EXTENSIONS OF VIRTUAL STABILITY

Mr. Pharunyou Chanthorn

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By	Mr. Pharunyou Chanthorn
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....Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

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....Thesis Advisor

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....Examiner

(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

....Examiner

(Associate Professor Songkiat Sumetkijakan, Ph.D.)

....External Examiner

(Annop Kaewkhao, Ph.D.)

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เรานำเสนอแบบแผนวิธีการทำซ้ำเชิงจุดตรึง ซึ่งรวมแนวคิดของกระบวนการทำซ้ำหลายแบบเข้าไว้ด้วยกัน จากนั้นเราขยายแนวคิด การส่งเสถียรภาพเสมือนไปสู่แบบแผนเสถียรภาพเสมือน ทำให้ได้ผลลัพธ์ของการหดตัวซึ่งนำไปสู่เกณฑ์การหดตัวสำหรับเซตจุดตรึงของการส่งในตัวและการส่งค่าเซตแบบไม่ขยาย ยิ่งไปกว่านั้นเรานำเสนอลำดับของการส่งในตัวซึ่งไม่เป็นแบบแผนวิธีการทำซ้ำเชิงจุดตรึง แต่ก่อให้เกิดผลลัพธ์ของการหดตัวคล้ายกับแบบแผนเสถียรภาพเสมือน ซึ่งถูกเรียกว่า การแยกส่วนประกอบเชิงจุดตรึง

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We present the notion of fixed point iteration schemes, which includes concepts of various iteration processes. Then we extend the concept of virtually stable maps to virtually stable schemes. This gives a retraction result inducing contractibility criterion for fixed point sets of some self-maps and set-valued maps of nonexpansive type. Moreover, we present a sequence of self-maps, which is not a fixed point iteration scheme but induces a retraction result similar to virtually stable schemes, called a fixed point resolution.

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CONTENTS

NOTATIONS

$\mathbb{C}$	the set of complex numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of nonnegative real numbers
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathcal{P}(X)$	the set of subsets of the set X
2^X	the set of nonempty subsets of the set X
$\mathcal{C}(X)$	the set of nonempty closed subsets of the set X
$\mathcal{CB}(X)$	the set of nonempty closed bounded subsets of the set X
$\mathcal{CC}(X)$	the set of nonempty closed convex subsets of the set X
$\mathcal{CCB}(X)$	the set of nonempty closed convex bounded subsets of the set X
$B(x;\varepsilon)$	the open ball of radius $\varepsilon > 0$ centered at the point x
$D(x;\varepsilon)$	the closed disc of radius $\varepsilon \geq 0$ centered at the point x
$\eta(A;\varepsilon)$	the union of all open balls of radius $\varepsilon > 0$ centered in A
∂A	the boundary of A
d	the fixed metric on the set X
$\|\cdot\|$	the fixed norm on the Banach space
$H(\cdot,\cdot)$	the Hausdorff metric.

CHAPTER I

INTRODUCTION

Fixed point theory is useful in many fields such as Economics, Physics, Biology, etc. For the structures of fixed point sets, we know that in a strictly convex Banach space, the fixed point set of a nonexpansive map is convex, but this may not be true in a general Banach space. So it motivates us to study the structures of fixed point sets in particular cases. In 1973, Bruck [5] gave an intriguing result on fixed point sets that used Zorn's lemma to show the existence of a retraction as follows :

Theorem 1.1. Let X be a locally weakly compact and convex subset of a Banach space. If $f : X \to X$ is a nonexpansive map satisfying the conditional fixed point property, then there is a nonexpansive retraction from X onto $\mathrm{Fix}(f)$; i.e., the fixed point set of f is a nonexpansive retract of X.

Some topological structures such as the connectedness and the contractibility of a space are preserved under the retraction. Later in 2001, Benavides and Ramirez [3] improved Bruck's work by considering asymptotically nonexpansive maps satisfying (ω)-fixed point property instead of nonexpansive maps satisfying the conditional fixed point property.

Theorem 1.2. Let X be a locally weakly compact and convex subset of a Banach space. If $f : X \to X$ is asymptotically nonexpansive maps satisfying (ω)-fixed point property, then there exists a nonexpansive retraction r from X onto $\mathrm{Fix}(f)$ satisfying $r \circ f = r$.

In [6], Chaoha presented the notion of virtually nonexpansive maps, which generalizes nonexpansive-type maps, and gave an explicit retraction as follows :

Theorem 1.3. Let X be a metric space and $f : X \to X$ a map. Let $\mathrm{C}(f) = \{x \in X : (f^n) \text{ converges }\}$. If f is a virtually nonexpansive map, then we obtain a retraction $f^\infty : \mathrm{C}(f) \to \mathrm{Fix}(f)$ given by

$$f^\infty(x) = \lim_{n\to\infty} f^n(x),$$

for each $x \in \mathrm{C}(f)$.

The result shows a connection between fixed point sets and fixed point iteration processes. Recently in [7], Chaoha and Atiponrat extended the notion of virtually nonexpansive map of metric spaces to virtually stable map of Hausdorff spaces, and obtained a retraction similar to Theorem 1.3 for regular spaces. In this dissertation, we give a notion of fixed point iteration schemes, which generalizes various iteration processes, and extend the concept of virtually stable maps to virtually stable schemes. Moreover, we obtain that in regular spaces, the fixed point set of a certain virtually stable scheme is a retract of its convergence set. Combined with the convergence result of the Mann iteration process in [16], we obtain a new contractibility criterion for the fixed point set of a certain Suzuki generalized nonexpansive self-map. For set-valued maps, we follow the concept of virtually stable schemes to induce retractions for the fixed point sets of set-valued maps in appropriate settings. Combined with convergence results in [8], [16], and [17], we obtain new contractibility criterions for fixed point sets of two certain set-valued maps in [8], a certain set-valued α-contraction, and a certain family of set-valued maps in [17], respectively. Following the construction of a set-valued analogue of the Mann iteration process in [15], we construct a sequence of self-maps which is not a fixed point iteration scheme, but give a similar retraction and contractibility results for a certain nonexpansive set-valued map. This leads us to introduce the concept of fixed point resolutions generalizing fixed point iteration schemes in the last chapter.

The dissertation is organized as follows. In Chapter 2, we recall some backgrounds in topology, set-valued analysis, and selection theory. In Chapter 3, we introduce notions of fixed point iteration schemes and virtually stable schemes on Hausdorff spaces, and present an interesting retraction result of virtually stable schemes on regular spaces. In Chapter 4, we combine the retraction result in Chapter 3 with convergence results of fixed point iterations to obtain contractibility criterion for fixed point sets of certain Suzuki generalized nonexpansive self-maps. In addition, we present a new concept of α-contractive schemes involving contractibility criterion for fixed point sets of some families of set-valued maps satisfying the Chebyshev condition. In Chapter 5, we introduce the notion of fixed point resolutions, and construct some resolutions generated by Michael's selection theorem [11]. Lastly, we construct a sequence of self-maps (motivated by the set-valued analogue of the Mann iteration process in [15]) that is not a fixed point iteration scheme, but we obtain retraction and contractibility results for the fixed point set of a certain nonexpansive set-valued map.

CHAPTER II

PRELIMINARIES

In this chapter, we recall some definitions and prove some theorems used in this dissertation. For a self-map f on a nonempty set X, we define the **fixed point set** of f by $\mathrm{Fix}(f) = \{x \in X : x = f(x)\}$. Let 2^X be the set of all nonempty subsets of X. For a set-valued map $F : X \to 2^X$, the **fixed point set of a set-valued map** F on X is defined by

$$\mathrm{Fix}(F) = \{x \in X : x \in F(x)\}.$$

In this dissertation, we always assume that every self-map and every set-valued map have a fixed point.

2.1 Topological spaces

We will omit the arguments of known theorems in this section as they can be found in [12].

Definition 2.1. A collection $\mathcal{T}$ of subsets of a nonempty set X is said to be a **topology** on X if it satisfies the following properties :

1. $\varnothing$ and X are in $\mathcal{T}$.
2. Every union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.
3. Every finite intersection of elements of $\mathcal{T}$ is also an element of $\mathcal{T}$.

Every element of $\mathcal{T}$ is called an **open set**. We simply say that the open set U of X is a **neighborhood** of a point $x \in X$ (a subset A of X) if U contains the point x (the set A). A subset A of X is said to be **closed** if the set $X - A$ is open.

Definition 2.2. Let A be a subset of a topological space X. Then the collection $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$ is a topology on A, and the topological space A with $\mathcal{T}_A$ is called a **subspace** of X.

A point $x \in X$ is said to be a **boundary point** of A if for every neighborhood U of x, the sets $U \cap A$ and $U \cap (X - A)$ are nonempty. The set of all boundary points of A is said to be the **boundary** of A (denoted by ∂A). The **closure** of the set A is the set $\overline{A} = A \cup \partial A$.

A collection $\mathcal{U}$ of (open) subsets of a topological space X is said to be a **(an open) covering** of X if the union of all elements of $\mathcal{U}$ is X. The subset A of X is said to be **compact** if every open covering of A has a finite subset which is a covering of A.

Proposition 2.3. ([12], Theorem 26.2.) Every closed subset of a compact set is compact.

Definition 2.4. Let X be a topological space. A sequence (x_n) in X is said to **converge** to a point $x \in X$ (denoted by $\lim\limits_{n\to\infty} x_n = x$) if every neighborhood U of x, there is $N \in \mathbb{N}$ such that $x_n \in U$, for each $n \geq N$.

It is clear that a sequence (x_n) converges to a point $x \in X$ if and only if every subsequence (x_{n_k}) of (x_n) converges to the point x.

Definition 2.5. Let X and Y be topological spaces. A map $f : X \to Y$ is said to be **continuous** at $x \in X$ if every neighborhood V of $f(x)$ in Y, there is a neighborhood U of x in X such that $f(U) \subseteq V$.

We say that a map $f : X \to Y$ is **continuous on** X or **continuous** in short if it is continuous at each $x \in X$.

For a subset A of X, a continuous map $f : X \to A$ is said to be a **retraction** if $f(a) = a$, for each $a \in A$. In addition, the set A is said to be a **retract** of X.

Definition 2.6. A topological space X is said to be **Hausdorff** if every pair $x, y \in X$ such that $x \neq y$, there are neighborhoods U of x and V of y, such that $U \cap V = \varnothing$.

Suppose that every one-point sets is closed in X. The space X is said to be **regular** if every pair consisting of a point $x \in X$ and a closed set A such that $x \notin A$, there are neighborhoods U of x and V of A, such that $U \cap V = \varnothing$.

Notice that a regular space is Hausdorff.

Proposition 2.7. Let X be a Hausdorff space. Then

1. Every convergent sequence in X converges to a unique point.

2. Every compact subset of X is closed.

Proof. (1) and (2) follow Theorem 17.10 and Theorem 26.3, respectively, in [12]. $\square$

Lemma 2.8 ([12], Lemma 31.1.)**.** A space X is regular if and only if every point x of X and every neighborhood U of x, there is a neighborhood V of x such that

$$x \in V \subseteq \overline{V} \subseteq U.$$

Definition 2.9. Let X be a nonempty set. A map $d : X \times X \to \mathbb{R}^+$ is said to be a **distance map** or simply a **metric** if it satisfies the following properties :

1. $d(x, y) = 0$ if and only if $x = y$,

2. $d(x,y) = d(y,x)$, and

3. $d(x,y) \leq d(x,z) + d(z,y)$,

for each $x, y, z \in X$.

For each $x \in X$, the **open ball** of radius $\varepsilon > 0$ centered at the point x is denoted by

$$B(x;\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}.$$

Let $\mathcal{U}$ be the collection of all finite intersections of open balls induced by the metric d. Then the collection $\mathcal{T}_d$ of all unions of elements of $\mathcal{U}$ is a topology on X and is called the **metric topology induced by** d. Notice that every subspace of a metric space is metric space, and every metric space is regular.

Later on, let X be a metric space.

Proposition 2.10. If $d(a, A) = 0$ and A is a closed subset of X, then $a \in A$.

Definition 2.11. A subset A of X is said to be **bounded** if there is $M \in \mathbb{N}$ such that $d(x,y) \leq M$, for each $x, y \in X$.

Notice that every compact subset of X is bounded, and every convergent sequence is bounded.

Proposition 2.12. A sequence (x_n) converges to a point x if and only if every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$, for each $n \geq N$.

Definition 2.13. A sequence (x_n) is said to be a **Cauchy sequence** if for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$, for each $m, n \geq N$. The metric space X is said to be **complete** if every Cauchy sequence in X converges.

Notice that every convergent sequence is always a Cauchy sequence.

Proposition 2.14. Every closed subset of a complete metric space is complete. Moreover, every compact subset of a metric space is complete.

Proposition 2.15. Let $\alpha \in [0, 1)$. Every sequence (x_n) in a metric space satisfying $d(x_{n+1}, x_n) < \alpha^n$, for each $n \in \mathbb{N}$, is a Cauchy sequence.

Theorem 2.16 ([12], Theorem 28.2.)**.** A subset A of a metric space X is compact if and only if every bounded sequence in A has a convergence subsequence.

Lemma 2.17. Let A be a compact subset of a metric space X, $p \in A$ and (x_n) a sequence in A. If each convergent subsequence of (x_n) converges to the same point p, then the sequence (x_n) converges to the point p.

Proof. Suppose for a contradiction that each convergent subsequence of (x_n) converges to the same point p, and (x_n) does not converge to the point p. Also, there exists a neighborhood U of p in A such that for each $n \in \mathbb{N}$, $x_m \notin U$, for some $m \geq n$. Then there is a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \notin U$, for each $k \in \mathbb{N}$.

By the compactness of A, the sequence (x_{n_k}) has a convergent subsequence, says (x_{m_k}). Also, (x_{m_k}) converges to a point q in X. Hence, $q = p$, by the assumption. Thus there is $x_{m_{k_0}} \in (x_{n_k})$ such that $x_{m_{k_0}} \in U$ which is impossible. $\square$

Lemma 2.18. Let X and Y be metric spaces and $f : X \to Y$ a map. Then the following are equivalent :

1. The map f is continuous at a point $x \in X$.

2. For each $\varepsilon > 0$, there is $\delta > 0$ such that $f(B(x; \delta)) \subseteq B(f(x); \varepsilon)$.

3. For each $\varepsilon > 0$ and sequence (x_n) in X such that (x_n) converges to the point x, there is $K \in \mathbb{N}$ such that $f(x_n) \in B(f(x); \varepsilon)$, for each $n \geq K$.

