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REGULARITY OF SOME TRANSFORMATION SEMIGROUPS WITH RESTRICTED RANGE

Miss Thanaporn Sumalroj

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By	Miss Thanaporn Sumalroj
Field of Study	Mathematics
Thesis Advisor	Teeraphong Phongpattanacharoen, Ph.D.
Thesis Co-advisor	Assistant Professor Sureeporn Chaopraknoi, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree.

.... Dean of the Faculty of Science

(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

.... Chairman

(Assistant Professor Sajee Pianskool, Ph.D.)

.... Thesis Advisor

(Teeraphong Phongpattanacharoen, Ph.D.)

.... Thesis Co-advisor

(Assistant Professor Sureeporn Chaopraknoi, Ph.D.)

.... Examiner

(Associate Professor Amorn Wasanawichit, Ph.D.)

.... External Examiner

(Nissara Sirasuntorn, Ph.D.)

ธนพร สุมาลย์โรจน์ : ความเป็นปกติของกึ่งกรุปการแปลงบางชนิดที่เรนจ์ถูกกำกัด (REGULARITY OF SOME TRANSFORMATION SEMIGROUPS WITH RESTRICTED RANGE) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ดร.ธีรพงษ์ พงษ์พัฒนเจริญ, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม: ผศ. ดร.สุรีย์พร ชาวแพรกน้อย, 35 หน้า.

กำหนดให้ X เป็นเซตไม่ว่าง และ $T(X)$ เป็นกึ่งกรุปการแปลงเต็มบน X เราให้ $AM(X)$ แทนเซตของการแปลงเกือบหนึ่งต่อหนึ่งบน X และ $AE(X)$ แทนเซตของการแปลงเกือบทั่วถึงบน X นอกจากนั้น เราให้ $OM(X)$ และ $OE(X)$ คือส่วนเติมเต็มของเซต $AM(X)$ และ $AE(X)$ ใน $T(X)$ ตามลำดับ เป็นที่รู้กันว่าทั้ง $AM(X)$ และ $AE(X)$ เป็นกึ่งกรุปปกติภายใต้เงื่อนไขบางประการ แต่ $OM(X)$ และ $OE(X)$ ไม่เป็นกึ่งกรุปปกติ ในวิทยานิพนธ์นี้ ได้แนะนำรูปแบบทั่วไปของกึ่งกรุป $T(X)$ และกึ่งกรุปย่อยของ $T(X)$ บางตัว ให้ Y คือเซตย่อยไม่ว่างของ X เรานิยาม $T(X,Y)$ คือเซตของการแปลงบน X ที่เรนจ์เป็นเซตย่อยของ Y ในทำนองเดียวกัน เรามีรูปแบบทั่วไปของ $AM(X)$, $AE(X)$, $OM(X)$ และ $OE(X)$ คือ $AM(X,Y) = AM(X) \cap T(X,Y)$, $AE(X,Y) = AE(X) \cap T(X,Y)$, $OM(X,Y) = OM(X) \cap T(X,Y)$ และ $OE(X,Y) = OE(X) \cap T(X,Y)$ จุดประสงค์หลักของวิทยานิพนธ์นี้คือ เพื่อศึกษาความเป็นปกติของกึ่งกรุปเหล่านี้ และกึ่งกรุปการแปลงเชิงเส้นที่เป็นคู่ขนานกับกึ่งกรุปเหล่านี้

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ปีการศึกษา 2559	ลายมือชื่อ อ.ที่ปรึกษาร่วม

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Let X be a nonempty set and $T(X)$ the full transformation semigroup on X. We denote by $AM(X)$ the set of almost one-to-one transformations on X and $AE(X)$ the set of almost onto transformations on X. Also, we define $OM(X)$ and $OE(X)$ to be the complement of the set $AM(X)$ and $AE(X)$ in $T(X)$, respectively. It is known that $AM(X)$ and $AE(X)$ are regular semigroups under certain conditions, but $OM(X)$ and $OE(X)$ are not regular semigroups. In this thesis, some generalisations of $T(X)$ and its subsemigroups are introduced. Let Y be a nonempty subset of X. We define $T(X,Y)$ to be the set of transformations on X whose range is a subset of Y. Likewise, we have a generalisation of $AM(X)$, $AE(X)$, $OM(X)$ and $OE(X)$, namely, $AM(X,Y) = AM(X) \cap T(X,Y)$, $AE(X,Y) = AE(X) \cap T(X,Y)$, $OM(X,Y) = OM(X) \cap T(X,Y)$ and $OE(X,Y) = OE(X) \cap T(X,Y)$. Our thesis is devoted to the study of regularity of these semigroups, and their parallels linear transformation semigroups.

Department : ...Mathematics and.....	Student's Signature : ....
...Computer Science...	Advisor's Signature : ....
Field of Study : ......Mathematics.....	Co-advisor's Signature : ....
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CONTENTS

CHAPTER I
INTRODUCTION AND PRELIMINARIES

Let $(S, \cdot)$ be a system consisting of a nonempty set S with binary operation $\cdot$ on S. If $(S, \cdot)$ satisfies the associative law, i.e., $\forall a, b, c \in S, (a \cdot b) \cdot c = a \cdot (b \cdot c)$, we say that $(S, \cdot)$ is a *semigroup*. For convenience, we write S for a semigroup $(S, \cdot)$ and ab for $a \cdot b$ where $a, b \in S$. For a semigroup S, we call an element a in S *regular* if there exists an element x in S such that $a = axa$. If every element in S is regular, then S is called a *regular semigroup*.

In 1951, J.A. Green introduced regular semigroup in his paper "On the structure of semigroups"; this was also the paper in which Green's relations were introduced. In semigroup theory, regular semigroups are very familiar and are one of the most extensively studied of semigroups.

A significant benefit of regularity can be found in the study of Green's relations and the natural partial order, which are important relations in semigroup theory. The relation between Green's relations and regular semigroups are difficult to be briefly mentioned here. However, we describe the relation between the natural partial order and regular semigroups.

The natural partial order $\leq$ on a semigroup S is defined by $a \leq b$ if and only if $a = xb = by$ and $a = ay$ for some $x, y \in S^1$ where S^1 is the semigroup S if S contains an identity; otherwise S^1 is the semigroup obtained from S by adjoining a new symbol 1 as its identity. It is known that any semigroup endowed with the natural partial order hands down the order to its regular subsemigroups.

Theorem 1.1. [2] *If T is a regular subsemigroup of a semigroup S and $a, b \in T$. Then $a \leq b$ on T if and only if $a \leq b$ on S.*

Moreover, there are many researches about regularity of transformation semigroups. For example, Y. Kemprasit studied regularity of generalized semigroups of linear transformations in [4] and studied regular elements of some transformation semigroups in [5]. The main purpose of this thesis is to investigate the regularity of certain transformation semigroups with restricted range.

In the rest of this chapter, we give precise definitions, notations and fundamental results which will be used throughout this thesis. We separate this chapter into two sections. The first section is to introduce necessary background and basic results of transformation semigroups, and the other section is to give definitions, notations and results of linear transformation semigroups, and provide some results needed in this thesis.

1.1 Transformation semigroups

Given a nonempty set X, the *full transformation semigroup on* X means the set of transformations on X, denoted by $T(X)$. That is,

$$T(X) = \{\alpha : \alpha \text{ is a function on } X\}.$$

Y. Kemprasit showed in [3, p. 109] that $T(X)$ is a regular semigroup under composition.

In this thesis, all maps are written on the right of the argument. For $\alpha \in T(X)$, the range of α is denoted by $\operatorname{ran}\alpha$, and the inverse relation of α is denoted by α^{-1}. Also, the inverse image of x under α is written by $x\alpha^{-1}$. Furthermore, let 1_X be the identity map on X and let $|X|$ be the cardinality of X.

For any transformation $\alpha \in T(X)$ and $x \in X$, α is said to be *one-to-one at* x if $|x\alpha\alpha^{-1}| = 1$. If $\{x \in X : |x\alpha\alpha^{-1}| > 1\}$ is finite, then α is called *almost one-to-one*. A transformation α in $T(X)$ is called *almost onto* if $X \smallsetminus \operatorname{ran}\alpha$ is finite. Then, a transformation α in $T(X)$ is one-to-one if and only if α is one-to-one at x for all $x \in X$. Moreover, every injection and surjection are almost one-to-one and almost onto, respectively. But its converse is not true; see Example 1.2 (*iii*). In this thesis, we study the regularity of a generalisation of the following transformation semigroups.

For a nonempty set X, let $AM(X)$ be the set of almost one-to-one transformations on X and $AE(X)$ the set of almost onto transformations on X, that is,

$$AM(X) = \big\{\alpha \in T(X) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is finite}\big\} \text{ and}$$

$$AE(X) = \big\{\alpha \in T(X) : X \smallsetminus \operatorname{ran}\alpha \text{ is finite}\big\}.$$

Both $AM(X)$ and $AE(X)$ are subsemigroups of $T(X)$ [3, p. 133], known as the *almost one-to-one transformation semigroup on* X and the *almost onto transformation semigroup on* X, respectively.

Example 1.2. (i) Every injection on a nonempty set X is contained in $AM(X)$.
(ii) Every surjection on a nonempty set X is contained in $AE(X)$.
(iii) Let $\mathbb{N}$ be the set of natural numbers. We define $\mu : \mathbb{N} \to \mathbb{N}$ by

$$x\mu = \begin{cases} 2 & \text{if } x = 1, \\ x & \text{otherwise.} \end{cases}$$

Then $2\mu^{-1} = \{1, 2\}$ and $x\mu^{-1} = \{x\}$ for all $x \in \mathbb{N} \smallsetminus \{2\}$. So $\{x \in \mathbb{N} : |x\mu\mu^{-1}| > 1\}$ $=\{1, 2\}$, and hence $\mu \in AM(\mathbb{N})$. Clearly, $\operatorname{ran}\mu = \mathbb{N} \smallsetminus \{1\}$, so $\mathbb{N} \smallsetminus \operatorname{ran}\mu = \{1\}$. Hence $\mu \in AE(\mathbb{N})$. But μ is neither injective nor surjective.

Note that if X is finite then $AM(X) = T(X) = AE(X)$, so it is regular. Actually, this is the only case for $AM(X)$ and also $AE(X)$ to be regular.

