CHAPTER IV

Examples

4.1 Initial value problem of heat equation with impulse

Consider the following initial value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(t,u), \quad 0 < x < 1, t > 0, t \neq t_i \ ; \\ \Delta u(t_i) = J_i(u(t_i)), \quad t = t_i, \ i \in \mathbb{N}; \\ u(t,0) = u(t,1) = 0, u_x(t,0) = u_x(t,1) \\ u(0,x) = u_0(x). \end{cases} \tag{4.1}$$

This problem represents the heat flow in a ring of length one with a temperature dependent *source*. In this section we will introduce some functions $f, J_i(i \in \mathbb{N})$ such that satisfies the condition (JF) for this problem and use the previous chapter implying the existence and uniqueness of PCAP classical solution such that asymptotically stable as $t \to +\infty$. Now, we assume that $\{t_i\}$ is a sequence such that $\{t_i^j\}$ is EAP and $\inf t_j^1 = \xi > 0$.

We start by introducing a convenient abstract frame. Let $X = L^2([0,1];\Re)$ denote the space of all L^2-integrable functions on $[0,1]$. Define the $|\cdot| = |\cdot|_2-$norm on $X = L^2([0,1];\Re)$ by

$$|u|_X = \left\{ \int_0^1 |u(x)|^p \right\}^{\frac{1}{p}}.$$

Then $X = L^2([0,1];\Re)$ is Banach space. Define the operator $u(t) : [0,1] \to \Re$ by

$$u(t)(x) = u(t,x)$$

and define an $A : X \to X$ by

$$Au = -u'', \tag{4.2}$$

i.e.,

$$Au(t)(x) = -\frac{\partial^2 u(t,x)}{\partial x^2}, \quad \text{for all } x \in (0,1) \quad \text{with } D(A) = \{u \in \Xi \subseteq X \mid u', u'' \in X\}$$

where $\Xi = \{u \in X \mid u(t,0) = u(t,1) = 0, u_x(t,0) = u_x(t,1)\}$.

Then A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup $S(t)$ (see [10]) and we have, (4.1) corresponding to the abstract nonlinear impulsive differential system:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t \neq t_i \ ; \\ \Delta u(t_i) = J_i(u(t_i)), & t = t_i \ , i \in \mathbb{N}; \\ u(0) = u_0. \end{cases} \tag{4.3}$$

where

$$u'(t)(x) = \lim_{h \to 0^+} \frac{u(t+h)(x) - u(t)(x)}{h}.$$

We now take $\alpha = 1/2$ and let $X_{1/2} = D(A^{1/2})$ with norm $|.|_{1/2}$. Define the function $f : \Re \times X_{1/2} \to X$ and $J_i(i \in \mathbb{N}) : X_{1/2} \to X$ by

$$f(t,u) = h(t)g(u') \ \text{ for each } t \in \Re - \{t_i\} \ \text{ and } \ J_i(u) = g_i(u') \ \text{ for each, } u \in X_{1/2}$$

where $h : \Re \to \Re$ is PCAP with a sequence $\{t_i\}$ such that $\{t_i^j\}$ is EAP and there exist $k > 0$ and $0 < \theta < 1$ satisfying

$$|h(t) - h(s)| \leq k|t-s|^\theta \quad \text{for all } \ t, s \in \Re - \{t_i\} \tag{4.4}$$

$g : X \to X$ and $g_i : X \to X$ are Lipschitz continuous on X. Concrete examples of functions g and g_i are

$$sin(u), \quad ku, \quad arctan(u)$$

First, we introduce some known result for the operator A and $A^{1/2}$ defined by (4.2). Let $u \in D(A)$ and $k \in \Re$ such that

$$Au = -u'' = ku \ \text{ that is } u'' + ku = 0. \tag{4.5}$$

We have

$$< Au, u >=< ku, u >; \quad \text{that is}$$

$$k|u|_X^2 = k < u, u >=< -u'', u >=< u', u' >= |u'|_X^2.$$

So $k \in \Re_+$, for convenient we let $k = \lambda^2$. Then the solution of equation (4.5) have the form

$$u(x) = B_1 cos(\lambda x) + B_2 sin(\lambda x).$$

We have $u(0) = u(1) = 0$ so $B_1 = 0$ and $\lambda = n\pi$, $n \in \mathbb{N}$. Put $\lambda_n = n\pi$. The solutions of equation (4.5) are

$$u_n(x) = B_2 sin(\lambda_n x) \quad , n \in \mathbb{N}.$$

We have $< u_n, u_m >= 0$ for $n \neq m$ if $< u_n, u_n >= 1$, then $B_2 = \sqrt{2}$ and