4. If (x_n) converges to the point x, then $(f(x_n))$ converges to $f(x)$.

Definition 2.19. Let X be a space, and Y be a metric space. A family $\mathcal{F}$ of self-maps of X is said to be **equicontinuous** at $x \in X$ if every $\varepsilon > 0$, there is a neighborhood U of x such that $f(U) \in B(f(x); \varepsilon)$, for each $f \in \mathcal{F}$.

2.2 Continuity of set-valued maps

For now, let X and Y be Hausdorff topological spaces.

Definition 2.20 ([9])**.** A map $F : X \to 2^Y$ is said to be

- **upper semi-continuous** if

$$\{x \in X : F(x) \subseteq U\} \text{ is open in } X, \text{ for each open set } U \text{ in } Y.$$

- **lower semi-continuous** if

$$\{x \in X : F(x) \cap U \neq \varnothing\} \text{ is open in } X, \text{ for each open set } U \text{ in } Y.$$

We say that the set-valued map $F : X \to Y$ is **continuous** if it is upper and lower semi-continuous. Notice that a constant set-valued map is continuous.

Example 2.21. For each map $f : X \to Y$, we define a set-valued map (in the trivial way) $\hat{f} : X \to 2^Y$ by

$$\hat{f}(x) = \{f(x)\}, \text{ for each } x \in X. \tag{2.1}$$

It is not difficult to see that the continuity of f and all above continuities of $\hat{f}$ are equivalent.

For general set-valued maps, the upper semi-continuity and lower semi-continuity do not imply each other as shown in the following examples.

Example 2.22. Define $F : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$F(x) = \begin{cases} \{0\}, & \text{if } x \leq 0, \\ [-1, 1], & \text{if } 0 < x. \end{cases}$$

Then the map F is lower semi-continuous, because

$$\{x \in X : F(x) \cap U \neq \varnothing\} = \begin{cases} \mathbb{R}, & \text{if } 0 \in U, \\ (0, \infty), & \text{if } 0 \notin U \text{ and } U \cap [-1, 1] \neq \varnothing, \\ \varnothing, & \text{if } U \cap [-1, 1] = \varnothing. \end{cases}$$

Since $\{x \in X : F(x) \subseteq (-1, 1)\} = (\infty, 0]$ is not an open set in $\mathbb{R}$, F is not upper semi-continuous.

Example 2.23. Define $F : \mathbb{R} \to 2^{\mathbb{R}}$ by

$$F(x) = \begin{cases} \{-1\}, & \text{if } x < 0 \\ \{-1, 1\}, & \text{if } x = 0, \\ \{1\}, & \text{if } 0 < x. \end{cases}$$

Since the set $\{x \in X : F(x) \cap (-2, 0) \neq \varnothing\} = (\infty, 0]$ is not an open set in $\mathbb{R}$, F is not lower semi-continuous. On the other hand,

$$\{x \in X : F(x) \subseteq U\} = \begin{cases} \mathbb{R}, & \text{if } -1 \in U \text{ and } 1 \in U, \\ (0, \infty), & \text{if } -1 \notin U \text{ and } 1 \in U, \\ (-\infty, 0), & \text{if } -1 \in U \text{ and } 1 \notin U, \\ \varnothing, & \text{if } -1 \notin U \text{ and } 1 \notin U, \end{cases}$$

so the map F is upper semi-continuous.

Definition 2.24. Let $x \in X$. A map $F : X \to 2^Y$ is said to be

- **upper semi-continuous** at x if every neighborhood V of $F(x)$ in Y, there is a neighborhood U of x in X such that $F(y) \subseteq V$, for each $y \in V$.
- **lower semi-continuous** at x if every neighborhood V of $F(x)$ in Y, there is a neighborhood U of x in X such that $F(y) \cap V \neq \varnothing$, for each $y \in V$.

We say that the set-valued map F is **continuous** at x if it is upper and lower semi-continuous at x. Moreover, the map $F : X \to 2^Y$ is said to be **(upper and lower semi-)continuous** if it is (upper and lower semi-, respectively)continuous at each $x \in X$.

Theorem 2.25 ([9], Proposition 5.3.6 and Proposition 5.3.15)**.** Let X and Y be metric spaces, $F : X \to 2^Y$ a map and $x \in X$. Then

1. the map F is upper semi-continuous at x if and only if every a sequence (x_n) in X converging to x and every neighborhood V of $F(x)$ in Y, there exists $N \in \mathbb{N}$ such that $F(x_n) \subseteq V$, for each $n \geq N$,
2. the map F is lower semi-continuous at x if and only if every a sequence (x_n) in X converging to x and every point $y \in F(x)$, there exists a sequence (y_n) in Y such that $y_n \in F(x_n)$ and (y_n) converges to y.

Next, we will give a definition of H-continuity of set-valued maps. Let X be a metric space. For a subset A of X, the union of all open balls of radius $\varepsilon > 0$ centered in A is denoted by $\eta(A; \varepsilon) = \bigcup_{a \in A} B(a; \varepsilon)$.

From [9], the map $h : 2^X \times 2^X \to [0, \infty]$ is defined by

$$h(A, B) = \inf\{\varepsilon > 0 : A \subseteq \eta(B; \varepsilon)\},$$

for each $A, B \in 2^X$.

Definition 2.26 ([9])**.** Let X and Y be metric spaces, and $x \in X$. A map $F : X \to 2^Y$ is said to be

- **H-upper semi-continuous** at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$h(F(y), F(x)) < \varepsilon, \text{ for each } y \in B(x, \delta),$$

- **H-lower semi-continuous** at x if for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$h(F(x), F(y))) < \varepsilon, \text{ for each } y \in B(x, \delta),$$

We say that the map F is **H-continuous (at x)** if F is H-upper and H-lower semi-continuous (at x).

Lemma 2.27 ([9], Proposition 5.3.43)**.** Let X and Y be metric spaces. If a map $F : X \to 2^Y$ is H-lower semi-continuous at x, then F is lower semi-continuous at x.

Lemma 2.28 ([9], Proposition 5.3.42)**.** Let X and Y be metric spaces, $F : X \to 2^Y$ a map with compact values. If F is H-upper semi-continuous at x, then F is upper semi-continuous at x.

Note that the compactness of values for the map in Lemma 2.28 is necessary as shown in Example 5.3.40 [9].

Proposition 2.29 ([9], Lemma 4.1.13)**.** Let X be a metric space. Then

$$h(A, B) = \sup_{a \in A} d(a, B),$$

for each $A, B \in 2^X$, where $d(a, B) = \inf_{b \in B} d(a, B)$.

The map h defined as above can induce the following metric.

Definition 2.30 ([15])**.** Let X be a metric space, and $\mathcal{CB}(X)$ the set of nonempty closed bounded subsets of X. The **Hausdorff metric** $H : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ is defined by

$$H(A, B) = \inf\{\varepsilon : A \subseteq \eta(B; \varepsilon) \text{ and } B \subseteq \eta(A; \varepsilon)\},$$

for each $A, B \in \mathcal{CB}(X)$.

Proposition 2.31 ([9], Proposition 4.1.14)**.** Let X be a metric space. Then

$$H(A,B) = \max\{h(A,B), h(B,A)\},$$

for each $A, B \in \mathcal{CB}(X)$.

Proposition 2.32. Let X and Y be metric spaces, and $F : X \to \mathcal{CB}(Y)$ be a map. Then the map F is H-continuous if and only if it is continuous with respect to the Hausdorff topology on $\mathcal{CB}(Y)$.

The following proposition is straightforward from the definition.

Proposition 2.33. Let X be a metric space, $x, y \in X$, and $A, B \subseteq X$. We have the following properties :

1. $d(x, A) \leq d(x, y) + d(y, A)$.
2. $d(x, A) \leq d(x, B) + h(B, A)$.
3. $d(x, A) \leq d(x, B) + H(B, A)$.
4. If $\varepsilon > 0$ and $A, B \in \mathcal{CB}(X)$, then every $x \in A$, there is $y \in B$ such that $d(x, y) < H(A, B) + \varepsilon$.

2.3 Normed spaces and Banach spaces

In this section, we will recall the definition of normed spaces. For a vector space E, we write " $+$ " and " $\cdot$ " for the additive operator and the multiplicative operator, respectively, of E.

Definition 2.34. Let E be a vector space over $\mathbb{R}$. A real-valued function $\|\cdot\| : E \to \mathbb{R}$ is said to be a **norm** on E if for each $x, y \in E$ and $\alpha \in \mathbb{R}$,

1. $\|x\| \geq 0$,

2. $\|x\| = 0$ if and only if $x = 0$,

3. $\|\alpha \cdot x\| = |\alpha| \ \|x\|$, and

4. $\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality).

A norm $\|\cdot\|$ on E induces a metric d on E which is given by

$$d(x, y) = \|x - y\|,$$

for each $x, y \in E$, and is called the **metric induced by the norm**. The pair $(E, \|\cdot\|)$ of the nonempty set E and the norm $\|\cdot\|$ is called the **normed space** written E in short. Moreover, a normed space E is called a **Banach space** if the metric space E induced by the norm is complete.

Example 2.35. The vector space $\ell_1 = \{(x_n) \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n| < \infty\}$ with the norm $\|(x_n)\| = \sum_{n=1}^{\infty} |x_n|$ is a Banach space.

Example 2.36. The vector space $\ell_p = \{(x_n) \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ with the norm $\|(x_n)\| = (\sum_{n=1}^{\infty} |x_n|)^{1/p}$ is a Banach space.

Example 2.37. The vector space $\ell_\infty = \{(x_n) \subseteq \mathbb{R} : \sup |x_n| < \infty\}$ with the norm $\|(x_n)\| = \sup_{n \in \mathbb{N}} |x_n|$ is a Banach space.

Proposition 2.38. In a normed space, the norm is continuous. Moreover, the addition, and the multiplication by scalar are continuous operators.

Definition 2.39. A subset A of a vector space E is said to be **convex** if every $x, y \in A$, the point $\alpha x + (1 - \alpha)y \in A$ for each $\alpha \in [0, 1]$. The set $\{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$ is called a **closed segment** with **boundary points** x and y. In addition, the smallest convex set containing a subset A of X is called the **convex hull** of A (denoted by $\mathrm{co}(A)$).

Proposition 2.40. The closure of every convex set in a Banach space is convex.

Recall that a metric space X is said to be **metrically convex** [1] if for each $x, y \in X$ with $x \neq y$, there exists an element $z \in X$, such that $x \neq z$, $y \neq z$, and $d(x, y) = d(x, z) + d(z, y)$.

Lemma 2.41 ([1])**.** Let X be a complete and metrically convex metric space. If A is a nonempty closed subset of X, then for each $x \in A$ and $y \notin A$, there exists $z \in \partial A$ (the boundary of A) such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.42. Let X be a complete and metrically convex metric space, $x, y \in X$ and $s, t \in [0, \infty)$. Then we have $H(D(x; s), D(x; t)) = |s - t|$. Moreover, if X is a normed space over $\mathbb{R}$, then $H(D(x; t), D(y; t)) = \|x - y\|$.

Proof. WLOG, we assume that $s \leq t$. Since $D(x; s) \subseteq D(x; t)$, we have

$$h(D(x; s), D(x; t)) = \sup_{a \in D(x;s)} d(a, D(x; t)) = 0.$$

Following Proposition 2.31,

$$\begin{aligned} H(D(x; t), D(x; s)) &= \max\{h(D(x; t), D(x; s)), h(D(x; s), D(x; t))\} \\ &= h(D(x; t), D(x; s)). \end{aligned}$$

It is easy to see that $t - s \leq d(y, z)$, for each $y \in S(x; t)$ and $z \in D(x; s)$. Also,

$$t - s \leq \inf_{z \in D(x;s)} d(y, z) = d(y, D(x; s)) \leq h(D(x; t), D(x; s)).$$

On the other hand, by Lemma 2.41, for each $y \in S(x; t)$, there exists $z \in S(x; s)$ such that $d(x, z) + d(z, y) = d(x, y)$. Consequently,

$$h(y, D(x; s)) \leq d(y, z) \leq t - s,$$

for each $y \in S(x;t)$. Then $h(D(x;t), D(x;s)) \leq t - s$.

Finally, let X be a normed space and $x \neq y$. Let $a = x + \frac{t}{\|x-y\|}(x-y) \in S(x;t)$. Then,

$$\begin{aligned}
\|x-y\| + t &= \|x-y\| + \frac{t\,\|x-y\|}{\|x-y\|} \\
&= \left(1 + \frac{t}{\|x-y\|}\right)\|x-y\| \\
&= \left\|\left(1 + \frac{t}{\|x-y\|}\right)(x-y)\right\| \\
&= \left\|(x-y) + \frac{t}{\|x-y\|}(x-y)\right\| \\
&= \|a-y\| \\
&\leq \|a-z\| + \|z-y\| \leq \|a-z\| + t,
\end{aligned}$$

for each $z \in D(y;t)$. Therefore, $\|x-y\| \leq \|a-z\|$, for each $z \in D(y;t)$, and hence

$$\|x-y\| \leq \inf_{z \in D(y;t)} \|a-z\| = d(a, D(y;t)) \leq h(D(x;t), D(y;t)) \leq H(D(x;t), D(y;t)).$$

On the other hand, let $a \in D(x;t)$ and $\varepsilon > 0$. Consider the point $b = a + y - x$. Then $\|a-b\| = \|x-y\|$ and $\|b-y\| = \|a-x\| \leq t$. Hence,

$$a \in B(b; \|x-y\| + \varepsilon),\ b \in D(y;t), \text{ and } a \in \eta(D(y;t); \|x-y\| + \varepsilon).$$

Consequently, $D(x;t) \subseteq \eta(D(y;t); \|x-y\| + \varepsilon)$. Similarly to the previous argument, we have $D(y;t) \subseteq \eta(D(x;t); \|x-y\| + \varepsilon)$. Thus, $H(D(x;t), D(y;t)) \leq \|x-y\| + \varepsilon$. Therefore, $H(D(x;t), D(y;t)) \leq \|x-y\|$. $\square$

In Lemma 2.42, the assumption that X is a normed space, is necessary. It is shown in the following examples.

Example 2.43. Consider $[0,1]$ with the standard metric. Let $x = 0$, $y = 1$, and $t = 1$. Then $H(D(x;t), D(y;t)) = 0$ but $|x-y| = 1$.