Theorem 1.3. [3, p. 133] *Let X be a nonempty set. The following statements are equivalent:*
(i) *X is finite,*
(ii) *$AM(X)$ is regular,*
(iii) *$AE(X)$ is regular.*

Next, for an infinite set X, define

$$OM(X) = \{\alpha \in T(X) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is infinite}\} \text{ and}$$

$$OE(X) = \{\alpha \in T(X) : X \smallsetminus \operatorname{ran}\alpha \text{ is infinite}\}.$$

Clearly, both are subsemigroups of $T(X)$, known as the *opposite semigroup of one-to-one transformation semigroup on* X and the *opposite semigroup of onto transformation semigroup on* X, respectively. These semigroups are intensively studied in [3].

Example 1.4. (i) Every constant map on an infinite set X is an element in $OM(X)$ and $OE(X)$.

(ii) Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}^+$ the set of positive integers and $\mathbb{Z}^-$ the set of negative integers. We define $\lambda : \mathbb{Z} \to \mathbb{Z}$ by

$$x\lambda = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $0\lambda^{-1} = \mathbb{Z}^- \cup \{0\}$ and $x\lambda^{-1} = \{x\}$ for all $x \in \mathbb{Z}^+$. So $\{x \in \mathbb{Z} : |x\lambda\lambda^{-1}| > 1\} = \mathbb{Z}^- \cup \{0\}$. Hence $\lambda \in OM(\mathbb{Z})$. Clearly, $\operatorname{ran}\lambda = \mathbb{Z}^+ \cup \{0\}$. Thus $\mathbb{Z} \smallsetminus \operatorname{ran}\lambda = \mathbb{Z}^-$, so $\lambda \in OE(\mathbb{Z})$.

From Theorem 1.3, there is a chance that $AM(X)$ and $AE(X)$ are regular semigroups. But the story becomes different in $OM(X)$ and $OE(X)$.

Theorem 1.5. [3, p. 135] *$OM(X)$ and $OE(X)$ are not regular.*

Now, we introduce a generalisation of the full transformation semigroup $T(X)$ on X. For a nonempty subset Y of X, let $T(X, Y)$ be the set of all transformations on X whose range is in Y, that is,

$$T(X, Y) = \{\alpha \in T(X) : \operatorname{ran}\alpha \subseteq Y\}.$$

This semigroup is studied in [5] and one can see that it is a subsemigroup of $T(X)$ and $T(X, X) = T(X)$. Then we may regard $T(X, Y)$ as a generalisation of $T(X)$.

We call $T(X,Y)$ the *full transformation semigroup on X with restricted range Y*. Clearly if $Y = X$ or $|Y| = 1$ then $T(X,Y)$ is regular. In addition, Y. Kemprasit et al. showed that it fails in other cases.

Theorem 1.6. [5] *For a set X and its nonempty subset Y, $T(X,Y)$ is regular if and only if $|Y| = 1$ or $Y = X$.*

We next introduce a generalisation of $AM(X)$. For a nonempty subset Y of X, we mean by $AM(X,Y)$ the set of all elements in $AM(X)$ whose range is contained in Y. That is,

$$AM(X,Y) = \big\{\alpha \in T(X,Y) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is finite}\big\}.$$

Likewise, we have a generalisation of $AE(X)$, defined by

$$AE(X,Y) = \big\{\alpha \in T(X,Y) : X \setminus \operatorname{ran}\alpha \text{ is finite}\big\}.$$

It is easy to see that $AM(X,Y) = T(X,Y) = AE(X,Y)$ when X is finite. Furthermore, if $Y = X$ then $AM(X,Y) = AM(X)$ and $AE(X,Y) = AE(X)$.

Note that $AM(X,Y) = T(X,Y) \cap AM(X)$ and $AE(X,Y) = T(X,Y) \cap AE(X)$. We notice that there is an occasion that $AM(X,Y)$ or $AE(X,Y)$ becomes the empty set, and we will discuss about this in the next chapter. If this is not the case, $AM(X,Y)$ and $AE(X,Y)$ are semigroups, called the *almost one-to-one transformation semigroup on X with restricted range Y* and the *almost onto transformation semigroup on X with restricted range Y*, respectively. Moreover, we show in Proposition 2.8 that these two semigroups are different under some conditions.

In the case that X is an infinite set and Y is a nonempty subset of X, we define

$$\begin{aligned} OM(X,Y) &= \big\{\alpha \in T(X,Y) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is infinite}\big\} \text{ and} \\ OE(X,Y) &= \big\{\alpha \in T(X,Y) : X \setminus \operatorname{ran}\alpha \text{ is infinite}\big\}. \end{aligned}$$

Obviously, $OM(X,Y)$ and $OE(X,Y)$ are not empty, containing all constant maps. Also, it is clear that both are semigroups as $OM(X,Y) = T(X,Y) \cap OM(X)$ and $OE(X,Y) = T(X,Y) \cap OE(X)$. We call $OM(X,Y)$ the *opposite semigroup of one-to-one transformation semigroup on X with restricted range Y* and $OE(X,Y)$ the *opposite semigroup of onto transformation semigroup on X with restricted range Y.* Clearly, $OM(X,Y)$ and $OE(X,Y)$ can be considered as generalisations of $OM(X)$ and $OE(X)$, respectively.

In Chapter II, we intensively study these semigroups; examples and characterisations of regularity are provided.

1.2 Linear transformation semigroups

Let V be a vector space over a division ring and $\mathcal{L}(V)$ the set of all linear transformations on V. Under composition $\mathcal{L}(V)$ is a regular semigroup [3, p. 145], known as the *full linear transformation semigroup on V.*

Throughout this thesis, we denote by $\dim(V)$ the dimension of a vector space V. For any subset A of a vector space V, the subspace spanned by A is denoted by $\langle A\rangle$. For a vector space V and a subspace W of V, we let V/W be the quotient space of V and W. For $\alpha \in \mathcal{L}(V)$, the kernel of linear transformation α is denoted by $\ker\alpha$, and α is said to be *almost one-to-one* if $\dim(\ker\alpha) < \infty$, and we call α *almost onto* if $\dim(V/\operatorname{ran}\alpha) < \infty$. In this thesis, we study the regularity of a generalisation of the following linear transformation semigroups.

For a vector space V over a division ring, let

$$\mathcal{AM}(V) = \{\alpha \in \mathcal{L}(V) : \dim(\ker\alpha) < \infty\} \text{ and}$$

$$\mathcal{AE}(V) = \{\alpha \in \mathcal{L}(V) : \dim(V/\operatorname{ran}\alpha) < \infty\}.$$

In [3], the author showed that these are subsemigroups of $\mathcal{L}(V)$, called the *almost one-to-one linear transformation semigroup on* V and the *almost onto linear transformation semigroup on V*, respectively.

Example 1.7. (i) Every monomorphism on a vector space V belongs to $\mathcal{AM}(V)$. (ii) Every epimorphism on a vector space V is contained in $\mathcal{AE}(V)$.

Note that if $\dim(V) < \infty$ then $\mathcal{AM}(V) = \mathcal{L}(V) = \mathcal{AE}(V)$. Y. Kemprasit showed that $\mathcal{AM}(V)$ and $\mathcal{AE}(V)$ are regular semigroups under a certain conditions.

Theorem 1.8. [3, p. 168] *Let V be a vector space over a division ring. The following statements are equivalent:*
(i) $\dim(V) < \infty$,
(ii) $\mathcal{AM}(V)$ *is regular,*
(iii) $\mathcal{AE}(V)$ *is regular.*

Let V be an infinite dimensional vector space over a division ring and let

$$\begin{aligned}\mathcal{OM}(V) &= \{\alpha \in \mathcal{L}(V) : \dim(\ker\alpha) \text{ is infinite}\} \text{ and}\\ \mathcal{OE}(V) &= \{\alpha \in \mathcal{L}(V) : \dim(V/\operatorname{ran}\alpha) \text{ is infinite}\},\end{aligned}$$

which have been defined and proved in [3, p. 170] that they are subsemigroups of $\mathcal{L}(V)$, called the *opposite semigroup of one-to-one linear transformation semigroup on V* and the *opposite semigroup of onto linear transformation semigroup on V*, respectively; however, we do not worry about the regularity of $\mathcal{OM}(V)$ and $\mathcal{OE}(V)$.

Theorem 1.9. [3, p. 171] $\mathcal{OM}(V)$ *and* $\mathcal{OE}(V)$ *are not regular.*

We now introduce a generalisation of the full linear transformation semigroup $\mathcal{L}(V)$ on V. Given a subspace W of V, we let

$$\mathcal{L}(V, W) = \{\alpha \in \mathcal{L}(V) : \operatorname{ran}\alpha \subseteq W\}.$$

Then $\mathcal{L}(V, W)$ is a subsemigroup of $\mathcal{L}(V)$. Clearly, $\mathcal{L}(V, W) = \mathcal{L}(V)$ when $W = V$.

Throughout this thesis, we let 0 be the zero element in a vector space V over a division ring, that is, $u + 0 = u$ for all $u \in V$. The proposition below is a direct consequence of Theorem 2.2 in [4]. For the sake of completeness, we provide the reader with a proof.

Proposition 1.10. *Let V be a vector space over a division ring and W a subspace of V. Then $\mathcal{L}(V, W)$ is regular if and only if $V = \{0\}$ or $W = \{0\}$ or $W = V$.*

Proof. Clearly, if $V = \{0\}$ or $W = \{0\}$ then $\mathcal{L}(V, W)$ is a singleton of the zero map, and hence $\mathcal{L}(V, W)$ is regular. In case $W = V$, we have $\mathcal{L}(V, W) = \mathcal{L}(V)$, which is done.

