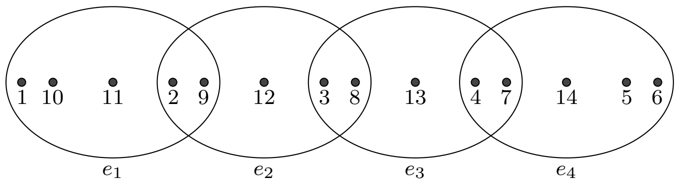

Figure 4.6: The vertex-labeling of ${}^2P_4^{(5)}$

The vertex-labeling of ${}^2P_4^{(5)}$ is shown in Figure 4.6 and we see that

$$\begin{aligned}\sum_{v \in e_1} f(v) &= f(w_{1,1}) + f(w_{2,1}) + f(w_{1,2}) + f(w_{2,2}) + f(v_{1,1}) \\ &= 1 + 2 + 10 + 9 + 11 \\ &= 33,\end{aligned}$$

$$\begin{aligned}\sum_{v \in e_2} f(v) &= f(w_{2,1}) + f(w_{3,1}) + f(w_{2,2}) + f(w_{3,2}) + f(v_{2,1}) \\ &= 2 + 3 + 9 + 8 + 12 \\ &= 34,\end{aligned}$$

$$\begin{aligned}\sum_{v \in e_3} f(v) &= f(w_{3,1}) + f(w_{4,1}) + f(w_{3,2}) + f(w_{4,2}) + f(v_{3,1}) \\ &= 3 + 4 + 8 + 7 + 13\end{aligned}$$

$$
\begin{aligned}
&= 35, \\
\sum_{v \in e_4} f(v) &= f(w_{4,1}) + f(w_{5,1}) + f(w_{4,2}) + f(w_{5,2}) + f(v_{4,1}) \\
&= 4 + 5 + 7 + 6 + 14 \\
&= 36.
\end{aligned}
$$

Then, $\{\sum_{v \in e_i} f(v) |\ i \in \{1, 2, 3, 4\}\} = \{33, 34, 35, 36\}$ is a set of four consecutive integers. Hence, by Theorem 2.8, ${}^2P_4^{(5)}$ is super edge-magic.

Now, we are ready to enlarge the small super edge-magic hyperpaths, ${}^1P_n^{(2)}$, ${}^1P_n^{(3)}$, ${}^2P_n^{(4)}$, and ${}^2P_n^{(5)}$ to bigger super edge-magic hyperpaths by either adding more vertices into the uniform hyperedges or adding more vertices into each node of a super edge-magic hyperpath.

Lemma 4.5. *If ${}^mP_n^{(k)}$ is super edge-magic, then ${}^mP_n^{(k+2)}$ is also super edge-magic.*

Proof. Let ${}^mP_n^{(k)} = (V, E)$. By Definition 2.5, we can construct ${}^mP_n^{(k+2)}$ by considering each $e_i \in E$, define $\tilde{e_i} = e_i \cup \{v_{i,k-2m+1}, v_{i,k-2m+2}\}$ for $i \in \{1, 2, 3, \ldots, n\}$. Then, ${}^mP_n^{(k+2)} = (\tilde{V}, \tilde{E})$ where $\tilde{E} = \{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \ldots, \tilde{e_n}\}$ and $\tilde{V} = \bigcup_{i=1}^{n} \tilde{e_i}$. Note that, $|V| = (n-1)(k-m) + k, |\tilde{V}| = (n-1)(k-m) + k + 2n,$ and $|E| = n = |\tilde{E}|$.

Assume that ${}^mP_n^{(k)}$ is super edge-magic. By Theorem 2.8, there is a bijection $f : V \to \{1, 2, 3, \ldots, (n-1)(k-m) + k\}$ such that $\{\sum_{v \in e} f(v) |\ e \in E\}$ is a set of n consecutive integers. We define a function $\hat{f}$ by

$$
\hat{f}(v_{i,j}) = \begin{cases} (n-1)(k-m) + k + i & \text{if } j = k - 2m + 1, \\ (n-1)(k-m) + k + 2n + 1 - i & \text{if } j = k - 2m + 2, \end{cases}
$$

for each $i \in \{1, 2, 3, \ldots, n\}$. Next, we define $\tilde{f} : \tilde{V} \to \{1, 2, 3, \ldots, (n-1)(k-m) +$

$k + 2n\}$ by

$$\tilde{f}(v) = \begin{cases} f(v) & \text{if } v \in V, \\ \hat{f}(v) & \text{if } v \in \tilde{V} - V, \end{cases}$$

for $v \in \tilde{V}$. Since $f(V) = \{1, 2, 3, \ldots, (n-1)(k-m)+k\}$ and $\hat{f}(\tilde{V} - V) = \{(n-1)(k-m)+k+1, (n-1)(k-m)+k+2, (n-1)(k-m)+k+3, \ldots, (n-1)(k-m)+k+2n\}$, $\tilde{f}$ is a bijection from $\tilde{V}$ to $\{1, 2, 3, \ldots, (n-1)(k-m)+k+2n\}$.

To see that $\tilde{f}$ is a super edge-magic labeling, let $\tilde{e}_i \in \tilde{E}$. Then, for $i \in \{1, 2, 3, \ldots, n\}$,

$$\begin{aligned} \sum_{v \in \tilde{e}_i} \tilde{f}(v) &= \sum_{v \in e_i} \tilde{f}(v) + \tilde{f}(v_{i,k-2m+1}) + \tilde{f}(v_{i,k-2m+2}) \\ &= \sum_{v \in e_i} f(v) + \hat{f}(v_{i,k-2m+1}) + \hat{f}(v_{i,k-2m+2}) \\ &= \sum_{v \in e_i} f(v) + ((n-1)(k-m)+k+i) + ((n-1)(k-m)+k+2n+1-i) \\ &= \sum_{v \in e_i} f(v) + 2(n-1)(k-m) + 2k + 2n + 1. \end{aligned}$$

Since $\{\sum_{v \in e} f(v) | \; e \in E\}$ is a set of n consecutive integers, similar goes for $\{\sum_{v \in \tilde{e}} \tilde{f}(v) | \; \tilde{e} \in \tilde{E}\}$. Consequently, by Theorem 2.8, ${}^mP_n^{(k+2)}$ is super edge-magic.

□

Example 4.7. By applying Lemma 4.5 with the vertex-labeling of ${}^1P_4^{(2)}$ (shown in Figure 4.2), we obtain the vertex-labeling for ${}^1P_4^{(4)}$ which is also super edge-magic labeling as shown in Figure 4.7.

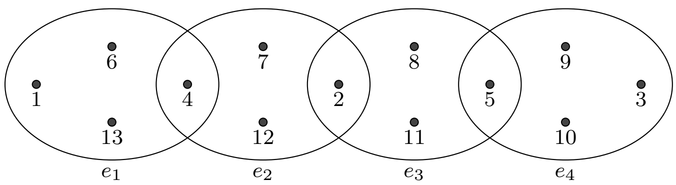

Figure 4.7: The vertex-labeling of ${}^1P_4^{(4)}$ obtained by applying Lemma 4.5 to the vertex-labeling of ${}^1P_4^{(2)}$

Lemma 4.6. *If* ${}^{m}P_n^{(k)}$ *is super edge-magic, then* ${}^{m+2}P_n^{(k+4)}$ *is also super edge-magic.*

Proof. Let ${}^{m}P_n^{(k)} = (V, E)$. By Definition 2.5, we can construct ${}^{m+2}P_n^{(k+4)}$ by considering each $e_i \in E$, define $\tilde{e_i} = e_i \cup \{w_{i,m+1}, w_{i,m+2}, w_{i+1,m+1}, w_{i+1,m+2}\}$ for $i \in \{1, 2, 3, \ldots, n\}$. Then, ${}^{m+2}P_n^{(k+4)} = (\tilde{V}, \tilde{E})$ where $\tilde{E} = \{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \ldots, \tilde{e_n}\}$ and $\tilde{V} = \bigcup_{i=1}^{n} \tilde{e_i}$. Note that, $|V| = (n-1)(k-m)+k, |\tilde{V}| = (n-1)(k-m)+k+2n+2$, and $|E| = n = |\tilde{E}|$.

Assume that ${}^{m}P_n^{(k)}$ is super edge-magic. By Theorem 2.8, there is a bijection $f : V \to \{1, 2, 3, \ldots, (n-1)(k-m)+k\}$ such that $\{\sum_{v\in e} f(v)|\ e \in E\}$ is a set of n consecutive integers. We define a function $\hat{f}$ by

$$\hat{f}(w_{i,j}) = \begin{cases} (n-1)(k-m)+k+i & \text{if } j = m+1, \\ (n-1)(k-m)+k+2n+3-i & \text{if } j = m+2, \end{cases}$$

for each $i \in \{1, 2, 3, \ldots, n+1\}$. Next, we define $\tilde{f} : \tilde{V} \to \{1, 2, 3, \ldots, 1, 2, 3, \ldots, (n-1)(k-m)+k+2n+2\}$ by

$$\tilde{f}(v) = \begin{cases} f(v) & \text{if } v \in V, \\ \hat{f}(v) & \text{if } v \in \tilde{V} - V, \end{cases}$$

for $v \in \tilde{V}$. Since $f(V) = \{1, 2, 3, \ldots, (n-1)(k-m)+k\}$ and $\hat{f}(\tilde{V} - V) = \{(n-1)(k-m)+k+1, (n-1)(k-m)+k+2, (n-1)(k-m)+k+3, \ldots, (n-1)(k-m)+k+2n+2\}$, $\tilde{f}$ is a bijection from $\tilde{V}$ to $\{1, 2, 3, \ldots, (n-1)(k-m)+k+2n+2\}$.

To see that $\tilde{f}$ is a super edge-magic labeling, let $\tilde{e_i} \in \tilde{E}$. Observe that, for $i \in \{1, 2, 3, \ldots, n\}$.

$$\begin{aligned} \hat{f}(w_{i,m+1}) + \hat{f}(w_{i,m+2}) &= ((n-1)(k-m)+k+i) \\ &\quad + ((n-1)(k-m)+k+2n+3-i) \end{aligned}$$

$$= 2(n-1)(k-m) + 2k + 2n + 3.$$

Then, for $i \in \{1, 2, 3, \ldots, n\}$.

$$\begin{aligned}
\sum_{v \in \tilde{e}_i} \tilde{f}(v) &= \sum_{v \in e_i} \tilde{f}(v) + \tilde{f}(w_{i,m+1}) + \tilde{f}(w_{i,m+2}) + \tilde{f}(w_{i+1,m+1}) + \tilde{f}(w_{i+1,m+2}) \\
&= \sum_{v \in e_i} f(v) + \hat{f}(w_{i,m+1}) + \hat{f}(w_{i,m+2}) + \hat{f}(w_{i+1,m+1}) + \hat{f}(v_{i+1,m+2}) \\
&= \sum_{v \in e_i} f(v) + (2(n-1)(k-m) + 2k + 2n + 3) \\
&\qquad + (2(n-1)(k-m) + 2k + 2n + 3) \\
&= \sum_{v \in e_i} f(v) + 4(n-1)(k-m) + 4k + 4n + 6.
\end{aligned}$$

Since $\{\sum_{v \in e} f(v) |\ e \in E\}$ is a set of n consecutive integers, similar goes for $\{\sum_{v \in \tilde{e}} \tilde{f}(v) |\ \tilde{e} \in \tilde{E}\}$. Consequently, by Theorem 2.8, ${}^{m+2}P_n^{(k+4)}$ is super edge-magic. $\square$

Example 4.8. By applying Lemma 4.6 with the vertex-labeling of ${}^{1}P_4^{(2)}$ (shown in Figure 4.2), we obtain the vertex-labeling for ${}^{3}P_4^{(6)}$ which is also super edge-magic labeling as shown in Figure 4.8.

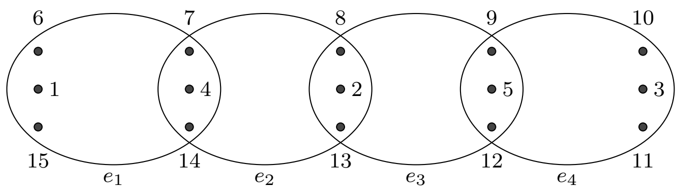

Figure 4.8: The vertex-labeling of ${}^{3}P_4^{(6)}$ obtained by applying Lemma 4.6 to the vertex-labeling of ${}^{1}P_4^{(2)}$

To sum up this chapter, by using both Lemma 4.5 and Lemma 4.6, we will show that all hyperpaths ${}^{m}P_n^{(k)}$ are super edge-magic. In general, to construct a super edge-magic hyperpath ${}^{m}P_n^{(k)}$, we first find the starting small hyperpath of which super edge-magic as follow:

- if m and k are odd, then the starting hyperpath is ${}^{1}P_n^{(3)}$,
- if m and k are even, then the starting hyperpath is ${}^{2}P_n^{(4)}$,
- if m is odd and k is even, then the starting hyperpath is ${}^{1}P_n^{(2)}$,
- if m is even and k is odd, then the starting hyperpath is ${}^{2}P_n^{(5)}$.