Definition 2.44. A topological space X is said to be **contractible** if there are a point $x_0 \in X$ and a continuous map $f : X \times [0,1] \to X$ such that $f(x,1) = x$ and $f(x,0) = x_0$, for all $x \in X$.

Note that a convex subset of a Banach space and a retract of a contractible space are always contractible.

2.4 Continuous selections of set-valued maps

Let $F : X \to 2^Y$ be a map. A map $f : X \to Y$ is said to be a **selection** [11] of F if $f(x) \in F(x)$, for each $x \in X$. Although a set-valued map is continuous or H-continuous, we may have a discontinuous selection as shown in the following example.

Example 2.45. We define a map $F : \mathbb{R} \to 2^{\mathbb{R}}$ by $F(x) = [0,1]$, for each $x \in \mathbb{R}$. It is continuous and H-continuous. Consider a map $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Then the map f is a selection of F but it is not continuous.

Definition 2.46. Let X be a topological space.

A covering $\mathcal{U}$ of X is said to be **locally finite** if every $x \in X$ has a neighborhood intersecting only finitely many $U \in \mathcal{U}$.

A collection $\mathcal{V}$ of subset of X is said to be a **refinement** of a covering $\mathcal{U}$ if $\mathcal{V}$ is a covering of X and every $V \in \mathcal{V}$ is a subset of some $U \in \mathcal{U}$.

A space X is called **paracompact** if every covering of X has a locally finite refinement.

It is not difficult to see that every compact space is paracompact. Note that every metric space is paracompact [11]. In particular, every subspace of a Banach space is paracompact.

The following theorem is implied directly from the classical Michael's selection theorem ([11], Theorem 3.2$''$).

Theorem 2.47. Let X be a paracompact space and Y a Banach space. If $\Phi : X \to 2^Y$ is lower semi-continuous with closed and convex values, then Φ admits a continuous selection.

Theorem 2.48 ([11], Proposition 2.3)**.** Let X and Y be metric spaces and $F : X \to 2^Y$ lower semi-continuous. If a map $G : X \to 2^Y$ is such that $\overline{G(x)} = \overline{F(x)}$, for each $x \in X$, then G is lower semi-continuous.

Theorem 2.49 ([9], Proposition 5.3.20)**.** Let X be a metric space and Y a Banach space. Assume that $F : X \to 2^Y$ and $G : X \to 2^Y$ are lower semi-continuous. If G has open convex values and $F(x) \cap G(x)$ is nonempty, for each $x \in X$, then the map $\Phi : X \to 2^Y$ defined by

$$\Phi(x) = F(x) \cap G(x), \text{ for each } x \in X,$$

is lower semi-continuous.

Lemma 2.50. Let X be a closed subset of a Banach space E. Assume that $f : E \to E$ and $\varphi : E \to (0, \infty)$ are continuous maps.

1. The map $\overline{\Phi} : E \to 2^E$ defined by $\overline{\Phi}(x) = D(f(x); \varphi(x)),$ for each $x \in E$, is lower semi-continuous.

2. The map $\Phi : E \to 2^E$ defined by $\Phi(x) = B(f(x); \varphi(x)),$ for each $x \in E$, is lower semi-continuous.

3. If $F : X \to \mathcal{CC}(X)$ is a lower semi-continuous map satisfying $\Phi(x) \cap F(x) \neq \varnothing$, for each $x \in X$, then there is a continuous map $g : X \to X$ such that $g(x) \in \overline{\Phi}(x) \cap F(x)$, for each $x \in X$.

Proof. (1) Let $x \in E$ and $\varepsilon > 0$. Since f and φ are continuous, there is $\delta > 0$ such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{2} \text{ and } |\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2},$$

for each $y \in B(x; \delta)$. Thus, from Lemma 2.42,

$$\begin{aligned} H(\overline{\Phi}(x), \overline{\Phi}(y)) =& H(D(f(x); \varphi(x)), D(f(y); \varphi(y))) \\ \leq& H(D(f(x); \varphi(x)), D(f(x); \varphi(y))) \\ &+ H(D(f(x); \varphi(y)), D(f(y); \varphi(y))) \\ =& |\varphi(x)) - \varphi(y)| + \|f(x) - f(y)\| < \varepsilon, \end{aligned}$$

for each $y \in B(x; \delta)$. Therefore, $\overline{\Phi}$ is H-continuous and hence lower semi-continuous by Lemma 2.25 (2).

(2) It follows directly from (1) and Lemma 2.48.

(3) By (2), the restriction $\Phi|_X : X \to 2^E$ is lower semi-continuous. Since $B(f(x); \varphi(x))$ and $F(x)$ are convex, for each $x \in X$, so is $\overline{\Phi(x) \cap F(x)}$ by Proposition 2.40. Following Lemma 2.48 and Lemma 2.49, the map $\Psi : X \to \mathcal{CC}(X) \subseteq \mathcal{CC}(E)$ defined by

$$\Psi(x) = \overline{\Phi(x) \cap F(x)},$$

for each $x \in X$, is lower semi-continuous. By Theorem 2.47, there is a continuous map $g : X \to E$ such that

$$g(x) \in \Psi(x) = \overline{\Phi(x) \cap F(x)} \subseteq \overline{\Phi}(x) \cap \overline{F(x)} = \overline{\Phi}(x) \cap F(x) \subseteq X,$$

for each $x \in X$. $\square$

2.5 Nonexpansive-type maps

Definition 2.51. Let X be a metric space and $\alpha \in (0,1]$. A map $F : X \to \mathcal{CB}(X)$ is said to be

- an **α-contraction** if for each $x, y \in X$, $H(F(x), F(y)) \leq \alpha d(x,y)$,
- **nonexpansive** if for each $x, y \in X$, $H(F(x), F(y)) \leq d(x,y)$,
- **quasi-nonexpansive** if for each $x \in X$ and $y \in \mathrm{Fix}(F)$,

$$H(F(x), F(y)) \leq d(x,y).$$

When F is single-valued, the above definitions of α-contraction, nonexpansive set-valued map and quasi-nonexpansive set-valued map coincide with the usual definitions for single-valued map.

It is easy to see that each α-contraction (set-valued) self-map is nonexpansive and (H-)continuous, while every nonexpansive (set-valued) self-map is quasi-nonexpansive.

Example 2.52. Consider the set $X = [-1,1] \times [0,1] \subseteq \mathbb{R}^2$ with the maximum norm $\|(x,y)\| = \max\{|x|, |y|\}$. Let $f : X \to X$ be defined by

$$f(x,y) = (x, |x|), \text{ for each } (x,y) \in X.$$

Then f is nonexpansive.

Example 2.53. Consider the set $X = \{re^{i\theta} \in \mathrm{C} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$. We define a map $f : X \to X$ by

$$f(z) = \begin{cases} re^{\frac{i\pi}{2}}, & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ re^{\frac{-i\pi}{2}}, & \text{if } -\frac{\pi}{2} < \theta < 0, \\ -re^{i\theta}, & \text{if } \theta \in \{\frac{\pi}{2}, -\frac{\pi}{2}\}, \end{cases}$$

for each $z = re^{i\theta} \in X$. We will show that f is quasi-nonexpansive but is not continuous; i.e., f is not nonexpansive.

First, we show that $\text{Fix}(f) = \{0\}$. From the definition of the map f, $f(0) = 0$. On the other hand, let $z \in X$ such that $z \neq 0$. By the definition of f, $z \neq f(z)$. Therefore, $\text{Fix}(f) = \{0\}$.

Next we show that f is quasi-nonexpansive. Let $z = re^{i\theta} \in X$. Then

$$\|f(z) - f(0)\| = \begin{cases} \left\| re^{\frac{i\pi}{2}} - 0 \right\|, & \text{if } 0 \leq \theta < \frac{\pi}{2}, \\ \left\| re^{\frac{-i\pi}{2}} - 0 \right\|, & \text{if } -\frac{\pi}{2} < \theta < 0, \\ \left\| -re^{i\theta} - 0 \right\|, & \text{if } \theta \in \{\frac{\pi}{2}, -\frac{\pi}{2}\}, \end{cases}$$

$$= |r| = \|z - 0\|$$

Therefore, f is quasi-nonexpansive.

To show that f is not continuous, we consider a sequence $\left(e^{\frac{-i}{n}}\right)$ in X. Then the sequence $\left(e^{\frac{-i}{n}}\right)$ converges to 1. Hence, $f\left(e^{\frac{-i}{n}}\right) = e^{\frac{-i\pi}{2}}$ but $f(1) = e^{\frac{i\pi}{2}}$. Therefore, f is not continuous.

For a set-valued map $F : X \to 2^X$, we define a map $P_F : X \to \mathcal{P}(X)$ by

$$P_F(x) = \{y \in F(x) : d(x, y) = d(x, F(x))\},$$

for each $x \in X$. Notice that $P_F(x)$ may be the empty set for some $x \in X$. The set-valued map $F : X \to 2^X$ is said to satisfy

- the **end point condition** [15] if $F(p) = \{p\}$, $p \in \text{Fix}(F)$,
- the **proximal condition** if $P_F(x) \neq \varnothing$, for each $x \in X$,
- the **Chebyshev condition** if $P_F(x)$ is a singleton, for each $x \in X$.

Notice that every self-map satisfies all above conditions.

Remark 2.54. In [10], it gives the concept of the proximal set and the Chebyshev set which is the motivation of the proximal condition and the Chebyshev condition. If a map $F : X \to 2^X$ satisfies the Chebyshev condition, then we identify $P_F(x)$ with its element; that is, we can consider P_F as a self-map on X. Note that every set-valued map with compact values always satisfies the proximal condition.

Lemma 2.55. Let $F : X \to 2^X$ be a map satisfing the proximal condition. Then we obtain the followings :

1. $\mathrm{Fix}(F) = \mathrm{Fix}(P_F)$.
2. P_F satisfies the end point condition.
3. $P_F(x) = F(x) \cap S(x; d(x, F(x)))$, for each $x \in X$.
4. If F has closed values, then $P_F(x)$ is closed and bounded, for each $x \in X$.

Proof. (1) and (2) are obvious. (3) is straightforward from the definition of P_F. Finally, (4) follows directly from (3) and $F(x)$ is closed. □

Definition 2.56 ([2])**.** A map $F : X \to 2^X$ is called $*$-nonexpansive if for all $x, y \in X$ and $u_x \in P_F(x)$, there is $u_y \in P_F(y)$ such that $d(u_x, u_y) \leq d(x, y)$.

Notice that the set-valued map $\hat{f}$ of every nonexpansive self-map is $*$-nonexpansive and nonexpansive.

A nonexpansive map and a $*$-nonexpansive map do not imply each other. The following example shows that a map F is $*$-nonexpansive but F is not nonexpansive.

Example 2.57. Consider the space $X = [-1, 1] \times [0, 1] \subseteq \mathbb{R}^2$ with the maximum norm $\|(x, y)\| = \max\{|x|, |y|\}$. We define a map $F : X \to \mathcal{CB}(X)$ by

$$F(x,y) = \begin{cases} \{(x, |x|)\}, & \text{if } (x,y) \neq \left(\frac{1}{2}, \frac{1}{2}\right), \\ \{(x, |x|), (1, 0)\}, & \text{if } (x,y) = \left(\frac{1}{2}, \frac{1}{2}\right), \end{cases}$$

for each $(x, y) \in X$. The map F is $*$-nonexpansive since $\theta_F(x, y) = (x, |x|)$, for each $(x, y) \in X$; i.e., θ_F is nonexpansive. Thus F is $*$-nonexpanisve.

To show that F is not nonexapansive, we consider points $u = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $v = (0, 0)$. Then $\|u - v\| = \left\|\left(\frac{1}{2}, \frac{1}{2}\right) - (0, 0)\right\| = \frac{1}{2}$ and

$$\begin{aligned} H(F(u), F(v)) &= H\left(F\left(\frac{1}{2}, \frac{1}{2}\right), F(0, 0)\right) \\ &= H\left(\left\{\left(\frac{1}{2}, \frac{1}{2}\right), (1, 0)\right\}, \{(0, 0)\}\right) = 1. \end{aligned}$$

Therefore, F is not nonexpansive.

The following example shows that a map F is nonexpansive but F is not $*$-nonexpansive.

Example 2.58. We define $F : \mathbb{R} \to \mathcal{CB}(\mathbb{R})$ by

$$F(x) = \{-2, 2\}, \text{ for each } x \in \mathbb{R}.$$

Since every constant map is a contraction map, F is a nonexpansive map. Next, we consider two points $x = 1$ and $y = -1$. Also, $\theta_F(x) = \{2\}$ and $\theta_F(y) = \{-2\}$. Thus $|u_x - u_y| = |2 - (-2)| = 4$ and $|x - y| = 2$. Therefore, F is not $*$-nonexpansive.

Lemma 2.59. Let X be a metric space. If $F : X \to 2^X$ is a $*$-nonexpansive map with closed values and satisfies the proximal condition, then the map $P_F : X \to \mathcal{CB}(X)$ is nonexpansive.

Proof. Let $x, y \in X$ and $\varepsilon > 0$. For each $z \in P_F(x)$, there is $u_y \in P_F(y)$ such that

$$z \in B(u_y; d(x, y) + \varepsilon) \subseteq \bigcup_{u \in P_F(y)} B(u; d(x, y) + \varepsilon) = \eta(P_F(y); d(x, y) + \varepsilon).$$

That is, $P_F(x) \subseteq \eta(P_F(y); d(x, y) + \varepsilon)$. By the similar argument,

$$P_F(y) \subseteq \eta(P_F(x); d(x, y) + \varepsilon).$$

Therefore, $H(P_F(x), P_F(y)) \leq d(x, y)$. $\square$

CHAPTER III

VIRTUALLY STABLE SCHEMES

In this chapter, we will present the concept of fixed point iteration schemes, which includes concepts of various iteration processes, and the notion of virtually stable schemes, which generalizes virtually stable self-maps.