To prove the necessity, by contrapositive, suppose that $V \neq \{0\}$, $W \neq \{0\}$ and $W \neq V$. Since $W \neq \{0\}$, it contains a nonzero vector, say w. Let B_1 be a basis of W and B_2 a basis of V such that $B_1 \subseteq B_2$. Since $W \neq V$, $B_2 \setminus B_1$ is not empty. Let $\alpha : V \to W$ be a linear transformation defined by

$$v\alpha = \begin{cases} 0 & \text{if } v \in B_1, \\ w & \text{if } v \in B_2 \setminus B_1. \end{cases}$$

Let $\beta \in \mathcal{L}(V, W)$ and $v \in B_2 \setminus B_1$. Then $v\alpha\beta\alpha = w\beta\alpha \in \langle B_1 \rangle \alpha = \{0\}$ and $v\alpha = w$. Thus $\alpha\beta\alpha \neq \alpha$, and hence α is not regular. □

Next, we introduce generalisations of $\mathcal{AM}(V)$ and $\mathcal{AE}(V)$. For a subspace W of V, by $\mathcal{AM}(V, W)$ we mean the set of all elements in $\mathcal{AM}(V)$ whose range is in W and $\mathcal{AE}(V, W)$ the set of all elements in $\mathcal{AE}(V)$ whose range is in W. That is,

$$\mathcal{AM}(V, W) = \{\alpha \in \mathcal{L}(V, W) : \dim(\ker \alpha) < \infty\} \text{ and}$$

$$\mathcal{AE}(V, W) = \{\alpha \in \mathcal{L}(V, W) : \dim(V/\operatorname{ran} \alpha) < \infty\}.$$

It is clear that if $\dim(V)$ is finite, then $\mathcal{AM}(V, W) = \mathcal{L}(V, W) = \mathcal{AE}(V, W)$. Moreover, if $W = V$ then $\mathcal{AM}(V, W) = \mathcal{AM}(V)$ and $\mathcal{AE}(V, W) = \mathcal{AE}(V)$. Notice that $\mathcal{AM}(V, W) = \mathcal{L}(V, W) \cap \mathcal{AM}(V)$ and $\mathcal{AE}(V, W) = \mathcal{L}(V, W) \cap \mathcal{AE}(V)$. We need to be aware that $\mathcal{AM}(V, W)$ and $\mathcal{AE}(V, W)$ are possibly empty. When this is not the case, we call $\mathcal{AM}(V, W)$ the *almost one-to-one linear transformation semigroup on V with restricted range W* and $\mathcal{AE}(V, W)$ the *almost onto linear transformation semigroup on V with restricted range W*.

When V is an infinite dimensional vector space and W is a subspace of V, we let

$$\mathcal{OM}(V, W) = \{\alpha \in \mathcal{L}(V, W) : \dim(\ker \alpha) \text{ is infinite}\} \text{ and}$$

$$\mathcal{OE}(V, W) = \{\alpha \in \mathcal{L}(V, W) : \dim(V/\operatorname{ran} \alpha) \text{ is infinite}\}.$$

Obviously, $\mathcal{OM}(V, W)$ and $\mathcal{OE}(V, W)$ are not empty, as they contain the zero map. Since $\mathcal{OM}(V, W) = \mathcal{L}(V, W) \cap \mathcal{OM}(V)$ and it is not empty, $\mathcal{OM}(V, W)$ is a semigroup. The set $\mathcal{OE}(V,W)$ can be considered similarly. We call $\mathcal{OM}(V,W)$ the *opposite semigroup of one-to-one linear transformation semigroup on* V *with restricted range* W and $\mathcal{OE}(V,W)$ the *opposite semigroup of onto linear transformation semigroup on* V *with restricted range* W. It is clear that if $W=V$ then $\mathcal{OM}(V,W)=\mathcal{OM}(V)$ and $\mathcal{OE}(V,W)=\mathcal{OE}(V)$.

Chapter III is devoted to the study of regularity of these linear transformation semigroups. In addition, we finish this present chapter with a list of background knowledge which is always used in this thesis.

Proposition 1.11. [3, p. 144] *Let* $\alpha \in \mathcal{L}(V)$. *If* B_1 *and* B_2 *are base of* $\ker \alpha$ *and* $\operatorname{ran} \alpha$, *respectively and for any* $v \in B_2$, $w_v \in v\alpha^{-1}$ *is fixed, then* $B_1 \cup \{w_v : v \in B_2\}$ *is a basis of* V.

Proposition 1.12. [3, p. 144] *Let* $\alpha \in \mathcal{L}(V)$. *If* U *is a subspace of* V, B_1 *is a basis of* U *and* B *is a basis of* V *with* $B_1 \subseteq B$, *then* $\dim(V/U) = |B \smallsetminus B_1|$.

CHAPTER II

REGULARITY OF TRANSFORMATION SEMIGROUPS

In this chapter, X is a nonempty set and Y is a nonempty subset of X. Our main purpose is to determine regularity of specific subsemigroups of the total transformation that are introduced in the previous chapter, namely,

$$AM(X,Y) = \big\{\alpha \in T(X,Y) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is finite}\big\},$$

$$AE(X,Y) \ = \big\{\alpha \in T(X,Y) : X \smallsetminus \operatorname{ran}\alpha \text{ is finite}\big\},$$

$$OM(X,Y) = \big\{\alpha \in T(X,Y) : \{x \in X : |x\alpha\alpha^{-1}| > 1\} \text{ is infinite}\big\},$$

$$OE(X,Y) \ = \big\{\alpha \in T(X,Y) : X \smallsetminus \operatorname{ran}\alpha \text{ is infinite}\big\},$$

and also $AM(X,Y) \cap AE(X,Y)$ and $OM(X,Y) \cap OE(X,Y)$.

Before that, we give a characterisation of regular elements in $T(X,Y)$, which is given by Y. Kemprasit in [5]. However, for convenience, we bring only a part of the statement of Theorem 2.1 in [5]. Actually, the proof we provide is different from the original one.

Theorem 2.1. [5] *For any transformation α in $T(X,Y)$, α is regular in $T(X,Y)$ if and only if $Y\alpha = \operatorname{ran}\alpha$.*

Proof. Let α be an element in $T(X,Y)$. First, we assume that α is a regular element in $T(X,Y)$. Then there exists a transformation β in $T(X,Y)$ such that $\alpha\beta\alpha = \alpha$. Thus $Y\alpha \subseteq \operatorname{ran}\alpha = X\alpha = X\alpha\beta\alpha = (X\alpha\beta)\alpha \subseteq Y\alpha$. Hence $Y\alpha = \operatorname{ran}\alpha$.

Conversely, we assume that $Y\alpha = \operatorname{ran}\alpha$. For each y in $\operatorname{ran}\alpha$ there is an element z_y in Y such that $z_y\alpha = y$. Let $b \in Y$ and define $\beta : X \to Y$ by

$$y\beta = \begin{cases} z_y & \text{if } y \in \operatorname{ran}\alpha, \\ b & \text{otherwise.} \end{cases}$$

For any $x \in X$, we have $x\alpha\beta\alpha = (x\alpha)\beta\alpha = z_{x\alpha}\alpha = x\alpha$, so $\alpha\beta\alpha = \alpha$. Hence α is regular in $T(X,Y)$. $\square$

Remark 2.2. Let $S(X,Y)$ be a subsemigroup of $T(X,Y)$ and $\alpha \in S(X,Y)$. If $Y\alpha \neq \operatorname{ran}\alpha$ then α is not regular in $S(X,Y)$.

The converse of Remark 2.2 is not true. That is, the condition $Y\alpha = \operatorname{ran}\alpha$ does not always imply that α is regular in $S(X,Y)$. Theorems 1.3, 1.5, 1.8 and 1.9 show that there exists an element α in S, when S is the semigroup $AM(X)$, $AE(X)$, $OM(X)$, $OE(X)$, $\mathcal{AM}(V)$, $\mathcal{AE}(V)$, $\mathcal{OM}(V)$ or $\mathcal{OE}(V)$, such that $X\alpha = \operatorname{ran}\alpha$ or $V\alpha = \operatorname{ran}\alpha$, but α is not regular in S.

However, some transformation semigroups satisfying the converse of Remark 2.2 are given in Corollaries 2.11 and 3.7.

Theorem 2.3. *If Y is a proper subset of a set X, then every injection in $T(X,Y)$ is not regular in $T(X,Y)$.*

Proof. Let α be an injection in $T(X,Y)$ where Y is a proper subset of X. Then $Y\alpha \subsetneq X\alpha = \operatorname{ran}\alpha$. By Theorem 2.1, α is not regular in $T(X,Y)$. $\square$

Remark 2.4. In case Y is a proper subset of a set X, every subsemigroup of $T(X,Y)$ containing an injection is not regular.

The next example shows that there exists a class of transformations in $T(X,Y)$ which are neither regular in $T(X,Y)$ nor injective, when Y is a proper subset of X with $|Y| \geq 2$.

Example 2.5. Let a and b be distinct elements in a proper subset Y of a set X. Define $\alpha \in T(X,Y)$ by

$$x\alpha = \begin{cases} a & \text{if } x \in Y, \\ b & \text{otherwise.} \end{cases}$$

Then α is not injective and $Y\alpha = \{a\} \neq \{a,b\} = \operatorname{ran}\alpha$. By Theorem 2.1, α is not regular in $T(X,Y)$.

2.1 Regularity of $AM(X, Y)$ and $AE(X, Y)$

As we mentioned before, there is a chance that $AM(X,Y)$ or $AE(X,Y)$ becomes the empty set. A condition that would help eliminate such a weak spot is in need.

Proposition 2.6. *Let X be an infinite set. Then*
(i) $AM(X,Y)$ is not the empty set if and only if $|X| = |Y|$,
(ii) $AE(X,Y)$ is not the empty set if and only if $X \smallsetminus Y$ is finite.

Proof. (i) We first assume that $AM(X,Y)$ is not the empty set. Then there exists a transformation α in $AM(X,Y)$. Since α is an element in $AM(X,Y)$, $\{x \in X : |x\alpha\alpha^{-1}| > 1\}$ is finite; hence for each $y \in \operatorname{ran}\alpha$, $y\alpha^{-1}$ is finite. Since $\{y\alpha^{-1} : y \in \operatorname{ran}\alpha\}$ is a partition of X, we have $X = \bigcup(y\alpha^{-1})$ where the union is taken over all y in $\operatorname{ran}\alpha$. Since X is an infinite set and $y\alpha^{-1}$ is a finite set for all $y \in \operatorname{ran}\alpha$, $\operatorname{ran}\alpha$ must be an infinite set with the same cardinality as X. Consequently, $|X| = |\operatorname{ran}\alpha| \leq |Y| \leq |X|$.

The other implication follows from the fact that if X and Y have the same cardinality, then there exists an injection from X to Y and it is clearly contained in $AM(X,Y)$.

(ii) Assume that $AE(X,Y)$ contains a transformation β. We have $X \smallsetminus Y$ is a subset of $X \smallsetminus \operatorname{ran}\beta$, which is finite since $\beta \in AE(X,Y)$. Therefore $X \smallsetminus Y$ is also finite.