Then, we apply Lemma 4.6 for $\lceil \frac{m-2}{2} \rceil$ times and Lemma 4.5 for $\lceil \frac{k-2m-1}{2} \rceil$ in any order to the starting hyperpath. We then obtain the super edge-magic labeling for ${}^{m}P_n^{(k)}$. Therefore, we have the main theorem.

Theorem 4.7. *All hyperpaths ${}^{m}P_n^{(k)}$ are super edge-magic.*

Example 4.9. To obtain the super edge-magic labeling of ${}^{5}P_4^{(13)}$ with $m = 5$ which is odd and $k = 13$ which is odd, we start with the super edge-magic labeling of ${}^{1}P_4^{(3)}$. Then, we apply Lemma 4.6 for $\lceil \frac{5-2}{2} \rceil = 2$ times and Lemma 4.5 for $\left\lceil \frac{13-2(5)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^{3}P_4^{(7)}, {}^{5}P_4^{(11)}$ and ${}^{5}P_4^{(13)}$, respectively.

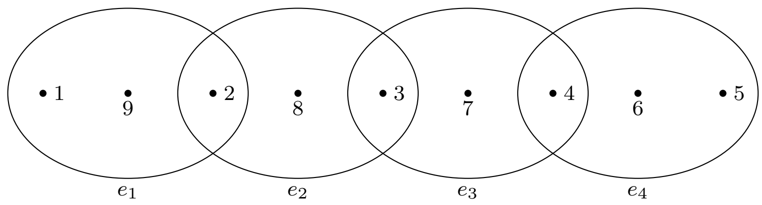

Figure 4.9: The super edge-magic labeling of ${}^{1}P_4^{(3)}$

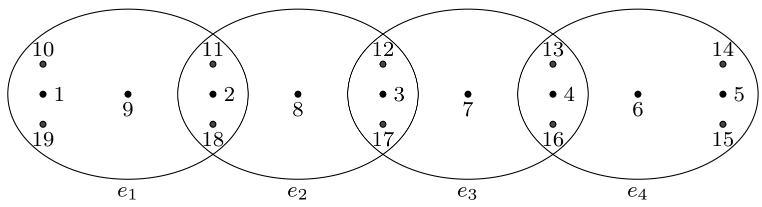

Figure 4.10: The super edge-magic labeling of ${}^{3}P_4^{(7)}$

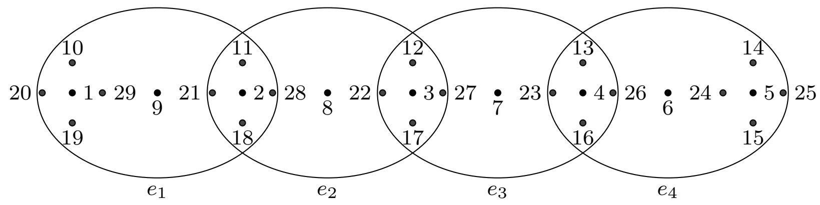

Figure 4.11: The super edge-magic labeling of ${}^5P_4^{(11)}$

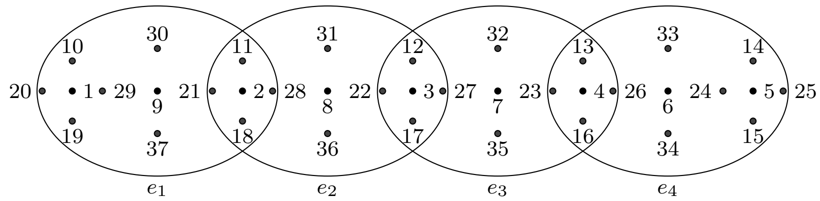

Figure 4.12: The super edge-magic labeling of ${}^5P_4^{(13)}$

Example 4.10. To obtain the super edge-magic labeling of ${}^4P_4^{(10)}$ with $m = 4$ which is even and $k = 10$ which is even, we start with the super edge-magic labeling of ${}^2P_4^{(4)}$. Then, we apply Lemma 4.6 for $\left\lceil \frac{4-2}{2} \right\rceil = 1$ time and Lemma 4.5 for $\left\lceil \frac{10-2(4)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^4P_4^{(8)}$ and ${}^4P_4^{(10)}$, respectively.

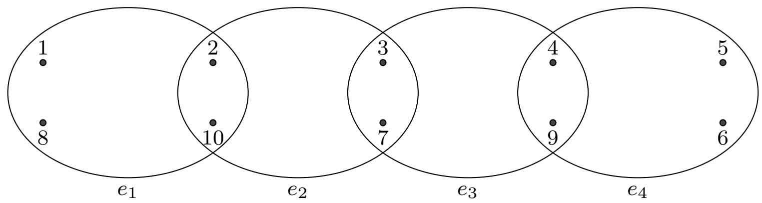

Figure 4.13: The super edge-magic labeling of ${}^2P_4^{(4)}$

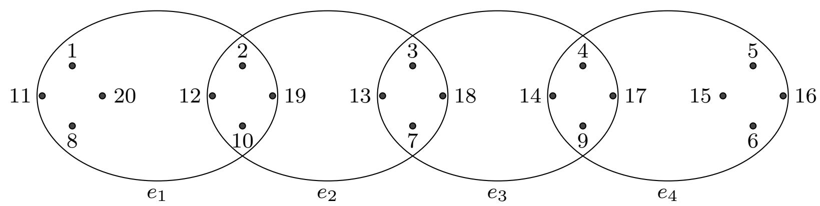

Figure 4.14: The super edge-magic labeling of ${}^4P_4^{(8)}$

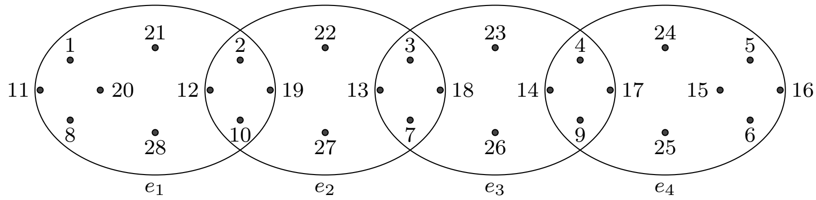

Figure 4.15: The super edge-magic labeling of ${}^4P_4^{(10)}$

Example 4.11. To obtain the super edge-magic labeling of ${}^5P_4^{(12)}$ with $m = 5$ which is odd and $k = 12$ which is even, we start with the super edge-magic labeling of ${}^1P_4^{(2)}$. Then, we apply Lemma 4.6 for $\lceil \frac{5-2}{2} \rceil = 2$ times and Lemma 4.5 for $\left\lceil \frac{12-2(5)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^3P_4^{(6)}, {}^5P_4^{(10)}$ and ${}^5P_4^{(12)}$, respectively.

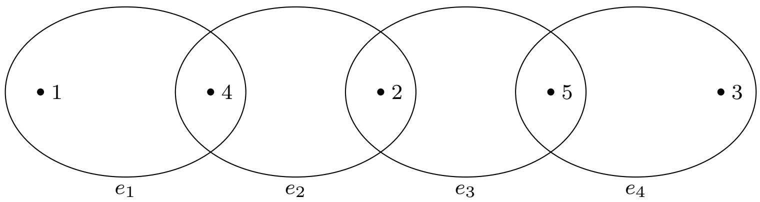

Figure 4.16: The super edge-magic labeling of ${}^1P_4^{(2)}$

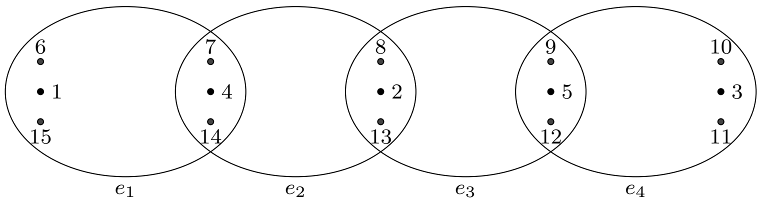

Figure 4.17: The super edge-magic labeling of ${}^3P_4^{(6)}$

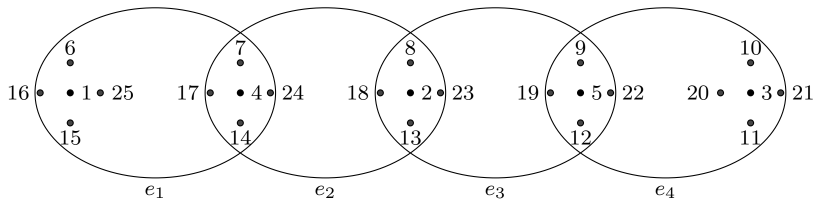

Figure 4.18: The super edge-magic labeling of ${}^5P_4^{(10)}$

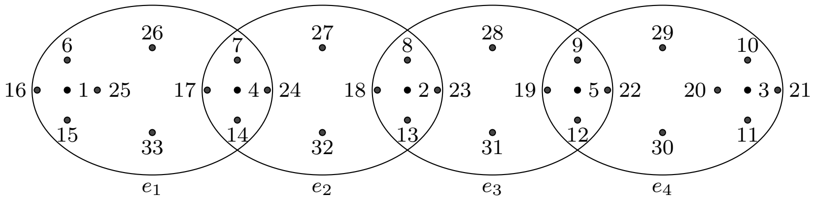

Figure 4.19: The super edge-magic labeling of ${}^5P_4^{(12)}$

Example 4.12. To obtain the super edge-magic labeling of ${}^4P_4^{(11)}$ with $m = 4$ which is even and $k = 11$ which is odd, we start with the super edge-magic labeling of ${}^2P_4^{(5)}$. Then, we apply Lemma 4.6 for $\lceil \frac{4-2}{2} \rceil = 1$ time and Lemma 4.5 for $\left\lceil \frac{11-2(4)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^4P_4^{(9)}$ and ${}^4P_4^{(11)}$, respectively.

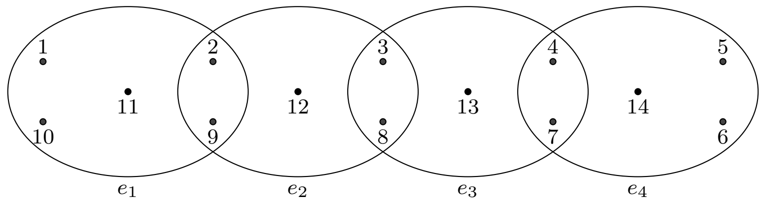

Figure 4.20: The super edge-magic labeling of ${}^2P_4^{(5)}$

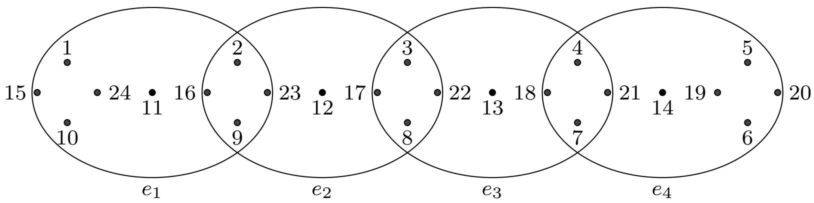

Figure 4.21: The super edge-magic labeling of ${}^4P_4^{(9)}$

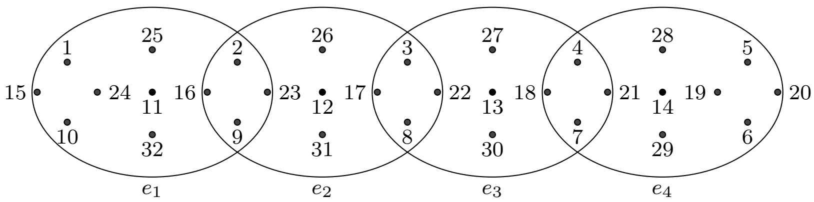

Figure 4.22: The super edge-magic labeling of ${}^4P_4^{(11)}$

CHAPTER V

m-NODE k-UNIFORM HYPERCYCLES

In this chapter, we will show that under some conditions on m, n and k, an m-node k-uniform hypercycle ${}^{m}C_n^{(k)}$, defined in Chapter 2, is super edge-magic. Note again that, m, n and k are always positive integers such that $n \geq 3$ and $k \geq 2m \geq 2$. For simplicity, throughout this chapter, we use a hypercycle ${}^{m}C_n^{(k)}$ instead of an m-node k-uniform hypercycle. According to [1], a hypercycle ${}^{1}C_n^{(2)}$, an ordinary cycle C_n in graph theory, is super edge-magic if and only if n is odd. Hence, we state this fact in the first theorem and omit the proof.

Theorem 5.1. [1] *A hypercycle ${}^{1}C_n^{(2)}$ is super edge-magic if and only if n is odd.*

Remark that, when n is odd, ${}^{1}C_n^{(2)}$ is super edge-magic by a bijection $f : V \to \{1, 2, 3, \ldots, n\}$ defined by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1, 3, 5, \ldots, n\}, \\ \dfrac{n+1+i}{2} & \text{if } i \in \{2, 4, 6, \ldots, n-1\}. \end{cases}$$

Next, we construct super edge-magic labelings for small hypercycles; ${}^{1}C_n^{(3)}$ (in Theorem 5.2) and ${}^{2}C_n^{(5)}$ for all n (in Theorem 5.3), ${}^{1}C_n^{(4)}$ for all even integers n (in Theorem 5.4), and ${}^{2}C_n^{(4)}$ for all odd integers n (in Theorem 5.5) .

