การประมาณค่าแบบกึ่งปกติสำหรับจำนวนครั้งที่กลับมายังจุดเริ่มต้นของแนวเดินแบบสุ่ม

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HALF-NORMAL APPROXIMATION FOR NUMBER OF RETURNS TO ORIGIN OF RANDOM WALKS

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ทัตพล ศิริประภารัตน์: การประมาณค่าแบบกึ่งปกติสำหรับจำนวนครั้งที่กลับมายังจุดเริ่มต้นของแนวเดินแบบสุ่ม (HALF-NORMAL APPROXIMATION FOR NUMBER OF RETURNS TO ORIGIN OF RANDOM WALKS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ศ. ดร.กฤษณะ เนียมมณี, 38 หน้า.

ให้ (X_n) เป็นลำดับของตัวแปรสุ่มที่เป็นอิสระต่อกันและมีการแจกแจงแบบเดียวกัน โดยที่ $P(X_1 = 1) = p, P(X_1 = -1) = 1 - p$ เมื่อ $0 < p < 1$ แนวเดินแบบสุ่มคือ กระบวนการสโตแคสติกแบบวิยุต $(S_n)_{n\geq 0}$ ซึ่งถูกนิยามโดย $S_0 = 0$ และ $S_n = \sum_{i=1}^{n} X_i$ เมื่อ $n \geq 1$ K_n ถูกเรียกว่าจำนวนครั้งที่กลับมายังจุดเริ่มต้น ถ้า $K_n = |\{k \in \mathbb{N} \mid 1 \leq k \leq n$ และ $S_k = 0\}|$ ในกรณีของแนวเดินแบบสุ่มสมมาตร นั่นคือ $p = \frac{1}{2}$ ดอปเลอร์ (2015) แสดงไว้ว่า การแจกแจงของ K_n สามารถประมาณโดยการแจกแจงแบบกึ่งปกติ และยังให้ขอบเขตแบบเอกรูปของการประมาณค่านี้ หลังจากนั้นสะมาแอและคณะ (2016) ให้ขอบเขตแบบไม่เอกรูป ในวิทยานิพนธ์ฉบับนี้เราปรับปรุงขอบเขตแบบไม่เอกรูปของสะมาแอและคณะ ในกรณีของแนวเดินแบบสุ่มอสมมาตร นั่นคือ $p \neq \frac{1}{2}$ เราให้การแจกแจงของ K_n และ แสดงว่ามันไม่ลู่เข้าสู่การแจกแจงแบบกึ่งปกติ

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต

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TATPON SIRIPRAPARAT : HALF-NORMAL APPROXIMATION FOR NUMBER OF RETURNS TO ORIGIN OF RANDOM WALKS.
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Let (X_n) be a sequence of independent identically distributed random variables with $P(X_1 = 1) = p, P(X_1 = -1) = 1 - p$ for $0 < p < 1$. A random walk is a discrete time stochastic process $(S_n)_{n\geq 0}$ defined by $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. K_n is called the number of returns to the origin if $K_n = |\{k \in \mathbb{N} | 1 \leq k \leq n \text{ and } S_k = 0\}|$. In case of symmetric random walk, i.e., $p = \frac{1}{2}$, Döbler (2015) showed that the distribution of K_n can be approximated by half-normal distribution and he also gave a uniform bound of this approximation. After that Sama-ae et.al. (2016) gave non-uniform bounds. In this thesis, we improve a non-uniform bound of Sama-ae et.al. In case of asymmetric random walk, i.e., $p \neq \frac{1}{2}$, we give a distribution of K_n and show that it is not convergent to half-normal distribution.

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CONTENTS

CHAPTER I

INTRODUCTION

Let (X_n) be a sequence of independent identically distributed random variables with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. A symmetric random walk is a discrete time stochastic process $(S_n)_{n\geq 0}$ defined by $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. **The number of returns to the origin** which is defined by

$$K_n = |\{k \in \mathbb{N} | 1 \leq k \leq n \quad \text{and} \quad S_k = 0\}|.$$

In 2015, Döbler [4] approximated the distribution of K_n by half-normal distribution. A distribution H is called half-normal if

$$H(z) = \begin{cases} 0 & \text{if } z < 0, \\ \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} dt & \text{if } z \geq 0. \end{cases}$$

Theorem 1.1 is his result.

Theorem 1.1. *([4]) Let n be an even positive integer. Then*

$$\sup_{z\geq 0} \left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right| \leq \frac{1}{\sqrt{n}} \left(\frac{3 + 2\sqrt{2}}{\sqrt{2\pi}} + \frac{3}{4} \right) + \frac{3}{2n}.$$

After that, A. Sama-ae et al. [11] improved Theorem 1.1 to the case of a non-uniform bound as follows.

Theorem 1.2. *([11]) Let n be an even positive integer. Then for $z \geq 0$*

$$\left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right|$$
$$\leq \frac{1}{(1+z)^3} \left(\frac{107.56185}{\sqrt{n}} + \frac{73.75519}{n} + \frac{43.14923}{n\sqrt{n}} + \frac{13.97885}{n^2} + \frac{2}{n^2\sqrt{n}} \right).$$

From Theorem 1.2, we observe that the exponent of z is 3. In this thesis, we improve the exponent of z to k where $k \in \mathbb{N}$ by using the Stein's method and the concentration inequality approach. Theorem 1.3 is our main result.

Theorem 1.3. *Let $W = \frac{K_n}{\sqrt{n}}$ and n be an even positive integer such that $n \geq 4$. For $z \geq 1$ and $k \in \mathbb{N}$, we have*

$$\left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right|$$
$$\leq \frac{1}{\sqrt{n}} \left[\frac{2.0918}{e^{\frac{7z^2}{32}}} + \frac{0.8946}{ze^{\frac{z^2}{2}}} + \frac{2.0958}{z^k} + \frac{1}{z^k}\left(2.9166\left(\frac{4}{3}\right)^k + 3 \cdot 2^k\right) EW^{k+1} \right].$$

From Theorem 1.3, we can see that the result has the form of EW^k for $k \in \mathbb{N}$. Therefore we give the bounds of EW^k as follows.

Proposition 1.4. *$EW^k \leq \prod_{i=0}^{\lfloor \frac{k}{2} \rfloor - 1} (k-2i)$ for $k = 2, 3, 4, \ldots$, where $\lfloor \frac{k}{2} \rfloor$ is the largest integer less than or equal to $\frac{k}{2}$. Futhermore, if k is even, then $EW^k \leq 2^{\frac{k}{2}} \left(\frac{k}{2}\right)!$.*

In case of asymmetric, i.e., $p \neq \frac{1}{2}$, we consider the number of returns to the origin $K_{n,p}$ defined by

$$K_{n,p} = |\{k \in \mathbb{N} | 1 \leq k \leq n \quad \text{and} \quad S_k = 0\}|.$$

Note that $K_{n,\frac{1}{2}} = K_n$. First, we give the distribution of $K_{n,p}$ in Theorem 1.5 and give the bounds for $P(K_{n,p} = r)$ in Theorem 1.6 and Theorem 1.7.

Theorem 1.5. *Let $n = 2m$, $r = 1, 2, \ldots, m$ and $q = 1 - p$.*

(i) $P(K_{n,p} = 0) = u_{2m}$,

(ii) $P(K_{n,p} = r)$

$$= 2^r \sum_{l=0}^{m-r-1} \frac{r}{r+2l} \binom{r+2l}{r+l} (pq)^{r+l} u_{2(m-r-l)} + \frac{r}{2m-r} \binom{2m-r}{m} 2^r (pq)^m$$

where

$$u_{2l} = \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1} \right] (pq)^{l-k-1} (p^{2k+2} + q^{2k+2})$$

and

$$\binom{s}{t} = 0 \quad \text{for} \quad t > s.$$

Theorem 1.6. *Let* $n = 2m$, $r = 1, 2, \ldots, m$ *and* $q = 1 - p$.

$$|P(K_{n,p} = 0) - |p - q|| \leq \triangle_{n,p}$$

where

$$\triangle_{n,p} = \frac{1}{\sqrt{2\pi m}} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^m.$$

Theorem 1.7. *Let* $n = 2m$, $r = 1, 2, \ldots, m$ *and* $q = 1 - p$.

$$|P(K_{n,p} = r) - (2pq)^r (p - q)| \leq \triangle_{n,p,r}$$

where

$$\triangle_{n,p,r} = \frac{\sqrt{2}\,(p-q)}{\sqrt{\pi r}(1-4pq)} (4pq)^r + \left(\frac{\sqrt{2}}{\sqrt{\pi m}} + \frac{1}{\pi} \left(\frac{p}{q} + \frac{q}{p} \right) \frac{m-r}{\sqrt{r}} \right) (4pq)^m.$$

Finally, we will show that the distribution of $\frac{K_{n,p}}{\sqrt{n}}$ does not converge to half-normal distribution. Theorem 1.8 and 1.9 are our results.

Theorem 1.8. $P\left(\frac{K_{n,p}}{\sqrt{n}} \leq 0 \right)$ *does not converge to* $H(0)$ *for* $p \neq \frac{1}{2}$.