For the sufficiency, we assume that $X \smallsetminus Y$ is finite. Since X is infinite and $X \smallsetminus Y$ is finite, Y is infinite and $|X| = |Y|$. Then we have a transformation from X onto Y and $AE(X,Y)$ contains this element. □

From Proposition 2.6, we have the following proposition.

Proposition 2.7. *Let X be an infinite set. Then*
(i) $AM(X,Y)$ is a semigroup if and only if $|X| = |Y|$,
(ii) $AE(X,Y)$ is a semigroup if and only if $X \smallsetminus Y$ is finite,
(iii) $AM(X,Y) \cap AE(X,Y)$ is a semigroup if and only if $X \smallsetminus Y$ is finite.

Proof. Note that the necessity of (i) and (ii) follow from Proposition 2.6.

(i) For the sufficiency, assume that $|X| = |Y|$. By Proposition 2.6, $AM(X,Y)$ is not empty. Then $AM(X,Y) = T(X,Y) \cap AM(X)$ is a subsemigroup of $T(X)$.

(ii) The sufficiency is obtained from Proposition 2.6 and the fact that $AE(X,Y)$ is $T(X,Y) \cap AE(X)$.

(iii) For the necessity, we assume that $AM(X,Y) \cap AE(X,Y)$ is a semigroup. Then $AM(X,Y) \cap AE(X,Y)$ is not empty. Hence $AE(X,Y)$ is not empty. By Proposition 2.6, $X \setminus Y$ is finite.

For the sufficiency, we assume that $X \setminus Y$ is finite. Since $X \setminus Y$ is finite and X is infinite, $|X| = |Y|$. Thus there exists a bijection from X to Y, which is contained in both $AM(X,Y)$ and $AE(X,Y)$. □

Proposition 2.8. *Given semigroups $AM(X,Y)$ and $AE(X,Y)$, if X is infinite, then neither $AM(X,Y) \setminus AE(X,Y)$ nor $AE(X,Y) \setminus AM(X,Y)$ is the empty set.*

Proof. Assume that X is infinite. Since $AM(X,Y)$ is a semigroup, by Proposition 2.7 (i), $|X| = |Y|$. Since $AE(X,Y)$ is a semigroup, by Proposition 2.7 (ii), $X \setminus Y$ is finite, which implies that $|X| = |Y|$. In either case, we have $|X| = |Y|$. Since Y is infinite, there exists an infinite subset Z of Y with $|Y| = |Z| = |Y \setminus Z|$. Choose $z \in Z$. Provided two bijections $\varphi : Y \to Y \setminus Z$ and $\psi : Z \to Y \setminus \{z\}$, we define $\alpha, \beta \in T(X,Y)$ by

$$x\alpha = \begin{cases} x\varphi & \text{if } x \in Y, \\ z & \text{otherwise}, \end{cases}$$

and

$$x\beta = \begin{cases} x\psi & \text{if } x \in Z, \\ z & \text{otherwise}. \end{cases}$$

First, we show that $\alpha \in AM(X,Y) \setminus AE(X,Y)$. We have $\operatorname{ran}\alpha = (Y \setminus Z) \cup \{z\}$. Thus $X \setminus \operatorname{ran}\alpha = [X \setminus (Y \setminus Z)] \setminus \{z\} = (X \setminus Y) \cup (Z \setminus \{z\})$. Since $Z \setminus \{z\}$ is infinite, $X \setminus \operatorname{ran}\alpha$ is infinite. Hence α is not in $AE(X,Y)$. Now, we have that $\{x \in X : |x\alpha\alpha^{-1}| > 1\} = X \setminus Y$, so $\{x \in X : |x\alpha\alpha^{-1}| > 1\}$ is finite, as $X \setminus Y$ is finite. Thus α belongs to $AM(X,Y)$.

Next, we show that β is in $AE(X,Y) \smallsetminus AM(X,Y)$. Clearly, $\operatorname{ran}\beta = Y$, so $X \smallsetminus \operatorname{ran}\beta = X \smallsetminus Y$, which is finite. Therefore β is in $AE(X,Y)$. We have $\{x \in X : |x\beta\beta^{-1}| > 1\} = X \smallsetminus Z \supseteq Y \smallsetminus Z$, which is infinite. Thus β is not in $AM(X,Y)$. □

If X is finite, we have $AM(X,Y) = T(X,Y) = AE(X,Y)$; otherwise, these semigroups are different. From these results we have $AM(X,Y) = AE(X,Y)$ if and only if X is finite.

For the semigroups $AM(X,Y)$ and $AE(X,Y)$, when $|Y| = 1$, these semigroups are the same and their unique element is a constant map. In this case, they are regular semigroups. For the other cases we have:

Theorem 2.9. *Let $S(X,Y)$ with $|Y| \geq 2$ be either the semigroup $AM(X,Y)$ or the semigroup $AE(X,Y)$. Then $S(X,Y)$ is regular if and only if X is finite and $Y = X$.*

Proof. The sufficiency is directly obtained from Theorem 1.3. To prove the necessity, by contrapositive, suppose that X is infinite or $Y \neq X$. We divide the situation into three cases.

Case 1: X is finite and $Y \neq X$. Then $S(X,Y) = T(X,Y)$. Since $|Y| \geq 2$ and $Y \neq X$, by Theorem 1.6, $S(X,Y)$ is not regular.

Case 2: X is infinite and $Y \neq X$. Then by the assumption that $S(X,Y)$ is a semigroup and by Proposition 2.7 (i) and (ii), $|X| = |Y|$, and hence there is a bijection from X to Y, say α. Clearly α is in $S(X,Y)$. Since α is an injection, by Theorem 2.3, α is not regular in $T(X,Y)$. Hence $S(X,Y)$ is not regular.

Case 3: X is infinite and $Y = X$. We can follow directly from Theorem 1.3.

Therefore the proof is complete. □

Theorem 2.10. *Let X be an infinite set. The semigroup $AM(X,Y) \cap AE(X,Y)$ is regular if and only if $Y = X$.*

Proof. To prove the necessity, by contrapositive, we suppose that $Y \neq X$ and $AM(X,Y) \cap AE(X,Y)$ is a semigroup. By Proposition 2.7 (iii), $X \smallsetminus Y$ is finite.

Consequently, $|X| = |Y|$. Thus there exists a bijection from X to Y and this map is in $AM(X,Y) \cap AE(X,Y)$. In addition, by Theorem 2.3, it is not regular even in $T(X,Y)$. Hence $AM(X,Y) \cap AE(X,Y)$ is not regular.

Conversely, we assume that $Y = X$. Let α be an element in $AM(X) \cap AE(X)$. For each $y \in \operatorname{ran}\alpha$, we choose $z_y \in y\alpha^{-1}$. By Theorem 2.1, α is regular in $T(X)$. Moreover, according to the proof of sufficiency of Theorem 2.1 we have $\beta \in T(X)$ such that

$$y\beta = \begin{cases} z_y & \text{if } y \in \operatorname{ran}\alpha, \\ b & \text{otherwise,} \end{cases}$$

where $b \in Y$ and $\alpha\beta\alpha = \alpha$. Claim that $\{x \in X : |x\beta\beta^{-1}| > 1\}$ is a subset of $(X \setminus \operatorname{ran}\alpha) \cup \{b\alpha\}$. Let $y \in X$ be such that $|y\beta\beta^{-1}| > 1$. Suppose that $y \in \operatorname{ran}\alpha$. Since $|y\beta\beta^{-1}| > 1$, there exists $t \in X \setminus \{y\}$ such that $t\beta = y\beta$. Then we have two cases to consider.

Case 1: $t \in \operatorname{ran}\alpha$. Then $z_t = t\beta = y\beta = z_y$, so $t = z_t\alpha = z_y\alpha = y$, which is a contradiction.

Case 2: $t \notin \operatorname{ran}\alpha$. Then $b = t\beta = y\beta = z_y$, which implies that $b\alpha = z_y\alpha = y$.

Then we have the claim. Since α is in $AE(X)$, $X \setminus \operatorname{ran}\alpha$ is finite. Therefore $\{x \in X : |x\beta\beta^{-1}| > 1\}$ is finite. Hence β belongs to $AM(X)$. To see that β belongs to $AE(X)$ we consider the set $X \setminus \operatorname{ran}\beta$. Since $\{z_x : x \in \operatorname{ran}\alpha\} \subseteq \operatorname{ran}\beta$, $X \setminus \operatorname{ran}\beta$ is a subset of $X \setminus \{z_x : x \in \operatorname{ran}\alpha\}$. Claim that $X \setminus \{z_x : x \in \operatorname{ran}\alpha\}$ is finite. It suffices to prove that $X \setminus \{z_x : x \in \operatorname{ran}\alpha\}$ is a subset of $\{x \in X : |x\alpha\alpha^{-1}| > 1\}$, since $\alpha \in AM(X)$. Let $y \in X \setminus \{z_x : x \in \operatorname{ran}\alpha\}$. Then $y\alpha \in \operatorname{ran}\alpha$ and $z_{y\alpha}\alpha = y\alpha$ but $y \neq z_{y\alpha}$. Consequently, $|y\alpha\alpha^{-1}| > 1$ and the claim is done. This implies that $X \setminus \operatorname{ran}\beta$ is finite. Hence $\beta \in AE(X)$. □

From Theorem 2.1 and applying the converse proof of Theorem 2.10, we have a subsemigroup of $T(X,Y)$ preserving the converse of Remark 2.2.

Corollary 2.11. *For any α in the semigroup $AM(X,Y) \cap AE(X,Y)$, α is regular in $AM(X,Y) \cap AE(X,Y)$ if and only if $Y\alpha = \operatorname{ran}\alpha$.*

Note that when X is finite, we have $AM(X,Y) \cap AE(X,Y) = T(X,Y)$, and by Theorem 1.6, it is regular if and only if $Y = X$ or Y is a singleton. In general, we have

Corollary 2.12. *$AM(X) \cap AE(X)$ is a regular semigroup.*

Theorem 2.3 showed that for any proper subset Y of X, every bijection from X to Y is not regular in $T(X,Y)$ and in its subsemigroups, including $AM(X,Y)$, $AE(X,Y)$ and $AM(X,Y) \cap AE(X,Y)$. We next show that, apart from the bijections, there is some other kind of nonregular elements.