Theorem 5.2. *A hypercycle ${}^{1}C_n^{(3)}$ is super edge-magic.*

Proof. Note that, ${}^{1}C_n^{(3)}$ has $p = 2n$ and $q = n$.

Case (i): n is odd. Define $f : V \to \{1, 2, 3, \ldots, 2n\}$ by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1, 3, 5, \ldots, n\}, \\ \dfrac{n+1+i}{2} & \text{if } i \in \{2, 4, 6, \ldots, n-1\}, \end{cases}$$

$$f(v_{i,1}) = \begin{cases} \dfrac{3n+1}{2} + i & \text{if } i \in \left\{1, 2, 3, \ldots, \dfrac{n-1}{2}\right\}, \\ \dfrac{n+1}{2} + i & \text{if } i \in \left\{\dfrac{n+1}{2}, \dfrac{n+3}{2}, \dfrac{n+5}{2}, \ldots, n\right\}. \end{cases}$$

To show that f is bijective, since $|V| = |\{1, 2, 3, \ldots, 2n\}|$, it suffices to show that f is surjective. Let $a \in \{1, 2, 3, \ldots, 2n\}$.

If $a \in \{1, 2, 3, \ldots, \frac{n+1}{2}\}$, then $2a - 1 \in \{1, 3, 5, \ldots, n\}$. Thus, $f(w_{2a-1,1}) = \frac{1+(2a-1)}{2} = a$.

If $a \in \{\frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \ldots, n\}$, then $2a - n - 1 \in \{2, 4, 6, \ldots, n-1\}$. Thus, $f(w_{2a-n-1,1}) = \frac{n+1+(2a-n-1)}{2} = a$.

If $a \in \{n+1, n+2, n+3, \ldots, \frac{3n+1}{2}\}$, then $a - \frac{n+1}{2} \in \{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n\}$. Thus, $f(v_{a-\frac{n+1}{2},1}) = \frac{n+1}{2} + (a - \frac{n+1}{2}) = a$.

If $a \in \{\frac{3n+3}{2}, \frac{3n+5}{2}, \frac{3n+7}{2}, \ldots, 2n\}$, then $a - \frac{3n+1}{2} \in \{1, 2, 3, \ldots, \frac{n-1}{2}\}$. Thus, $f(v_{a-\frac{3n+1}{2},1}) = \frac{3n+1}{2} + (a - \frac{3n+1}{2}) = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, v_{i,1}\} \in E$. Since for $i \in \{1, 3, 5, \ldots, n-2\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{1+i}{2}\right) + \left(\frac{n+1+(i+1)}{2}\right) \\ &= \frac{n+3}{2} + i, \end{aligned}$$

and for $i \in \{2, 4, 6, \ldots, n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{n+1+i}{2}\right) + \left(\frac{1+(i+1)}{2}\right) \\ &= \frac{n+3}{2} + i. \end{aligned}$$

Hence, for $i \in \{1, 2, 3, \ldots, \frac{n-1}{2}\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(v_{i,1}) \\ &= \left(\frac{n+3}{2} + i\right) + \left(\frac{3n+1}{2} + i\right) \\ &= 2n + 2 + 2i.\end{aligned}$$

For $i \in \{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, n-1\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(v_{i,1}) \\ &= \left(\frac{n+3}{2} + i\right) + \left(\frac{n+1}{2} + i\right) \\ &= n + 2 + 2i.\end{aligned}$$

For $i = n$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{n,1}) + f(w_{1,1}) + f(v_{n,1}) \\ &= \left(\frac{1+n}{2}\right) + \left(\frac{1+1}{2}\right) + \left(\frac{n+1}{2} + n\right) \\ &= 2n + 2.\end{aligned}$$

Thus, $\{\sum_{v \in e_i} f(v) |\, i \in \{1, 2, 3, \ldots, n\}\} = \{2n{+}2, 2n{+}3, 2n{+}4, \ldots, 3n{+}1\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = 5n + 2$.

Case (ii): n is even. Define $f : V \to \{1, 2, 3, \ldots, 2n\}$ by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1, 3, 5, \ldots, n-1\}, \\ \dfrac{n+i}{2} & \text{if } i \in \{2, 4, 6, \ldots, n\}, \end{cases}$$

$$f(v_{i,1}) = \begin{cases} \dfrac{3n}{2} + i & \text{if } i \in \left\{1, 2, 3, \ldots, \dfrac{n}{2}\right\}, \\ \dfrac{n+2}{2} + i & \text{if } i \in \left\{\dfrac{n+2}{2}, \dfrac{n+4}{2}, \dfrac{n+6}{2}, \ldots, n-1\right\}, \\ n+1 & \text{if } i = n. \end{cases}$$

To show that f is bijective, since $|V| = |\{1, 2, 3, \ldots, 2n\}|$, it suffices to show that f is surjective. Let $a \in \{1, 2, 3, \ldots, 2n\}$.

If $a \in \{1, 2, 3, \ldots, \frac{n}{2}\}$, then $2a - 1 \in \{1, 3, 5, \ldots, n-1\}$. Thus, $f(w_{2a-1,1}) = \frac{1+(2a-1)}{2} = a$.

If $a \in \{\frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}, \ldots, n\}$, then $2a-n \in \{2, 4, 6, \ldots, n\}$. Thus, $f(w_{2a-n,1}) = \frac{n+(2a-n)}{2} = a$.

If $a = n+1$, then $f(v_{n,1}) = n + 1 = a$.

If $a \in \{n+2, n+3, n+4, \ldots, \frac{3n}{2}\}$, then $a - \frac{n+2}{2} \in \{\frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}, \ldots, n-1\}$. Thus, $f(v_{a-\frac{n+2}{2},1}) = \frac{n+2}{2} + (a - \frac{n+2}{2}) = a$.

If $a \in \{\frac{3n+2}{2}, \frac{3n+4}{2}, \frac{3n+6}{2}, \ldots, 2n\}$, then $a - \frac{3n}{2} \in \{1, 2, 3, \ldots, \frac{n}{2}\}$. Thus, $f(v_{a-\frac{3n}{2},1}) = \frac{3n}{2} + (a - \frac{3n}{2}) = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, let $e_i = \{w_{i,1}, w_{i+1,1}, v_{i,1}\} \in E$. Since for $i \in \{1, 3, 5, \ldots, n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{1+i}{2}\right) + \left(\frac{n+(i+1)}{2}\right) \\ &= \frac{n+2}{2} + i, \end{aligned}$$

and for $i \in \{2, 4, 6, \ldots, n-2\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= \left(\frac{n+i}{2}\right) + \left(\frac{1+(i+1)}{2}\right) \\ &= \frac{n+2}{2} + i. \end{aligned}$$

Hence, for $i \in \{1, 2, 3, \ldots, \frac{n}{2}\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(v_{i,1}) \\ &= \left(\frac{n+2}{2} + i\right) + \left(\frac{3n}{2} + i\right) \\ &= 2n + 1 + 2i. \end{aligned}$$

For $i \in \{\frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}, \ldots, n-1\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(v_{i,1}) \\ &= \left(\frac{n+2}{2} + i\right) + \left(\frac{n+2}{2} + i\right) \\ &= n + 2 + 2i.\end{aligned}$$

For $i = n$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{n,1}) + f(w_{1,1}) + f(v_{n,1}) \\ &= \left(\frac{n+n}{2}\right) + \left(\frac{1+1}{2}\right) + (n+1) \\ &= 2n + 2.\end{aligned}$$

Thus, $\{\sum_{v \in e_i} f(v) | \, i \in \{1, 2, 3, \ldots, n\}\} = \{2n+2, 2n+3, 2n+4, \ldots, 3n+1\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = 5n + 2$. Consequently, the hypercycle ${}^1C_n^{(3)}$ is super edge-magic. □

Example 5.1. For the hypergraph ${}^1C_5^{(3)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

$$v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}, v_{5,1},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, v_{1,1}\}, e_2 = \{w_{2,1}, w_{3,1}, v_{2,1}\}, e_3 = \{w_{3,1}, w_{4,1}, v_{3,1}\},$$

$$e_4 = \{w_{4,1}, w_{5,1}, v_{4,1}\}, e_5 = \{w_{5,1}, w_{1,1}, v_{5,1}\},$$

by Theorem 5.2, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 4, f(w_{3,1}) = 2, f(w_{4,1}) = 5, f(w_{5,1}) = 3,$$

$$f(v_{1,1}) = 9, f(v_{2,1}) = 10, f(v_{3,1}) = 6, f(v_{4,1}) = 7, f(v_{5,1}) = 8.$$

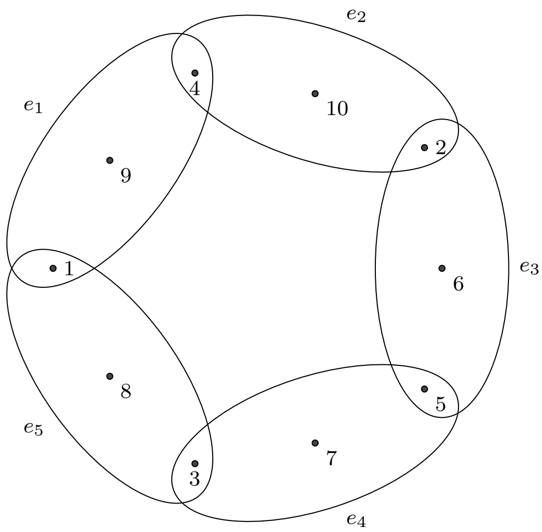

Figure 5.1: The vertex-labeling of ${}^1C_5^{(3)}$

The vertex-labeling of ${}^1C_5^{(3)}$ is shown in Figure 5.1 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(v_{1,1}) = 1 + 4 + 9 = 14,$$

$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(v_{2,1}) = 4 + 2 + 10 = 16,$$

$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(v_{3,1}) = 2 + 5 + 6 = 13,$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(v_{4,1}) = 5 + 3 + 7 = 15,$$

$$\sum_{v\in e_5} f(v) = f(w_{5,1}) + f(w_{1,1}) + f(v_{5,1}) = 3 + 1 + 8 = 12.$$

Then, $\{\sum_{v\in e_i} f(v)|\ i \in \{1,2,3,4,5\}\} = \{12,13,14,15,16\}$ is a set of five consecutive integers. Hence, by Theorem 2.8, ${}^1C_5^{(3)}$ is super edge-magic.

Example 5.2. For the hypergraph ${}^1C_4^{(3)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1},$$

$$v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, v_{1,1}\}, e_2 = \{w_{2,1}, w_{3,1}, v_{2,1}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, v_{3,1}\}, e_4 = \{w_{4,1}, w_{1,1}, v_{4,1}\},$$

by Theorem 5.2, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 3, f(w_{3,1}) = 2, f(w_{4,1}) = 4,$$

$$f(v_{1,1}) = 7, f(v_{2,1}) = 8, f(v_{3,1}) = 6, f(v_{4,1}) = 5.$$

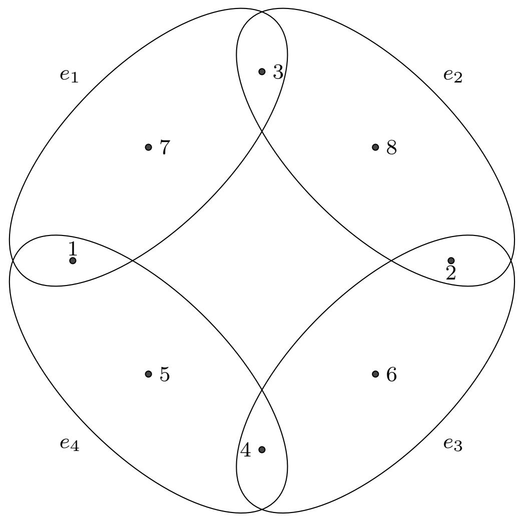

Figure 5.2: The vertex-labeling of ${}^1C_4^{(3)}$

The vertex-labeling of ${}^1C_4^{(3)}$ is shown in Figure 5.2 and we see that

$$\sum_{v \in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(v_{1,1}) = 1 + 3 + 7 = 11,$$

$$\sum_{v \in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(v_{2,1}) = 3 + 2 + 8 = 13,$$

$$\sum_{v \in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(v_{3,1}) = 2 + 4 + 6 = 12,$$

$$\sum_{v \in e_4} f(v) = f(w_{4,1}) + f(w_{1,1}) + f(v_{4,1}) = 4 + 1 + 5 = 10.$$

Then, $\{\sum_{v\in e_i} f(v)|\ i \in \{1,2,3,4\}\} = \{10,11,12,13\}$ is a set of four consecutive integers. Hence, by Theorem 2.8, ${}^1C_4^{(3)}$ is super edge-magic.

Theorem 5.3. *A hypercycle* ${}^2C_n^{(5)}$ *is super edge-magic.*

Proof. Note that, ${}^2C_n^{(5)}$ has $p = 3n$ and $q = n$.