Theorem 1.9. *For* $z > 0$, $P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z \right)$ *does not converge to* $H(z)$ *for* $p > \Phi(z)$ *or* $p < 1 - \Phi(z)$.

We organize this thesis as follows. We improve the bound of K_n in case of symmetric in chapter 2. In chapter 3, we find the distribution of $K_{n,p}$ in case of asymmetric, give the bound of its and show that the distribution of $\frac{K_{n,p}}{\sqrt{n}}$ does not converge in distribution to half-normal distribution. In chapter 4 we present the idea for future research.

CHAPTER II

BOUNDS IN SYMMETRIC CASE

Define the sequence $(S_n)_{n\geq 0}$ by $S_0 = 0$ and $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$, where $X_1, X_2, ...,$ are independent identically distributed random variables with $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$. In this thesis, let n be an even positive integer, say $n = 2m$, and K_n denote the number of returns to the origin, i.e.,

$$K_n = |\{k \in \mathbb{N} | 1 \leq k \leq n \quad \text{and} \quad S_k = 0\}|.$$

Let $W = \frac{K_n}{\sqrt{n}}$. It is known ([5], p.96) that for each $r \in \{0, 1, \dots, m\}$

$$p(r) := P(K_n = r) = \frac{1}{2^{n-r}} \binom{n-r}{\frac{n}{2}} = \frac{1}{2^{2m-r}} \binom{2m-r}{m} \tag{2.1}$$

and a random variable X with support $[0, m] \cap \mathbb{Z}$ has probability mass function p if and only if

$$E[(2m - X + 1)(g(X) - g(X-1)) - (X+1)g(X)] = 0 \tag{2.2}$$

for all function $g : [-1, m] \cap \mathbb{Z} \to \mathbb{R}$ such that $g(-1) = 0$ ([4], p.178).
From (2.2) Döbler [4] showed that

$$0 \leq EK_n = (2m+1)P(K_n = 0) - 1 \leq \sqrt{\frac{2n}{\pi}}, \tag{2.3}$$

and hence,

$$EW \leq \sqrt{\frac{2}{\pi}}. \tag{2.4}$$

From (2.2), Sama-ae et al. ([11], p.5) showed that

$$0 \leq EK_n^2 = 2m + 3 - 3(2m+1)P(K_n = 0) \leq 2m = n, \tag{2.5}$$

and hence,

$$EW^2 \leq 1. \tag{2.6}$$

In 2015, Döbler [4] approximated the distribution of K_n by half-normal distribution. A distribution H is called half-normal if

$$H(z) = \begin{cases} 0 & \text{if } z < 0, \\ \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{\frac{-t^2}{2}} dt & \text{if } z \geq 0. \end{cases}$$

Theorem 2.1 is his result.

Theorem 2.1. *([4]) Let n be an even positive integer. Then*

$$\sup_{z \geq 0} \left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right| \leq \frac{1}{\sqrt{n}} \left(\frac{3 + 2\sqrt{2}}{\sqrt{2\pi}} + \frac{3}{4} \right) + \frac{3}{2n}.$$

After that, A. Sama-ae et al. [11] improved Theorem 2.1 to the case of a non-uniform bound as follows.

Theorem 2.2. *([11]) Let n be an even positive integer. Then for $z \geq 0$*

$$\left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right| \leq \frac{\delta_n}{(1+z)^3}$$

where

$$\delta_n = \left(\frac{107.56185}{\sqrt{n}} + \frac{73.75519}{n} + \frac{43.14923}{n\sqrt{n}} + \frac{13.97885}{n^2} + \frac{2}{n^2\sqrt{n}} \right).$$

From Theorem 2.2, we observe that the exponent of z is 3. In this chapter, we improve the exponent of z to k where $k \in \mathbb{N}$ by using the Stein's method and the concentration inequality approach. To do this, we first need to know k^{th} moment and concentration inequality for our main result.

2.1 k^{th} moment of W

In this section, we give bounds of the k^{th} moment of W as follows.

Proposition 2.3. *$EW^k \leq \prod_{i=0}^{\lfloor \frac{k}{2} \rfloor - 1} (k-2i)$ for $k = 2, 3, 4, \ldots$, where $\lfloor \frac{k}{2} \rfloor$ is the largest integer less than or equal to $\frac{k}{2}$. Futhermore, if k is even, then $EW^k \leq 2^{\frac{k}{2}} \left(\frac{k}{2}\right)!$.*

Proof. From (2.5) we know that $EK_n^2 \leq n \leq n^{\frac{k}{2}} \prod_{i=0}^{\lfloor \frac{k}{2} \rfloor - 1} (k-2i)$ for $k = 2$. Hence, we will prove the proposition for $k \geq 3$.
Let $g : [-1, m] \cap \mathbb{Z} \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} t^{k-1} & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Note that

$$\begin{aligned} Eg(K_n - 1) &= \sum_{r=0}^{m} g(r-1) P(K_n = r) \\ &= \sum_{r=1}^{m} (r-1)^{k-1} P(K_n = r) \\ &= \sum_{r=1}^{m} \sum_{l=0}^{k-1} \binom{k-1}{l} r^l (-1)^{k-1-l} P(K_n = r) \\ &= \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} \sum_{r=1}^{m} r^l P(K_n = r) \\ &= \sum_{l=1}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} \sum_{r=0}^{m} r^l P(K_n = r) \\ &\quad + (-1)^{k-1} \sum_{r=1}^{m} P(K_n = r) \\ &= \sum_{l=1}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} EK_n^l + (-1)^{k-1}(1 - P(K_n = 0)) \end{aligned} \tag{2.7}$$

and
$$EK_ng(K_n-1) = \sum_{r=0}^{m} rg(r-1)P(K_n=r)$$
$$= \sum_{r=1}^{m} r(r-1)^{k-1}P(K_n=r)$$
$$= \sum_{r=1}^{m} r \sum_{l=0}^{k-1} \binom{k-1}{l} r^l (-1)^{k-1-l} P(K_n=r)$$
$$= \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^{k-1-l} EK_n^{l+1}. \tag{2.8}$$

From (2.2), (2.7) and (2.8),

$$\begin{aligned}
0 &= E[(2m-K_n+1)(g(K_n)-g(K_n-1))-(K_n+1)g(K_n)] \\
&= E[(2m-K_n+1)(K_n^{k-1}-g(K_n-1))-(K_n+1)K_n^{k-1}] \\
&= 2mEK_n^{k-1} - 2mEg(K_n-1) + EK_ng(K_n-1) - Eg(K_n-1) - 2EK_n^k \\
&= 2mEK_n^{k-1} - (2m+1)Eg(K_n-1) + EK_ng(K_n-1) - 2EK_n^k \\
&= 2mEK_n^{k-1} - (2m+1)\left(\sum_{l=1}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + (-1)^{k-1}(1-P(K_n=0))\right) \\
&\quad + \sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} - 2EK_n^k \\
&= 2mEK_n^{k-1} - (2m+1)\left(\sum_{l=1}^{k-1}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + (-1)^{k-1}(1-P(K_n=0))\right) \\
&\quad + \sum_{l=0}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} + EK_n^k - 2EK_n^k \\
&= 2mEK_n^{k-1} - (2m+1)\Bigg(\sum_{l=1}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + EK_n^{k-1} \\
&\quad + (-1)^{k-1}(1-P(K_n=0))\Bigg) + \sum_{l=0}^{k-3}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} - \binom{k-1}{k-2}EK_n^{k-1} \\
&\quad - EK_n^k
\end{aligned}$$

$$= \left(2m - (2m+1) - \binom{k-1}{k-2}\right) EK_n^{k-1} - (2m+1)\left(\sum_{l=1}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l\right.$$
$$\left. + (-1)^{k-1}(1 - P(K_n = 0))\right) + \sum_{l=0}^{k-3}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} - EK_n^k$$
$$= -kEK_n^{k-1} - (n+1)\left(\sum_{l=1}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + (-1)^{k-1}(1 - P(K_n = 0))\right)$$
$$+ \sum_{l=0}^{k-3}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} - EK_n^k.$$

Therefore,

$$EK_n^k$$
$$= -kEK_n^{k-1} - (n+1)\left(\sum_{l=1}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + (-1)^{k-1}(1 - P(K_n = 0))\right)$$
$$+ \sum_{l=0}^{k-3}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1}. \tag{2.9}$$