Proposition 2.13. *Let X be an infinite set and Y a proper subset of X such that $AM(X,Y) \cap AE(X,Y)$ is a semigroup. Then there are infinitely many elements in $AM(X,Y) \cap AE(X,Y)$ which are not regular in $T(X,Y)$ and which are neither injective nor surjective.*

Proof. Since $AM(X,Y) \cap AE(X,Y)$ is a semigroup and Y is a proper subset of X, by Proposition 2.7 (iii), $X \setminus Y$ is a nonempty finite set. We know that X is infinite, so is Y. Let B be a finite subset of Y with $|B| \geq 3$. Let $b_1, b_2 \in B$ and $y_1, y_2 \in Y$ be distinct. Since B is a finite subset of an infinite set Y, $|Y \setminus \{y_1, y_2\}| = |Y \setminus B|$. Choose a bijection φ from $Y \setminus \{y_1, y_2\}$ to $Y \setminus B$. Define $\alpha \in T(X,Y)$ by

$$x\alpha = \begin{cases} x\varphi & \text{if } x \in Y \setminus \{y_1, y_2\}, \\ b_1 & \text{if } x \in \{y_1, y_2\}, \\ b_2 & \text{otherwise.} \end{cases}$$

Then $\operatorname{ran}\alpha = (Y \setminus B) \cup \{b_1, b_2\} \neq Y$ and $X \setminus \operatorname{ran}\alpha = (X \setminus Y) \cup (B \setminus \{b_1, b_2\})$, which is finite, so α is not surjective and $\alpha \in AE(X,Y)$. It is easy to see that

$$y_1, y_2 \in \{x \in X : |x\alpha\alpha^{-1}| > 1\} \subseteq (X \setminus Y) \cup \{y_1, y_2\},$$

these show that α is not injective and $\alpha \in AM(X,Y)$, since $X \setminus Y$ is finite. We have $Y\alpha = (Y \setminus B) \cup \{b_1\} \neq \operatorname{ran}\alpha$, by Theorem 2.1, α is not regular in $T(X,Y)$.

Notice that if B has only two elements then the nonregular element α in $T(X,Y)$ is surjective and it is still contained in $AM(X,Y) \cap AE(X,Y)$. □

2.2 Regularity of $OM(X, Y)$ and $OE(X, Y)$

Throughout this section, X is an infinite set. Recall that $OM(X,Y)$, $OE(X,Y)$ and their intersection are always semigroups. In addition, let us note that under the condition in Proposition 2.8, Y is an infinite proper subset of X, we have $OM(X,Y)\smallsetminus OE(X,Y)$ and $OE(X,Y)\smallsetminus OM(X,Y)$ are not the empty sets, as both $\{AM(X,Y), OM(X,Y)\}$ and $\{AE(X,Y), OE(X,Y)\}$ are partitions of $T(X,Y)$. First of all, if $|Y| = 1$ then $OM(X,Y) = OE(X,Y) = OM(X,Y)\cap OE(X,Y)$, which is a singleton of one constant map; in this case, the semigroup clearly is regular. Otherwise, by Theorem 2.14, $OM(X,Y)\cap OE(X,Y)$ contains a nonregular element in $T(X,Y)$.

Theorem 2.14. *Let Y be a proper subset of a set X with $|Y| \geq 2$. Then the semigroups $OM(X,Y)$, $OE(X,Y)$ and its intersection have infinitely many nonregular elements in $T(X,Y)$. In particular, all $OM(X,Y)$, $OE(X,Y)$ and $OM(X,Y)\cap OE(X,Y)$ are nonregular semigroups.*

Proof. Let $y_1, y_2 \in Y$ be distinct. Let $m \in X \smallsetminus Y$. Define $\alpha \in T(X,Y)$ by

$$x\alpha = \begin{cases} y_1 & \text{if } x = m, \\ y_2 & \text{otherwise.} \end{cases}$$

Then $\{x \in X : |x\alpha\alpha^{-1}| > 1\} = X \smallsetminus \{m\}$ and $X \smallsetminus \operatorname{ran}\alpha = X \smallsetminus \{y_1, y_2\}$, which implies that $\alpha \in OM(X,Y)\cap OE(X,Y)$. Since $Y\alpha = \{y_2\} \neq \{y_1, y_2\} = \operatorname{ran}\alpha$, by Theorem 2.1, α is not regular in $T(X,Y)$. □

Theorem 2.15. *$OM(X,Y)\cap OE(X,Y)$ is regular if and only if either $Y = X$ or $|Y| = 1$.*

Proof. For the sufficiency, we have two cases to consider.

Case 1: $Y = X$. Let α belong to $OM(X)\cap OE(X)$ and let $a \in X$. For each $x \in \operatorname{ran}\alpha$, we choose an element $z_x \in x\alpha^{-1}$ and a transformation on X defined in the proof of Theorem 2.1, that is,

$$x\beta = \begin{cases} z_x & \text{if } x \in \operatorname{ran}\alpha, \\ a & \text{otherwise,} \end{cases}$$

and clearly we have $\alpha\beta\alpha = \alpha$. Since $\alpha \in OE(X)$ and $X \smallsetminus \operatorname{ran}\alpha$ is a subset of $\{x \in X : |x\beta\beta^{-1}| > 1\}$, the set $\{x \in X : |x\beta\beta^{-1}| > 1\}$ is infinite. Thus $\beta \in OM(X)$.

Next, we let $T = \{z_x : x \in \operatorname{ran}\alpha\}$. Then $X \smallsetminus \operatorname{ran}\beta = X \smallsetminus (T \cup \{a\})$. To show that $\beta \in OE(X)$, we prove that $X \smallsetminus T$ is infinite. One can see that $X \smallsetminus T = \bigcup(z_x\alpha\alpha^{-1} \smallsetminus \{z_x\})$ where the union is taken over all x in $\operatorname{ran}\alpha$. Since $\alpha \in OM(X)$, there exists m in $\operatorname{ran}\alpha$ such that $m\alpha^{-1}$ is infinite, or $\{z_x \in T : |z_x\alpha\alpha^{-1}| > 1\}$ is infinite. In either case, we get that $X \smallsetminus T$ is an infinite set, and so is $X \smallsetminus (T \cup \{a\})$. Hence β belongs to $OE(X)$.

Case 2: $|Y| = 1$. Then $OM(X, Y) \cap OE(X, Y)$ is a singleton, containing exactly one constant map. Obviously, $OM(X, Y) \cap OE(X, Y)$ is regular.

The necessity follows directly from Theorem 2.14. □

From Theorem 2.15, we have the following corollary.

Corollary 2.16. *$OM(X) \cap OE(X)$ is a regular semigroup.*

CHAPTER III
REGULARITY OF LINEAR TRANSFORMATION SEMIGROUPS

Previously, we investigated regularity of the semigroups $AM(X,Y)$, $AE(X,Y)$, $OM(X,Y)$ and $OE(X,Y)$. One may question what will happen if we switch over to vector spaces. Throughout, let V be a vector space over a division ring and W a subspace of V. We now recall the semigroups of our interest, namely,

$$\begin{aligned}
\mathcal{AM}(V,W) &= \{\alpha \in \mathcal{L}(V,W) : \dim(\ker\alpha) < \infty\},\\
\mathcal{AE}(V,W) &= \{\alpha \in \mathcal{L}(V,W) : \dim(V/\operatorname{ran}\alpha) < \infty\},\\
\mathcal{OM}(V,W) &= \{\alpha \in \mathcal{L}(V,W) : \dim(\ker\alpha) \text{ is infinite}\} \text{ and}\\
\mathcal{OE}(V,W) &= \{\alpha \in \mathcal{L}(V,W) : \dim(V/\operatorname{ran}\alpha) \text{ is infinite}\}.
\end{aligned}$$

The purpose of this chapter is to determine the regularity of the above sets, and also $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ and $\mathcal{OM}(V,W) \cap \mathcal{OE}(V,W)$ whenever they are semigroups. This chapter comprises two parts: the first is concerned with regularity of $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$, while the second is dedicated to the study of regularity of $\mathcal{OM}(V,W)$ and $\mathcal{OE}(V,W)$. Note that if $\dim(V) < \infty$ then $\mathcal{L}(V,W)$, $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$ are the same semigroup. Therefore, in the rest of this chapter, infinite dimensional vector spaces become of particular interest. Moreover, one can see that every subsemigroup of $\mathcal{L}(V,W)$ is also a subsemigroup of $T(V,W)$. That means we can apply and take advantage of Remarks 2.2 and 2.4, and Theorem 2.3 in this chapter.

3.1 Regularity of $\mathcal{AM}(V, W)$ and $\mathcal{AE}(V, W)$

Note that it is not certain that $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$ will be semigroups even though $\dim(V)$ is infinite. So we need characterisations for $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$ to be semigroups.

Proposition 3.1. *Let V be an infinite dimensional vector space. Then*
(i) $\mathcal{AM}(V,W)$ is not the empty set if and only if $\dim(V) = \dim(W)$,
(ii) $\mathcal{AE}(V,W)$ is not the empty set if and only if $\dim(V/W) < \infty$.

Proof. To show that (i) holds, we first assume that $\dim(V) = \dim(W)$. Thus there exists an isomorphism from V to W, which is contained in $\mathcal{AM}(V,W)$.

Next, we assume that there exists a linear transformation β in $\mathcal{AM}(V,W)$. Thus $\dim(\ker\beta)$ is finite. Since $\dim(V) = \dim(\ker\beta) + \dim(\operatorname{ran}\beta)$, $\dim(V)$ is infinite and $\dim(\ker\beta)$ is finite, we have $\dim(V) = \dim(\operatorname{ran}\beta) \leq \dim(W)$. Hence $\dim(V) = \dim(W)$.

(ii) Assume that $\mathcal{AE}(V,W)$ is not empty and let φ be a linear transformation in $\mathcal{AE}(V,W)$. Since $\varphi \in \mathcal{AE}(V,W)$, $\dim(V/W) \leq \dim(V/\operatorname{ran}\varphi) < \infty$.

Conversely, we assume that $\dim(V/W) < \infty$. Let B_1 be a basis of W. Then we extend B_1 to a basis B of V. By Proposition 1.12, $|B \smallsetminus B_1| = \dim(V/W) < \infty$. Since $B_1 \subseteq B$, B is infinite and $B \smallsetminus B_1$ is finite, we have B_1 is infinite and $|B| = |B_1|$. Then there exists a bijection from B to B_1, which induces an isomorphism from V to W. Obviously, it is contained in $\mathcal{AE}(V,W)$. □

Below is a consequence of Proposition 3.1 and the fact that $\mathcal{AM}(V,W)$ is an intersection of $\mathcal{L}(V,W)$ and $\mathcal{AM}(V)$. Similar arguments can be applied to $\mathcal{AE}(V,W)$.