3.1 Fixed point iteration schemes

Let X be a Hausdorff space and $\mathcal{F}$ a family of self-maps of X. We denote the **fixed point set of** $\mathcal{F}$ by $\mathrm{Fix}(\mathcal{F}) = \bigcap_{f\in\mathcal{F}} \mathrm{Fix}(f)$. For each sequence $\mathcal{S} = (s_n) \subseteq \mathcal{F}$, we denote the **convergence set of** $\mathcal{S}$ by $\mathrm{C}(\mathcal{S}) = \{x \in X : (s_n(x)) \text{ converges}\}$. Following Proposition 2.7, the sequence $(s_n(x))$ of each point $x \in \mathrm{C}(\mathcal{S})$ converges to a unique point. Consequently, we define a natural map $r : \mathrm{C}(\mathcal{S}) \to X$ by

$$r(x) = \lim_{n\to\infty} s_n(x).$$

Notice that we always have $\mathrm{Fix}(\mathcal{S}) \subseteq r(\mathrm{C}(\mathcal{S}))$ since $s_n(x) = x$, for each $n \in \mathbb{N}$ and $x \in \mathrm{Fix}(\mathcal{S})$. Throughout this work, for each self-maps $f_j, f_{j+1}, \ldots, f_n$, we write $\prod_{i=j}^{n} f_i$ for the composition $f_n \circ f_{n-1} \circ \cdots \circ f_j$.

Proposition 3.1. If $\mathcal{F} = (f_n)$ is a sequence of self-maps of X and $\mathcal{S} = (\prod_{i=1}^{n} f_i)$, then $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(\mathcal{F})$.

Proof. Let $p \in \mathrm{Fix}(\mathcal{F})$ and $n \in \mathbb{N}$. Then $f_i(p) = p$, for each $i \in \mathbb{N}$, and

$$\prod_{i=1}^{n} f_i(p) = \prod_{i=2}^{n} f_i \circ f_1(p) = \prod_{i=2}^{n} f_i(p) = \ldots = p.$$

So, $\text{Fix}(\mathcal{F}) \subseteq \text{Fix}(\mathcal{S})$.

On the other hand, let $p \in \text{Fix}(\mathcal{S})$. Then $p = \prod_{i=1}^{n} f_i(p)$, for each $n \in \mathbb{N}$. Hence, $p = \prod_{i=1}^{1} f_i(p) = f_1(p)$ and $p = f_2 \circ f_1(p) = f_2(p)$. Inductively, $p = f_n(p)$, for each $n \in \mathbb{N}$. Therefore, $\text{Fix}(\mathcal{S}) \subseteq \text{Fix}(\mathcal{F})$. $\square$

Definition 3.2. Let (f_n) be a sequence of self-maps of X, and $\mathcal{S} = (s_n) = (\prod_{i=1}^{n} f_i)$. The sequence $\mathcal{S}$ is said to be a **fixed point iteration scheme**, or simply a **scheme**, on X if $\text{Fix}(\mathcal{S}) = r(\text{C}(\mathcal{S}))$. In addition, $\mathcal{S}$ is said to have a **continuous subsequence** if there is a subsequence (s_{n_k}) of $\mathcal{S}$ consisting of continuous maps.

Notice that if $r(\text{C}(\mathcal{S})) \subseteq \text{Fix}(\mathcal{S})$, then $\mathcal{S}$ is a scheme.

Example 3.3. Let $f : X \to X$ be continuous. If $\lim_{n\to\infty} f^n(x)$ exists, then

$$\lim_{n\to\infty} f^n(x) = f \lim_{n\to\infty} f^{n-1}(x) = f(\lim_{n\to\infty} f^n(x));$$

i.e., $\lim_{n\to\infty} f^n(x) \in \text{Fix}(f)$, for each $x \in \text{C}(\mathcal{S})$. Therefore, the Picard iteration sequence (f^n) is a scheme.

Later, we will present sequences of self-maps motivated by the Mann iteration process and the Ishikawa iteration process (for more detail, see [4]) defined as the following :

Let X be a convex subset of a Banach space and $f : X \to X$.
For sequences $(\alpha_n), (\beta_n)$ in $[0, 1]$, and $x \in X$, let

$$x_1 = (1 - \alpha_1)x + \alpha_1 f((1 - \beta_1)x + \beta_1 f(x)),$$

and for each $n \in \mathbb{N}$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f((1 - \beta_n)x_n + \beta_n f(x_n)).$$

The sequence (x_n) is called the **Ishikawa iteration process**. In the case that $(\beta_n) = (0)$, the sequence (x_n) is called the **Mann iteration process**. Moreover, the sequence (x_n) is the Picard iteration process when $(\alpha_n) = (\beta_n) = (0)$.

Definition 3.4. Let X be a convex subset of a normed space, and (α_n) and (β_n) sequences in $[0,1]$. For each map $f : X \to X$ and $i \in \mathbb{N}$, we define $f_i : X \to X$ by

$$f_i(x) = \left[(1-\alpha_i)I + \alpha_i f((1-\beta_i)I + \beta_i f)\right](x), \tag{3.1}$$

for each $x \in X$, where I is the identity map on X. The sequence $\mathcal{S} = (\prod_{i=1}^{n} f_i)$ is said to be the **Ishikawa iteration sequence** for f associated to sequences (α_n) and (β_n). If $(\beta_n) = (0)$, then $\mathcal{S}$ is said to be the **Mann iteration sequence** for f associated to a sequence (α_n).

Notice that for each Ishikawa iteration sequence $\mathcal{S} = (s_n)$ for f associated to sequences (α_n) and (β_n), we obtain that for each $p \in \mathrm{Fix}(f)$ and $n \in \mathbb{N}$,

$$f_n(p) = (1-\alpha_{n-1})p + \alpha_{n-1} f((1-\beta_{n-1})p + \beta_{n-1} f(p)) = p$$

and

$$s_n(p) = \left(\prod_{i=1}^{n} f_i\right)(p) = \left(\prod_{i=2}^{n} f_i\right) f_1(p) = \left(\prod_{i=2}^{n} f_i\right)(p) = \ldots = p.$$

In particular, $r(p) = \lim_{n\to\infty} s_n(x) = p$, for each $p \in \mathrm{Fix}(f)$; i.e., $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(\mathcal{S})$.

Proposition 3.5. If $\mathcal{S} = (s_n)$ is the Mann iteration sequence for f associated to sequences (α_n), then $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(f)$.

Proof. Following the above argument, it is enough to show that $\mathrm{Fix}(\mathcal{S}) \subseteq \mathrm{Fix}(f)$. Let $p \in \mathrm{Fix}(\mathcal{S})$. WLOG, we assume that k is the first number that $\alpha_k \neq 0$. Then $p = s_k(p) = f_k(p) = (1-\alpha_k)p + \alpha_k f(p)$ and $p = f(p)$; that is, $p \in \mathrm{Fix}(f)$. $\square$

Following Theorem 4.1 and Theorem 4.2 (c) in [4], it implies the following theorem.

Theorem 3.6. Let X be a closed convex subset of a Banach space, $f : X \to X$ continuous, and (α_n) a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\mathcal{S} = (s_n)$ is the Mann iteration sequence for f associated to (α_n). If $x \in \mathrm{C}(\mathcal{S})$, then $r(x) = \lim_{n\to\infty} s_n(x) \in \mathrm{Fix}(f)$.

Corollary 3.7. Let X be a closed convex subset of a Banach space, $f : X \to X$ continuous, and (α_n) a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then the Mann iteration sequence for f associated to a sequence (α_n) is a scheme.

Proof. It follows directly from Theorem 3.6. $\square$

3.2 Virtually stable schemes

In this section, we present the notion of the virtual stability for schemes, which is a generalization of virtually stable self-maps, and now let us recall the notion of virtually stable self-maps [7].

A map $f : X \to X$ is said to be **virtually stable** if for each fixed point $p \in \mathrm{Fix}(f)$ and each neighborhood U of p, there exist a neighborhood V of p and a subsequence (n_k) of (n) such that $f^{n_k}(V) \in U$, for all $k \in \mathbb{N}$. Moreover, a map $f : X \to X$ is said to be **uniformly virtually stable with respect to a subsequence** (n_k) of (n) if for each fixed point $p \in \mathrm{Fix}(f)$ and each neighborhood U of p, there exists a neighborhood V of p such that $f^{n_k}(V) \in U$, for all $k \in \mathbb{N}$.

Definition 3.8. Let (f_n) be a sequence of self-maps of X, and $\mathcal{S} = (\prod_{i=1}^{n} f_i)$ a scheme. The scheme $\mathcal{S}$ is said to be **virtually stable** if for each fixed point $p \in \mathrm{Fix}(\mathcal{S})$ and each neighborhood U of p, there exist a neighborhood V of p and a subsequence (n_k) of (n) such that $\displaystyle\prod_{i=j}^{n_k} f_i(V) \in U$, for all $k \in \mathbb{N}$ and for each $j \leq n_k$. In addition, the scheme $\mathcal{S}$ is also said to be **uniformly virtually stable with respect to the sequence** (n_k) if for each fixed point $p \in \mathrm{Fix}(\mathcal{S})$ and each neighborhood U of p, there exist a neighborhood V of p such that $\prod_{i=j}^{n_k} f_i(V) \in U$, for all $k \in \mathbb{N}$ and for each $j \leq n_k$.

It is easy to see that every uniformly virtually stable scheme is virtually stable. Moreover, if $f : X \to X$ is (uniformly) virtually stable (with respect to (n_k)), then so is the scheme (f^n).

Example 3.9. Let X be a metric space and $f : X \to X$ a quasi-nonexpansive map. It is easy to verify that

$$d(f^n(x), p) \leq \ldots \leq d(f(x), p) \leq d(x, p),$$

for each $x \in X$, $p \in \mathrm{Fix}(f)$ and $n \in \mathbb{N}$. We obtain that for each $\varepsilon > 0$ and $x \in B(p; \varepsilon)$,

$$\left(\prod_{i=j}^{n} f_i\right)(B(p;\varepsilon)) \subseteq f^n(B(p;\varepsilon)) \subseteq B(p;\varepsilon),$$

for each $n \in \mathbb{N}$ and $j \leq n$. Therefore, the Picard iteration scheme (f^n) is uniformly virtually stable with respect to (n).

Lemma 3.10. Let X be a convex subset of a Banach space, and $f : X \to X$ is a quasi-nonexpansive map. If the Ishikawa iteration sequence $\mathcal{S} = (s_n)$ for f associated to (α_n) and (β_n) is a scheme, then $\mathcal{S}$ is uniformly virtually stable with respect to (n). In particular, the Mann iteration sequence $\mathcal{S} = (s_n)$ for f associated to (α_n) is virtually stable.

Proof. Assume that $\mathcal{S}$ is the Ishikawa iteration scheme for f associated to (α_n) and (β_n). Let $x \in X$, $n \in \mathbb{N}$ and $p \in \mathrm{Fix}(f)$. Consider

$$\begin{aligned}
\|f_n(x) - p\| &= \|(1-\alpha_{n-1})x + \alpha_{n-1} f((1-\beta_{n-1})x + \beta_{n-1} f(x)) - p\| \\
&\leq (1-\alpha_{n-1}) \|x - p\| + \alpha_{n-1} \|f((1-\beta_{n-1})x + \beta_{n-1} f(x)) - p\| \\
&\leq (1-\alpha_{n-1}) \|x - p\| + \alpha_{n-1} \|(1-\beta_{n-1})x + \beta_{n-1} f(x) - p\|
\end{aligned}$$

$$\leq (1-\alpha_{n-1})\|x-p\| + \alpha_{n-1}\left((1-\beta_{n-1})\|x-p\| + \beta_{n-1}\|f(x)-p\|\right)$$

$$= \|x-p\|.$$

Consequently, for each $j \leq n$,

$$\left\| \left(\prod_{i=j}^{n} f_i \right)(x) - p \right\| = \left\| f_n \left(\left(\prod_{i=j}^{n-1} f_i \right)(x) \right) - p \right\|$$
$$\leq \left\| \left(\prod_{i=j}^{n-1} f_i \right)(x) - p \right\| \leq \ldots \leq \|x-p\|.$$

This implies that for each $\varepsilon > 0$ and $x \in B(p;\varepsilon)$, we obtain

$$\left(\prod_{i=j}^{n} f_i \right)(B(p;\varepsilon)) \subseteq B(p;\varepsilon),$$

for each $n \in \mathbb{N}$ and $j \leq n$. Therefore, $\mathcal{S}$ is uniformly virtually stable with respect to (n). □

Theorem 3.11. Let X be a regular space, (f_n) a sequence of self-maps of X, and $\mathcal{S} = (s_n) = (\prod_{i=1}^{n} f_i)$ a virtually stable scheme having a continuous subsequence. Then the map $r : \mathrm{C}(\mathcal{S}) \to \mathrm{Fix}(\mathcal{S})$ is continuous, and hence $\mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S})$.

Proof. Let $x \in \mathrm{C}(\mathcal{S})$ and U be a neighborhood of $r(x)$ in $\mathrm{Fix}(\mathcal{S})$. Then $U = U' \cap \mathrm{Fix}(\mathcal{S})$ for some a neighborhood U' of $r(x)$ in X. Since X is regular, there is a neighborhood W of $r(x)$ in X such that $r(x) \in W \cap \mathrm{Fix}(\mathcal{S}) \subseteq \overline{W} \cap \mathrm{Fix}(\mathcal{S}) \subseteq U' \cap \mathrm{Fix}(\mathcal{S}) = U$. From virtual stability of $\mathcal{S}$, there exist a neighborhood V of $r(x)$ in X and a subsequence (n_k) of (n) such that $\prod_{i=j}^{n_k} f_i(V) \subseteq W$, for all $k \in \mathbb{N}$ and $j \leq n_k$. For each $j \in \mathbb{N}$, let $A_j = (s_j)^{-1}(V) \cap \mathrm{C}(\mathcal{S})$. Hence, for each $a \in A_j$,

$$r(a) = \lim_{n\to\infty} s_n(a) = \lim_{k\to\infty} s_{n_k}(a)$$
$$= \lim_{k\to\infty} \prod_{i=1}^{n_k} f_i(a)$$

$$= \lim_{k\to\infty} \prod_{i=j}^{n_k} f_i \left(\prod_{i=1}^{j} f_i(a) \right) \qquad \text{(when } n_k \text{ is more than } j\text{)}$$

$$= \lim_{k\to\infty} \prod_{i=j}^{n_k} f_i \left(s_j(a) \right) \in \overline{W} \subseteq U$$

since $\prod_{i=j}^{n_k} f_i(s_j(a)) \in \prod_{i=j}^{n_k} f_i(V) \subseteq W$, for each $n_k \in \mathbb{N}$. That is, $r(A_j) \subseteq U$. Following the assumption, suppose (s_{n_m}) is the subsequence of (s_n) consisting of continuous maps. Since V is a neighborhood of $r(x) = \lim_{n\to\infty} s_n(x) = \lim_{m\to\infty} s_{n_m}(x)$, there is $M \in \mathbb{N}$ such that $s_{n_m}(x) \in V$, for each $n_m \geq M$. By the continuity of s_{n_m}, the inverse image $(s_{n_m})^{-1}(V)$ is a neighborhood of x in X. Then the set $A_{n_m} = (s_{n_m})^{-1}(V) \cap \mathrm{C}(\mathcal{S})$ is a neighborhood of x in $\mathrm{C}(\mathcal{S})$ with $r(A_{n_m}) \subseteq U$. Therefore, r is continuous and $\mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S})$. $\square$

The above theorem is a generalization of Theorem 2.6 [7], so we have as a corollary below.