We can see that

$$\begin{aligned} EK_n^3 &= -3EK_n^2 - (n+1)(-2EK_n + 1 - P(K_n = 0)) + EK_n \\ &= -3EK_n^2 + 2(n+1)EK_n - (n+1)(1 - P(K_n = 0)) + EK_n \\ &\le -3EK_n^2 + 2(n+1)EK_n + EK_n \\ &= -3EK_n^2 + 2nEK_n + 3EK_n \\ &\le -3EK_n^2 + 2nEK_n + 3EK_n^2 \\ &= 2nEK_n \end{aligned} \tag{2.10}$$

where we have used the fact that

$$K_n^k \le K_n^{k+1} \quad \text{for} \quad k \in \mathbb{N} \tag{2.11}$$

in the last inequality. Since

$$\begin{aligned}1 - P(K_n = 0) &= \sum_{r=1}^{m} P(K_n = r) \\ &\leq \sum_{r=1}^{m} r^k P(K_n = r) \\ &= EK_n^k \quad \text{for all} \quad k \in \mathbb{N}, \end{aligned} \tag{2.12}$$

so we get

$$\begin{aligned} & EK_n^4 \\ &= -4EK_n^3 - (n+1)(3EK_n - 3EK_n^2 - 1 + P(K_n = 0)) - EK_n + 3EK_n^2 \\ &= -4EK_n^3 - 3(n+1)EK_n + 3(n+1)EK_n^2 + (n+1)(1 - P(K_n = 0)) - EK_n \\ &\quad + 3EK_n^2 \\ &= 3nEK_n^2 - 4EK_n^3 + 6EK_n^2 - 3nEK_n - 4EK_n + n(1 - P(K_n = 0)) \\ &\quad + (1 - P(K_n = 0)) \\ &\leq 3nEK_n^2 - 4EK_n^3 + 6EK_n^2 - 3nEK_n - 4EK_n + nEK_n^2 + EK_n \\ &\leq 4nEK_n^2 - 4EK_n^3 + 6EK_n^2 - 3nEK_n. \end{aligned} \tag{2.13}$$

By (2.11) and the fact that

$$EK_n^{l+1} \leq nEK_n^l \quad \text{for} \quad l \in \mathbb{N}, \tag{2.14}$$

we get

$$\begin{aligned} EK_n^4 &\leq 4nEK_n^2 - 4EK_n^3 + 6EK_n^2 - 3EK_n^2 \\ &= 4nEK_n^2 - 4EK_n^3 + 3EK_n^2 \\ &\leq 4nEK_n^2 - 4EK_n^3 + 3EK_n^3 \\ &\leq 4nEK_n^2. \end{aligned} \tag{2.15}$$

For $k \geq 5$, by (2.9),

$$
\begin{aligned}
&EK_n^k \\
&= -kEK_n^{k-1} - (n+1)\left(\sum_{l=1}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + (-1)^{k-1}(1-P(K_n=0))\right) \\
&\quad + \sum_{l=0}^{k-3}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} \\
&= A_k + B_k + C_k + D_k, \qquad (2.16)
\end{aligned}
$$

where

$$
\begin{aligned}
A_k &= n\binom{k-1}{k-2}EK_n^{k-2}, \\
B_k &= -\left(kEK_n^{k-1} + n\binom{k-1}{k-3}EK_n^{k-3}\right), \\
C_k &= -(n+1)(-1)^{k-1}(1-P(K_n=0)), \\
D_k &= -n\sum_{l=1}^{k-4}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l - \sum_{l=1}^{k-2}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l \\
&\quad + \sum_{l=0}^{k-3}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1}.
\end{aligned}
$$

We next estimate D_k, by (2.11), (2.14) and (2.16), we get

$$
\begin{aligned}
&D_k \\
&= -n\sum_{l=1}^{k-4}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + \binom{k-1}{k-2}EK_n^{k-2} - \binom{k-1}{k-3}EK_n^{k-3} \\
&\quad - \sum_{l=1}^{k-4}\binom{k-1}{l}(-1)^{k-1-l}EK_n^l + \binom{k-1}{k-3}EK_n^{k-2} + \sum_{l=0}^{k-4}\binom{k-1}{l}(-1)^{k-1-l}EK_n^{l+1} \\
&= \left[\binom{k-1}{k-2}EK_n^{k-2} - \binom{k-1}{k-3}EK_n^{k-3} + \binom{k-1}{k-3}EK_n^{k-2} + (-1)^{k-1}EK_n\right] \\
&\quad + \sum_{l=1}^{k-4}\binom{k-1}{l}(-1)^{k-1-l}(-nEK_n^l - EK_n^l + EK_n^{l+1})
\end{aligned}
$$

$$\leq \left[\binom{k-1}{k-2}EK_n^{k-2} + \binom{k-1}{k-3}EK_n^{k-2}\right] + \sum_{l=1}^{k-4}\binom{k-1}{l}\left(-(n+1)EK_n^l + (n+1)EK_n^l\right)$$
$$= \left[\binom{k-1}{k-2} + \binom{k-1}{k-3}\right]EK_n^{k-2}. \tag{2.17}$$

Therefore, by (2.16) and (2.17),

$$\begin{aligned}
&EK_n^k \\
&\leq n\binom{k-1}{k-2}EK_n^{k-2} - kEK_n^{k-1} - n\binom{k-1}{k-3}EK_n^{k-3} + \left[\binom{k-1}{k-2} + \binom{k-1}{k-3}\right]EK_n^{k-2} \\
&\quad - (n+1)(-1)^{k-1}(1 - P(K_n = 0)) \\
&\leq n\binom{k-1}{k-2}EK_n^{k-2} - kEK_n^{k-1} - n\binom{k-1}{k-3}EK_n^{k-3} + \left[\binom{k-1}{k-2} + \binom{k-1}{k-3}\right]EK_n^{k-2} \\
&\quad + (n+1)(1 - P(K_n = 0)).
\end{aligned} \tag{2.18}$$

By (2.11) and (2.14),

$$\begin{aligned}
kEK_n^{k-1} - \binom{k-1}{k-2}EK_n^{k-2} &= kEK_n^{k-1} - (k-1)EK_n^{k-2} \\
&= kE(K_n^{k-1} - K_n^{k-2}) + EK_n^{k-2} \\
&\geq EK_n^{k-2},
\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}
n\binom{k-1}{k-3}EK_n^{k-3} - \binom{k-1}{k-3}EK_n^{k-2} &= \binom{k-1}{k-3}E(nK_n^{k-3} - K_n^{k-2}) \\
&\geq \binom{k-1}{k-3}E(K_n^{k-2} - K_n^{k-2}) \\
&= 0.
\end{aligned} \tag{2.20}$$

Thus, by (2.12), (2.18), (2.19) and (2.20),

$$\begin{aligned} EK_n^k &\le n\binom{k-1}{k-2}EK_n^{k-2} - EK_n^{k-2} + (n+1)(1-P(K_n=0)) \\ &\le n(k-1)EK_n^{k-2} - EK_n^{k-2} + (n+1)EK_n^{k-2} \\ &= n(k-1)EK_n^{k-2} - EK_n^{k-2} + nEK_n^{k-2} + EK_n^{k-2} \\ &= nkEK_n^{k-2} \quad \text{for} \quad k = 5,6,7,\dots. \end{aligned}$$

From this fact and (2.10), (2.15) , we have

$$EK_n^k \le nkEK_n^{k-2} \quad \text{for} \quad k = 3,4,5,\dots. \tag{2.21}$$

Next we will show that

$$EK_n^{2k} \le n^k \prod_{i=0}^{k-1}(2k-2i) \quad \text{for} \quad k = 2,3,4,\dots \tag{2.22}$$

and

$$EK_n^{2k+1} \le n^{k+\frac{1}{2}} \prod_{i=0}^{k-1}(2k-2i+1) \quad \text{for} \quad k = 1,2,3,\dots. \tag{2.23}$$

From (2.5) and (2.15), we see that

$$EK_n^4 \le 4n^2 \le 8n^2 = n^k \prod_{i=0}^{k-1}(2k-2i) \quad \text{for} \quad k = 2.$$

Assume that $EK_n^{2k_0} \le n^{k_0} \prod_{i=0}^{k_0-1}(2k_0-2i)$ is true for $k_0 \in \mathbb{N}\backslash\{1\}$. Thus, by (2.21),

$$\begin{aligned} EK_n^{2k_0+2} &\le n(2k_0+2)EK_n^{2k_0} \\ &\le n(2k_0+2)n_0^k \prod_{i=0}^{k_0-1}(2k-2i) \\ &= n^{k_0+1}\prod_{i=0}^{k_0}(2k_0-2i+2). \end{aligned}$$

By Mathematical Induction, we have (2.22).

Similarly, from (2.3) and (2.10) we can see that

$$EK_n^3 \leq 2\sqrt{\frac{2}{\pi}}n\sqrt{n} \leq 3n\sqrt{n} = n^{k+\frac{1}{2}}\prod_{i=0}^{k-1}(2k-2i+1) \quad \text{for} \quad k=1.$$

Hence (2.23) is true for $k=1$. To use mathematical induction,we assume that $EK_n^{2k_0+1} \leq n^{k_0+\frac{1}{2}}\prod_{i=0}^{k_0-1}(2k_0-2i+1)$ is true for $k_0 \in \mathbb{N}$. Therefore, by (2.21),

$$\begin{aligned} EK_n^{2k_0+3} &\leq n(2k_0+3)EK_n^{2k_0+1} \\ &\leq n(2k_0+3)n^{k_0+\frac{1}{2}}\prod_{i=0}^{k_0-1}(2k_0-2i+1) \\ &= n^{k_0+\frac{3}{2}}\prod_{i=0}^{k_0}(2k_0-2i+3). \end{aligned}$$

By Mathematical Induction, we have (2.23).