Proposition 3.2. *Let V be an infinite dimensional vector space. Then*
(i) $\mathcal{AM}(V,W)$ is a semigroup if and only if $\dim(V) = \dim(W)$,
(ii) $\mathcal{AE}(V,W)$ is a semigroup if and only if $\dim(V/W) < \infty$,
(iii) $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ is a semigroup if and only if $\dim(V/W) < \infty$.

Proof. It remains to show that if $\dim(V/W) < \infty$, then $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ is not empty. Assume that $\dim(V/W) < \infty$. By the proof of the sufficiency of Proposition 3.1 (*ii*), we have an isomorphism from V to W, which is contained in $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$. Thus $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ is not empty. □

Proposition 3.3. *Given semigroups $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$, if $\dim(V)$ is infinite, then neither $\mathcal{AM}(V,W) \smallsetminus \mathcal{AE}(V,W)$ nor $\mathcal{AE}(V,W) \smallsetminus \mathcal{AM}(V,W)$ is the empty set.*

Proof. Assume that $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$ are semigroups and $\dim(V)$ is infinite. Let B_1 be a basis of W. Then we extend B_1 to a basis B of V. By Proposition 1.12, $|B \smallsetminus B_1| = \dim(V/W) < \infty$. Since B is infinite, we have B_1 is infinite and there exists an infinite subset Z of B_1 with $|B_1| = |Z| = |B_1 \smallsetminus Z|$. We now are able to provide two bijections $\varphi : B_1 \to Z$ and $\psi : Z \to B_1$. Then we define $\alpha, \beta \in \mathcal{L}(V,W)$ by

$$x\alpha = \begin{cases} x\varphi & \text{if } x \in B_1, \\ 0 & \text{if } x \in B \smallsetminus B_1, \end{cases}$$

and

$$x\beta = \begin{cases} x\psi & \text{if } x \in Z, \\ 0 & \text{if } x \in B \smallsetminus Z. \end{cases}$$

First, we show that $\alpha \in \mathcal{AM}(V,W) \smallsetminus \mathcal{AE}(V,W)$. We have $\dim(\ker\alpha) = |B \smallsetminus B_1| < \infty$. Hence $\alpha \in \mathcal{AM}(V,W)$. By the definition of α, we have $\operatorname{ran}\alpha = \langle Z \rangle$. By Proposition 1.12, $\dim(V/\operatorname{ran}\alpha) = |B \smallsetminus Z|$, which is infinite, since $B_1 \smallsetminus Z$ is an infinite subset of $B \smallsetminus Z$. Thus $\alpha \notin \mathcal{AE}(V,W)$. Therefore α belongs to $\mathcal{AM}(V,W) \smallsetminus \mathcal{AE}(V,W)$.

Next, we show that $\beta \in \mathcal{AE}(V,W) \smallsetminus \mathcal{AM}(V,W)$. By the definition of β, $\dim(\ker\beta) = |B \smallsetminus Z|$ and $\operatorname{ran}\beta = W$. Since $|B \smallsetminus Z|$ is infinite, so is $\dim(\ker\beta)$, and hence $\beta \notin \mathcal{AM}(V,W)$. Since $\operatorname{ran}\beta = W$, $\dim(V/\operatorname{ran}\beta) = \dim(V/W) < \infty$, we have $\beta \in \mathcal{AE}(V,W)$. Therefore $\beta \in \mathcal{AE}(V,W) \smallsetminus \mathcal{AM}(V,W)$. □

From Proposition 3.2 (*i*) and (*ii*), if $\mathcal{AE}(V,W)$ is a semigroup then $\mathcal{AM}(V,W)$

is also a semigroup, but the converse is not true. It follows from Proposition 3.3 that the semigroup $\mathcal{AM}(V,W) = \mathcal{AE}(V,W)$ if and only if $\dim(V)$ is finite.

Consider semigroups $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$. From Proposition 1.10, if $\dim(V)$ is finite and W is the zero subspace of V then $\mathcal{AM}(V,W) = \mathcal{L}(V,W) = \mathcal{AE}(V,W)$ is regular.

Theorem 3.4. *Let W be a nonzero subspace of V and let $S(V,W)$ be either the semigroup $\mathcal{AM}(V,W)$ or the semigroup $\mathcal{AE}(V,W)$. Then $S(V,W)$ is regular if and only if* $\dim(V) < \infty$ *and* $W = V$.

Proof. For the sufficiency, we assume that $\dim(V) < \infty$ and $W = V$. Then $\mathcal{AM}(V,W) = \mathcal{AM}(V)$ and $\mathcal{AE}(V,W) = \mathcal{AE}(V)$. By Theorem 1.8, $S(V,W)$ is regular, as $\dim(V) < \infty$. To prove the necessity, by contrapositive, suppose that $\dim(V)$ is infinite or $W \neq V$.

Case 1: $\dim(V)$ is infinite and $W = V$. By Theorem 1.8, $S(V,W)$ is not regular.

Case 2: $\dim(V) < \infty$ and $W \neq V$. Then $S(V,W) = \mathcal{L}(V,W)$. Since $V \neq \{0\}$, $W \neq \{0\}$ and $W \neq V$, by Proposition 1.10, $S(V,W)$ is not regular.

Case 3: $\dim(V)$ is infinite and $W \neq V$. Since $S(V,W)$ is a semigroup, by Proposition 3.2 (i) and (ii), $\dim(V) = \dim(W)$. Then there exists an isomorphism α from V to W, which clearly is in $S(V,W)$; the reader is reminded that, when $S(V,W) = \mathcal{AE}(V,W)$, $\dim(V/W)$ is finite. By Theorem 2.3, α is not regular in $S(V,W)$. □

We use Theorem 3.4 to generalise Theorem 1.8 as follows.

Corollary 3.5. *Let $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$ be semigroups with W a nonzero subspace of V. The following statements are equivalent:*

(i) $\dim(V) < \infty$ *and* $W = V$,

(ii) $\mathcal{AM}(V,W)$ *is regular,*

(iii) $\mathcal{AE}(V,W)$ *is regular.*

It is clear that $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W) = \mathcal{L}(V,W)$ when $\dim(V) < \infty$. We therefore consider only the case when V is an infinite dimensional vector space, and give a necessary and sufficient condition for $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ to be regular.

Theorem 3.6. *Let V be an infinite dimensional vector space. The semigroup $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ is regular if and only if $W = V$.*

Proof. To prove the necessity, by contrapositive, we suppose that W is a proper subspace of V. Since $W \neq V$ and $\dim V$ is infinite, by the proof of Theorem 3.4, it is clear that the linear transformation α that we have in the third case of the proof is also a nonregular element in the semigroups $\mathcal{AM}(V,W)$ and $\mathcal{AE}(V,W)$. Therefore $\mathcal{AM}(V,W) \cap \mathcal{AE}(V,W)$ is not regular.

For the sufficiency, we assume that $W = V$. Thus $\mathcal{AM}(V,W) = \mathcal{AM}(V)$ and $\mathcal{AE}(V,W) = \mathcal{AE}(V)$. Let $\alpha \in \mathcal{AM}(V) \cap \mathcal{AE}(V)$. Let K be a basis of $\ker\alpha$ and P a basis of $\operatorname{ran}\alpha$. For each $v \in P$, we choose $z_v \in v\alpha^{-1}$. By Proposition 1.11, $K \cup \{z_v : v \in P\}$ is a basis for V. Let B be a basis for V with $P \subseteq B$. Then we define $\beta \in \mathcal{L}(V)$ by

$$v\beta = \begin{cases} z_v & \text{if } v \in P, \\ 0 & \text{if } v \in B \smallsetminus P. \end{cases}$$

We have

$$\begin{aligned} \dim(\ker\beta) &= |B \smallsetminus P| \\ &= \dim(V/\operatorname{ran}\alpha) && \text{(by Proposition 1.12)} \\ &< \infty && (\text{as } \alpha \in \mathcal{AE}(V)). \end{aligned}$$

Thus $\beta \in \mathcal{AM}(V)$. We then show that $\beta \in \mathcal{AE}(V)$. We know that $K \cup \{z_v : v \in P\}$ is a basis of V. By Proposition 1.12,

$$\begin{aligned} \dim(V/\operatorname{ran}\beta) &= |K| \\ &= \dim(\ker\alpha) \\ &< \infty && (\text{as } \alpha \in \mathcal{AM}(V)). \end{aligned}$$

That is, β belongs to $\mathcal{AE}(V)$. Next, claim that $\alpha\beta\alpha = \alpha$. Let $t \in K \cup \{z_v : v \in P\}$.

If $t \in K$, then $t\alpha\beta\alpha = 0 = t\alpha$. Otherwise, $t = z_u$ for some $u \in P$. Then $t\alpha\beta\alpha = z_u\alpha\beta\alpha = u\beta\alpha = z_u\alpha = t\alpha$. Hence we have the claim. Therefore the proof is complete. $\square$

Corollary 3.7. *For any α in the semigroup $\mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$, α is regular in $\mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$ if and only if $W\alpha = \operatorname{ran}\alpha$.*

Proof. It is obtained from Theorem 2.1 and the converse proof of Theorem 3.6 step by step. $\square$

From Theorem 3.6 and the fact that $\mathcal{AM}(V) = \mathcal{L}(V) = \mathcal{AE}(V)$ when $\dim(V)$ is finite, we have the following corollary.

Corollary 3.8. *$\mathcal{AM}(V) \cap \mathcal{AE}(V)$ is a regular semigroup.*

By Theorem 2.3, for any proper subspace W of V, every isomorphism from V to W is not regular in $T(V, W)$ and certainly in its subsemigroups, including $\mathcal{AM}(V, W)$, $\mathcal{AE}(V, W)$ and $\mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$. The next proposition shows that, apart from the isomorphisms, there is some other kind of nonregular elements.