Corollary 3.12. Let X be a regular space and $f : X \to X$ a continuous map. If f is virtually stable, then the map $f^\infty : \mathrm{C}(f) \to X$ given by

$$f^\infty(x) = \lim_{n\to\infty} f^n(x)$$

is continuous, and $\mathrm{Fix}(f)$ is a retract of $\mathrm{C}(f)$, where $\mathrm{C}(f) = \{x : (f^n) \text{ converges}\}$.

Proof. Following the definition, $\mathcal{S} = (f^n)$ is a virtually stable scheme. Combining the previous theoem and the fact that each f^n is continuous implies that

$$f^\infty(x) = \lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} \prod_{i=1}^{n} f(x) = r(x),$$

for each $x \in \mathrm{C}(f) = \mathrm{C}(\mathcal{S})$, and hence $\mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S})$. Finally, $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(\{f^n : n \in \mathbb{N}\}) = \mathrm{Fix}(f)$ by Proposition 3.1. $\square$

CHAPTER IV

APPLICATIONS OF VIRTUALLY STABLE SCHEMES

In this chapter, we combine the retraction result of virtually stable schemes with convergence results of fixed point iterations to obtain contractibility criterion for fixed point sets of some self-maps and set-valued maps.

4.1 Suzuki generalized nonexpansive self-maps

In this section, we study the Suzuki generalized nonexpansive self-map in [16] which shows the convergence of the Mann iteration process in an appropriate situation.

Definition 4.1. Let (X, d) be a metric space. A map $f : X \to X$ is said to be a **Suzuki generalized nonexpansive map** if for every $x, y \in X$,

$$\frac{1}{2}d(x, f(x)) \leq d(x, y) \Rightarrow d(f(x), f(y)) \leq d(x, y).$$

It is easy to see that every nonexpansive self-map is Suzuki generalized nonexpansive. In addition, every Suzuki generalized nonexpansive self-map is quasi-nonexpansive since $\frac{1}{2}d(p, f(p)) = \frac{1}{2}d(p, p) = 0 \leq d(p, y)$, for each $p \in \mathrm{Fix}(f)$ and $y \in X$. A Suzuki generalized nonexpansive map need not be continuous as shown in the next example.

Example 4.2. Consider the set $X = [-1, 1] \times [0, 1]$ in $\mathbb{R}^2$ with the maximum norm $\|(x,y)\| = \max\{|x|,|y|\}$. Define $f : X \to X$ by

$$f(x,y) = \begin{cases} (x,|x|), & (x,y) \neq (0,1), \\ \left(0,\frac{1}{3}\right), & (x,y) = (0,1). \end{cases}$$

This map is not continuous at the point $(0,1)$ and hence, not nonexpansive. To show that f is Suzuki generalized nonexpansive, it is enough to consider the point $(0,1)$ and an arbitrary point $(x,y) \in X - \{(0,1)\}$ since the restriction $f|_{X-\{(0,1)\}}$ is nonexpansive. Then

$$\begin{aligned}
\|f(0,1) - f(x,y)\| &= \left\|\left(0,\frac{1}{3}\right) - (x,|x|)\right\| \\
&= \max\left\{|x|, \left|\frac{1}{3} - |x|\right|\right\} \\
&= \begin{cases} \max\left\{|x|, |x| - \frac{1}{3}\right\} & \text{if } \frac{1}{3} \geq |x|, \\ \max\left\{|x|, \frac{1}{3} - |x|\right\} & \text{if } \frac{1}{6} \leq |x| < \frac{1}{3}, \\ \max\left\{|x|, \frac{1}{3} - |x|\right\} & \text{if } |x| < \frac{1}{6} \end{cases} \\
&= \begin{cases} |x| & \text{if } \frac{1}{3} \geq |x|, \\ |x| & \text{if } \frac{1}{6} \leq |x| < \frac{1}{3}, \\ \frac{1}{3} - |x| & \text{if } |x| < \frac{1}{6} \end{cases} \\
&\leq \begin{cases} \max\{|x|, |1-y|\} & \text{if } \frac{1}{6} \geq |x|, \\ \frac{1}{3} & \text{if } |x| < \frac{1}{6}. \end{cases} \\
&= \begin{cases} \|(0,1) - (x,y)\| & \text{if } \frac{1}{6} \geq |x|, \\ \frac{1}{3} & \text{if } |x| < \frac{1}{6}. \end{cases}
\end{aligned}$$

Next, we consider $(x,y) \in (-\frac{1}{6}, \frac{1}{6}) \times [0,1]$.

Case 1. $\frac{1}{2}\|(0,1) - f(0,1)\| \leq \|(0,1) - (x,y)\|$. So,

$$\|f(0,1) - f(x,y)\| \leq \frac{1}{3} = \frac{1}{2}\left\|(0,1) - \left(0,\frac{1}{3}\right)\right\|$$

$$= \frac{1}{2} \|(0,1) - f(0,1)\| \leq \|(0,1) - (x,y)\|.$$

Case 2. $\frac{1}{2}\|(x,y) - f(x,y)\| \leq \|(x,y) - (0,1)\|$. Suppose for a contradiction that $\|f(x,y) - f(0,1)\| > \|(x,y) - (0,1)\|$. Then

$$\max\{|x|, 1-y\} = \max\{|x|, |1-y|\} = \|(x,y) - (0,1)\| < \|f(x,y) - f(0,1)\| = \frac{1}{3} - |x|.$$

This implies $1 - y \leq \frac{1}{3} - |x|$ and hence

$$\frac{1}{3} \leq \frac{y - |x|}{2} = \left|\frac{y - |x|}{2}\right| = \frac{1}{2}\|(x,y) - f(x,y)\| \leq \|(x,y) - (0,1)\| < \frac{1}{3} - |x| \leq \frac{1}{3}.$$

Therefore, f is Suzuki generalized nonexpansive.

Next, we recall the convergence result for Suzuki generalized nonexpansive self-maps in [16] under the notion of schemes as the following.

Theorem 4.3 ([16], Theorem 2)**.** Let X be a compact convex subset of a Banach space, $f : X \to X$ Suzuki generalized nonexpansive, and $\alpha \in [1/2, 1)$. If $\mathcal{S} = (s_n)$ is the Mann iteration sequence for f associated to the constant sequence (α), then for each $x \in X$, $(s_n(x))$ converges to a fixed point $p \in \mathrm{Fix}(f)$.

Theorem 4.4. Let X be a compact convex subset of a Banach space, $f : X \to X$ Suzuki generalized nonexpansive and $\alpha \in [1/2, 1)$. If f is continuous, then $\mathrm{Fix}(f)$ is a retract of X and hence $\mathrm{Fix}(f)$ is contractible.

Proof. Let $\mathcal{S}$ be the Mann iteration sequence for f associated to (α). Note that $\mathcal{S} = ([(1-\alpha)I + \alpha f]^n)$. Since f is continuous, we have that $[(1-\alpha)I + \alpha f]$ is continuous, and so is each $[(1-\alpha)I + \alpha f]^n$. Following Theorem 4.3, $\mathrm{C}(\mathcal{S}) = X$ and $r(\mathrm{C}(\mathcal{S})) = \mathrm{Fix}(\mathcal{S})$; i.e., $\mathcal{S}$ is a scheme. By Proposition 3.5, $\mathrm{Fix}(f) = \mathrm{Fix}(\mathcal{S})$. Since every Suzuki generalized nonexpansive map is quasi-nonexpansive, by Lemma 3.10, $\mathcal{S}$ is virtually stable. Then $\mathrm{Fix}(f) = \mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S}) = X$ by Theorem 3.11. Therefore, $\mathrm{Fix}(f)$ is contractible by the convexity of X. $\square$

In the previous theorem, the compactness of the space X is necessary as shown in the following example.

Example 4.5. Consider the Banach space $c_0 = \{(x_1, x_2, \ldots) \subseteq \mathbb{R} :$ there exists $m \in \mathbb{N}$ such that $x_n = 0$, for each $n \geq m\}$ with the supremum norm $\|(x_1, x_2, \ldots)\| = \sup\{x_n : n \in \mathbb{N}\}$. Define $f : c_0 \to c_0$ by

$$f(x_1, x_2, x_3 \ldots) = (x_1, 1 - |x_1|, x_2, \ldots),$$

for each $(x_1, x_2, \ldots) \in c_0$. Following the inequality

$$\begin{aligned}\|f(x_1, x_2, x_3 \ldots) - f(y_1, y_2, y_3 \ldots)\| &= \|(x_1 - y_1, (1 - |x_1|) - (1 - |y_1|), x_2 - y_2, \ldots)\| \\ &= \|(x_1 - y_1, -|x_1| + |y_1|, x_2 - y_2, \ldots)\| \\ &= \max\{\|(x_1, x_2, \ldots) - (y_1, y_2, \ldots)\|, ||\, x_1 \mid - \mid y_1 \,||\} \\ &\leq \|(x_1, x_2, \ldots) - (y_1, y_2, \ldots)\|,\end{aligned}$$

f is nonexpansive and hence Suzuki generalized nonexpansive. Therefore, f is a Suzuki generalized nonexpansive map, and $\mathrm{Fix}(f) = \{(1, 0, 0, \ldots), (-1, 0, 0, \ldots)\}$ is not contractible.

4.2 α-contractive set-valued maps

In this section, we present an α-contractive scheme which is a virtually stable scheme. Then, we also obtain a contractibility criterion for the fixed point set of a certain α-contraction set-valued map.

Definition 4.6. For each $\alpha \in [0, 1)$, a scheme $\mathcal{S} = (\prod_{i=1}^{n} f_i)$ on X is said to be **α-contractive** if it satisfies the following conditions :

1. For each sequence $\mathcal{T} = (t_n) \in \{(\prod_{i=0}^{n} f_{k+i}) : k \in \mathbb{N}\}$,

$$d(t_{n+1}(x), t_n(x)) \leq \alpha d(t_n(x), t_{n-1}(x)),$$

for each $n \in \mathbb{N}$, where $t_0(x) = x$ for each $x \in X$.

2. The set $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is equicontinuous on $\mathrm{Fix}(\mathcal{F})$.

Example 4.7. If $f : X \to X$ an α-contraction, then the scheme (f^n) is α-contractive.

Theorem 4.8. Every α-contractive scheme $\mathcal{S}$ on a metric space X is virtually stable. Moreover, if X is complete, then $\mathrm{C}(\mathcal{S}) = X$.

Proof. Suppose that $\mathcal{S} = (\prod_{i=1}^{n} f_i)$ is an α-contractive scheme on X. Note that $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(\mathcal{F})$ by Proposition 3.1. To show that $\mathcal{S}$ is virtually stable, let $p \in \mathrm{Fix}(\mathcal{S})$, $\varepsilon > 0$, and $\varepsilon_0 = \min\left\{\dfrac{\varepsilon(1-\alpha)}{4}, \dfrac{\varepsilon}{4}\right\}$. Since $\mathcal{F}$ is equicontinuous at p, there exists $\delta > 0$ such that $f_i(B(p;\delta)) \subseteq B(f_i(p);\varepsilon_0)$, for each $f_i \in \mathcal{F}$. WLOG., we assume that $\delta \leq \varepsilon_0$.

Let $n \in \mathbb{N}$, $j \leq n$, $\mathcal{T} = (t_n) = (\prod_{i=0}^{n} f_{j+i})$, and $y \in B(p;\delta)$. Since $p \in \mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(\mathcal{F}) \subseteq \mathrm{Fix}(\mathcal{T})$,

$$t_1(y) = f_j(y) \in B(f_j(p);\varepsilon_0) = B(p;\varepsilon_0),$$

and

$$d(y, t_1(y)) \leq d(y,p) + d(p, t_1(y)) < \delta + \varepsilon_0 \leq 2\varepsilon_0 \leq \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned}
d(t_1(y), t_{n-j+1}(y)) &\leq \sum_{i=1}^{n-j} d(t_i(y), t_{i-1}(y)) \\
&\leq \sum_{i=n}^{n-j} \alpha^{i-1} d(t_1(y), y) \\
&\leq d(t_1(y), y) \sum_{i=1}^{\infty} \alpha^{i-1} \\
&\leq 2\varepsilon_0 \left(\frac{1}{1-\alpha}\right) \\
&\leq 2\left(\frac{\varepsilon(1-\alpha)}{4}\right)\left(\frac{1}{1-\alpha}\right) \leq \frac{\varepsilon}{2}.
\end{aligned}$$

Combining the both inequalities gives that

$$d\left(p,\ \prod_{i=j}^{n} f_i(y)\right) \le d\left(p, t_1(y)\right) + d\left(t_1(y),\ \prod_{i=j}^{n} f_i(y)\right)$$

$$= d\left(p, t_1(y)\right) + d\left(t_1(y), t_{n-j+1}(y)\right) < \varepsilon.$$

Hence, for each $p \in \mathrm{Fix}(\mathcal{S})$ and $\varepsilon > 0$, there is $\delta > 0$ such that

$$\prod_{i=j}^{n} f_i(B(p;\delta)) \subseteq B(p;\varepsilon),$$

for all $n \in \mathbb{N}$ and $j \le n$. Therefore, $\mathcal{S}$ is virtually stable.

Finally, recall the fact that every sequence (x_n) in a metric space satisfying $d(x_{n+1}, x_n) \le \alpha^n$, for each $n \in \mathbb{N}$, is a Cauchy sequence. This implies the convergence of the sequence $(s_n(x))$ when X is complete. □

Corollary 4.9. Let X be a metric space and $f : X \to X$ a map. If $\mathcal{S} = (f^n)$ is an α-contractive scheme, then $\mathcal{S}$ is virtually stable.