Define $f : V \to \{1,2,3,\ldots,3n\}$ by

$$f(w_{i,j}) = \begin{cases} i & \text{if } j = 1, \\ 2n+1-i & \text{if } j = 2, \end{cases}$$

$$f(v_{i1}) = \quad 2n + i,$$

for every $i \in \{1,2,3,\ldots,n\}$.

To show that f is bijective, since $|V| = |\{1,2,3,\ldots,3n\}|$, it suffices to show that f is surjective. Let $a \in \{1,2,3,\ldots,3n\}$.

If $a \in \{1,2,3,\ldots,n\}$, then $f(w_{a,1}) = a$.

If $a \in \{n+1, n+2, n+3, \ldots, 2n\}$, then $2n+1-a \in \{1,2,3,\ldots,n\}$. Thus, $f(w_{2n+1-a,2}) = 2n+1-(2n+1-a) = a$.

If $a \in \{2n+1, 2n+2, 2n+3, \ldots, 3n\}$, then $a - 2n \in \{1,2,3,\ldots,n\}$. Thus, $f(v_{a-2n,1}) = 2n + (a-2n) = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1,2,3,\ldots,n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, w_{i,2}, w_{i+1,2}, v_{i,1}\} \in E$. Since for $i \in \{1,2,3,\ldots,n\}$, we observe that

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= i + (2n+1-i) \\ &= 2n+1. \end{aligned}$$

Hence, for $i \in \{1,2,3,\ldots,n-1\}$,

$$\sum_{v\in e_i} f(v) = f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) + f(v_{i,1})$$

$$= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) + f(v_{i,1})$$

$$= (2n + 1) + (2n + 1) + (2n + i)$$

$$= 6n + 2 + i.$$

For $i = n$,

$$\sum_{v \in e_i} f(v) = f(w_{n,1}) + f(w_{1,1}) + f(w_{n,2}) + f(w_{1,2}) + f(v_{i,1})$$

$$= (f(w_{n,1}) + f(w_{n,2})) + (f(w_{1,1}) + f(w_{1,2})) + f(v_{n,1})$$

$$= (2n + 1) + (2n + 1) + (2n + n)$$

$$= 7n + 2.$$

Thus, $\{\sum_{v \in e_i} f(v) | \, i \in \{1, 2, 3, \ldots, n\}\} = \{6n+3, 6n+4, 6n+5, \ldots, 7n+2\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = 10n + 3$. $\square$

Example 5.3. For the hypergraph ${}^2C_5^{(5)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

$$w_{1,2}, w_{2,2}, w_{3,2}, w_{4,2}, w_{5,2},$$

$$v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}, v_{5,1},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}, v_{1,1}\}, e_2 = \{w_{2,1}, w_{3,1}, w_{2,2}, w_{3,2}, v_{2,1}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, w_{3,2}, w_{4,2}, v_{3,1}\}, e_4 = \{w_{4,1}, w_{5,1}, w_{4,2}, w_{5,2}, v_{4,1}\},$$

$$e_5 = \{w_{5,1}, w_{1,1}, w_{5,2}, w_{1,2}, v_{5,1}\},$$

by Theorem 5.3, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 2, f(w_{3,1}) = 3, f(w_{4,1}) = 4, f(w_{5,1}) = 5,$$

$$f(w_{1,2}) = 10, f(w_{2,2}) = 9, f(w_{3,2}) = 8, f(w_{4,2}) = 7, f(w_{5,2}) = 6,$$

$$f(v_{1,1}) = 11, f(v_{2,1}) = 12, f(v_{3,1}) = 13, f(v_{4,1}) = 14, f(v_{5,1}) = 15.$$

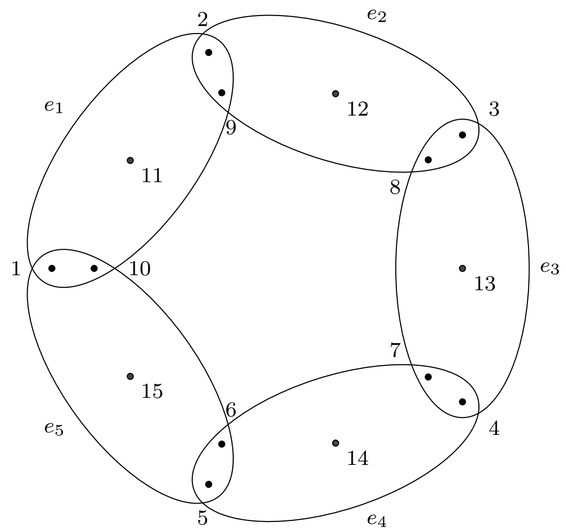

Figure 5.3: The vertex-labeling of ${}^2C_5^{(5)}$

The vertex-labeling of ${}^2C_5^{(5)}$ is shown in Figure 5.3 and we see that

$$\begin{aligned}
\sum_{v\in e_1} f(v) &= f(w_{1,1}) + f(w_{2,1}) + f(w_{1,2}) + f(w_{2,2}) + f(v_{1,1}) \\
&= 1 + 2 + 10 + 9 + 11 \\
&= 33, \\
\sum_{v\in e_2} f(v) &= f(w_{2,1}) + f(w_{3,1}) + f(w_{2,2}) + f(w_{3,2}) + f(v_{2,1}) \\
&= 2 + 3 + 9 + 8 + 12 \\
&= 34, \\
\sum_{v\in e_3} f(v) &= f(w_{3,1}) + f(w_{4,1}) + f(w_{3,2}) + f(w_{4,2}) + f(v_{3,1}) \\
&= 3 + 4 + 8 + 7 + 13 \\
&= 35,
\end{aligned}$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(w_{4,2}) + f(w_{5,2}) + f(v_{4,1})$$
$$= 4 + 5 + 7 + 6 + 14$$
$$= 36,$$
$$\sum_{v\in e_5} f(v) = f(w_{5,1}) + f(w_{1,1}) + f(w_{5,2}) + f(w_{1,2}) + f(v_{5,1})$$
$$= 5 + 1 + 6 + 10 + 15$$
$$= 37.$$

Then, $\{\sum_{v\in e_i} f(v)|\ i \in \{1,2,3,4,5\}\} = \{33,34,35,36,37\}$ is a set of five consecutive integers. Hence, by Theorem 2.8, ${}^2C_5^{(5)}$ is super edge-magic.

Theorem 5.4. *Let n be an even integer. A hypercycle ${}^1C_n^{(4)}$ is super edge-magic.*

Proof. Note that, ${}^1C_n^{(4)}$ has $p = 3n$ and $q = n$.

Define $f : V \to \{1,2,3,\ldots,3n\}$ by

$$f(w_{i,1}) = \begin{cases} 1+i & \text{if } i \in \{1,3,5,\ldots,n-1\}, \\ n+i & \text{if } i \in \{2,4,6,\ldots,n\}, \end{cases}$$

$$f(v_{i,1}) = \begin{cases} 2n-1-2i & \text{if } i \in \{1,2,3,\ldots,n-1\}, \\ 2n-1 & \text{if } i = n. \end{cases}$$

$$f(v_{i,2}) = \begin{cases} 2n+1+i & \text{if } i \in \{1,2,3,\ldots,n-1\}, \\ 2n+1 & \text{if } i = n. \end{cases}$$

To show that f is bijective, since $|V| = |\{1,2,3,\ldots,3n\}|$, it suffices to show that f is surjective. Let $a \in \{1,2,3,\ldots,3n\}$.

If $a \in \{1,3,5,\ldots,2n-3\}$, then $\frac{2n-1-a}{2} \in \{1,2,3,\ldots,n-1\}$. Thus, $f(v_{\frac{2n-1-a}{2},1}) = 2n-1-2(\frac{2n-1-a}{2}) = a$.

If $a = 2n-1$, then $f(v_{n,1}) = 2n-1 = a$.

If $a \in \{2,4,6,\ldots,n\}$, then $a-1 \in \{1,3,5,\ldots,n-1\}$. Thus, $f(w_{a-1,1}) = 1+(a-1) = a$.

If $a \in \{n+2, n+4, n+6, \ldots, 2n\}$, then $a-n \in \{2,4,6,\ldots,n\}$. Thus, $f(w_{a-n,1}) = n+(a-n) = a$.

If $a = 2n+1$, then $f(v_{n,2}) = 2n+1 = a$.

If $a \in \{2n+2, 2n+3, 2n+4, \ldots, 3n\}$, then $a-2n-1 \in \{1,2,3,\ldots,n-1\}$. Thus, $f(v_{a-2n-1,2}) = 2n+1-(a-2n-1) = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1,2,3,\ldots,n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, v_{i,1}, v_{i,2}\} \in E$. Since for $i \in \{1,3,5,\ldots,n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= (1+i) + (n+(i+1)) \\ &= n+2+2i, \end{aligned}$$

and for $i \in \{2,4,6,\ldots,n-2\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i+1,1}) &= (n+i) + (1+(i+1)) \\ &= n+2+2i. \end{aligned}$$

Hence, for $i \in \{1,2,3,\ldots,n-1\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(v_{i,1}) + f(v_{i,2}) \\ &= (n+2+2i) + (2n-1-2i) + (2n+1+i) \\ &= 5n+2+i. \end{aligned}$$

For $i = n$,

$$\sum_{v \in e_i} f(v) = f(w_{n,1}) + f(w_{1,1}) + f(v_{n,1}) + f(v_{n,2})$$

$$= (n+n) + (1+1) + (2n-1) + (2n+1)$$
$$= 6n+2.$$

Thus, $\{\sum_{v \in e_i} f(v) | \, i \in \{1,2,3,\ldots,n\}\} = \{5n+3, 5n+4, 5n+5, \ldots, 6n+2\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = 9n+3$. $\square$

Example 5.4. For the hypergraph ${}^1C_6^{(4)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1}, w_{6,1},$$
$$v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1}, v_{5,1}, v_{6,1},$$
$$v_{1,2}, v_{2,2}, v_{3,2}, v_{4,2}, v_{5,2}, v_{6,2},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, v_{1,1}, v_{1,2}\}, e_2 = \{w_{2,1}, w_{3,1}, v_{2,1}, v_{2,2}\},$$
$$e_3 = \{w_{3,1}, w_{4,1}, v_{3,1}, v_{3,2}\}, e_4 = \{w_{4,1}, w_{5,1}, v_{4,1}, v_{4,2}\},$$
$$e_5 = \{w_{5,1}, w_{6,1}, v_{5,1}, v_{5,2}\}, e_6 = \{w_{6,1}, w_{1,1}, v_{6,1}, v_{6,2}\},$$

by Theorem 5.4, we label each vertex as follows:

$$f(w_{1,1}) = 2, f(w_{2,1}) = 8, f(w_{3,1}) = 4, f(w_{4,1}) = 10, f(w_{5,1}) = 6, f(w_{6,1}) = 12,$$
$$f(v_{1,1}) = 9, f(v_{2,1}) = 7, f(v_{3,1}) = 5, f(v_{4,1}) = 3, f(v_{5,1}) = 1, f(v_{6,1}) = 11,$$
$$f(v_{1,2}) = 14, f(v_{2,2}) = 15, f(v_{3,2}) = 16, f(v_{4,2}) = 17, f(v_{5,2}) = 18, f(v_{6,2}) = 13.$$

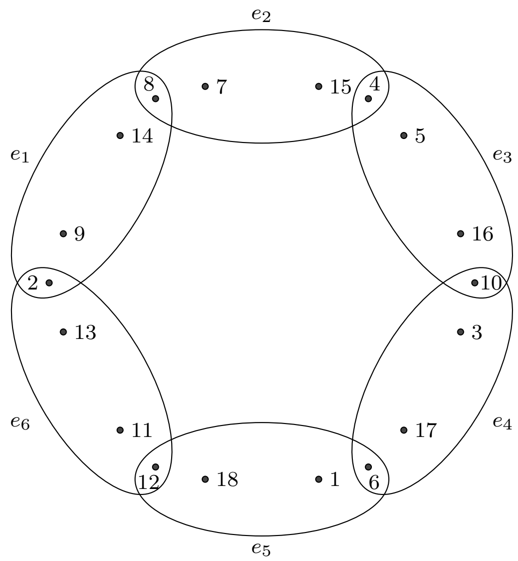

Figure 5.4: The vertex-labeling of ${}^1C_6^{(4)}$

The vertex-labeling of ${}^1C_6^{(4)}$ is shown in Figure 5.4 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(v_{1,1}) + f(v_{1,2}) = 2+8+9+14 = 33,$$

$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(v_{2,1}) + f(v_{2,2}) = 8+4+7+15 = 34,$$

$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(v_{3,1}) + f(v_{3,2}) = 4+10+5+16 = 35,$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(v_{4,1}) + f(v_{4,2}) = 10+6+3+17 = 36,$$

$$\sum_{v\in e_5} f(v) = f(w_{5,1}) + f(w_{6,1}) + f(v_{5,1}) + f(v_{5,2}) = 6+12+1+18 = 37,$$

$$\sum_{v\in e_6} f(v) = f(w_{6,1}) + f(w_{1,1}) + f(v_{6,1}) + f(v_{6,2}) = 12+2+11+13 = 38.$$

Then, $\{\sum_{v\in e_i} f(v) |\ i \in \{1,2,3,4,5,6\}\} = \{33,34,35,36,37,38\}$ is a set of six consecutive integers. Hence, by Theorem 2.8, ${}^1C_6^{(4)}$ is super edge-magic.