Proposition 3.9. *Let V be an infinite dimensional vector space and W a proper subspace of V such that $\mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$ is a semigroup. Then there are infinitely many elements in $\mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$ which are not regular in $\mathcal{L}(V, W)$ and which are neither injective nor surjective.*

Proof. Let W be a proper subspace of V such that $\mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$ is a semigroup. Then $\dim(V/W)$ is finite. Let B_1 be a basis of W. Then we extend B_1 to a basis B of V. By Proposition 1.12, $|B \smallsetminus B_1| = \dim(V/W) < \infty$. Since B is infinite and $B \smallsetminus B_1$ is finite, we have B_1 is infinite. Since $W \neq V$, $B \smallsetminus B_1$ is not empty. Let $a \in B \smallsetminus B_1$. Let Z be a finite subset of B_1 with $|Z| \geq 2$. Choose $z \in Z$. Since B_1 is infinite and Z is a finite subset of B_1, $B_1 \smallsetminus Z$ is infinite and $|B_1 \smallsetminus Z| = |B_1|$. Let $u_1, u_2 \in B_1$ be distinct. Thus $|B_1 \smallsetminus Z| = |B_1 \smallsetminus \{u_1, u_2\}|$.

Hence there exists a bijection $\gamma : B_1 \setminus \{u_1, u_2\} \to B_1 \setminus Z$. Define $\alpha \in \mathcal{L}(V, W)$ by

$$x\alpha = \begin{cases} x\gamma & \text{if } x \in B_1 \setminus \{u_1, u_2\}, \\ 0 & \text{if } x \in \{u_1, u_2\} \cup [B \setminus (B_1 \cup \{a\})], \\ z & \text{if } x = a. \end{cases}$$

Then $\operatorname{ran}\alpha = \langle (B_1 \setminus Z) \cup \{z\} \rangle$. By Proposition 1.12,

$$\begin{aligned} \dim (V/\operatorname{ran}\alpha) &= |B \setminus [(B_1 \setminus Z) \cup \{z\}]| \\ &= |(B \setminus B_1) \cup (Z \setminus \{z\})| \\ &= |B \setminus B_1| + |Z \setminus \{z\}| \\ &< \infty. \end{aligned}$$

Thus α is not surjective and α belongs to $\mathcal{AE}(V, W)$. We have

$$\begin{aligned} \dim (\ker\alpha) &= |\{u_1, u_2\} \cup [B \setminus (B_1 \cup \{a\})]| \\ &= |\{u_1, u_2\}| + |B \setminus (B_1 \cup \{a\})| \\ &< \infty. \end{aligned}$$

Thus α is not injective and $\alpha \in \mathcal{AM}(V, W)$. Hence $\alpha \in \mathcal{AM}(V, W) \cap \mathcal{AE}(V, W)$. By the definition of α, $W\alpha = \langle B_1\alpha \rangle = \langle B_1 \setminus Z \rangle \neq \langle (B_1 \setminus Z) \cup \{z\} \rangle = \operatorname{ran}\alpha$. By Theorem 2.1, α is not regular in $\mathcal{L}(V, W)$. □

3.2 Regularity of $\mathcal{OM}(V, W)$ and $\mathcal{OE}(V, W)$

Throughout this section, we let V be an infinite dimensional vector space over a division ring. The objective of this section is to investigate the regularity of linear transformation semigroups $\mathcal{OM}(V, W)$ and $\mathcal{OE}(V, W)$. We note that under the constraint in Proposition 3.3, we have $\mathcal{OM}(V, W) \setminus \mathcal{OE}(V, W)$ and $\mathcal{OE}(V, W) \setminus \mathcal{OM}(V, W)$ are not the empty sets, as both $\{\mathcal{AM}(V, W), \mathcal{OM}(V, W)\}$ and $\{\mathcal{AE}(V, W), \mathcal{OE}(V, W)\}$ are partitions of $\mathcal{L}(V, W)$.

Theorem 3.10. *For any nontrivial subspace W of V, the semigroups $\mathcal{OM}(V,W)$, $\mathcal{OE}(V,W)$ and its intersection have infinitely many nonregular elements in $\mathcal{L}(V,W)$. In particular, all $\mathcal{OM}(V,W)$, $\mathcal{OE}(V,W)$ and its intersection are nonregular semigroups.*

Proof. Let B_1 be a basis of W. Then we extend B_1 to a basis B of V. Let $b \in B_1$ and let $m \in B \setminus B_1$. Define $\alpha \in \mathcal{L}(V,W)$ by

$$v\alpha = \begin{cases} b & \text{if } v = m, \\ 0 & \text{if } v \in B \setminus \{m\}. \end{cases}$$

Then $\dim(\ker\alpha) = |B \setminus \{m\}|$, which is infinite. Hence $\alpha \in \mathcal{OM}(V,W)$. We have $\operatorname{ran}\alpha = \langle\{b\}\rangle$. Thus $\dim(V/\operatorname{ran}\alpha) = |B \setminus \{b\}|$, and hence $\alpha \in \mathcal{OE}(V,W)$. Therefore $\alpha \in \mathcal{OM}(V,W) \cap \mathcal{OE}(V,W)$. Since $W\alpha = \langle B_1\alpha\rangle = \{0\} \neq \langle\{b\}\rangle = \operatorname{ran}\alpha$, by Theorem 2.1, α is not regular in $\mathcal{L}(V,W)$. □

Theorem 3.11. *$\mathcal{OM}(V,W) \cap \mathcal{OE}(V,W)$ is regular if and only if either $W = V$ or $W = \{0\}$.*

Proof. Clearly, if $W = \{0\}$ then the zero transformation is the only one element in $\mathcal{OM}(V,W) \cap \mathcal{OE}(V,W)$, and hence $\mathcal{OM}(V,W) \cap \mathcal{OE}(V,W)$ is regular. Assume that $W = V$. Then $\mathcal{OM}(V,W) = \mathcal{OM}(V)$ and $\mathcal{OE}(V,W) = \mathcal{OE}(V)$. Let α be in $\mathcal{OM}(V) \cap \mathcal{OE}(V)$. Let K be a basis of ker α and P a basis of ran α. For each $v \in P$, we choose $z_v \in v\alpha^{-1}$. By Proposition 1.11, $K \cup \{z_v : v \in P\}$ is a basis of V. Let B be a basis for V with $P \subseteq B$. Then we let γ be the linear transformation defined by

$$v\gamma = \begin{cases} z_v & \text{if } v \in P, \\ 0 & \text{if } v \in B \setminus P. \end{cases}$$

Thus $\dim(\ker\gamma) = |B \setminus P| = \dim(V/\operatorname{ran}\alpha)$, which is infinite, since $\alpha \in \mathcal{OE}(V)$. Thus $\gamma \in \mathcal{OM}(V)$. We show that $\gamma \in \mathcal{OE}(V)$. We have $K \cup \{z_v : v \in P\}$ is a basis of V. Then $\dim(V/\operatorname{ran}\gamma) = |K| = \dim(\ker\alpha)$, which is infinite, since $\alpha \in \mathcal{OM}(V)$. Hence $\gamma \in \mathcal{OE}(V)$. Thus $\gamma \in \mathcal{OM}(V) \cap \mathcal{OE}(V)$. It is straightforward to see that $\alpha\gamma\alpha = \alpha$, and the proof is then complete.

The neccessity follows directly from Theorem 3.10 □

Corollary 3.12. $\mathcal{OM}(V) \cap \mathcal{OE}(V)$ *is a regular semigroup.*

CHAPTER IV
SUPPLEMENTARY COMMENTS

In Chapter II, the transformations under consideration are contained in $T(X,Y)$; the codomain of each map is Y. Possibly, there is one likely to put a question why the conditions of elements α in $AE(X,Y)$ is not "$Y \smallsetminus \operatorname{ran}\alpha$ is finite".

We show that both $AE(X,Y)$ and

$$\overline{AE}(X,Y) = \{\alpha \in T(X,Y) : Y \smallsetminus \operatorname{ran}\alpha \text{ is finite}\},$$

are the same semigroups under certain conditions. We also discuss in which way they differ to each other. In addition, $AE(X,Y)$ and $\overline{AE}(X,Y)$ are identical whenever they both are semigroups. Then it is enough only to study the regularity of the semigroup $AE(X,Y)$.

Furthermore, in the last section, we discuss its analogous problem in case of linear transformations.

4.1 The relation between $\boldsymbol{AE(X,Y)}$ and $\boldsymbol{\overline{AE}(X,Y)}$

In this section, we assume that Y is a nonempty subset of X. By the definition of $AE(X,Y)$ and $\overline{AE}(X,Y)$, we have that $AE(X,Y)$ is a subset of $\overline{AE}(X,Y)$, and we see that if X is finite then $AE(X,Y) = T(X,Y) = \overline{AE}(X,Y)$. In general, we have

Proposition 4.1. $AE(X,Y) = \overline{AE}(X,Y)$ *if and only if* $X \smallsetminus Y$ *is finite.*

Proof. Suppose that $X \setminus Y$ is infinite. Let $a \in Y$. Define $\alpha \in T(X,Y)$ by

$$x\alpha = \begin{cases} x & \text{if } x \in Y, \\ a & \text{otherwise.} \end{cases}$$

Then $\operatorname{ran}\alpha = Y$. Thus $\alpha \notin AE(X,Y)$ but $\alpha \in \overline{AE}(X,Y)$. Hence $AE(X,Y)$ and $\overline{AE}(X,Y)$ are different, and $\overline{AE}(X,Y)$ is not empty.

Conversely, assume that $X \setminus Y$ is finite. Clearly, $AE(X,Y) \subseteq \overline{AE}(X,Y)$. Let $\alpha \in \overline{AE}(X,Y)$. Then $Y \setminus \operatorname{ran}\alpha$ is finite. Since $X \setminus \operatorname{ran}\alpha = (X \setminus Y) \cup (Y \setminus \operatorname{ran}\alpha)$, $X \setminus Y$ and $Y \setminus \operatorname{ran}\alpha$ are also finite, we have $X \setminus \operatorname{ran}\alpha$ is finite. Hence α is in $AE(X,Y)$. Therefore $AE(X,Y) = \overline{AE}(X,Y)$. □

Propositions 2.7 (*ii*) and 4.1 show that if $AE(X,Y)$ is a semigroup then so is $\overline{AE}(X,Y)$. The converse holds whenever Y is infinite.

Proposition 4.2. *Let Y be an infinite subset of X. Then $\overline{AE}(X,Y)$ is a semigroup if and only if $X \setminus Y$ is finite.*

Proof. For the sufficiency, assume that $X \setminus Y$ is finite. Whether X is finite or infinite, by Proposition 2.7 (*ii*) and the fact that $AE(X,Y) = T(X,Y)$ when X is finite, one can see that $AE(X,Y)$ is a semigroup. By the assumption and Proposition 4.1 , $\overline{AE}(X,Y)$ is a semigroup.