By (2.22) and (2.23), we get

$$EK_n^k \leq n^{\frac{k}{2}}\prod_{i=0}^{\lfloor\frac{k}{2}\rfloor-1}(k-2i) \quad \text{for} \quad k \geq 2. \tag{2.24}$$

Hence,

$$EW^k \leq \prod_{i=0}^{\lfloor\frac{k}{2}\rfloor-1}(k-2i) \quad \text{for} \quad k=2,3,4,\ldots.$$

□

2.2 Non-uniform concentration inequality

In this section, we use the idea of Sama-ae et al.([11], pp.6–8) to obtain the following non-uniform concentration inequality.

Proposition 2.4. (Non-uniform concentration inequality). *For $z > 0$ and $k \in \mathbb{N}$,*

$$P\left(z < W \leq z + \frac{1}{\sqrt{n}}\right) \leq \frac{2^k}{\sqrt{n}z^k}\left[3EW^{k+1} + \frac{1}{(\sqrt{n})^k}\left(2\sqrt{\frac{2}{\pi}} + \frac{1}{\sqrt{n}}\right)\right].$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{if} \quad t < z - \frac{1}{\sqrt{n}}, \\ (t + \frac{1}{\sqrt{n}})^k(t - z + \frac{1}{\sqrt{n}}) & \text{if} \quad z - \frac{1}{\sqrt{n}} \leq t \leq z + \frac{1}{\sqrt{n}}, \\ \frac{2}{\sqrt{n}}(t + \frac{1}{\sqrt{n}})^k & \text{if} \quad t > z + \frac{1}{\sqrt{n}}. \end{cases}$$

Then

$$f'(t) \geq \begin{cases} z^k & \text{if} \quad z - \frac{1}{\sqrt{n}} < t < z + \frac{1}{\sqrt{n}}, \\ 0 & \text{if} \quad t < z - \frac{1}{\sqrt{n}} \ \text{ or } \ t > z + \frac{1}{\sqrt{n}}. \end{cases}$$

We can follow the argument of Sama-ae et al. ([11], p.7) to show that

$$P\left(z < W \leq z + \frac{1}{\sqrt{n}}\right) \leq \frac{1}{z^k}\left(2E[Wf(W)] + \frac{1}{\sqrt{n}}E[f(W)]\right).$$

Note that,

$$\begin{aligned} |E[Wf(W)]| &\leq \frac{2}{\sqrt{n}}\left|EW\left(W + \frac{1}{\sqrt{n}}\right)^k\right| \\ &\leq \frac{2^k}{\sqrt{n}}EW\left(W^k + \frac{1}{(\sqrt{n})^k}\right) \\ &= \frac{2^k}{\sqrt{n}}\left(EW^{k+1} + \frac{1}{(\sqrt{n})^k}EW\right) \end{aligned}$$

and

$$\begin{aligned}|E[f(W)]| &\leq \frac{2}{\sqrt{n}}\left|E\left(W+\frac{1}{\sqrt{n}}\right)^k\right|\\ &\leq \frac{2^k}{\sqrt{n}}E\left(W^k+\frac{1}{(\sqrt{n})^k}\right)\\ &= \frac{2^k}{\sqrt{n}}\left(EW^k+\frac{1}{(\sqrt{n})^k}\right).\end{aligned}$$

By (2.11),

$$\frac{1}{\sqrt{n}}EW^k = \frac{1}{(\sqrt{n})^{k+1}}EK_n^k \leq \frac{1}{(\sqrt{n})^{k+1}}EK_n^{k+1} = EW^{k+1}.$$

This implies

$$|E[f(W)]| \leq 2^k\left(EW^{k+1}+\frac{1}{(\sqrt{n})^{k+1}}\right).$$

Hence, by (2.4),

$$\begin{aligned}&P\left(z<W\leq z+\frac{1}{\sqrt{n}}\right)\\ &\leq \frac{1}{z^k}\left[\frac{2^{k+1}}{\sqrt{n}}\left(EW^{k+1}+\frac{1}{(\sqrt{n})^k}EW\right)+\frac{2^k}{\sqrt{n}}\left(EW^{k+1}+\frac{1}{(\sqrt{n})^{k+1}}\right)\right]\\ &\leq \frac{2^k}{\sqrt{n}z^k}\left[2EW^{k+1}+\frac{2}{(\sqrt{n})^k}\sqrt{\frac{2}{\pi}}+EW^{k+1}+\frac{1}{(\sqrt{n})^{k+1}}\right]\\ &= \frac{2^k}{\sqrt{n}z^k}\left[3EW^{k+1}+\frac{1}{(\sqrt{n})^k}\left(2\sqrt{\frac{2}{\pi}}+\frac{1}{\sqrt{n}}\right)\right].\end{aligned}$$

□

2.3 Non–uniform bounds

Stein's method of obtaining the bound in the normal approximation for dependent random variables was investigated by Stein ([12]). Stein's technique is free of Fourier technique and relied instead on the differential equation. It was first developed to the Poisson approximation by Chen ([2]). Nowadays, the method were developed on other distributions (see [1], [3], [6], [7], [8], [9], [10] for examples). In 2015, Döbler ([4]) applied Stein's method in order to approximate to distribution of K_n by half-normal distribution. He gave Stein's equation for standard half-normal approximaton,

$$f'(x) - xf(x) = h(x) - H(z) \tag{2.25}$$

where f and h are a continuous, piecewise differentiable functions on $[0, \infty)$. Let $z \geq 0$ and define $h_z : [0, \infty) \to \mathbb{R}$ by

$$h_z(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq z, \\ 0 & \text{if } x > z. \end{cases}$$

Then the solution of equation (2.25) is $f_z : [0, \infty) \to \mathbb{R}$ given by

$$f_z(x) = \begin{cases} \sqrt{2\pi}e^{\frac{x^2}{2}}(1 - \Phi(z))(2\Phi(x) - 1) & \text{if } x \leq z, \\ \sqrt{2\pi}e^{\frac{x^2}{2}}(1 - \Phi(x))(2\Phi(z) - 1) & \text{if } x > z. \end{cases} \tag{2.26}$$

We see that f_z is not differentiable at $x = z$, then we define derivative of f_z at $x = z$ from (2.25).
Hence,

$$f_z'(z) = z\sqrt{2\pi}e^{\frac{z^2}{2}}(1 - \Phi(z))(2\Phi(z) - 1) + 2(1 - \Phi(z)).$$

and

$$|f_z'(x)| \leq 1 \text{ for all } x \geq 0 \qquad ([4], \text{p.177}). \tag{2.27}$$

From (2.25), for any random variable W, we get

$$E(f_z'(W)) - E(Wf_z(W)) = P(W \leq z) - H(z).$$

This implies that, we can bound $|E(f_z'(W)) - E(Wf_z(W))|$ instead of $|P(W \leq z) - H(z)|$. We call this technique Stein's method. In this section, we give a non–uniform bound for K_n. From now on, we use f to represent f_z.

Theorem 2.5. *Let $W = \frac{K_n}{\sqrt{n}}$ and n be an even positive integer such that $n \geq 4$. For $z \geq 1$ and $k \in \mathbb{N}$, we have*

$$\left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right|$$
$$\leq \frac{1}{\sqrt{n}} \left[\frac{2.0918}{e^{\frac{7z^2}{32}}} + \frac{0.8946}{ze^{\frac{z^2}{2}}} + \frac{2.0958}{z^k} + \frac{1}{z^k}\left(2.9166\left(\frac{4}{3}\right)^k + 3 \cdot 2^k\right) EW^{k+1} \right].$$

Proof. Let $k \in \mathbb{N}$. Döbler [4] and Sama-ae et al. [11] used Stein's method to show that

$$|P(W \leq z) - H(z)| \leq |A_1| + |A_2| + |A_3| \tag{2.28}$$

where

$$|A_1| \leq \left| E\left[W\left(f(W) - f(W - \frac{1}{\sqrt{n}}) \right) \right] \right| + \left| \frac{1}{\sqrt{n}} E[f(W)] \right| \quad \text{([4], p.179)}, \tag{2.29}$$

$$|A_2| \leq \sqrt{n} \left| E\left[\int_{W-\frac{1}{\sqrt{n}}}^{W} \int_t^W (f(s) + sf'(s)) ds dt \right] \right| \quad \text{([11], p.9, [4], p.179)}, \tag{2.30}$$

$$|A_3| \leq P(z < W \leq z + \frac{1}{\sqrt{n}}) \quad \text{([11], p.11, [4], p.179)}. \tag{2.31}$$

By Proposition 2.4,

$$|A_3| \leq \frac{2^k}{\sqrt{n} z^k} \left[3EW^{k+1} + \frac{1}{(\sqrt{n})^k} \left(2\sqrt{\frac{2}{\pi}} + \frac{1}{\sqrt{n}} \right) \right]. \tag{2.32}$$

Next, we will bound $|A_1|$. Sama-ae. et al. ([11], p.8) showed that

$$f'(x) \leq \frac{3}{4e^{\frac{7z^2}{32}}} + \frac{\sqrt{2}}{z\sqrt{\pi} e^{\frac{z^2}{2}}} \quad \text{for} \quad x < \frac{3z}{4}. \tag{2.33}$$

By (2.27) and (2.33),

$$\begin{aligned}
&\left|E\left[W\left(f(W)-f(W-\frac{1}{\sqrt{n}})\right)\right]\right| \\
&\le EW\int_{W-\frac{1}{\sqrt{n}}}^{W}|f'(t)|dt \\
&= EW\left[\int_{W-\frac{1}{\sqrt{n}}}^{W}|f'(t)|\mathbb{I}(W<\frac{3z}{4})dt\right]+EW\left[\int_{W-\frac{1}{\sqrt{n}}}^{W}|f'(t)|\mathbb{I}(W\ge\frac{3z}{4})dt\right] \\
&\le \frac{1}{\sqrt{n}}\left[\left(\frac{3}{4e^{\frac{7z^2}{32}}}+\frac{\sqrt{2}}{z\sqrt{\pi}e^{\frac{z^2}{2}}}\right)EW+EW\mathbb{I}(W\ge\frac{3z}{4})\right] \\
&\le \frac{1}{\sqrt{n}}\left[\left(\frac{3}{4e^{\frac{7z^2}{32}}}+\frac{\sqrt{2}}{z\sqrt{\pi}e^{\frac{z^2}{2}}}\right)\sqrt{\frac{2}{\pi}}+EW\mathbb{I}(W\ge\frac{3z}{4})\right] \qquad (2.34)
\end{aligned}$$

where we have used (2.4) in the last inequality.