$$u_n(x) = \sqrt{2} sin(\lambda_n x) \ , \ n \in \mathbb{N}.$$

Then $\{u_n(x) = \sqrt{2} sin(\lambda_n x)\}_{n\in\mathbb{N}}$ is orthonormal basis. Thus for $u \in D(A)$, there exists a sequence of real $\{\alpha_n\}$ such

$$u(x) = \sum_{n\in\mathbb{N}} \alpha_n u_n(x),$$

and

$$-u''(x) = \sum_{n\in\mathbb{N}} (\lambda_n)^2 \alpha_n u_n(x).$$

By using Bessel 's inequality,

$$\sum_{n\in\mathbb{N}} (\alpha_n)^2 < +\infty \ , \quad \sum_{n\in\mathbb{N}} (\lambda_n)^4 (\alpha_n)^2 < +\infty.$$

By Theorem 2.17, we have

$$A^{1/2} u(x) = \sum_{n\in\mathbb{N}} \lambda_n \alpha_n u_n(x)$$

with $u \in D(A^{1/2})$.

Next, we show now that f and $J_i (i \in \mathbb{N})$ are satisfying condition (JF). Let $t_1, t_2 \in \Re$

and $u_1, u_2 \in X_{1/2}$, we obtain

$$\begin{aligned} f(t_1, u_1) - f(t_2, u_2) &= h(t_1)g(u_1') - h(t_2)g(u_2') \\ &= [h(t_1) - h(t_2)]g(u_1') + h(t_2)[g(u_1') - g(u_2')] \end{aligned} \tag{4.6}$$

so

$$\begin{aligned} |f(t_1, u_1) - f(t_2, u_2)|_X &\leq |h(t_1) - h(t_2)||g(u_1')| + |h(t_2)||g(u_1') - g(u_2')| \\ &\leq |g|_\infty |h(t_1) - h(t_2)| + |g|_{Lip}|h(t_2)||u_1' - u_2'|_X \end{aligned} \tag{4.7}$$

since h is PCAP, there exist $k_2 > 0$ such that

$$|h(t_2)| < k_2.$$

We using the fact that $g(u')$ is Lipchitz on $X_{1/2}$ (see more detail [5], page 75), we have

$$\begin{aligned} |f(t_1, u_1) - f(t_2, u_2)|_X &\leq k_1|g|_\infty |t_1 - t_2|^\theta + k_2|g|_{Lip}|u_1 - u|_{1/2} \\ &\leq L(|t_1 - t_2|^\theta + |u_1 - u|_{1/2}) \end{aligned} \tag{4.8}$$

where $0 < \theta < 1$.
Similarly, for $i \in \mathbb{N}$, we have

$$\begin{aligned} |J_i(u_1(t)) - J_i(u_2(t))|_X &= |g_i(u_1'(t) - g_i(u_2'(t)))|_X \\ &\leq |g_i|_{Lip}|u_1' - u_2'|_X \\ &\leq L|u_1 - u_2|_{1/2}. \end{aligned} \tag{4.9}$$

Therefore f and $J_i (i \in \mathbb{N})$ satisfy the condition (JF) with
$L = max\{k_1|g|_\infty, k_2|g|_{Lip}, sup_{i\in\mathbb{N}}|g_i|_{Lip}\}$. Moreover, we are following in the proof of Theorem 3.15., if $L < \frac{N}{K(\frac{\alpha}{2})} < \frac{N\sqrt{\pi}}{M_{1/2}}$ with $K(\alpha) = M_\alpha \left(\frac{\beta\pi}{\Gamma(\alpha) sin\pi\alpha} + \frac{\xi^{-\alpha}}{1 - e^{-\beta\xi}}\right)$, then this theorem implies the existence and uniqueness of a classical PCAP solution for this system (4.3).

4.2 An impulsive logistic equation

Consider nonautomous logistic system:

$$\begin{cases} x'(t) = r(t)x(t)\left[1 - \frac{x(t)}{K(t)}\right], & t \neq t_k, t > 0 \\ \Delta x(t) = B_k(x(t)), & t = t_k, k \in \mathbb{N} \end{cases} \tag{4.10}$$

in which $r(t)$ is nonnegative and $K(t)$ is a strictly positive continuous function and B_k are bounded operators.

By changing variable $x = \frac{1}{z}$ from (4.10) become to the nonautomous impulsive logistic system:

$$\begin{cases} z'(t) + Az(t) = f(t), & t \neq t_k, t > 0 \\ \Delta z(t) = J_k(z(t)), & t = t_k, k \in \mathbb{N} \end{cases} \tag{4.11}$$

where $Az(t) = -r(t)z(t)$ and $f(t) = \frac{r(t)}{K(t)}$. We can see that the behavior or properties of solution for this problem depend on f and J_k which we put in our systems and one example of them is same as before.