To prove the necessity, by contrapositive, we suppose that $X \setminus Y$ is infinite. By the contrapositive proof of Proposition 4.1, $\overline{AE}(X,Y)$ is not empty. It suffices to show that $\overline{AE}(X,Y)$ is not closed. We have two cases to consider.

Case 1: $|X \setminus Y| < |Y|$. Since $X \setminus Y$ is infinite and $|X \setminus Y| < |Y|$, there exists a subset Z of Y with $|X \setminus Y| = |Z|$. Then there exists a bijection α from $X \setminus Y$ to Z. Clearly, we have a surjection β from Y to $Y \setminus Z$. Now, we define $\varphi : X \to Y$ by

$$w\varphi = \begin{cases} w\alpha & \text{if } w \in X \setminus Y, \\ w\beta & \text{otherwise.} \end{cases}$$

Then $\operatorname{ran}\varphi = Y$. Thus φ belongs to $\overline{AE}(X,Y)$. We have $\operatorname{ran}\varphi^2 = Y \smallsetminus Z$. Hence $Y \smallsetminus \operatorname{ran}\varphi^2 = Z$, which is infinite. Thus $\varphi^2 \notin \overline{AE}(X,Y)$. Hence $\overline{AE}(X,Y)$ is not closed.

Case 2: $|X \smallsetminus Y| \geq |Y|$. Then there exists a surjection γ from $X \smallsetminus Y$ to Y. Let $a \in Y$. We define $\mu : X \to Y$ by

$$x\mu = \begin{cases} x\gamma & \text{if } x \in X \smallsetminus Y, \\ a & \text{otherwise.} \end{cases}$$

Then $\operatorname{ran}\mu = Y$, so μ is in $\overline{AE}(X,Y)$. Clearly, $\operatorname{ran}\mu^2 = \{a\}$. Thus $Y \smallsetminus \operatorname{ran}\mu^2 = Y \smallsetminus \{a\}$, which is infinite. Hence $\mu^2 \notin \overline{AE}(X,Y)$. Therefore $\overline{AE}(X,Y)$ is not closed. □

The first main theorem follows from Propositions 2.7 (ii), 4.1 and 4.2.

Theorem 4.3. *Let Y be an infinite subset of X. The following are equivalent:*
(i) $X \smallsetminus Y$ is finite,
(ii) $AE(X,Y)$ is a semigroup,
(iii) $\overline{AE}(X,Y)$ is a semigroup,
(iv) $\overline{AE}(X,Y) = AE(X,Y)$.
In particular, $OE(X,Y) = \{\alpha \in T(X,Y) : Y \smallsetminus \operatorname{ran}\alpha$ is infinite$\}$ if and only if $X \smallsetminus Y$ is finite.

From Proposition 4.2 and the fact that $\overline{AE}(X,Y) = T(X,Y)$ when Y is finite, we have the following theorem.

Theorem 4.4. *$\overline{AE}(X,Y)$ is a semigroup if and only if either Y is finite or Y is infinite and $X \smallsetminus Y$ is finite.*

Proof. We first assume that $\overline{AE}(X,Y)$ is a semigroup and Y is an infinite set. By Proposition 4.2, we have $X \smallsetminus Y$ is finite.

The converse is obtained from Proposition 4.2 and the fact that $\overline{AE}(X,Y)$ is $T(X,Y)$ when Y is finite. □

Therefore the semigroup $\overline{AE}(X,Y)$ is either $T(X,Y)$ or $AE(X,Y)$. This is a reason why it suffices only to study the regularity of the semigroup $AE(X,Y)$, but not both.

4.2 The relation between $\mathcal{AE}(V,W)$ and $\overline{\mathcal{AE}}(V,W)$

Let $\overline{\mathcal{AE}}(V,W) = \{\beta \in \mathcal{L}(V,W) : \dim(W/\operatorname{ran}\beta) < \infty\}$. In this section, we study the relation between $\mathcal{AE}(V,W)$ and $\overline{\mathcal{AE}}(V,W)$ in the same fashion as we have done in the previous section. Throughout, W is a subspace of a vector space V.

Theorem 4.5. *For a basis B_1 of W and a basis B of V containing B_1, we let $f \in T(B,B_1)$ and $\alpha \in \mathcal{L}(V,W)$ be such that $\alpha|_B = f$. Then $f \in \overline{AE}(B,B_1)$ if and only if $\alpha \in \overline{\mathcal{AE}}(V,W)$*

Proof. Let $f \in T(B,B_1)$ and $\alpha \in \mathcal{L}(V,W)$ be such that $\alpha|_B = f$. Then $\operatorname{ran} f \subseteq B_1$ and it is easy to see that $\operatorname{ran} f$ is a basis of $\operatorname{ran}\alpha$. Consequently, $\dim(W/\operatorname{ran}\alpha) = |B_1 \setminus \operatorname{ran} f|$. The proof is then complete from this fact. □

Proposition 4.6. $\mathcal{AE}(V,W) = \overline{\mathcal{AE}}(V,W)$ *if and only if* $\dim(V/W) < \infty$.

Proof. For the necessity, we prove by contrapositive. Let B_1 be a basis of W. Then we extend B_1 to a basis B of V. Suppose that $\dim(V/W)$ is infinite. Define $\beta \in \mathcal{L}(V,W)$ by

$$x\beta = \begin{cases} x & \text{if } x \in B_1, \\ 0 & \text{if } x \in B \setminus B_1. \end{cases}$$

Then $\operatorname{ran}\beta = W$. Thus $\beta \in \overline{\mathcal{AE}}(V,W)$ but $\beta \notin \mathcal{AE}(V,W)$. Hence $\overline{\mathcal{AE}}(V,W)$ is not empty and $\mathcal{AE}(V,W) \neq \overline{\mathcal{AE}}(V,W)$.

To prove the sufficiency, we assume that $\dim(V/W) < \infty$. It is clear that $\mathcal{AE}(V,W) \subseteq \overline{\mathcal{AE}}(V,W)$. Let $\alpha \in \overline{\mathcal{AE}}(V,W)$. Let P be a basis of $\operatorname{ran}\alpha$. We extend P to a basis B_1 of W, and we then extend B_1 to a basis B of V. By

Proposition 1.12,

$$|B \smallsetminus B_1| = \dim(V/W) < \infty,$$
$$|B_1 \smallsetminus P| = \dim(W/\operatorname{ran}\alpha) < \infty \quad (\text{as } \alpha \in \overline{\mathcal{AE}}(V,W)) \quad \text{and}$$
$$|B \smallsetminus P| = \dim(V/\operatorname{ran}\alpha).$$

Since $B \smallsetminus P = (B \smallsetminus B_1) \cup (B_1 \smallsetminus P)$ is finite, we have $\dim(V/\operatorname{ran}\alpha)$ is finite. Hence $\alpha \in \mathcal{AE}(V,W)$. □

Next, we will show that $\mathcal{AE}(V,W)$ and $\overline{\mathcal{AE}}(V,W)$ are semigroups in the same time. Note that if $\dim(V)$ is infinite and $\dim(W)$ is finite then $\mathcal{AE}(V,W)$ is empty and $\overline{\mathcal{AE}}(V,W) = \mathcal{L}(V,W)$.

Proposition 4.7. *Let W be an infinite dimensional subspace of V. Then $\overline{\mathcal{AE}}(V,W)$ is a semigroup if and only if $\dim(V/W) < \infty$.*

Proof. Assume that $\dim(V/W) < \infty$. By Proposition 4.6, $\mathcal{AE}(V,W) = \overline{\mathcal{AE}}(V,W)$. By Proposition 3.2 (*ii*), $\overline{\mathcal{AE}}(V,W)$ is a semigroup.

Conversely, suppose that $\dim(V/W)$ is infinite. Let B_1 be a basis of W and B a basis of V containing B_1. By assumption and Proposition 1.12, $B \smallsetminus B_1$ is infinite. By the necessary proof of Proposition 4.2, there exists $f \in \overline{AE}(B,B_1)$ but $f^2 \notin \overline{AE}(B,B_1)$. Extend f to a linear transformation $\alpha \in \mathcal{L}(V,W)$. We have $\alpha^2|_B = f^2$. From these facts and Theorem 4.5, we have $\alpha^2 \notin \overline{\mathcal{AE}}(V,W)$ when $\alpha \in \overline{\mathcal{AE}}(V,W)$. Therefore a nonempty set $\overline{\mathcal{AE}}(V,W)$ is not closed, and hence it is not a semigroup. □

These results give us an interesting fact.

Theorem 4.8. *Let W be an infinite dimensional subspace of V. The following are equivalent:*

(*i*) $\dim(V/W) < \infty$,

(*ii*) $\mathcal{AE}(V,W)$ *is a semigroup,*

(*iii*) $\overline{\mathcal{AE}}(V,W)$ *is a semigroup,*

(*iv*) $\overline{\mathcal{AE}}(V,W) = \mathcal{AE}(V,W)$.

In particular, $\mathcal{OE}(V,W) = \{\alpha \in \mathcal{L}(V,W) : \dim(W/\operatorname{ran}\alpha)$ *is infinite*$\}$ *if and only if* $\dim(V/W) < \infty$.

Below is a consequence of Proposition 4.7 and the fact that $\overline{\mathcal{AE}}(V,W) = \mathcal{L}(V,W)$ when $\dim(W)$ is finite.

Theorem 4.9. $\overline{\mathcal{AE}}(V,W)$ *is a semigroup if and only if either* $\dim(W)$ *is finite or* $\dim(W)$ *is infinite and* $\dim(V/W) < \infty$.

Proof. For the forward implication, we assume that $\overline{\mathcal{AE}}(V,W)$ is a semigroup and $\dim(W)$ is infinite. By Proposition 4.7, we have $\dim(V/W) < \infty$.

The other implication follows from Proposition 4.7 and the fact that $\overline{\mathcal{AE}}(V,W)$ is $\mathcal{L}(V,W)$ when $\dim(W)$ is finite. □

From the above theorem, it is reasonable to only study the regularity of the semigroup $\mathcal{AE}(V,W)$.

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VITA

Name	Miss Thanaporn Sumalroj
Date of Birth	1 July 1991
Place of Birth	Nakhon Pathom, Thailand
Education	B.Sc. (Mathematics), Silpakorn University, 2013
Scholarship	Development and Promotion of Science and Technology Talents Project (DPST)