Theorem 4.10. Let X be a compact metric space, and $F : X \to \mathcal{CB}(X)$ be an α-contraction set-valued map satisfying the Chebyshev condition. Then the sequence $\mathcal{S} = (s_n)$ of self-maps of X defined by

$$s_0(x) = x \text{ and } s_n(x) = P_F(s_{n-1}(x)) = (P_F)^n(x),$$

for each $n = 1, 2, \ldots,$ and $x \in X$, is an α-contractive scheme with $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(F)$, and hence $\mathrm{Fix}(F)$ is a retract of X.

Proof. Let $x \in X$ and $n \ge 1$. By Proposition 2.31,

$$d(s_{n-1}(x), s_n(x)) = d(s_{n-1}(x), F \circ s_{n-1}(x))$$

$$\le \sup_{a \in F \circ s_{n-2}(x)} d(a, F \circ s_{n-1}(x))$$

$$= h(F \circ s_{n-2}(x), F \circ s_{n-1}(x))$$

$$\leq H(F \circ s_{n-2}(x), F \circ s_{n-1}(x))$$
$$\leq \alpha d(s_{n-2}(x), s_{n-1}(x)).$$

It is easy to see that $(s_n(x))$ is a Cauchy sequence, and hence $(s_n(x))$ converges to a point $p \in X$. Consider the following inequality

$$\begin{aligned} d(p, F(p)) &\leq d(p, F \circ s_n(x)) + H(F \circ s_n(x), F(p)) \\ &\leq d(p, s_{n+1}(x)) + H(F \circ s_n(x), F(p)). \end{aligned}$$

Since $(s_n(x))$ converges to p and F is H-continuous, $(F \circ s_n(x))$ converges to $F(p)$. Consequently, $d(p, F(p)) = 0$, and since $F(p)$ is closed, $p \in F(p)$. This implies $r(\mathrm{C}(\mathcal{S})) \subseteq \mathrm{Fix}(F)$. By Lemma 2.55 (1),

$$\mathrm{Fix}(F) = \mathrm{Fix}(P_F) = \mathrm{Fix}(\{(P_F)^n : n \in \mathbb{N}\}) = \mathrm{Fix}(\mathcal{S}) \subseteq r(\mathrm{C}(\mathcal{S})).$$

It follows that the sequence $\mathcal{S}$ is a scheme satisfying the condition in Corollary 4.9 with $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(F)$. Therefore, the scheme $\mathcal{S}$ is α-contractive with $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(F)$, and hence $\mathrm{Fix}(F) = \mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S}) = X$ by Theorem 4.8 and Theorem 3.11. $\square$

Remark 4.11. The sequence $((P_F)^n)$ in the proof of Theorem 4.10 is motivated by the iteration process defined in Theorem 5 [13]. In addition, Neammanee and Kaewkhao [14] studied the convergence result of an iteration sequence for a certain (a, L)-weak contraction set-valued map F where $a < 1$ and $L \in \mathbb{R}$. Following that, we know that the sequence $\mathcal{S} = ((P_F)^n)$ is an α-contractive scheme for some $\alpha \in [0, 1)$ if F satisfies the Chebyshev condition. Similar to the argument of Theorem 4.10, the fixed point set $\mathrm{Fix}(F)$ is a retract of X when X is compact.

Corollary 4.12. Let X be a compact and contractible subset of a Banach space. If $F : X \to \mathcal{CB}(X)$ is an α-contraction set-valued map satisfying the Chebyshev condition, then the fixed point set of F is contractible.

Proof. It follows directly from Theorem 4.10. □

The Chebyshev condition in Corollary 4.12 is necessary as shown in the following example.

Example 4.13 ([13], Example 1.)**.** Let $X = [0,1]$ with the usual metric and $F : X \to \mathcal{CB}(X)$ be given by

$$F(x) = \begin{cases} \left\{\frac{1}{2}x + \frac{1}{2}\right\} \cup \{0\}, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \left\{-\frac{1}{2}x + 1\right\} \cup \{0\}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then F is a $\frac{1}{2}$-contraction set-valued map, and does not satisfy the Chebyshev condition at $x = \frac{1}{3}$ since $P_F\left(\frac{1}{3}\right) = \left\{0, \frac{2}{3}\right\}$. Moreover, $\mathrm{Fix}(F) = \left\{0, \frac{2}{3}\right\}$ is not contractible.

We will show that the fixed point set of an α-contraction set-valued map may not be convex but contractible, in the example below.

Example 4.14. Consider the subset $X = [-1,1] \times [0,1]$ of the Euclidean space $\mathbb{R}^2$. Define $F : X \to \mathcal{CB}(X)$ by

$$F(x,y) = [-1,1] \times \left\{\left|\frac{x}{2}\right|\right\}, \text{ for each } (x,y) \in X.$$

We obtain

$$\begin{aligned} H(F(x_1,y_1), F(x_2,y_2)) &= H\left([-1,1] \times \left\{\left|\frac{x_1}{2}\right|\right\}, [-1,1] \times \left\{\left|\frac{x_2}{2}\right|\right\}\right) \\ &= \left|\left|\frac{x_1}{2}\right| - \left|\frac{x_2}{2}\right|\right| \\ &\leq \frac{1}{2}|x_1 - x_2| \leq \frac{1}{2}\|(x_1,y_1) - (x_2,y_2)\|, \end{aligned}$$

and $P_F(x,y) = (x, \left|\frac{x}{2}\right|)$, for each (x,y), (x_1,y_1), and $(x_2,y_2) \in X$. Therefore, the map F is a $\frac{1}{2}$-contraction satisfying the Chebyshev condition with $\mathrm{Fix}(F) = \left\{(x,y) : y = \left|\frac{x}{2}\right|\right\}$.

4.3 Families of set-valued maps

In this section, we will present contractibility criterions for fixed point sets of families of set-valued maps. In [8], Dimri, Singh and Bhatt present two set-valued maps F_1 and F_2 on a metric space (X, d) satisfying the relation,

$$H(F_1(x), F_2(y)) \leq ad(x, F_1(x)) + bd(y, F_2(y)) + cd(x, y), \tag{4.1}$$

for all $x, y \in X$ and $a, b, c \geq 0$, where $a+b+c < 1$. Note that every α-contraction set-valued map F on a metric space satisfies (4.1) by setting $F = F_1 = F_2$, $a = 0 = b$, and $c = \alpha$. We obtain that the both maps F_1 and F_2 can induce the similar retraction result as the α-contraction set-valued maps.

Theorem 4.15. Let X be a compact subset of a metric space. If two H-continuous maps $F_1, F_2 : X \to \mathcal{CB}(X)$ satisfy the relation (4.1), then the sequence $\mathcal{S} = (s_n)$ of self-maps defined by $s_0(x) = x$,

$$s_{2n-1}(x) = P_{F_1} \circ s_{2n-2}(x) \text{ and } s_{2n}(x) = P_{F_2} \circ s_{2n-1}(x),$$

for each $n \in \mathbb{N}$, and $x \in X$, is an α-contractive scheme. Moreover, $\mathrm{Fix}(\{F_1, F_2\})$ is a retract of X.

Proof. Let $x \in X$ and $n \in \mathbb{N}$. Since $a, b, c \geq 0$ and $a + b + c < 1$, $1 - a > 0$ and $1 - b > 0$. Let $\alpha = \max\left\{\frac{b+c}{1-a}, \frac{a+c}{1-b}\right\}$. It is not difficult to see that $0 \leq \alpha < 1$. Following [8], we have

$$\begin{aligned}
d(s_{2n}(x), s_{2n+1}(x)) =& d(s_{2n}(x), P_{F_1} \circ s_{2n}(x)) \\
=& d(s_{2n}(x), F_1 \circ s_{2n}(x)) \\
\leq& h(F_2 \circ s_{2n-1}(x), F_1 \circ s_{2n}(x)) \\
\leq& H(F_2 \circ s_{2n-1}(x), F_1 \circ s_{2n}(x)) \\
\leq& ad(s_{2n}(x), F_1 \circ s_{2n}(x)) + bd(s_{2n-1}(x), F_2 \circ s_{2n-1}(x))
\end{aligned}$$

$$
\begin{aligned}
&+ cd(s_{2n}(x), s_{2n-1}(x)) \\
=&ad(s_{2n}(x), s_{2n+1}(x)) + bd(s_{2n-1}(x), s_{2n}(x)) \\
&+ cd(s_{2n}(x), s_{2n-1}(x)).
\end{aligned}
$$

Consequently,

$$(1-a)d(s_{2n}(x), s_{2n+1}(x)) \leq (b+c)d(s_{2n}(x), s_{2n-1}(x))$$

and

$$d(s_{2n}(x), s_{2n+1}(x)) \leq \frac{b+c}{1-a} d(s_{2n-1}(x), s_{2n}(x)).$$

Similar to the previous method, we have

$$d(s_{2n-1}(x), s_{2n}(x)) \leq \frac{a+c}{1-b} d(s_{2n-2}(x), s_{2n-1}(x)).$$

Therefore,

$$d(s_n(x), s_{n-1}(x)) \leq \alpha d(s_{n-1}(x), s_{n-2}(x)).$$

Similar to Theorem 4.10, the sequence $(s_n(x))$ converges to a point $p \in X$. By the H-continuity of F_1, $(F_1 \circ s_{2n-2}(x))$ converges to $F_1(p)$. Then

$$
\begin{aligned}
d(p, F_1(p)) &\leq d(p, F_1 \circ s_{2n-2}(x)) + H(F_1 \circ s_{2n-2}(x), F_1(p)) \\
&\leq d(p, P_{F_1} \circ s_{2n-2}(x)) + H(F_1 \circ s_{2n-2}(x), F_1(p)) \\
&= d(p, s_{2n-1}(x)) + H(F_1 \circ s_{2n-2}(x), F_1(p)).
\end{aligned}
$$

Since $(s_{2n-1}(x))$ and $(F_1 \circ s_{2n-2}(x))$ converge to p and $F_1(p)$, respectively, $d(p, F(p)) = 0$. Thus $p \in F_1(p)$ since $F_1(p)$ is closed. Similarly, we have $p \in F_2(p)$. This implies that $\mathrm{C}(\mathcal{S}) = X$ and $r(\mathrm{C}(\mathcal{S})) \subseteq \mathrm{Fix}(F)$.

By Lemma 2.55, $\mathrm{Fix}(P_{F_i}) = \mathrm{Fix}(F_i)$, for $i = 1, 2$. Consequently,

$$\mathrm{Fix}(\{F_1, F_2\}) = \mathrm{Fix}(\{P_{F_1}, P_{F_2}\}) = \mathrm{Fix}(\mathcal{S}) \subseteq r(\mathrm{C}(\mathcal{S})),$$

by Lemma 3.1. Therefore, $\mathcal{S}$ is a scheme and $\mathrm{Fix}(\{F_1, F_2\}) = \mathrm{Fix}(\mathcal{S})$.

Now we consider sequences (f_n) and (g_n) of maps on X such that

$$f_n = \begin{cases} P_{F_1}, & \text{if } n \text{ is odd,} \\ P_{F_2}, & \text{if } n \text{ is even,} \end{cases}$$

and

$$g_n = \begin{cases} P_{F_2}, & \text{if } n \text{ is odd,} \\ P_{F_1}, & \text{if } n \text{ is even,} \end{cases}$$

respectively. Since P_{F_1} and P_{F_2} are continuous, (f_n) is equicontinuous on X.

Let $\mathcal{T} = (t_n) = (\prod_{i=1}^{n} g_n)$. Similar to the sequence $\mathcal{S}$, the sequence $\mathcal{T}$ satisfies

$$d(t_n(x), t_{n-1}(x)) \leq \alpha d(t_{n-1}(x), t_{n-2}(x)),$$

for each $x \in X$. Moreover, the set $\{(\prod_{i=0}^{n} f_{k+i}) : k \in \mathbb{N}\} = \{\mathcal{S}, \mathcal{T}\}$. Therefore, $\mathcal{S}$ is an α-contractive scheme with $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(\{F_1, F_2\})$, and hence $\mathrm{Fix}(\{F_1, F_2\}) = \mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S}) = X$ by Theorem 4.8 and Theorem 3.11.

□

We end this chapter with a retraction result for a sequence (F_n) of set-valued maps on a metric space satisfying the relation [17],

$$\begin{aligned} H(F_i(x), F_j(y)) \leq\ & ad(x,y) + b[d(x, F_i(x)) + d(y, F_j(y))] \\ & + c[d(x, F_j(y)) + d(y, F_i(x))], \end{aligned} \tag{4.2}$$

for all $x, y \in X$ and $i, j \in \mathbb{N}$, where $a, b, c \geq 0$ such that $a + (a+3)(b+c) < 1$. Note that every α-contraction multi-valued map F satisfies (4.2) by setting $a = \alpha$, $0 = b = c$, and $F_i = F$, for each $i \in \mathbb{N}$.

Theorem 4.16. Let X be a compact subset of a metric space and (F_n) a sequence of set-valued maps from X into $\mathcal{CB}(X)$ satisfying the Chebyshev condition. Assume that the sequence (F_n) satisfies the relation (4.2). Then

1. $\mathcal{F} = (P_{F_n})$ is equicontinuous on $\mathrm{Fix}(\mathcal{F})$.

2. A sequence $\mathcal{S} = (s_n)$ of self-maps defined by $s_0(x) = x$,

$$s_n(x) = P_{F_n} \circ s_{n-1}(x) = \prod_{i=1}^{n} P_{F_i}(x),$$

   for each $n \in \mathbb{N}$ and $x \in X$, is an α-contractive scheme.

3. $\mathrm{Fix}(\{F_n : n \in \mathbb{N}\}) = \mathrm{Fix}(\mathcal{F})$ is a retract of X.

Proof. Since $a, b, c \geq 0$ such that $a + (a+3)(b+c) < 1$, we obtain $b + c < 1$.