Theorem 5.5. *Let n be an odd integer. A hypercycle ${}^2C_n^{(4)}$ is super edge-magic.*

Proof. Note that, ${}^2C_n^{(4)}$ has $p = 2n$ and $q = n$.

Case (i): $n \equiv 1 \pmod 4$. Define $f: V \to \{1, 2, 3, \ldots, 2n\}$ by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1, 3, 5, \ldots, n\}, \\ \dfrac{n+1+i}{2} & \text{if } i \in \{2, 4, 6, \ldots, n-1\}, \end{cases}$$

$$f(w_{i,2}) = \begin{cases} \dfrac{7n+3+2i}{4} & \text{if } i \in \left\{1, 3, 5, \ldots, \dfrac{n-3}{2}\right\}, \\ \dfrac{3n+3+2i}{4} & \text{if } i \in \left\{\dfrac{n+1}{2}, \dfrac{n+5}{2}, \dfrac{n+9}{2}, \ldots, n\right\}, \\ \dfrac{5n+3+2i}{4} & \text{if } i \in \{2, 4, 6, \ldots, n-1\}. \end{cases}$$

To show that f is bijective, since $|V| = |\{1, 2, 3, \ldots, 2n\}|$, it suffices to show that f is surjective. Let $a \in \{1, 2, 3, \ldots, 2n\}$.

If $a \in \{1, 2, 3, \ldots, \frac{n+1}{2}\}$, then $2a - 1 \in \{1, 3, 5, \ldots, n\}$. Thus, $f(w_{2a-1,1}) = \frac{1+(2a-1)}{2} = a$.

If $a \in \{\frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \ldots, n\}$, then $2a - n - 1 \in \{2, 4, 6, \ldots, n-1\}$. Thus, $f(w_{2a-n-1,1}) = \frac{n+1+(2a-n-1)}{2} = a$.

If $a \in \{n+1, n+2, n+3, \ldots, \frac{5n+3}{4}\}$, then $\frac{4a-3n-3}{2} \in \{\frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n\}$. Thus, $f(w_{\frac{4a-3n-3}{2},2}) = \frac{3n+3+2(\frac{4a-3n-3}{2})}{4} = a$.

If $a \in \{\frac{5n+7}{4}, \frac{5n+11}{4}, \frac{5n+15}{4}, \ldots, \frac{7n+1}{4}\}$, then $\frac{4a-5n-3}{2} \in \{2, 4, 6, \ldots, n-1\}$. Thus, $f(w_{\frac{4a-5n-3}{2},2}) = \frac{5n+3+2(\frac{4a-5n-3}{2})}{4} = a$;

If $a \in \{\frac{7n+5}{4}, \frac{7n+9}{4}, \frac{7n+13}{4}, \ldots, 2n\}$, then $\frac{4a-7n-3}{2} \in \{1, 3, 5, \ldots, \frac{n-3}{2}\}$. Thus, $f(w_{\frac{4a-7n-3}{2},2}) = \frac{7n+3+2(\frac{4a-7n-3}{2})}{4} = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, w_{i,2}, w_{i+1,2}\} \in E$. Observe that, for $i \in \{1, 3, 5, \ldots, \frac{n-3}{2}\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= \left(\frac{1+i}{2}\right) + \left(\frac{7n+3+2i}{4}\right) \\ &= \frac{7n+5}{4} + i, \end{aligned}$$

for $i \in \{\frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= \left(\frac{1+i}{2}\right) + \left(\frac{3n+3+2i}{4}\right) \\ &= \frac{3n+5}{4} + i, \end{aligned}$$

for $i \in \{2, 4, 6, \ldots, n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= \left(\frac{n+1+i}{2}\right) + \left(\frac{5n+3+2i}{4}\right) \\ &= \frac{7n+5}{4} + i. \end{aligned}$$

Hence, for $i \in \{1, 3, 5, \ldots, \frac{n-3}{2}\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \\ &= \left(\frac{7n+5}{4} + i\right) + \left(\frac{7n+5}{4} + (i+1)\right) \\ &= \frac{7n+7}{2} + 2i. \end{aligned}$$

For $i \in \{\frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n-2\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \\ &= \left(\frac{3n+5}{4} + i\right) + \left(\frac{7n+5}{4} + (i+1)\right) \\ &= \frac{5n+7}{2} + 2i. \end{aligned}$$

For $i \in \{2, 4, 6, \ldots, \frac{n-5}{2}\}$,

$$\begin{aligned} \sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \end{aligned}$$

$$= \left(\frac{7n+5}{4} + i\right) + \left(\frac{7n+5}{4} + (i+1)\right)$$
$$= \frac{7n+7}{2} + 2i.$$

For $i \in \{\frac{n-1}{2}, \frac{n+3}{2}, \frac{n+7}{2}, \ldots, n-1\}$,

$$\sum_{v \in e_i} f(v) = f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2})$$
$$= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2}))$$
$$= \left(\frac{7n+5}{4} + i\right) + \left(\frac{3n+5}{4} + (i+1)\right)$$
$$= \frac{5n+7}{2} + 2i.$$

For $i = n$,

$$\sum_{v \in e_i} f(v) = f(w_{n,1}) + f(w_{1,1}) + f(w_{n,2}) + f(w_{1,2})$$
$$= (f(w_{n,1}) + f(w_{n,2})) + (f(w_{1,1}) + f(w_{1,2}))$$
$$= \left(\frac{3n+5}{4} + n\right) + \left(\frac{7n+5}{4} + 1\right)$$
$$= \frac{7n+7}{2}.$$

Thus, $\{\sum_{v \in e_i} f(v) |\ i \in \{1, 2, 3, \ldots, n\}\} = \{\frac{7n+5}{2}, \frac{7n+7}{2}, \frac{7n+9}{2}, \ldots, \frac{9n+3}{2}\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = \frac{13n+5}{2}$.

Case (ii): $n \equiv 3 \pmod 4$. Define $f : V \to \{1, 2, 3, \ldots, 2n\}$ by

$$f(w_{i,1}) = \begin{cases} \dfrac{1+i}{2} & \text{if } i \in \{1, 3, 5, \ldots, n\}, \\ \dfrac{n+1+i}{2} & \text{if } i \in \{2, 4, 6, \ldots, n-1\}, \end{cases}$$

$$f(w_{i,2}) = \begin{cases} \dfrac{5n+3+2i}{4} & \text{if } i \in \{1, 3, 5, \ldots, n\}, \\ \dfrac{7n+3+2i}{4} & \text{if } i \in \{2, 4, 6, \ldots, \dfrac{n-3}{2}\}, \\ \dfrac{3n+3+2i}{4} & \text{if } i \in \{\dfrac{n+1}{2}, \dfrac{n+5}{2}, \dfrac{n+9}{2}, \ldots, n-1\}. \end{cases}$$

To show that f is bijective, since $|V| = |\{1, 2, 3, \ldots, 2n\}|$, it suffices to show that f is surjective. Let $a \in \{1, 2, 3, \ldots, 2n\}$.

If $a \in \{1, 2, 3, \ldots, \frac{n+1}{2}\}$, then $2a - 1 \in \{1, 3, 5, \ldots, n\}$. Thus, $f(w_{2a-1,1}) = \frac{1+(2a-1)}{2} = a$.

If $a \in \{\frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, \ldots, n\}$, then $2a - n - 1 \in \{2, 4, 6, \ldots, n-1\}$. Thus, $f(w_{2a-n-1,1}) = \frac{n+1+(2a-n-1)}{2} = a$.

If $a \in \{n+1, n+2, n+3, \ldots, \frac{5n+1}{4}\}$, then $\frac{4a-3n-3}{2} \in \{\frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n-1\}$. Thus, $f(w_{\frac{4a-3n-3}{2},2}) = \frac{3n+3+2(\frac{4a-3n-3}{2})}{4} = a$.

If $a \in \{\frac{5n+5}{4}, \frac{5n+9}{4}, \frac{5n+13}{4}, \ldots, \frac{7n+3}{4}\}$, then $\frac{4a-5n-3}{2} \in \{1, 3, 5, \ldots, n\}$. Thus, $f(w_{\frac{4a-5n-3}{2},2}) = \frac{5n+3+2(\frac{4a-5n-3}{2})}{4} = a$;

If $a \in \{\frac{7n+7}{4}, \frac{7n+11}{4}, \frac{7n+15}{4}, \ldots, 2n\}$, then $\frac{4a-7n-3}{2} \in \{2, 4, 6, \ldots, \frac{n-3}{2}\}$. Thus, $f(w_{\frac{4a-7n-3}{2},2}) = \frac{7n+3+2(\frac{4a-7n-3}{2})}{4} = a$.

Therefore, f is surjective.

To see that f is a super edge-magic labeling, for $i \in \{1, 2, 3, \ldots, n\}$, let $e_i = \{w_{i,1}, w_{i+1,1}, w_{i,2}, w_{i+1,2}\} \in E$. Observe that, for $i \in \{1, 3, 5, \ldots, n\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= \left(\frac{1+i}{2}\right) + \left(\frac{5n+3+2i}{4}\right) \\ &= \frac{5n+5}{4} + i, \end{aligned}$$

for $i \in \{2, 4, 6, \ldots, \frac{n-3}{2}\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= \left(\frac{n+1+i}{2}\right) + \left(\frac{7n+3+2i}{4}\right) \\ &= \frac{9n+5}{4} + i, \end{aligned}$$

for $i \in \{\frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n-1\}$,

$$\begin{aligned} f(w_{i,1}) + f(w_{i,2}) &= \left(\frac{n+1+i}{2}\right) + \left(\frac{3n+3+2i}{4}\right) \\ &= \frac{5n+5}{4} + i. \end{aligned}$$

Hence, for $i \in \{1, 3, 5, \ldots, \frac{n-5}{2}\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \\ &= \left(\frac{5n+5}{4} + i\right) + \left(\frac{9n+5}{4} + (i+1)\right) \\ &= \frac{7n+7}{2} + 2i.\end{aligned}$$

For $i \in \{\frac{n-1}{2}, \frac{n+3}{2}, \frac{n+7}{2}, \ldots, n-2\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \\ &= \left(\frac{5n+5}{4} + i\right) + \left(\frac{5n+5}{4} + (i+1)\right) \\ &= \frac{5n+7}{2} + 2i.\end{aligned}$$

For $i \in \{2, 4, 6, \ldots, \frac{n-3}{2}\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \\ &= \left(\frac{9n+5}{4} + i\right) + \left(\frac{5n+5}{4} + (i+1)\right) \\ &= \frac{7n+7}{2} + 2i.\end{aligned}$$

For $i \in \{\frac{n+1}{2}, \frac{n+5}{2}, \frac{n+9}{2}, \ldots, n-1\}$,

$$\begin{aligned}\sum_{v \in e_i} f(v) &= f(w_{i,1}) + f(w_{i+1,1}) + f(w_{i,2}) + f(w_{i+1,2}) \\ &= (f(w_{i,1}) + f(w_{i,2})) + (f(w_{i+1,1}) + f(w_{i+1,2})) \\ &= \left(\frac{5n+5}{4} + i\right) + \left(\frac{5n+5}{4} + (i+1)\right)\end{aligned}$$

$$= \frac{5n+7}{2} + 2i.$$

For $i = n$,

$$\begin{aligned}\sum_{v\in e_i} f(v) &= f(w_{n,1}) + f(w_{i,1}) + f(w_{n,2}) + f(w_{1,2}) \\ &= (f(w_{n,1}) + f(w_{n,2})) + (f(w_{i,1}) + f(w_{i,2})) \\ &= \left(\frac{5n+5}{4} + n\right) + \left(\frac{5n+5}{4} + 1\right) \\ &= \frac{7n+7}{2}.\end{aligned}$$

Thus, $\{\sum_{v\in e_i} f(v) |\ i \in \{1, 2, 3, \ldots, n\}\} = \{\frac{7n+5}{2}, \frac{7n+7}{2}, \frac{7n+9}{2}, \ldots, \frac{9n+3}{2}\}$ consists of n consecutive integers. By Theorem 2.8, f is a super edge-magic labeling with $\Lambda = \frac{13n+5}{2}$. Consequently, the hypercycle ${}^2C_n^{(4)}$ is super edge-magic. □

Example 5.5. For the hypergraph ${}^2C_5^{(4)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1}, w_{4,1}, w_{5,1},$$

$$w_{1,2}, w_{2,2}, w_{3,2}, w_{4,2}, w_{5,2},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}\}, e_2 = \{w_{2,1}, w_{3,1}, w_{2,2}, w_{3,2}\},$$

$$e_3 = \{w_{3,1}, w_{4,1}, w_{3,2}, w_{4,2}\}, e_4 = \{w_{4,1}, w_{5,1}, w_{4,2}, w_{5,2}\},$$

$$e_5 = \{w_{5,1}, w_{1,1}, w_{5,2}, w_{1,2}\},$$

by Theorem 5.5, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 4, f(w_{3,1}) = 2, f(w_{4,1}) = 5, f(w_{5,1}) = 3,$$

$$f(w_{1,2}) = 10, f(w_{2,2}) = 8, f(w_{3,2}) = 6, f(w_{4,2}) = 9, f(w_{5,2}) = 7.$$