Note that,

$$P(W\ge\frac{3z}{4})\le\left(\frac{4}{3}\right)^{k+1}\frac{EW^{k+1}}{z^{k+1}} \qquad (2.35)$$

and

$$EW\mathbb{I}(W\ge\frac{3z}{4})\le(EW^{k+1})^{\frac{1}{k+1}}\left(P(W\ge\frac{3z}{4})\right)^{\frac{k}{k+1}}\le\frac{1}{z^k}\left(\frac{4}{3}\right)^{k}EW^{k+1}. \qquad (2.36)$$

From Sama-ae et al. ([11], p.4), we have

$$E|f(W)|\le\frac{1}{ze^{\frac{7z^2}{32}}}+\frac{1}{z}P\left(W\ge\frac{3z}{4}\right). \qquad (2.37)$$

From (2.29), (2.34), (2.35), (2.36) and (2.37),

$$|A_1|\le\frac{1}{\sqrt{n}}\left[\frac{1}{ze^{\frac{7z^2}{32}}}+\left(\frac{3}{4e^{\frac{7z^2}{32}}}+\frac{\sqrt{2}}{z\sqrt{\pi}e^{\frac{z^2}{2}}}\right)\sqrt{\frac{2}{\pi}}+\frac{1}{z^k}\left(\frac{4}{3}\right)^{k}EW^{k+1}\left(\frac{4}{3z^2}+1\right)\right]. \qquad (2.38)$$

To bound $|A_2|$, Sama-ae et al.([11], pp.9-10) wrote A_2 in the form of

$$\left|E\left[\int_{W-\frac{1}{\sqrt{n}}}^{W}\int_{t}^{W}(f(s)+sf'(s))dsdt\right]\right| \leq |A_{21}|+|A_{22}|+|A_{23}| \tag{2.39}$$

where

$$\begin{aligned}
A_{21} &:= \frac{1}{2nze^{\frac{7z^2}{32}}} && ([11],\ \text{pp.}9-10),\\
A_{22} &:= E\left[\int_{W-\frac{1}{\sqrt{n}}}^{W}\int_{t}^{W}|sf'(s)|\mathbb{I}(W<\frac{3z}{4})dsdt\right] && ([11],\ \text{pp.}9-10),\\
A_{23} &:= \frac{1}{2n}\left[\frac{1}{z}P(W\geq\frac{3z}{4})+EW\mathbb{I}(W\geq\frac{3z}{4})\right] && ([11],\ \text{pp.}9-10).
\end{aligned}$$

Using (2.4) and (2.33), we have

$$\begin{aligned}
|A_{22}| &\leq \left(\frac{3}{4e^{\frac{7z^2}{32}}}+\frac{\sqrt{2}}{z\sqrt{\pi}e^{\frac{z^2}{2}}}\right)E\left[\int_{W-\frac{1}{\sqrt{n}}}^{W}\int_{t}^{W}\max\{\frac{1}{\sqrt{n}},W\}\mathbb{I}(W<\frac{3z}{4})dsdt\right]\\
&\leq \frac{1}{2}\left(\frac{3}{4e^{\frac{7z^2}{32}}}+\frac{\sqrt{2}}{z\sqrt{\pi}e^{\frac{z^2}{2}}}\right)\left(\frac{1}{n\sqrt{n}}+\frac{1}{n}\sqrt{\frac{2}{\pi}}\right).
\end{aligned} \tag{2.40}$$

Thus by (2.35) and (2.36), it follows that

$$\begin{aligned}
|A_{23}| &\leq \frac{1}{2n}\left[\frac{1}{z}\left(\frac{4}{3}\right)^{k+1}\frac{EW^{k+1}}{z^{k+1}}+\frac{1}{z^k}\left(\frac{4}{3}\right)^{k}EW^{k+1}\right]\\
&\leq \frac{1}{2nz^k}\left(\frac{4}{3}\right)^{k}EW^{k+1}\left(\frac{4}{3z^2}+1\right).
\end{aligned} \tag{2.41}$$

Therefore, by (2.30), (2.39), (2.40) and (2.41), we conclude

$$\begin{aligned}
&|A_2|\\
&\leq \frac{1}{2n}\left[\frac{1}{ze^{\frac{7z^2}{32}}}+\left(\frac{3}{4e^{\frac{7z^2}{32}}}+\frac{\sqrt{2}}{z\sqrt{\pi}e^{\frac{z^2}{2}}}\right)\left(\frac{1}{\sqrt{n}}+\sqrt{\frac{2}{\pi}}\right)+\frac{1}{z^k}\left(\frac{4}{3}\right)^{k}EW^{k+1}\left(\frac{4}{3z^2}+1\right)\right].
\end{aligned} \tag{2.42}$$

For $n \geq 4$ and $z \geq 1$, by (2.32), (2.38), (2.42), we get that

$$|A_1| \leq \frac{1}{\sqrt{n}} \left(\frac{1.5984}{e^{\frac{7z^2}{32}}} + \frac{0.6366}{ze^{\frac{z^2}{2}}} + \frac{2.3333}{z^k} \left(\frac{4}{3}\right)^k EW^{k+1} \right),$$

$$|A_2| \leq \frac{1}{\sqrt{n}} \left(\frac{0.4934}{e^{\frac{7z^2}{32}}} + \frac{0.2589}{ze^{\frac{z^2}{2}}} + \frac{0.5833}{z^k} \left(\frac{4}{3}\right)^k EW^{k+1} \right),$$

$$|A_3| \leq \frac{1}{\sqrt{n}} \left(3 \left(\frac{2}{z}\right)^k EW^{k+1} + \frac{2.0958}{z^k} \right).$$

By (2.28),

$$\left| P\left(\frac{K_n}{\sqrt{n}} \leq z\right) - H(z) \right|$$
$$\leq \frac{1}{\sqrt{n}} \left[\frac{2.0918}{e^{\frac{7z^2}{32}}} + \frac{0.8946}{ze^{\frac{z^2}{2}}} + \frac{2.0958}{z^k} + \frac{1}{z^k} \left(2.9166 \left(\frac{4}{3}\right)^k + 3 \cdot 2^k \right) EW^{k+1} \right].$$

□

CHAPTER III

BOUNDS IN ASYMMETRIC CASE

In chapter II, we consider the number of returns to the origin, K_n, in case of symmetric random walk, $p = \frac{1}{2}$. In this case we know that the distribution of K_n converge in distribution to half-normal distribution. In this chapter we investigate an asymmetric random walk. Let (X_n) be independent identically distributed random variables such that $P(X_n = 1) = p = 1 - P(X_n = 0)$ for $p \neq \dfrac{1}{2}$ and $S_n = \sum_{i=1}^{n} X_i$. Let

$$K_{n,p} = |\{k \in \mathbb{N} | 1 \leq k \leq n \quad \text{and} \quad S_k = 0\}|$$

be the number of returns to the origin. Note that $K_{n,\frac{1}{2}} = K_n$. For asymmetric case, we will show that the distribution of $\dfrac{K_{n,p}}{\sqrt{n}}$ does not converge in distribution to half-normal distribution. We organize this chapter as follows. The distribution of $K_{n,p}$ is in section 3.1 while the bounds for $P(K_{n,p} = r)$ are in section 3.2. In section 3.3 we show that the distribution of $K_{n,p}$ does not converge to half normal distribution.

3.1 Distribution of $K_{n,p}$

Let $n = 2m$ and $r \in \{0, 1, \dots, m\}$. In this section, we give distribution of $K_{n,p}$.

Theorem 3.1. *Let* $n = 2m$, $r = 1, 2, \ldots, m$ *and* $q = 1 - p$.