To show **(1)**, let $p \in \mathrm{Fix}(\mathcal{F})$, $y \in X$ and $i, j \in \mathbb{N}$. Then

$$\begin{aligned}
H(F_i(p), F_j(y)) \leq & ad(p,y) + b[d(p, F_i(p)) + d(y, F_j(y))] \\
& + c[d(p, F_j(y)) + d(y, F_i(p))] \\
\leq & ad(p,y) + bd(y, F_j(y)) \\
& + c[d(p, F_j(y)) + d(y, F_i(p))] \\
\leq & ad(p,y) + b[d(y, F_i(p)) + H(F_i(p), F_j(y))] \\
& + c[d(p, F_i(p)) + H(F_i(p), F_j(y)) + d(y, F_i(p))] \\
= & ad(p,y) + (b+c)d(y, F_i(p)) + (b+c)H(F_i(p), F_j(y)).
\end{aligned}$$

Hence,

$$(1 - (b+c))H(F_i(p), F_j(y)) \leq ad(p,y) + (b+c)d(y, F_i(p))$$

and

$$H(F_i(p), F_j(y)) \leq \frac{a}{1-(b+c)} d(p,y) + \frac{b+c}{1-(b+c)} d(y, F_i(p)).$$

We have that $d(y, F_i(p)) \leq d(y,p)$ since $p \in F_i(p)$. Consequently,

$$H(F_i(p), F_j(y)) \leq \frac{a}{1-(b+c)} d(p,y) + \frac{b+c}{1-(b+c)} d(p,y) = kd(p,y),$$

where $k = \left[\dfrac{a}{1-(b+c)} + \dfrac{b+c}{1-(b+c)}\right]$.

Now we will show $d(y, P_{F_j}(y)) \leq (1+k)d(p,y)$. Let $\gamma > 0$. By Proposition 2.33 (4) and $p \in F_i(p)$, there is a point $u \in F_j(y)$ such that $d(p,u) < H(F_i(p), F_j(y)) + \gamma \leq kd(p,y) + \gamma$. Consequently,

$$\begin{aligned}
d(y, P_{F_j}(y)) &= d(y, F_j(y)) \\
&= \inf_{a \in F_j(y)} d(y,a) \\
&\leq d(y,u) && \text{(since } u \in F_j(y)\text{)} \\
&\leq d(y,p) + d(p,u) \\
&< d(y,p) + kd(p,y) + \gamma \\
&= (1+k)d(p,y) + \gamma.
\end{aligned}$$

Since γ is arbitrary, $d(y, P_{F_j}(y)) \leq (1+k)d(p,y)$.

Let $\varepsilon > 0$ and $\delta = \dfrac{\varepsilon}{2+k}$. For each $y \in B(p;\delta)$,

$$d(p, P_{F_j}(y)) \leq d(p,y) + d(y, P_{F_j}(y)) < d(p,y) + (1+k)d(p,y) = (2+k)d(p,y) < \varepsilon.$$

Therefore, $\mathcal{F}$ is equicontinuous on $\mathrm{Fix}(\mathcal{F})$.

To show **(2)**, let $x \in X$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
d(s_{n-1}(x), s_n(x)) &= d(s_{n-1}(x), P_{F_n} \circ s_{n-1}(x)) \\
&= d(s_{n-1}(x), F_n \circ s_{n-1}(x)) \\
&\leq \sup_{a \in F_{n-1} \circ s_{n-2}(x)} d(a, F_n \circ s_{n-1}(x)) \\
&= h(F_{n-1} \circ s_{n-2}(x), F_n \circ s_{n-1}(x)) \\
&\leq H(F_{n-1} \circ s_{n-2}(x), F_n \circ s_{n-1}(x)) \\
&\leq ad(s_{n-2}(x), s_{n-1}(x)) \\
&\quad + b[d(s_{n-2}(x), F_{n-1} \circ s_{n-2}(x)) + d(s_{n-1}(x), F_n \circ s_{n-1}(x))]
\end{aligned}$$

$$
\begin{aligned}
&+ c[d(s_{n-2}(x), F_n \circ s_{n-1}(x)) + d(s_{n-1}(x), F_{n-1} \circ s_{n-2}(x))] \\
\leq\ & ad(s_{n-2}(x), s_{n-1}(x)) \\
&+ b[d(s_{n-2}(x), s_{n-1}(x)) + d(s_{n-1}(x), F_{n-1} \circ s_{n-2}(x))] \\
&+ b[d(s_{n-1}(x), s_n(x)) + d(s_n(x), F_n \circ s_{n-1}(x))] \\
&+ c[d(s_{n-2}(x), s_n(x)) + d(s_n(x), F_n \circ s_{n-1}(x))] \\
=\ & ad(s_{n-2}(x), s_{n-1}(x)) \\
&+ b[d(s_{n-2}(x), s_{n-1}(x)) + d(s_{n-1}(x), s_n(x))] \\
&+ c[d(s_{n-2}(x), s_n(x))] \\
\leq\ & ad(s_{n-2}(x), s_{n-1}(x)) \\
&+ b[d(s_{n-2}(x), s_{n-1}(x)) + d(s_{n-1}(x), s_n(x))] \\
&+ c[d(s_{n-2}(x), s_{n-1}(x)) + d(s_{n-1}(x), s_n(x))].
\end{aligned}
$$

Consequently,

$$(1-(b+c))d(s_{n-1}(x), s_n(x)) \leq (a+b+c)d(s_{n-2}(x), s_{n-1}(x))$$

and

$$d(s_{n-1}(x), s_n(x)) \leq \alpha d(s_{n-2}(x), s_{n-1}(x)),$$

where $\alpha = \dfrac{a+b+c}{1-(b+c)}$.

Next, we consider $a + (a+3)(b+c) < 1$. Then $a + ab + ac + 3b + 3c < 1$ and

$$a + b + c \leq a + ab + ac + 2b + 2c < 1 - (b+c).$$

Therefore, $\alpha = \dfrac{a+b+c}{1-(b+c)} \in [0, 1)$.

Following the argument in Theorem 4.10, the sequence $(s_n(x))$ converges to a point $p \in X$. We have that

$$d(s_n(x), F_1(p)) \leq H(F_n \circ s_{n-1}(x), F_1(p))$$

$$
\begin{aligned}
&\leq ad(s_{n-1}(x), p) + b\,[d(s_{n-1}(x), F_n \circ s_{n-1}(x)) + d(p, F_1(p))] \\
&\quad + c[d(s_{n-1}(x), F_1(p)) + d(p, F_n \circ s_{n-1}(x))] \\
&\leq a[d(s_{n-1}(x), s_n(x)) + d(s_n(x), p)] \\
&\quad + b\,[d(s_{n-1}(x), s_n(x)) + d(p, s_n(x)) + d(s_n(x), F_1(p))] \\
&\quad + c[d(s_{n-1}(x), s_n(x)) + d(s_n(x), F_1(p)) + d(p, s_n(x))]
\end{aligned}
$$

Hence,

$$
d(s_n(x), F_1(p)) \leq \frac{a+b+c}{1-b-c}[d(s_{n-1}(x), s_n(x)) + d(s_n(x), p)]
$$

and

$$
\begin{aligned}
d(p, F_1(p)) &\leq d(p, s_n(x)) + d(s_n(x), F_1(p)) \\
&= d(p, s_n(x)) + \frac{a+b+c}{1-b-c}[d(s_{n-1}(x), s_n(x)) + d(s_n(x), p)].
\end{aligned}
$$

The previous argument is directly motivated by K. Yanagi' work. Since the sequence $(s_n(x))$ converges to p, $d(p, F_1(p)) = 0$. Therefore, $p \in \text{Fix}(F_1(p))$ since $F(p)$ is closed. We apply the argument of F_1 to all map F_i and obtain that $p \in \text{Fix}(F_i(p))$, for each $i \in \mathbb{N}$. That is, $C(\mathcal{S}) = X$ and $r(C(\mathcal{S})) \subseteq \text{Fix}(\{F_i : i \in \mathbb{N}\})$.

By Lemma 2.55, $\text{Fix}(P_{F_i}) = \text{Fix}(F_i)$, for $i \in \mathbb{N}$. Consequently,

$$
\text{Fix}(\{F_i : i \in \mathbb{N}\}) = \text{Fix}(\{P_{F_i} : i \in \mathbb{N}\}) = \text{Fix}(\mathcal{S}) \subseteq r(C(\mathcal{S}))
$$

by Lemma 3.1. Therefore, $\mathcal{S}$ is a scheme and $\text{Fix}(\{F_i : i \in \mathbb{N}\}) = \text{Fix}(\mathcal{S})$.

Let $\mathcal{T} = (t_n) = \left(\prod_{i=0}^{n} P_{F_{k+i}}\right)$, for some $k \in \mathbb{N}$. Consider $\mathcal{H} = (F_{k+n})$. Since $\mathcal{H} \subseteq (F_n)$, $\mathcal{H}$ satisfies the relation (4.2). By the same process of $\mathcal{S}$, the sequence $\mathcal{T}$ satisfies the following,

$$
d(t_n(x), t_{n-1}(x)) \leq \alpha d(t_{n-1}(x), t_{n-2}(x)),
$$

for each $x \in X$. From (1), $\mathcal{F}$ is equicontinuous on $\text{Fix}(\mathcal{F})$, where $\mathcal{F} = (P_{F_n})$. Therefore, $\mathcal{S}$ is an α-contraction scheme. This implies **(3)**; i.e., $\text{Fix}(\mathcal{F}) = \text{Fix}(\mathcal{S})$ is a retract $\text{C}(\mathcal{S}) = X$. $\square$

CHAPTER V

FIXED POINT RESOLUTIONS

In [15], it has an interesting iteration sequence, called the multivalued version of the modified Mann iteration process, which is not a scheme but induces a similar retraction as the virtually stable scheme. This leads us to define the notion of fixed point resolutions which also generalizes the notion of virtually stable schemes as follows :

Definition 5.1. For a Hausdorff space X, a sequence $\mathcal{S} = (s_n)$ of self-maps of X is said to be a **fixed point resolution**, or a **resolution** in short, if the map $r : \mathrm{C}(\mathcal{S}) \to X$, given by

$$r(x) = \lim_{n\to\infty} s_n(x),$$

is continuous and $r(\mathrm{C}(\mathcal{S})) = \mathrm{Fix}(\mathcal{S})$; that is, $\mathrm{Fix}(\mathcal{S})$ is a retract of $\mathrm{C}(\mathcal{S})$.

From Theorem 3.11, it implies directly that in a regular space, every virtually stable scheme having a continuous subsequence is a resolution.

Proposition 5.2. Let $F : X \to 2^X$ be a map satisfying the end point condition. If $f : X \to X$ is a selection of F, then $\mathrm{Fix}(f) = \mathrm{Fix}(F)$.

Proof. Since $f(x) \in F(x)$, $\mathrm{Fix}(f) \subseteq \mathrm{Fix}(F)$. On the other hand, we get that $f(p) \in F(p) = \{p\}$, for each $p \in \mathrm{Fix}(F)$; i.e., $\mathrm{Fix}(F) \subseteq \mathrm{Fix}(f)$. $\square$

Lemma 5.3. Let X be a metric space. If $F : X \to \mathcal{CB}(X)$ is a quasi-nonexpansive map satisfying the end point condition, then every selection of F is quasi-nonexpansive.

Proof. Suppose that $f : X \to X$ is a selection of F. Let $p \in \mathrm{Fix}(f)$ and $x \in X$. Since F is quasi-nonexpansive and by Proposition 2.31,

$$d(f(x), p) = d(f(x), F(p)) \leq \sup_{y \in F(x)} d(y, F(p)) \leq H(F(x), F(p)) \leq d(x, p).$$

Therefore, f is quasi-nonexpansive. □

Later on, let $\mathcal{CCB}(X)$ be the set of nonempty closed convex bounded subsets of X, and $\mathcal{CC}(X)$ the set of nonempty closed convex subsets of X. Following the Michael's selection theorem [11], we obtain the following resolutions.

Theorem 5.4. Let X be a closed subset of a Banach space E. If $F : X \to \mathcal{CCB}(X)$ is a lower semi-continuous quasi-nonexpansive map satisfying the end point condition, then there is a virtually stable scheme $\mathcal{S}$ such that $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(F)$.

Proof. Note that $\mathcal{CC}(X) \subseteq \mathcal{CC}(E)$, because X is closed in E. Since F is lower semi-continuous, by Theorem 2.47, F admits a continuous selection, says $f : X \to X$. By Lemma 5.3, f is quasi-nonexpansive. Consequently, the scheme $\mathcal{S} = (f^n)$ is virtually stable by Example 3.9, and $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(f) = \mathrm{Fix}(F)$ by Proposition 5.2. □

Corollary 5.5. Let X be a closed subset of a Banach space E. If $F : X \to \mathcal{CC}(X)$ is a $*$-nonexpansive map satisfying the proximal condition, then there is a virtually stable scheme $\mathcal{S}$ such that $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(F)$.

Proof. From Lemma 2.55 (1), Lemma 2.55 (2) and Lemma 2.59, the set-valued map $P_F : X \to \mathcal{CCB}(X)$ is nonexpansive satisfying the end point condition and $\mathrm{Fix}(F) = \mathrm{Fix}(P_F)$. Moreover, P_F is lower semi-continuous by Lemma 2.25 (2). Following Theorem 5.4, the sequence $\mathcal{S} = ((P_F)^n)$ is a virtually stable scheme with $\mathrm{Fix}(\mathcal{S}) = \mathrm{Fix}(P_F) = \mathrm{Fix}(F)$, where f is a continuous selection of P_F. □

Now we construct the multivalued version of the Mann iteration sequence motivated by the iteration process (1.3) in [15] as follows:

Let X be a closed convex subset of a Banach space, $F : X \to \mathcal{CCB}(X)$ H-continuous, $(\alpha_n) \subseteq (0,1)$, and $(\gamma_n) \subseteq (0,+\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$.

Following Lemma 2.27 and Theorem 2.47, F admits a continuous selection, says $g_1 : X \to X$. We define maps $s_1 : X \to X$ and $s_2 : X \to X$ by

$$s_1(x) = x, \text{ and } s_2(x) = (1-\alpha_1)s_1(x) + \alpha_1 g_1(x),$$

for each $x \in X$, respectively.

Next, we will define s_3 as follows : Define the map $\varphi : X \to (0,\infty)$ by

$$\varphi(x) = H(F \circ s_2(x), F \circ s_1(x)) + \gamma_1,$$

for each $x \in X$.

We will show that $B(g_1(x); \varphi(x)) \cap F \circ s_2(x)$ is nonempty, for each $x \in X$. Let $x \in X$. By Proposition 2.33 (4) and $g_1(x) \in F(x) = F \circ s_1(x)$, there exists $a \in F \circ s_2(x)$ such that

$$\|a - g_1(x)\| < H(F \circ s_2(x), F \circ s_1(x)) + \gamma_1 = \varphi(x).$$

Thus $\phi \neq B(g_1(x); \varphi(x)) \cap F \circ s_2(x)$.