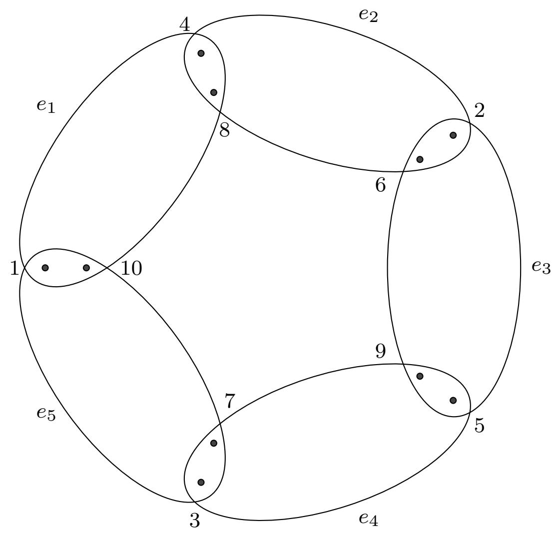

Figure 5.5: The vertex-labeling of ${}^2C_5^{(4)}$

The vertex-labeling of ${}^2C_5^{(4)}$ is shown in Figure 5.5 and we see that

$$\sum_{v\in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(w_{1,2}) + f(w_{2,2}) = 1+4+10+8 = 23,$$

$$\sum_{v\in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(w_{2,2}) + f(w_{3,2}) = 4+2+8+6 = 20,$$

$$\sum_{v\in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(w_{3,2}) + f(w_{4,2}) = 2+5+6+9 = 22,$$

$$\sum_{v\in e_4} f(v) = f(w_{4,1}) + f(w_{5,1}) + f(w_{4,2}) + f(w_{5,2}) = 5+3+9+7 = 24,$$

$$\sum_{v\in e_5} f(v) = f(w_{5,1}) + f(w_{1,1}) + f(w_{5,2}) + f(w_{1,2}) = 3+1+7+10 = 21.$$

Then, $\{\sum_{v\in e_i} f(v) |\ i \in \{1,2,3,4,5\}\} = \{20,21,22,23,24\}$ is a set of five consecutive integers. Hence, by Theorem 2.8, ${}^2C_5^{(4)}$ is super edge-magic.

Example 5.6. For the hypergraph ${}^2C_3^{(4)}$ with vertices

$$w_{1,1}, w_{2,1}, w_{3,1},$$

$$w_{1,2}, w_{2,2}, w_{3,2},$$

and hyperedges

$$e_1 = \{w_{1,1}, w_{2,1}, w_{1,2}, w_{2,2}\}, e_2 = \{w_{2,1}, w_{3,1}, w_{2,2}, w_{3,2}\},$$
$$e_3 = \{w_{3,1}, w_{1,1}, w_{3,2}, w_{1,2}\},$$

by Theorem 5.5, we label each vertex as follows:

$$f(w_{1,1}) = 1, f(w_{2,1}) = 3, f(w_{3,1}) = 2,$$
$$f(w_{1,2}) = 5, f(w_{2,2}) = 4, f(w_{3,2}) = 6.$$

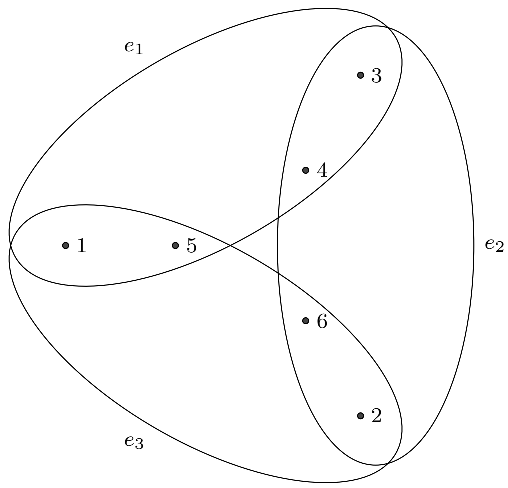

Figure 5.6: The vertex-labeling of ${}^2C_3^{(4)}$

The vertex-labeling of ${}^2C_3^{(4)}$ is shown in Figure 5.6 and we see that

$$\sum_{v \in e_1} f(v) = f(w_{1,1}) + f(w_{2,1}) + f(w_{1,2}) + f(w_{2,2}) = 1 + 3 + 5 + 4 = 13,$$
$$\sum_{v \in e_2} f(v) = f(w_{2,1}) + f(w_{3,1}) + f(w_{2,2}) + f(w_{3,2}) = 3 + 2 + 4 + 6 = 15,$$
$$\sum_{v \in e_3} f(v) = f(w_{3,1}) + f(w_{4,1}) + f(w_{3,2}) + f(w_{4,2}) = 2 + 1 + 6 + 5 = 14.$$

Then, $\{\sum_{v \in e_i} f(v) | \ i \in \{1, 2, 3\}\} = \{13, 14, 15\}$ is a set of three consecutive integers. Hence, by Theorem 2.8, ${}^2C_3^{(4)}$ is super edge-magic.

Now, if we have a small super edge-magic hypercycle ${}^{m}C_n^{(k)}$, we can use its super edge-magic labeling to construct new super-edge labelings for larger hypercycles ${}^{m}C_n^{(k+2)}$ and ${}^{m+2}C_n^{(k+4)}$.

Lemma 5.6. *If ${}^{m}C_n^{(k)}$ is super edge-magic, then ${}^{m}C_n^{(k+2)}$ is also super edge-magic.*

Proof. Let ${}^{m}C_n^{(k)} = (V, E)$. By Definition 2.6, we can construct ${}^{m}C_n^{(k+2)}$ by considering each $e_i \in E$, define $\tilde{e_i} = e_i \cup \{v_{i,k-2m+1}, v_{i,k-2m+2}\}$ for $i \in \{1, 2, 3, \ldots, n\}$. Then, ${}^{m}C_n^{(k+2)} = (\tilde{V}, \tilde{E})$ where $\tilde{E} = \{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \ldots, \tilde{e_n}\}$ and $\tilde{V} = \bigcup_{i=1}^{n} \tilde{e_i}$. Note that, $|V| = n(k-m), |\tilde{V}| = n(k-m) + 2n$, and $|E| = n = |\tilde{E}|$.

Assume that ${}^{m}C_n^{(k)}$ is super edge-magic. By Theorem 2.8, there is a bijection $f : V \to \{1, 2, 3, \ldots, n(k-m)\}$ such that $\{\sum_{v \in e} f(v) |\ e \in E\}$ is a set of n consecutive integers. We define a function $\hat{f}$ by

$$\hat{f}(v_{i,j}) = \begin{cases} n(k-m) + i & \text{if } j = k - 2m + 1, \\ n(k-m) + 2n + 1 - i & \text{if } j = k - 2m + 2, \end{cases}$$

for each $i \in \{1, 2, 3, \ldots, n\}$. Next, we define $\tilde{f} : \tilde{V} \to \{1, 2, 3, \ldots, n(k-m) + 2n\}$ by

$$\tilde{f}(v) = \begin{cases} f(v) & \text{if } v \in V, \\ \hat{f}(v) & \text{if } v \in \tilde{V} - V, \end{cases}$$

for $v \in \tilde{V}$. Since $f(V) = \{1, 2, 3, \ldots, n(k-m)\}$ and $\hat{f}(\tilde{V} - V) = \{n(k-m) + 1, n(k-m) + 2, n(k-m) + 3, \ldots, n(k-m) + 2n\}$, $\tilde{f}$ is a bijection from $\tilde{V}$ to $\{1, 2, 3, \ldots, n(k-m) + 2n\}$.

To see that $\tilde{f}$ is a super edge-magic labeling, let $\tilde{e_i} \in \tilde{E}$. Then, for $i \in \{1, 2, 3, \ldots, n\}$.

$$\sum_{v \in \tilde{e_i}} \tilde{f}(v) = \sum_{v \in e_i} \tilde{f}(v) + \tilde{f}(v_{i,k-2m+1}) + \tilde{f}(v_{i,k-2m+2})$$

$$= \sum_{v \in e_i} f(v) + \hat{f}(v_{i,k-2m+1}) + \hat{f}(v_{i,k-2m+2})$$

$$= \sum_{v \in e_i} f(v) + (n(k-m)+i) + (n(k-m)+2n+1-i)$$

$$= \sum_{v \in e_i} f(v) + 2n(k-m) + 2n + 1.$$

Since $\{\sum_{v \in e} f(v) |\ e \in E\}$ is a set of n consecutive integers, similar goes for $\{\sum_{v \in \tilde{e}} \tilde{f}(v) |\ \tilde{e} \in \tilde{E}\}$. Consequently, by Theorem 2.8, ${}^{m}C_n^{(k+2)}$ is super edge-magic.

□

Example 5.7. By applying Lemma 5.6 with the vertex-labeling of ${}^{2}C_5^{(4)}$ (shown in Figure 5.5), we obtain the vertex-labeling for ${}^{2}C_5^{(6)}$ which is also super edge-magic labeling as shown in Figure 5.7.

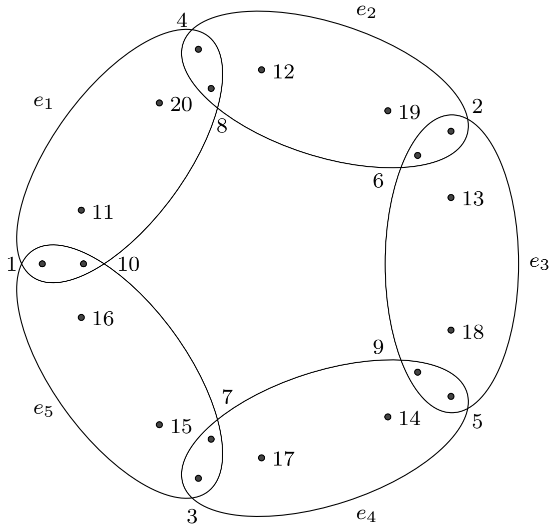

Figure 5.7: The vertex-labeling of ${}^{2}C_5^{(6)}$ obtained by applying Lemma 5.6 to the vertex-labeling of ${}^{2}C_5^{(4)}$

Lemma 5.7. *If ${}^{m}C_n^{(k)}$ is super edge-magic, then ${}^{m+2}C_n^{(k+4)}$ is also super edge-magic.*

Proof. Let ${}^{m}C_n^{(k)} = (V, E)$. By Definition 2.6, we can construct ${}^{m+2}C_n^{(k+4)}$ by considering each $e_i \in E$, define $\tilde{e_i} = e_i \cup \{w_{i,m+1}, w_{i,m+2}, w_{i+1,m+1}, w_{i+1,m+2}\}$ for

$i \in \{1, 2, 3, \ldots, n\}$. Then, ${}^{m+2}C_n^{(k+4)} = (\tilde{V}, \tilde{E})$ where $\tilde{E} = \{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}, \ldots, \tilde{e_n}\}$ and $\tilde{V} = \bigcup\limits_{i=1}^{n} \tilde{e_i}$. Note that, $|V| = n(k-m), |\tilde{V}| = n(k-m) + 2n$, and $|E| = n = |\tilde{E}|$.

Assume that ${}^{m}C_n^{(k)}$ is super edge-magic. By Theorem 2.8, there is a bijection $f : V \to \{1, 2, 3, \ldots, n(k-m)\}$ such that $\{\sum_{v \in e} f(v) |\ e \in E\}$ is a set of n consecutive integers. We define a function $\hat{f}$ by

$$\hat{f}(w_{ij}) = \begin{cases} n(k-m) + i & \text{if } j = m+1, \\ n(k-m) + 2n + 1 - i & \text{if } j = m+2, \end{cases}$$

for each $i \in \{1, 2, 3, \ldots, n\}$. Next, we define $\tilde{f} : \tilde{V} \to \{1, 2, 3, \ldots, n(k-m) + 2n\}$ by

$$\tilde{f}(v) = \begin{cases} f(v) & \text{if } v \in V, \\ \hat{f}(v) & \text{if } v \in \tilde{V} - V, \end{cases}$$

for $v \in \tilde{V}$. Since $f(V) = \{1, 2, 3, \ldots, n(k-m)\}$ and $\hat{f}(\tilde{V} - V) = \{n(k-m) + 1, n(k-m) + 2, n(k-m) + 3, \ldots, n(k-m) + 2n\}$, $\tilde{f}$ is a bijection from $\tilde{V}$ to $\{1, 2, 3, \ldots, n(k-m) + 2n\}$.