(i) $P(K_{n,p} = 0) = u_{2m}$,

(ii) $P(K_{n,p} = r)$

$$= 2^r \sum_{l=0}^{m-r-1} \frac{r}{r+2l} \binom{r+2l}{r+l} (pq)^{r+l} u_{2(m-r-l)} + \frac{r}{2m-r} \binom{2m-r}{m} 2^r (pq)^m$$

where

$$u_{2l} = \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1} \right] (pq)^{l-k-1} (p^{2k+2} + q^{2k+2})$$

and

$$\binom{s}{t} = 0 \quad \textit{for} \quad t > s.$$

Proof. Let $l = r, r+1, \ldots, m$,
$\rho_{r,2l}$ be the probability that the r^{th} return occurs at step $2l$ and
u_{2l} be the probability that no return occurs on $2l$ steps, i.e.,

$$u_{2l} = P(S_1 \neq 0, \ldots, S_{2l} \neq 0).$$

Hence $\rho_{r,2l} u_{2m-2l}$ is the probability that r^{th} return occurs at step $2l$ and does not occur at origin in the remaining $2m - 2l$ steps. This implies

$$P(K_{n,p} = r) = \sum_{l=0}^{m-r-1} \rho_{r,2r+2l} u_{2m-2r-2l} + \rho_{r,2m}. \tag{3.1}$$

Feller showed that

$$\rho_{r,2l} = \frac{r}{2l-r} \binom{2l-r}{l} 2^r (pq)^l \quad \text{for} \quad l = r, r+1, \ldots, m \quad \text{and} \quad \rho_{0,0} = 1 \qquad ([5], \text{p.275}) \tag{3.2}$$

and

$$P(S_1 > 0, \ldots, S_{2l} > 0) = \sum_{k=1}^{l} P(S_1 > 0, \ldots, S_{2l-1} > 0, S_{2l} = 2k) \qquad ([5], \text{p.}77)$$

with the number of paths satisfying the condition indicated on the right side equals

$$\binom{2l-1}{l+k-1} - \binom{2l-1}{l+k} \qquad ([5], \text{p.}73),$$

and we can see that the probability of each number in these paths equals $p^{l+k}q^{l-k}$. Therefore,

$$\begin{aligned} &P(S_1 > 0, \ldots, S_{2l} > 0) \\ &= \left[\binom{2l-1}{l} - \binom{2l-1}{l+1}\right] p^{l+1}q^{l-1} + \left[\binom{2l-1}{2l+1} - \binom{2l-1}{l+2}\right] p^{l+2}q^{l-2} \\ &\quad + \ldots + \left[\binom{2l-1}{2l-1} - \binom{2l-1}{2l}\right] p^{2l}q^{0} \\ &= \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1}\right] p^{l+k+1}q^{l-k-1}. \end{aligned} \tag{3.3}$$

Similarly, we get that

$$P(S_1 < 0, \ldots, S_{2l} < 0) = \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1}\right] p^{l-k-1}q^{l+k+1}. \tag{3.4}$$

Thus, by (3.3) and (3.4),

$$\begin{aligned} u_{2l} &= P(S_1 \neq 0, \ldots, S_{2l} \neq 0) \\ &= P(S_1 > 0, \ldots, S_{2l} > 0) + P(S_1 < 0, \ldots, S_{2l} < 0) \\ &= \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1}\right] p^{l+k+1}q^{l-k-1} \\ &\quad + \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1}\right] p^{l-k-1}q^{l+k+1} \end{aligned}$$

$$= \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1} \right] (pq)^{l-k-1} (p^{2k+2} + q^{2k+2}). \tag{3.5}$$

By (3.1), (3.2) and (3.5), we have Theorem 3.1. □

3.2 Approximation of $P(K_{n,p} = r)$

In this section, we give the bounds of $P(K_{n,p} = r)$. Theorem 3.2 and Theorem 3.3 are our results.

Theorem 3.2. *Let* $n = 2m$, $r = 1, 2, \ldots, m$ *and* $q = 1 - p$.

$$|P(K_{n,p} = 0) - |p - q|| \leq \triangle_{n,p}$$

where

$$\triangle_{n,p} = \frac{1}{\sqrt{2\pi m}} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^m.$$

Proof. First, we will prove theorem in case of $p \geq q$. By Theorem 3.1, we have

$$u_{2l} = (pq)^l \left(\frac{p}{q} \right) A + (pq)^l \left(\frac{q}{p} \right) B \quad \text{for} \quad l = 1, 2, \ldots, m \tag{3.6}$$

where

$$A = \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1} \right] \left(\frac{p}{q} \right)^k,$$

$$B = \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1} \right] \left(\frac{q}{p} \right)^k.$$

Therefore,

$$\begin{aligned}
A &= \left[\binom{2l-1}{l} - \binom{2l-1}{l+1}\right]\left(\frac{p}{q}\right)^0 + \left[\binom{2l-1}{l+1} - \binom{2l-1}{l+2}\right]\left(\frac{p}{q}\right)^1 \\
&\quad + \ldots + \left[\binom{2l-1}{2l-2} - \binom{2l-1}{2l-1}\right]\left(\frac{p}{q}\right)^{l-2} + \binom{2l-1}{2l-1}\left(\frac{p}{q}\right)^{l-1} \\
&= \binom{2l-1}{l}\left(\frac{p}{q}\right)^0 + \binom{2l-1}{l+1}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}\right)^0 + \binom{2l-1}{l+2}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}\right)^1 \\
&\quad + \ldots + \binom{2l-1}{2l-1}\left(\frac{p}{q}-1\right)\left(\frac{p}{q}\right)^{l-2} \\
&= \binom{2l-1}{l}\left(\frac{p}{q}\right)^0 + \left(\frac{p}{q}-1\right)\sum_{k=1}^{l-1}\binom{2l-1}{l+k}\left(\frac{p}{q}\right)^{k-1} \\
&= \binom{2l-1}{l} + \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\sum_{k=1}^{l-1}\binom{2l-1}{l+k}\left(\frac{p}{q}\right)^{l+k}. \qquad (3.7)
\end{aligned}$$

Since,

$$\begin{aligned}
\sum_{k=1}^{l-1}\binom{2l-1}{l+k}\left(\frac{p}{q}\right)^{l+k} &= \sum_{k=0}^{2l-1}\binom{2l-1}{k}\left(\frac{p}{q}\right)^k - \sum_{k=0}^{l}\binom{2l-1}{k}\left(\frac{p}{q}\right)^k \\
&\leq \sum_{k=0}^{2l-1}\binom{2l-1}{k}\left(\frac{p}{q}\right)^k \\
&= \left(1+\frac{p}{q}\right)^{2l-1} \\
&= \left(\frac{1}{q}\right)^{2l-1},
\end{aligned}$$

so we get

$$\begin{aligned}
A &\leq \binom{2l-1}{l} + \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\left(\frac{1}{q}\right)^{2l-1} \\
&= \binom{2l-1}{l} + \frac{q}{p}(p-q)\left(\frac{1}{pq}\right)^l. \qquad (3.8)
\end{aligned}$$

Since $0 \leq \frac{q}{p} \leq 1$,

$$B \leq \sum_{k=0}^{l-1} \left[\binom{2l-1}{l+k} - \binom{2l-1}{l+k+1} \right]$$
$$= \binom{2l-1}{l}. \tag{3.9}$$

From (3.6), (3.8), (3.9) and the fact that

$$\binom{2l-1}{l} = \frac{1}{2}\binom{2l}{l},$$

so we get

$$u_{2l} \leq (pq)^l \left(\frac{p}{q}\right) \binom{2l-1}{l} + (pq)^l \left(\frac{p}{q}\right) \frac{q}{p}(p-q) \left(\frac{1}{pq}\right)^l$$
$$+ (pq)^l \left(\frac{q}{p}\right) \binom{2l-1}{l}$$
$$= (pq)^l \binom{2l-1}{l} \left(\frac{p}{q} + \frac{q}{p}\right) + (p-q)$$
$$= \frac{1}{2}(pq)^l \binom{2l}{l} \left(\frac{p}{q} + \frac{q}{p}\right) + (p-q)\,. \tag{3.10}$$

By Stirling's formula ([5], p.54):

$$\sqrt{2\pi} l^{l+\frac{1}{2}} e^{-l} e^{\frac{1}{12l+1}} \leq l! \leq \sqrt{2\pi} l^{l+\frac{1}{2}} e^{-l} e^{\frac{1}{12l}}, \tag{3.11}$$

we have

$$\binom{2l}{l} = \frac{(2l)!}{l!l!}$$
$$\leq \frac{\sqrt{2\pi}(2l)^{2l+\frac{1}{2}} e^{-2l} e^{\frac{1}{24l}}}{2\pi l^{l+\frac{1}{2}} e^{-l} e^{\frac{1}{12l+1}} l^{l+\frac{1}{2}} e^{-l} e^{\frac{1}{12l+1}}}$$
$$= \frac{4^l}{\sqrt{\pi l}} e^{\frac{1}{24l} - \frac{2}{12l+1}}$$
$$\leq \frac{4^l}{\sqrt{\pi l}}$$

where we have used the fact that $\dfrac{1}{24l} \le \dfrac{2}{12l+1}$ in the last inequality.
By (3.10) and this fact, we get

$$\begin{aligned} u_{2l} &\le (p-q) + \frac{1}{2\sqrt{\pi l}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^l \\ &\le (p-q) + \frac{1}{\sqrt{2\pi l}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^l \\ &= (p-q) + \triangle_{2l,p}. \end{aligned} \tag{3.12}$$