Then we will show that the map φ is continuous. Let $x \in X$ and $\varepsilon > 0$. Since s_2 and s_1 are continuous and F is H-continuous, $F \circ s_2$ and $F \circ s_1$ are H-continuous. Then there is $\delta > 0$ such that

$$H(F \circ s_2(x), F \circ s_2(y)) < \frac{\varepsilon}{2} \text{ and } H(F \circ s_1(y), F \circ s_1(x)) < \frac{\varepsilon}{2},$$

for each $y \in B(x,\delta)$. Let $y \in B(x;\delta)$.

Case 1. $\varphi(x) - \varphi(y) \geq 0$. Then

$$\varphi(x) - \varphi(y) = [H(F \circ s_2(x), F \circ s_1(x)) + \gamma_1] - [H(F \circ s_2(y), F \circ s_1(y)) + \gamma_1]$$

$$
\begin{aligned}
&= H(F \circ s_2(x), F \circ s_1(x)) - H(F \circ s_2(y), F \circ s_1(y)) \\
&\leq H(F \circ s_2(x), F \circ s_2(y)) + H(F \circ s_2(y), F \circ s_1(y)) \\
&\quad + H(F \circ s_1(y), F \circ s_1(x)) - H(F \circ s_2(y), F \circ s_1(y)) \\
&= H(F \circ s_2(x), F \circ s_2(y)) + H(F \circ s_1(y), F \circ s_1(x)) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

Case 2. $\varphi(y) - \varphi(x) \geq 0$. It is similar to Case 1.

Therefore, φ is continuous.

From Lemma 2.50, there is a continuous map $g_2 : X \to X$ such that

$$g_2(x) \in D(g_1(x); \varphi(x)) \cap F \circ s_2(x),$$

for each $x \in X$. Thus g_2 is a continuous selection of $F \circ s_2$ satisfying

$$\|g_2(x) - g_1(x)\| \leq H(F \circ s_2(x), F \circ s_1(x)) + \gamma_2, \text{ for each } x \in X.$$

Now, we define a map $s_3 : X \to X$ by

$$s_3(x) = (1 - \alpha_2)s_2(x) + \alpha_2 g_2(x),$$

for each $x \in X$.

Inductively, for each $n = 4, 5, \ldots$, we obtain a map $s_n : X \to X$ given by

$$s_n(x) = (1 - \alpha_{n-1})s_{n-1}(x) + \alpha_{n-1} g_{n-1}(x), \tag{5.1}$$

for each $x \in X$, where $g_{n-1} : X \to X$ is a continuous selection of $F \circ s_{n-1}$ satisfying

$$\|g_{n-1}(x) - g_{n-2}(x)\| \leq H(F \circ s_{n-1}(x), F \circ s_{n-2}(x)) + \gamma_{n-1},$$

for each $x \in X$.

Lemma 5.6. Let X be a closed convex subset of a Banach space and $F : X \to \mathcal{CCB}(X)$ an H-continuous map. If F satisfies the end point condition, then the sequence $\mathcal{S} = (s_n)$ defined as (5.1) satisfies the following :

1. $\mathrm{Fix}(F) = \mathrm{Fix}(\mathcal{S})$.

2. If $(\alpha_n) \subseteq [a, 1) \subseteq (0, 1)$, then $\mathrm{Fix}(F) = r(\mathrm{C}(\mathcal{S}))$.

3. If F is quasi-nonexpansive, then for each $x \in X$, $p \in \mathrm{Fix}(F)$, and $n \in \mathbb{N}$,

$$\|s_{n+1}(x) - p\| \leq \|s_n(x) - p\| .$$

Proof. To show **(1)**, let $p \in \mathrm{Fix}(\mathcal{S}) = \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(s_n) \subseteq \mathrm{Fix}(s_2)$. Then

$$p = s_2(p) = (1 - \alpha_1)s_1(p) + \alpha_1 g_1(p) = (1 - \alpha_1)p + \alpha_1 g_1(p)$$

and $p = g_1(p) \in F(p) = \{p\}$; i.e., $p \in \mathrm{Fix}(F)$.

On the other hand, let $p \in \mathrm{Fix}(F)$. Then $s_1(p) = p$, $g_1(p) \in F(p) = \{p\}$ and

$$s_2(p) = (1 - \alpha_1)p + \alpha_1 g_1(p) = p.$$

Consequently, $g_2(p) \in F \circ s_2(p) = F(p) = \{p\}$ and

$$s_3(p) = (1 - \alpha_2)s_2(p) + \alpha_2 g_2(p) = p.$$

Inductively, $s_n(p) = p$, for each $n \in \mathbb{N}$. Therefore, $p \in \bigcap_{n\in\mathbb{N}} \mathrm{Fix}(s_n) = \mathrm{Fix}(\mathcal{S})$.

To show **(2)**, it is enough to show that $r(\mathrm{C}(\mathcal{S})) \subseteq \mathrm{Fix}(F)$ since $\mathrm{Fix}(F) = \mathrm{Fix}(\mathcal{S}) \subseteq r(\mathrm{C}(\mathcal{S}))$. Assume that $(\alpha_n) \subseteq [a, 1) \subseteq (0, 1)$.

Let $x \in \mathrm{C}(\mathcal{S})$ and $p = r(x) = \lim_{n\to\infty} s_n(x)$. Consider the equation $s_{n+1}(x) = (1 - \alpha_n)s_n(x) + \alpha_n g_n(x)$. Then

$$s_{n+1}(x) - s_n(x) = \alpha_n(g_n(x) - s_n(x)).$$

Since $\alpha_n \in [a, 1)$ and $(s_n(x))$ is a convergent sequence,

$$\begin{aligned}\limsup_{n\to\infty} \|g_n(x) - s_n(x)\| &= \limsup_{n\to\infty} \frac{\|s_{n+1}(x) - s_n(x)\|}{\alpha_n} \\ &\leq \limsup_{n\to\infty} \frac{\|s_{n+1}(x) - s_n(x)\|}{a} = 0.\end{aligned}$$

This implies $\lim_{n\to\infty} \|g_n(x) - s_n(x)\| = 0$.

Since $(s_n(x))$ converges to p and F is H-continuous, we have that for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $H(F \circ s_n(x), F(p)) < \varepsilon$, for each $n \geq N$. This implies that for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$g_n(x) \in F \circ s_n(x) \subseteq \eta(F(p); \varepsilon), \text{ for each } n \geq N.$$

That is, $\lim\limits_{n\to\infty} d(g_n(x), F(p)) = 0$. Consequently, there is a sequence $(y_n) \subseteq F(p)$, such that $\lim\limits_{n\to\infty} \|g_n(x) - y_n\| = 0$. So,

$$\begin{aligned} d(p, F(p)) &\leq \limsup_{n\to\infty} [\|p - s_n(x)\| + \|s_n(x) - g_n(x)\| + d(g_n(x), F(p))] \\ &\leq \limsup_{n\to\infty} [\|p - s_n(x)\| + \|s_n(x) - g_n(x)\| + \|g_n(x) - y_n\|] \end{aligned}$$

and hence $d(p, F(p)) = 0$. Then $p \in \mathrm{Fix}(\mathcal{S})$ since $F(p)$ is closed. Therefore, $r(\mathrm{C}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$.

To show **(3)**, suppose F is a quasi-nonexpansive map. Let $x \in X$, $p \in \mathrm{Fix}(F)$ and $n \in \mathbb{N}$. Since F is quasi-nonexpansive, by Proposition 2.29 and Proposition 2.31,

$$\begin{aligned} \|g_n(x) - p\| &\leq \sup_{y \in F\circ s_n(x)} \|y - p\| && (\text{since } g_n(x) \in F \circ s_n(x)) \\ &= h(F \circ s_n(x), \{p\}) \\ &\leq H(F \circ s_n(x), \{p\}) \\ &= H(F \circ s_n(x), F(p)) \\ &\leq \|s_n(x) - p\| \end{aligned}$$

and

$$\begin{aligned} \|s_{n+1}(x) - p\| &= (1 - \alpha_n) \|s_n(x) - p\| + \alpha_n \|g_n(x) - p\| \\ &\leq (1 - \alpha_n) \|s_n(x) - p\| + \alpha_n \|s_n(x) - p\| = \|s_n(x) - p\|. \end{aligned}$$

Therefore, $\|s_{n+1}(x) - p\| \leq \|s_n(x) - p\|$. $\square$

Theorem 5.7. Let X be a closed convex subset of a Banach space, and $F : X \to \mathcal{CCB}(X)$ an H-continuous map. If F is quasi-nonexpansive satisfying the end point condition and $(\alpha_n) \subseteq [a, 1) \subseteq (0, 1)$, then the sequence $\mathcal{S}$ defined as (5.1) is a resolution with $\mathrm{Fix}(F) = \mathrm{Fix}(\mathcal{S})$.

Proof. From Lemma 5.6 (1) and (2),

$$\mathrm{Fix}(F) = \mathrm{Fix}(\mathcal{S}) = r(\mathrm{C}(\mathcal{S})).$$

To show that the map r is continuous, let $x \in C(\mathcal{S})$ and $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that $\|s_N(x) - r(x)\| < \frac{\varepsilon}{3}$. Since s_N is continuous, there is $\delta > 0$ such that if $\|x - y\| < \delta$, then

$$\|s_N(x) - s_N(y)\| < \frac{\varepsilon}{3}.$$

Since $r(x) \in \mathrm{Fix}(F)$ and by Lemma 5.6,

$$\begin{aligned} \|r(x) - s_m(y)\| &\leq \|r(x) - s_N(y)\| \\ &\leq \|r(x) - s_N(x)\| + \|s_N(x) - s_N(y)\| < \frac{2\varepsilon}{3}, \end{aligned}$$

for each $m \geq N$. Let $y \in \mathrm{C}(\mathcal{S}) \cap B(x, \delta)$. Then there is $M \in \mathbb{N}$ such that $M \geq N$ and $\|s_M(y) - r(y)\| < \frac{\varepsilon}{3}$. Hence

$$\|r(x) - r(y)\| \leq \|r(x) - s_M(y)\| + \|s_M(x) - r(y)\| < \varepsilon.$$

Therefore, r is continuous and hence $\mathcal{S}$ is a resolution. $\square$

Next we recall the convergence result of Song and Wang [15] in the notion of the sequence defined as (5.1).

Theorem 5.8 ([15], Theorem 2.3)**.** Let X be a nonempty compact convex subset of a Banach space, $F : X \to \mathcal{CB}(X)$ a nonexpansive map satisfying the end point condition, and $(\alpha_n) \subseteq [a, b] \subseteq (0, 1)$. If $\mathcal{S} = (s_n)$ is a sequence defined as (5.1), then $\mathrm{C}(\mathcal{S}) = X$ and $\lim_{n\to\infty} s_n(x) \in \mathrm{Fix}(F)$, for each $x \in \mathrm{C}(\mathcal{S})$.

Corollary 5.9. Let X be a nonempty compact convex subset of a Banach space. If $F : X \to \mathcal{CCB}(X)$ a nonexpansive map satisfying the end point condition, then $\mathrm{Fix}(F)$ is a retract of X and hence contractible.

Proof. It is straightforward from Theorem 5.7 and Theorem 5.8. □

Corollary 5.10. Let X be a nonempty compact convex subset of a Banach space. If $F : X \to \mathcal{CC}(X)$ is a $*$-nonexpansive map satisfying the proximal condition, then $\mathrm{Fix}(F)$ is a retract of X and hence contractible.

Proof. Since the map $P_F : X \to \mathcal{CCB}(X)$ is a nonexpansive map satisfying the end point condition with $\mathrm{Fix}(P_F) = \mathrm{Fix}(F)$, by Corollary 5.9, $\mathrm{Fix}(F)$ is a retract of X and hence contractible. □

REFERENCES

[1] Assad N. A., Kirk W. A.: Fixed point theorems for set-valued mappings of contractive type, *Pacific J. Math.*, 43(3), 553–562 (1972).

[2] Beg I., Khan A. R., Hussain N.: Approximation of $*$-nonexpansive random multivalued operators on Banach spaces. *J. Aust. Math. Soc.*, 76, 51–66 (2004).

[3] Benavides T. D. , Ramirez P. L.: Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings. *Proc. Amer. Math. Soc.*, 129(12), 3549–3557 (2001).

[4] Berinde V.: *Iterative approximation of fixed points.* Lecture Notes in Math. Springer-Verlag Berlin Heidelberg, second edition, 2007.

[5] Bruck R. E.: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. *Trans. Amer. Math. Soc.*, 179, 251–262 (1973).

[6] Chaoha P.: Virtually nonexpansive maps and their convergence sets. *J. Math. Anal. Appl.*, 326, 390–397 (2007).

[7] Chaoha P., Atiponrat W.: Virtually stable maps and their fixed point sets. *J. Math. Anal. Appl.*, 359, 536–542 (2009).

[8] Dimri R., Singh A., Bhat S.: Common fixed point theorem for multivalued maps in cone metric spaces. *International Mathematical Forum*, 5(46), 2271–2278 (2010).

[9] Geletu A.: *Introduction to topological spaces and set-valued maps.* Lecture Notes in Math. Institute of Mathematics, Department of Operations Research & Stochestics, Ilmenau University of Technology, 2006.

[10] Li J.: The metric projection and its applications to solving variational inequalities in Banach spaces. *Fixed Point Theory*, 5(2), 285–298 (2004).

[11] Michael E.: Continuous selection. I. *Ann. Math.*, 63(2), 361–382 (1956).

[12] Munkres J. R.: *Topology.* Prentice Hall, Inc., second edition, 2000.

[13] Nalder S. B.: Multi-valued contraction mappings. *Pacific J. Math.*, 30(2), 475–488 (1969).

[14] Neammanee K., Kaewkhao A.: On multi-valued weak contraction mappings. *Jour. Math. Res.*, 3(2) (2011).

[15] Song Y.,Wang H.: Convergence of iterative algorithms for multivalued mappings in Banach spaces. *Nonlinear Anal.*, 70, 1547–1556 (2009).

[16] Suzuki T.: Fixed point theorems for set-valued mappings of contractive type. *J. Math. Anal. Appl.*, 340, 1088–1095 (2008).

[17] Yanagi K.: A common fixed point theorem for a sequence of multivalued mappings. *Publ. RIMS, Kyoto University*, 15, 47–52 (1979).

VITA

Name	Mr. Pharunyou Chanthorn
Date of Birth	27 October 1982
Place of Birth	Chiang Mai, Thailand
Education	M.Sc. (Mathematics), Chulalongkorn University, 2008