To see that $\tilde{f}$ is a super edge-magic labeling, let $\tilde{e_i} \in \tilde{E}$. Observe that, for $i \in \{1, 2, 3, \ldots, n\}$.

$$\begin{aligned} \hat{f}(w_{i,m+1}) + \hat{f}(w_{i,m+2}) &= (n(k-m) + i) + (n(k-m) + 2n + 1 - i) \\ &= 2n(k-m) + 2n + 1. \end{aligned}$$

Then, for $i \in \{1, 2, 3, \ldots, n-1\}$.

$$\begin{aligned} \sum_{v \in \tilde{e_i}} \tilde{f}(v) &= \textstyle\sum_{v \in e_i} \tilde{f}(v) + \tilde{f}(w_{i,m+1}) + \tilde{f}(w_{i,m+2}) + \tilde{f}(w_{i+1,m+1}) + \tilde{f}(w_{i+1,m+2}) \\ &= \sum_{v \in e_i} f(v) + \hat{f}(w_{i,m+1}) + \hat{f}(w_{i,m+2}) + \hat{f}(w_{i+1,m+1}) + \hat{f}(w_{i+1,m+2}) \\ &= \sum_{v \in e_i} f(v) + (2n(k-m) + 2n + 1) + (2n(k-m) + 2n + 1) \end{aligned}$$

$$= \sum_{v \in e_i} f(v) + 4n(k-m) + 4n + 2,$$

and for $i = n$

$$\begin{aligned}
\sum_{v \in \tilde{e_i}} \tilde{f}(v) &= \sum_{v \in e_i} \tilde{f}(v) + \tilde{f}(w_{n,m+1}) + \tilde{f}(w_{n,m+2}) + \tilde{f}(w_{1,m+1}) + \tilde{f}(w_{1,m+2}) \\
&= \sum_{v \in e_i} f(v) + \hat{f}(w_{n,m+1}) + \hat{f}(w_{n,m+2}) + \hat{f}(w_{1,m+1}) + \hat{f}(w_{1,m+2}) \\
&= \sum_{v \in e_i} f(v) + (2n(k-m) + 2n + 1) + (2n(k-m) + 2n + 1) \\
&= \sum_{v \in e_i} f(v) + 4n(k-m) + 4n + 2.
\end{aligned}$$

Since $\{\sum_{v \in e} f(v)|\ e \in E\}$ is a set of n consecutive integers, similar goes for $\{\sum_{v \in \tilde{e}} \tilde{f}(v)|\ \tilde{e} \in \tilde{E}\}$. Consequently, by Theorem 2.8, ${}^{m+2}C_n^{(k+4)}$ is super edge-magic. $\square$

Example 5.8. By applying Lemma 5.7 with the vertex-labeling of ${}^1C_4^{(3)}$ (shown in Figure 5.2), we obtain the vertex-labeling for ${}^3C_4^{(7)}$ which is also super edge-magic labeling as shown in Figure 5.8.

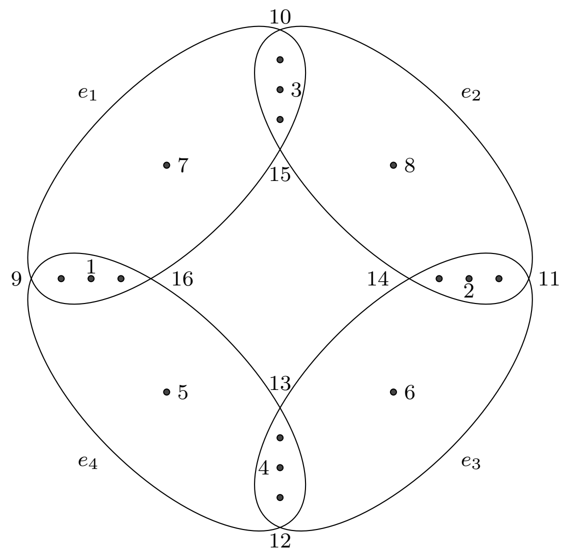

Figure 5.8: The vertex-labeling of ${}^3C_4^{(7)}$ obtained by applying Lemma 5.7 to the vertex-labeling of ${}^1C_4^{(3)}$

Althrough, we can construct a lot of larger super edge-magic hypercycles by using Lemma 5.6 and Lemma 5.7 from some smaller super edge-magic hypercycles. However, by observing some calculations, there is a family of hypercycles which always not super edge-magic.

Theorem 5.8. *If a hypercycle* ${}^{m}C_n^{(2m)}$ *is super edge-magic, then* n *is odd.*

Proof. Note that, ${}^{m}C_n^{(2m)}$ has $p = mn$ and $q = n$. Suppose that ${}^{m}C_n^{(2m)}$ is super edge-magic. Then, there is a bijection $f : V \cup E \to \{1, 2, 3, \ldots, mn + n\}$ and a constant Λ such that $\sum_{v \in e_i} f(v) + f(e_i) = \Lambda$ for all $i \in \{1, 2, 3, \ldots, n\}$ and $f(V) = \{1, 2, 3, \ldots, mn\}$. Since each vertex of ${}^{m}C_n^{(2m)}$ is contained exactly in 2 hyperedges, we obtain

$$n\Lambda = \sum_{i=1}^{n} \left(\sum_{v \in e_i} f(v) + f(e_i) \right)$$

$$n\Lambda = 2 \sum_{v \in V} f(v) + \sum_{e \in E} f(e)$$

$$n\Lambda = 2 \sum_{j=1}^{mn} j + \sum_{j=mn+1}^{mn+n} j$$

$$n\Lambda = (mn)(mn + 1) + \frac{n(2mn + n + 1)}{2}.$$

That is $\Lambda = m(mn + 1) + mn + \frac{n+1}{2}$. Since Λ is an interger, n must be odd. □

To sum up this chapter, by using both Lemma 5.6 and Lemma 5.7, we will show that all hypercycles ${}^{m}C_n^{(k)}$ are super edge-magic except the hypercycles of the form ${}^{m}C_n^{(2m)}$ where n is an even integer. In general, to construct a super edge-magic hypercycle ${}^{m}C_n^{(k)}$, we first find the starting small hypercycle of which super edge-magic as follow:

- if m and k are odd, then the starting hypercycle is ${}^{1}C_n^{(3)}$,
- if m, k are even and n is odd, then the starting hypercycle is ${}^{2}C_n^{(4)}$,

- if m, n are odd and k is even, then the starting hypercycle is ${}^{1}C_n^{(2)}$,
- if m is odd and k, n are even, then the starting hypercycle is ${}^{1}C_n^{(4)}$,
- if m is even and k is odd, then the starting hypercycle is ${}^{2}C_n^{(5)}$.

Then, we apply Lemma 5.7 for $\lceil \frac{m-2}{2} \rceil$ times and Lemma 5.6 for $\lceil \frac{k-2m-1}{2} \rceil$ times, in any order, to the starting hypercycle. Note that for the hypercycles in the fourth case, we apply Lemma 5.6 for only $\lceil \frac{k-2m-1}{2} \rceil - 1$ times. We then obtain the super edge-magic labeling for ${}^{m}C_n^{(k)}$. Therefore, we have the main theorem.

Theorem 5.9. *A hypercycle* ${}^{m}C_n^{(k)}$ *is super edge-magic if and only if* n *is odd or* $k \neq 2m$.

Example 5.9. To obtain the super edge-magic labeling of ${}^{5}C_3^{(13)}$ with $m = 5$ which is odd and $k = 13$ which is odd, we start with the super edge-magic labeling of ${}^{1}C_3^{(3)}$ (Figure 5.9). Then, we apply Lemma 5.7 for $\lceil \frac{5-2}{2} \rceil = 2$ times and Lemma 5.6 for $\left\lceil \frac{13-2(5)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^{3}C_3^{(7)}$ (Figure 5.10), ${}^{5}C_3^{(11)}$ (Figure 5.11) and ${}^{5}C_3^{(13)}$ (Figure 5.12), respectively.

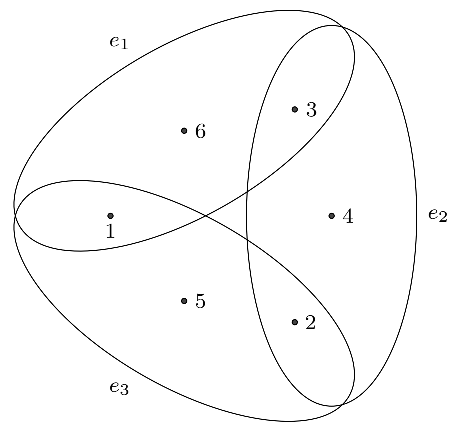

Figure 5.9: The super edge-magic labeling of ${}^{1}C_3^{(3)}$

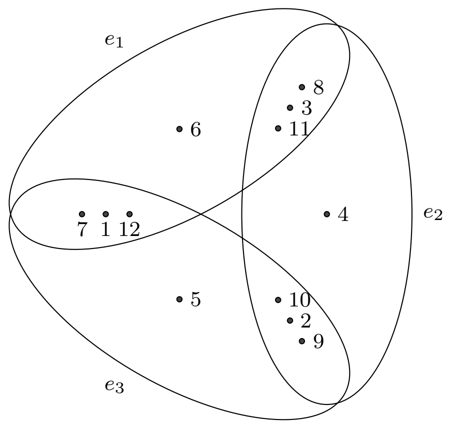

Figure 5.10: The super edge-magic labeling of ${}^{3}C_3^{(7)}$

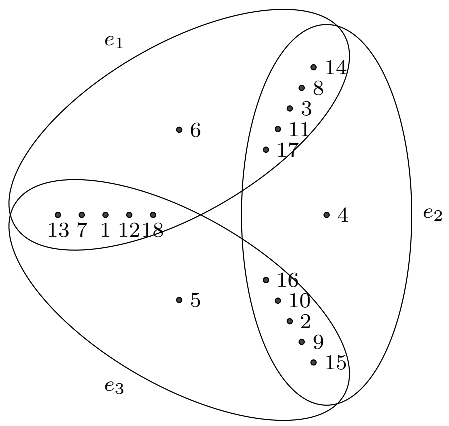

Figure 5.11: The super edge-magic labeling of ${}^{5}C_3^{(11)}$

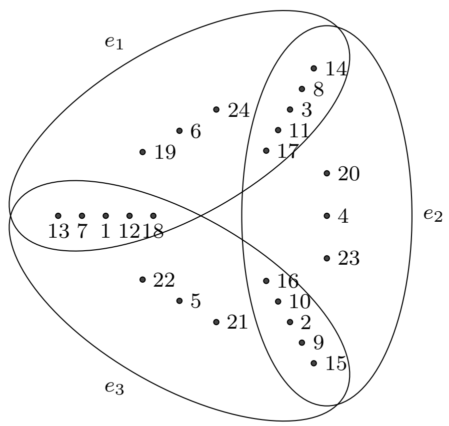

Figure 5.12: The super edge-magic labeling of ${}^{5}C_3^{(13)}$

Example 5.10. To obtain the super edge-magic labeling of ${}^{4}C_3^{(10)}$ with $m = 4$ which is even, $k = 10$ which is even and $n = 3$ which is odd, we start with the super edge-magic labeling of ${}^{2}C_3^{(4)}$ (Figure 5.13). Then, we apply Lemma 5.7 for $\left\lceil \frac{4-2}{2} \right\rceil = 1$ time and Lemma 5.6 for $\left\lceil \frac{10-2(4)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^{4}C_3^{(8)}$ (Figure 5.14) and ${}^{4}C_3^{(10)}$ (Figure 5.15), respectively.

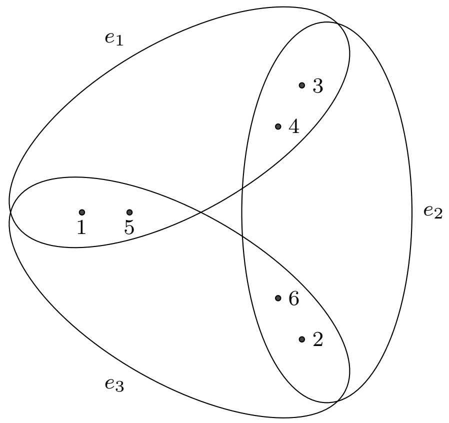

Figure 5.13: The super edge-magic labeling of ${}^{2}C_3^{(4)}$

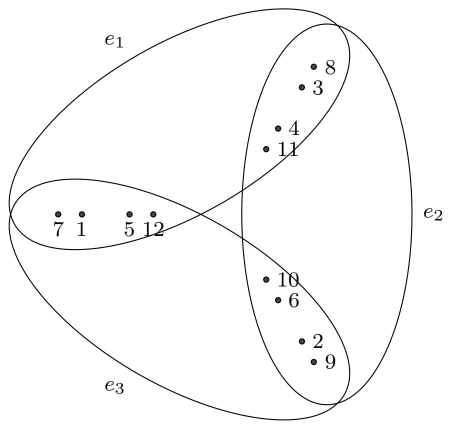

Figure 5.14: The super edge-magic labeling of ${}^{4}C_3^{(8)}$

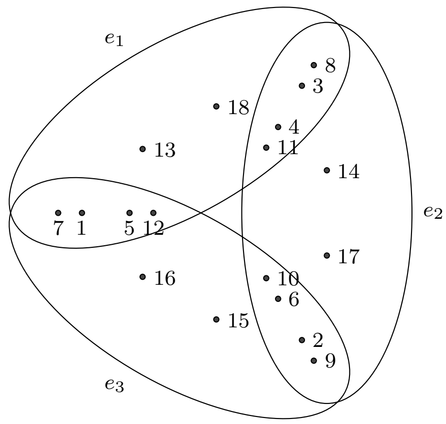

Figure 5.15: The super edge-magic labeling of ${}^4C_3^{(10)}$

Example 5.11. To obtain the super edge-magic labeling of ${}^5C_3^{(12)}$ with $m = 5$ which is odd, $k = 12$ which is even and $n = 3$ which is odd, we start with the super edge-magic labeling of ${}^1C_3^{(2)}$ (Figure 5.16). Then, we apply Lemma 5.7 for $\left\lceil \frac{5-2}{2} \right\rceil = 2$ times and Lemma 5.6 for $\left\lceil \frac{12-2(5)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^3C_3^{(6)}$ (Figure 5.17), ${}^5C_3^{(10)}$ (Figure 5.18) and ${}^5C_3^{(12)}$ (Figure 5.19), respectively.