By (3.7), we get that

$$\begin{aligned} A &\ge \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\sum_{k=1}^{l-1}\binom{2l-1}{l+k}\left(\frac{p}{q}\right)^{l+k} \\ &= \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\left[\sum_{k=0}^{2l-1}\binom{2l-1}{k}\left(\frac{p}{q}\right)^{k} - \sum_{k=0}^{l}\binom{2l-1}{k}\left(\frac{p}{q}\right)^{k}\right] \\ &= \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\left[\left(1+\frac{p}{q}\right)^{2l-1} - \sum_{k=0}^{l}\binom{2l-1}{k}\left(\frac{p}{q}\right)^{k}\right]. \end{aligned}$$

Since

$$\binom{2l-1}{k} \le \binom{2l-1}{l} \qquad \text{for} \qquad k = 0, 1, \ldots, l \qquad ([4], \text{p.181}), \tag{3.13}$$

so we get

$$\begin{aligned} A &\ge \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\left[\left(1+\frac{p}{q}\right)^{2l-1} - \sum_{k=0}^{l}\binom{2l-1}{l}\left(\frac{p}{q}\right)^{k}\right] \\ &= \left(\frac{p}{q}-1\right)\left(\frac{q}{p}\right)^{l+1}\left[\left(\frac{1}{q}\right)^{2l-1} - \binom{2l-1}{l}\frac{\left(\frac{p}{q}\right)^{l+1}-1}{\left(\frac{p}{q}\right)-1}\right] \\ &= \left(\frac{p-q}{q}\right)\left(\frac{q}{p}\right)^{l+1}\left[\left(\frac{1}{q}\right)^{2l-1} - \binom{2l-1}{l}\left(\frac{q}{p-q}\right)\left(\left(\frac{p}{q}\right)^{l+1}-1\right)\right]. \end{aligned} \tag{3.14}$$

By (3.13),

$$B \geq 0. \tag{3.15}$$

By (3.6), (3.14), (3.15) and the fact that

$$\binom{2l-v}{l} \leq \frac{2^{2l-v}\sqrt{2}}{\sqrt{\pi l}} \quad \text{for} \quad v = 1, 2, \ldots, l \tag{3.16}$$

([4], p. 181),

$$\begin{aligned}
u_{2l} &\geq (pq)^l \left(\frac{p}{q}\right)\left(\frac{p-q}{q}\right)\left(\frac{q}{p}\right)^{l+1}\left[\left(\frac{1}{q}\right)^{2l-1} - \frac{2^{2l-1}\sqrt{2}}{\sqrt{\pi l}}\left(\frac{q}{p-q}\right)\left(\left(\frac{p}{q}\right)^{l+1} - 1\right)\right] \\
&= (p-q) - q^{2l}\frac{2^{2l-1}\sqrt{2}}{\sqrt{\pi l}}\left(\left(\frac{p}{q}\right)^{l+1} - 1\right) \\
&\geq (p-q) - q^{2l}\frac{2^{2l-1}\sqrt{2}}{\sqrt{\pi l}}\left(\frac{p}{q}\right)^{l}\left(\frac{p}{q} - 1\right) \\
&= (p-q) - \frac{1}{\sqrt{2\pi l}}\left(\frac{p}{q} - 1\right)(4pq)^l \\
&\geq (p-q) - \frac{1}{\sqrt{2\pi l}}\left(\frac{p}{q} + \frac{q}{p}\right)(4pq)^l \\
&= (p-q) - \triangle_{2l,p}.
\end{aligned} \tag{3.17}$$

By (3.12) and (3.17), we follow that

$$|u_{2l} - (p-q)| \leq \triangle_{2l,p} \quad \text{for} \quad l = 1, 2, \ldots, m. \tag{3.18}$$

Similarly, if $p < q$, we can get that

$$|u_{2l} - (q-p)| \leq \triangle_{2l,p} \quad \text{for} \quad l = 1, 2, \ldots, m. \tag{3.19}$$

Hence, by (3.18) and (3.19),

$$|u_{2l} - |p-q|| \leq \triangle_{2l,p} \quad \text{for} \quad l = 1, 2, \ldots, m. \tag{3.20}$$

By Theorem 3.1 (i), we get that

$$|P(K_{n,p}=0)-|p-q||\leq \triangle_{n,p}.$$

□

Theorem 3.3. *Let $n=2m$, $r=1,2,\ldots,m$ and $q=1-p$.*

$$|P(K_{n,p}=r)-(2pq)^r(p-q)|\leq \triangle_{n,p,r}$$

where

$$\triangle_{n,p,r}=\frac{\sqrt{2}\,(p-q)}{\sqrt{\pi r}(1-4pq)}(4pq)^r+\left(\frac{\sqrt{2}}{\sqrt{\pi m}}+\frac{1}{\pi}\left(\frac{p}{q}+\frac{q}{p}\right)\frac{m-r}{\sqrt{r}}\right)(4pq)^m.$$

Proof. By Theorem 3.1 (ii) and (3.12),

$$\begin{aligned}
&P(K_{n,p}=r)\\
&=2^r\sum_{l=0}^{m-r-1}\frac{r}{r+2l}\binom{r+2l}{r+l}(pq)^{r+l}u_{2(m-r-l)}+\frac{r}{2m-r}\binom{2m-r}{m}2^r(pq)^m\\
&\leq 2^r\sum_{l=0}^{m-r-1}\frac{r}{r+2l}\binom{r+2l}{r+l}(pq)^{r+l}\left(\frac{1}{\sqrt{2\pi(m-r-l)}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^{m-r-l}+(p-q)\right)\\
&\quad+\frac{r}{2m-r}\binom{2m-r}{m}2^r(pq)^m\\
&=2^r\sum_{l=0}^{m-r-1}\frac{r}{r+2l}\binom{r+2l}{r+l}(pq)^{r+l}\left(\frac{1}{\sqrt{2\pi(m-r-l)}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^{m-r-l}\right)\\
&\quad+2^r\sum_{l=0}^{m-r-1}\frac{r}{r+2l}\binom{r+2l}{r+l}(pq)^{r+l}(p-q)+\frac{r}{2m-r}\binom{2m-r}{m}2^r(pq)^m\\
&=A_1+A_2+A_3 \qquad (3.21)
\end{aligned}$$

where

$$A_1 = 2^r \sum_{l=0}^{m-r-1} \frac{r}{r+2l} \binom{r+2l}{r+l} (pq)^{r+l} \left(\frac{1}{\sqrt{2\pi(m-r-l)}} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^{m-r-l} \right),$$

$$A_2 = 2^r \sum_{l=0}^{m-r-1} \frac{r}{r+2l} \binom{r+2l}{r+l} (pq)^{r+l} (p-q),$$

$$A_3 = \frac{r}{2m-r} \binom{2m-r}{m} 2^r (pq)^m.$$

By (3.16), we have

$$\begin{aligned} A_1 &\leq 2^r \left(\frac{p}{q} + \frac{q}{p} \right) \sum_{l=0}^{m-r-1} \frac{r}{r+2l} \frac{2^{r+2l}\sqrt{2}}{\sqrt{\pi(r+l)}} (pq)^{r+l} \frac{1}{\sqrt{2\pi(m-r-l)}} (4pq)^{m-r-l} \\ &= \frac{1}{\pi} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^m \sum_{l=0}^{m-r-1} \frac{r}{(r+2l)\sqrt{(r+l)(m-r-l)}} \\ &\leq \frac{1}{\pi} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^m \sum_{l=0}^{m-r-1} \frac{r}{(r+2l)\sqrt{r+l}} \\ &\leq \frac{1}{\pi} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^m \sum_{l=0}^{m-r-1} \frac{r}{r\sqrt{r}} \\ &= \frac{1}{\pi} \left(\frac{p}{q} + \frac{q}{p} \right) (4pq)^m \frac{m-r}{\sqrt{r}}. \end{aligned} \tag{3.22}$$

By (3.16), we get

$$\begin{aligned} A_2 &= (2pq)^r (p-q) + 2^r \sum_{l=1}^{m-r-1} \frac{r}{r+2l} \binom{r+2l}{r+l} (pq)^{r+l} (p-q) \\ &\leq (2pq)^r (p-q) + 2^r \sum_{l=1}^{m-r-1} \frac{r}{r+2l} \frac{2^{r+2l}\sqrt{2}}{\sqrt{\pi(r+l)}} (pq)^{r+l} (p-q) \\ &= (2pq)^r (p-q) + \frac{\sqrt{2}r(4pq)^r}{\sqrt{\pi}} (p-q) \sum_{l=1}^{m-r-1} \frac{1}{(r+2l)\sqrt{r+l}} (4pq)^l \\ &\leq (2pq)^r (p-q) + \frac{\sqrt{2}r(4pq)^r(p-q)}{\sqrt{\pi}} \sum_{l=1}^{m-r-1} \frac{1}{r\sqrt{r}} (4pq)^l \\ &= (2pq)^r (p-q) + \frac{\sqrt{2}(4pq)^r(p-q)}{\sqrt{\pi r}} \sum_{l=1}^{m-r-1} (4pq)^l \end{aligned}$$

$$\leq (2pq)^r(p-q) + \frac{\sqrt{2}(4pq)^r(p-q)}{\sqrt{\pi r}} \frac{1}{1-4pq}. \tag{3.23}$$

We next estimate A_3, by (3.16), we obtain

$$\begin{aligned} A_3 &\leq \frac{r}{2m-r} \frac{2^{2m-r}\sqrt{2}}{\sqrt{\pi m}} 2^r (pq)^m \\ &= \frac{r}{2m-r} \frac{\sqrt{2}}{\sqrt{\pi m}} (4pq)^m \\ &\leq \frac{\sqrt{2}}{\sqrt{\pi m}} (4pq)^m. \end{aligned} \tag{3.24}$$

Hence, by (3.21), (3.22), (3.23) and (3.24),

$$\begin{aligned} &P(K_{n,p} = r) \\ &\leq \frac{1}{\pi}\left(\frac{p}{q}+\frac{q}{p}\right)\frac{m-r}{\sqrt{r}}(4pq)^m + (2pq)^r(p-q) + \frac{\sqrt{2}(4pq)^r(p-q)}{\sqrt{\pi r}}\frac{1}{1-4pq} \\ &\quad + \frac{\sqrt{2}}{\sqrt{\pi m}}(4pq)^m \\ &= (2pq)^r(p-q) + \frac{\sqrt{2}\,(p-q)}{\sqrt{\pi r}(1-4pq)}(4pq)^r + \left(\frac{\sqrt{2}}{\sqrt{\pi m}} + \frac{1}{\pi}\left(\frac{p}{q}+\frac{q}{p}\right)\frac{m-r}{\sqrt{r}}\right)(4pq)^m. \end{aligned} \tag{3.25}$$

By Theorem 3.1 (ii) and (3.17),

$$\begin{aligned} &P(K_{n,p} = r) \\ &\geq 2^r \sum_{l=0}^{m-r-1} \frac{r}{r+2l}\binom{r+2l}{r+l}(pq)^{r+l}\left(p-q-\frac{1}{\sqrt{2\pi(m-r-l)}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^{m-r-l}\right) \\ &= (2pq)^r(p-q) - \frac{1}{2^r\sqrt{2\pi(m-r)}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^m \\ &\quad + 2^r \sum_{l=1}^{m-r-1} \frac{r}{r+2l}\binom{r+2l}{r+l}(pq)^{r+l}\left(p-q-\frac{1}{\sqrt{2\pi(m-r-l)}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^{m-r-l}\right) \\ &\geq (2pq)^r(p-q) - \frac{1}{2^r\sqrt{2\pi(m-r)}}\left(\frac{p}{q}+\frac{q}{p}\right)(4pq)^m \end{aligned}$$

$$\geq (2pq)^r(p-q) - \frac{\sqrt{2}\,(p-q)}{\sqrt{\pi r}(1-4pq)}(4pq)^r - \left(\frac{\sqrt{2}}{\sqrt{\pi m}} + \frac{1}{\pi}\left(\frac{p}{q}+\frac{q}{p}\right)\frac{m-r}{\sqrt{r}}\right)(4pq)^m. \tag{3.26}$$

By (3.25) and (3.26), we get that

$$|P(K_{n,p}=r) - (2pq)^r(p-q)| \leq \triangle_{n,p,r}$$

where

$$\triangle_{n,p,r} = \frac{\sqrt{2}\,(p-q)}{\sqrt{\pi r}(1-4pq)}(4pq)^r + \left(\frac{\sqrt{2}}{\sqrt{\pi m}} + \frac{1}{\pi}\left(\frac{p}{q}+\frac{q}{p}\right)\frac{m-r}{\sqrt{r}}\right)(4pq)^m.$$

□

3.3 Convergence to Half-normal Distribution

In case of symmetric, i.e., $p = q = \frac{1}{2}$, we know from [4] that

$$P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right) \to H(z) \quad \text{for} \quad z \geq 0$$

where H is a half-normal distribution defined by

$$H(z) = 2\Phi(z) - 1.$$

In this section we will show that $\dfrac{K_{n,p}}{\sqrt{n}}$ does not converge in distribution to H in case of asymmetric, i.e., $p \neq q$.

Theorem 3.4. $P\left(\dfrac{K_{n,p}}{\sqrt{n}} \leq 0\right)$ *does not converge to* $H(0)$ *for* $p \neq \dfrac{1}{2}$.

Proof. Since $H(0) = 2\Phi(0) - 1 = 0$,

$$\left|P\left(\frac{K_{n,p}}{\sqrt{n}} \leq 0\right) - H(0)\right| = P(K_{n,p}=0) \geq |p-q| - \triangle_{n,p}$$

where $\triangle_{n,p}$ be defined in Theorem 3.2.

If $P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right) \to H(0)$, then

$$0 < |p-q| \leq \lim_{n\to\infty}\left|P\left(\frac{K_{n,p}}{\sqrt{n}} \leq 0\right) - H(0)\right| + \lim_{n\to\infty} \triangle_{n,p} = 0$$

with is a contradiction.
Hence, $P\left(\frac{K_{n,p}}{\sqrt{n}} \leq 0\right)$ does not converge to $H(0)$ for $p \neq \frac{1}{2}$. □

Theorem 3.5. *For $z > 0$, $P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right)$ does not converge to $H(z)$ for $p > \Phi(z)$ or $p < 1 - \Phi(z)$.*

Proof. Note that

$$\begin{aligned} P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right) - H(z) &\geq P\left(K_{n,p} = 0\right) - H(z) \\ &\geq |p-q| - H(z) - \triangle_{n,p} \\ &= |2p-1| - (2\Phi(z)-1) - \triangle_{n,p} \end{aligned}$$

where $\triangle_{n,p}$ be defined in Theorem 3.2 and

$$\begin{aligned} |2p-1| - (2\Phi(z)-1) > 0 &\Longleftrightarrow |2p-1| > 2\Phi(z) - 1 \\ &\Longleftrightarrow 2p-1 > 2\Phi(z)-1 \quad \text{or} \quad 2p-1 < 1-2\Phi(z) \\ &\Longleftrightarrow p > \Phi(z) \quad \text{or} \quad p < 1-\Phi(z). \end{aligned}$$

Let p be such that $p > \Phi(z)$ or $p < 1 - \Phi(z)$.
Suppose that $\lim_{n\to\infty} P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right) = H(z)$.
Hence,

$$0 < |2p-1| - H(z) \leq \lim_{n\to\infty}\left|P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right) - H(z)\right| + \lim_{n\to\infty} \triangle_{n,p} = 0$$

which is contradiction. Then we conclude that $P\left(\frac{K_{n,p}}{\sqrt{n}} \leq z\right)$ does not converge to $H(z)$ for $p > \Phi(z)$ or $p < 1 - \Phi(z)$. □

CHAPTER IV

FUTURE RESEARCH

In this thesis, we investigate the statistic of random walk in 2 directions.
I) Find a non-uniform bound in half-normal approximation H for the number of return to the origin (K_n) in case of symmetric, i.e., $p = \frac{1}{2}$.
II) We find the probability mass function of $K_{n,p}$ in case of asymmetric random walk and give its bound and show that the distribution of $K_{n,p}$ does not converge to $H(z)$.

This work can be extended to other statistics, that is, the maximum value (M_n) and the number of sign changes (C_n) defined by

$$M_n = \max_{0 \le k \le n} S_k,$$
$$C_n = C_{2m+1} = |\{1 \le k \le 2m : S_{k-1} \cdot S_{k+1} = -1\}|$$

respectively.
We suggest 2 directions of future research.
Direction 1. we may investigate the non-uniform bound of M_n and C_n in case of symmetric random walk.

In our work, we gave the non-uniform bound of K_n in case of symmetric. The important tool is we have to bound EK_n^k by positive constant (depends on k). In order to be find this, we need the following lemma.

Lemma 4.1. *For all $g : [-1, m] \cap \mathbb{Z} \to \mathbb{R}$ such that $g(-1) = 0$,*

$$E\left[(2m - K_n + 1)(g(K_n) - g(K_n - 1)) - (K_n + 1)g(K_n)\right] = 0.$$

To give the non-uniform bounds of M_n and C_n, one can use our idea with following lemmas.

Lemma 4.2. *For all* $g : [-1, m] \cap \mathbb{Z} \to \mathbb{R}$ *such that* $g(-1) = 0$*,*

$$E\left[(m + M_n)(g(M_n) - g(M_n - 1)) - 2M_n g(M_n)\right] = 0.$$

Lemma 4.3. *For all* $g : [-1, m] \cap \mathbb{Z} \to \mathbb{R}$ *such that* $g(-1) = 0$*,*

$$E\left[(m + 1 + C_n)(g(C_n) - g(C_n - 1)) - 2(C_n + 1)g(C_n)\right] = 0.$$

Direction 2. We can follow argument in chapter III to show that the distribution of M_n and C_n do not converge to half-normal distribution in case of asymmetric random walk.

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