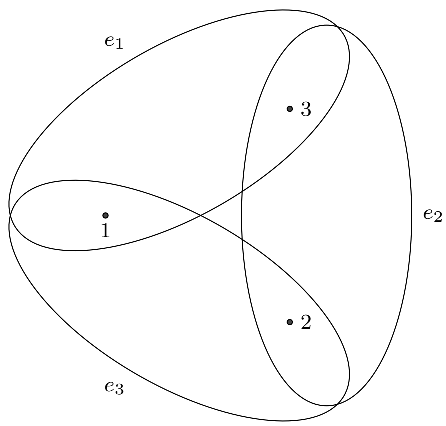

Figure 5.16: The super edge-magic labeling of ${}^1C_3^{(2)}$

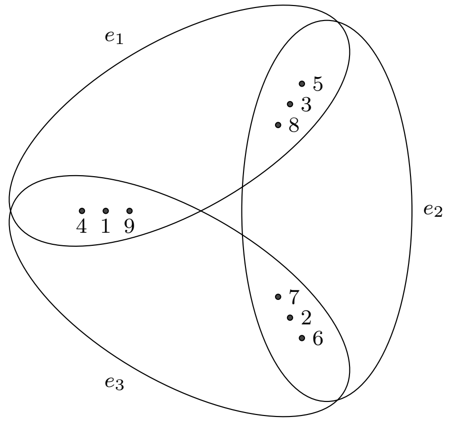

Figure 5.17: The super edge-magic labeling of ${}^3C_3^{(6)}$

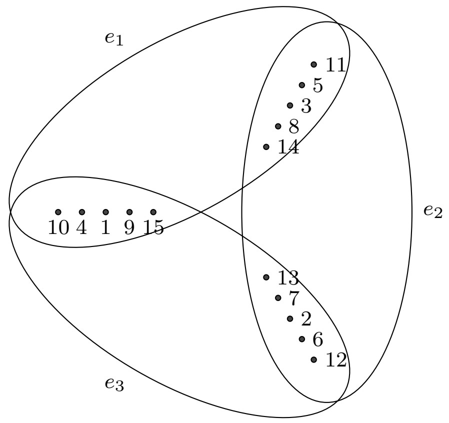

Figure 5.18: The super edge-magic labeling of ${}^5C_3^{(10)}$

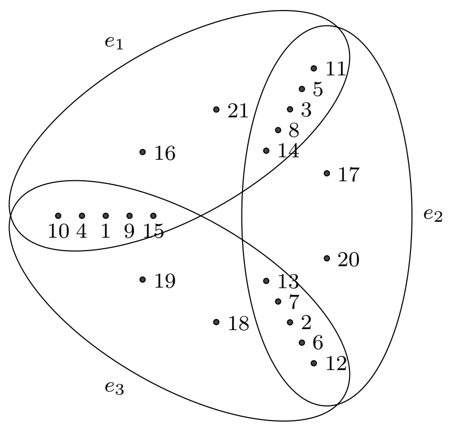

Figure 5.19: The super edge-magic labeling of ${}^5C_3^{(12)}$

Example 5.12. To obtain the super edge-magic labeling of ${}^3C_4^{(10)}$ with $m = 3$ which is odd, $k = 10$ which is even and $n = 4$ which is even, we start with the super edge-magic labeling of ${}^1C_4^{(4)}$ (Figure 5.20). Then, we apply Lemma 5.7 for $\left\lceil \frac{3-2}{2} \right\rceil = 1$ time and Lemma 5.6 for $\left\lceil \frac{10-2(3)-1}{2} \right\rceil - 1 = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^3C_4^{(8)}$ (Figure 5.21) and ${}^3C_4^{(10)}$ (Figure 5.22), respectively.

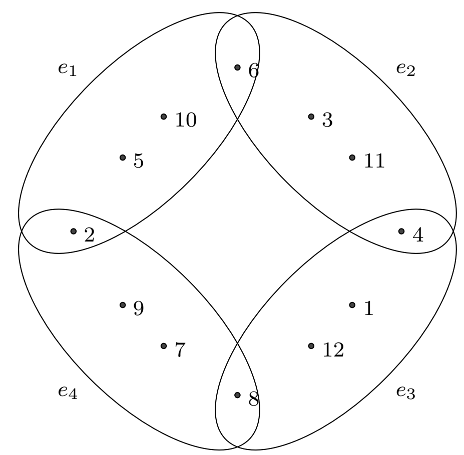

Figure 5.20: The super edge-magic labeling of ${}^1C_4^{(4)}$

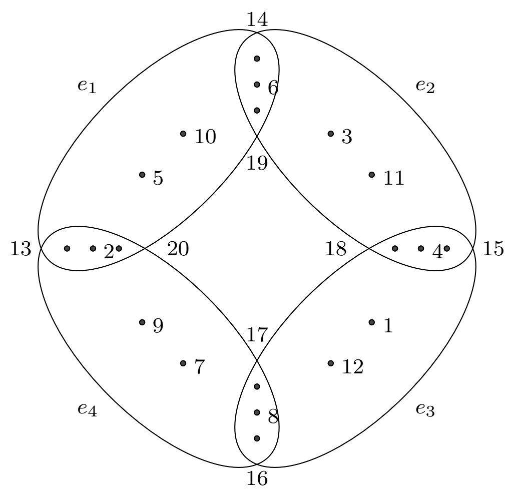

Figure 5.21: The super edge-magic labeling of ${}^3C_4^{(8)}$

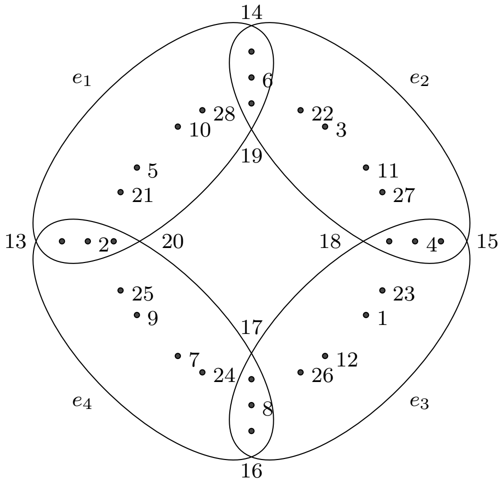

Figure 5.22: The super edge-magic labeling of ${}^{3}C_4^{(10)}$

Example 5.13. To obtain the super edge-magic labeling of ${}^{4}C_3^{(11)}$ with $m = 4$ which is even and $k = 11$ which is odd, we start with the super edge-magic labeling of ${}^{2}C_3^{(5)}$ (Figure 5.23). Then, we apply Lemma 5.7 for $\lceil \frac{4-2}{2} \rceil = 1$ time and Lemma 5.6 for $\left\lceil \frac{11-2(4)-1}{2} \right\rceil = 1$ time, respectively. Therefore, we have the super edge-magic labeling of ${}^{4}C_3^{(9)}$ (Figure 5.24) and ${}^{4}C_3^{(11)}$ (Figure 5.25), respectively.

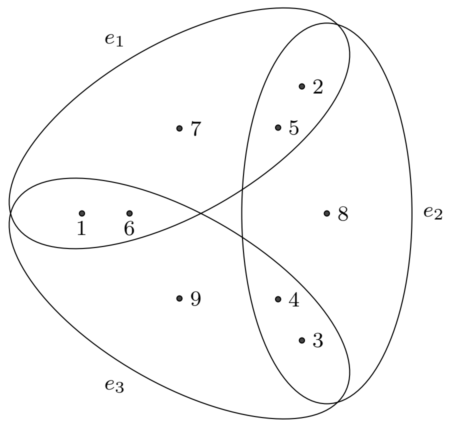

Figure 5.23: The super edge-magic labeling of ${}^{2}C_3^{(5)}$

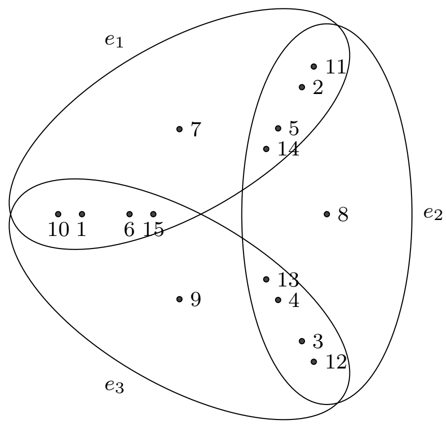

Figure 5.24: The super edge-magic labeling of ${}^4C_3^{(9)}$

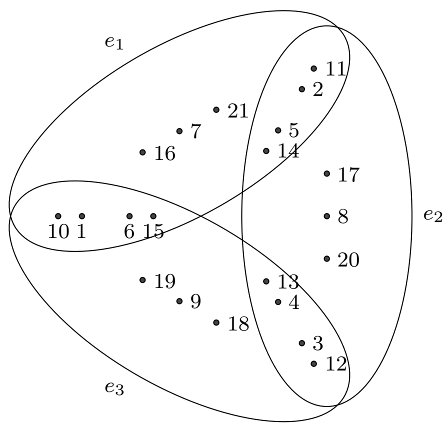

Figure 5.25: The super edge-magic labeling of ${}^4C_3^{(11)}$

CHAPTER VI

CONCLUSION AND DISCUSSION

In Chapter 2, we define three new classes of k-uniform hypergraphs including complete k-uniform hypergraphs, m-node k-uniform hyperpaths and m-node k-uniform hypercycles. Notice that there are varieties of definitions of those hypergraphs.

In Chapter 2, Theorem 2.10, we investigate the extremal value of Λ to obtain the necessary condition, $q \leq kp - k^2 + 1$ for a k-uniform hypergraph H being super edge-magic. However, we can also using this idea to establish the necessary condition for all hypergraphs. Since each hyperedge of a hypergraph contains at least 0 vertex and at most p vertices, we can mimick the same proof as in Theorem 2.10 to obtain the inequality $0 + (p+q) \leq 1 + 2 + 3 + \ldots + p + (p+1)$. Thus, the condition $q \leq \frac{p(p+1)}{2} + 1$ is a necessary condition for a hypergraph being super edge-magic.

In Chapter 3, we consider only the complete k-uniform hypergraph defined in Definition 2.4 with the condition $n \geq k$ and we found that $K_n^{(0)}, K_n^{(1)}, K_n^{(n-1)}$, and $K_n^{(n)}$ are super edge-magic. For the case of the complete k-uniform hypergraph with $n < k$, we can easily see that those hypergraphs have no hyperedges. Hence, by Corollary 2.9, the complete k-uniform hypergraph in this case is super edge-magic.

In Chapter 4 and 5, we finally give algorithm-liked-Theorem for vertex labelings of all ${}^mP_n^{(k)}$ and some classes of ${}^mC_n^{(k)}$. These labelings induce super edge-magic labelings for those hypergraphs. By the algorithms, one can actually give an explicit form of super edge-magic labelings for fixed parameters m, n and k in ${}^mP_n^{(k)}$ and ${}^mC_n^{(k)}$.

Notice that there are some varieties of definitions of hyperpaths and hypercycles. In [4], a *k-uniform hypercycle of order n* has vertex sequence $v_1, v_2, v_3, \ldots, v_n$ such that every adjacent vertices v_i, v_{i+1} are contained in a hyperedge of degree k. Moreover, it said to be *tight* if every k consecutive vertices form a hyperedge and it said to be *loose* if every adjacent hyperedges intersect exactly one vertex. Hyperpaths are defined in a similar way. In [6], they defined hyperpaths and hypercycles in a similar way in graph theory, i.e., hyperpath is an alternative finite sequence of vertices and hyperedges, $v_1, e_1, v_2, e_2, \ldots, e_n, v_{n+1}$, such that for every $i \in \{1, 2, 3, \ldots, n\}$, a hyperedge e_i contains v_i and v_{i+1} and occurs at most one in sequence. Furthermore, if $v_{n+1} = v_1$, then the alternative sequence becomes hypercycle instead of hyperpath. However, our m-node k-uniform hyperpaths and m-node k-uniform hypercyles, that defined in Defintion 2.5 and Definition 2.6, agree with those defined in [6].

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Publication
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Conference **Speaker**
- *Super Edge-Magic Labeling of k-Uniform Hypercycles* at the 20$^{\text{th}}$ Annual Meeting in Mathematics, 27-29 May 2015 at Silpakorn University, Sanam Chandra Palace Campus
- *Super Edge-Magic Labeling of Complete k-Uniform Hypergraphs* at the Annual Pure and Applied Mathematics Conference 2015, 4-5 June 2015 at Chulalongkorn University