การดึงดูดเฉพาะที่และลักษณะการมีคาบของสมการผลต่างตรรกยะ

นางสาวบุษยพัชร์ ใจห้าว

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2559

ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

บทคัดย่อและแฟ้มข้อมูลฉบับเต็มของวิทยานิพนธ์ตั้งแต่ปีการศึกษา 2554 ที่ให้บริการในคลังปัญญาจุฬาฯ (CUIR)
เป็นแฟ้มข้อมูลของนิสิตเจ้าของวิทยานิพนธ์ที่ส่งผ่านทางบัณฑิตวิทยาลัย
The abstract and full text of theses from the academic year 2011 in Chulalongkorn University Intellectual Repository(CUIR)
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LOCAL ATTRACTIVITY AND PERIODIC CHARACTER OF RATIONAL DIFFERENCE EQUATION

Miss Butsayapat Chaihao

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2016

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บุษยพัชร์ ใจห้าว : การดึงดูดเฉพาะที่และลักษณะการมีคาบของสมการผลต่างตรรกยะ. (LOCAL ATTRACTIVITY AND PERIODIC CHARACTER OF RATIONAL DIFFERENCE EQUATION) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.สุจินต์ คมฤทัย, 58 หน้า.

ในวิทยานิพนธ์ฉบับนี้ ได้ตรวจสอบพฤติกรรมบางอย่างของสมการผลต่างตรรกยะ

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + B_2 x_{n-2} + \cdots + B_k x_{n-k}}, \quad n = 0, 1, 2, \ldots$$

โดยที่ $k \in \mathbb{N}$ พารามิเตอร์ $A, B_0, B_1, B_2, \ldots, B_k$ และเงื่อนไขเริ่มต้น $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0$ เป็นจำนวนจริงที่ไม่เป็นลบ เราได้จัดกลุ่มผลเฉลยที่เป็นคาบซึ่งมีลักษณะเป็นคาบเฉพาะสองและคาบเฉพาะสาม แล้วนำเสนอการมีอยู่จริงของผลเฉลยที่เป็นคาบซึ่งมีลักษณะเป็นคาบเฉพาะ p โดยที่ p เป็นจำนวนเฉพาะ นอกจากนี้เราแสดงการลู่เข้าของผลเฉลยไปยังผลเฉลยที่เป็นคาบ $k+1$ และสุดท้ายแสดงการดึงดูดเฉพาะที่ของผลเฉลยรอบผลเฉลยที่เป็นคาบเฉพาะสองและสาม

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ปีการศึกษา 2559

5772036523 : MAJOR MATHEMATICS

KEYWORDS : RATIONAL DIFFERENCE EQUATION/ PERIODICITY/ LOCAL STABILITY

BUTSAYAPAT CHAIHAO : LOCAL ATTRACTIVITY AND PERIODIC CHARACTER OF RATIONAL DIFFERENCE EQUATION.
ADVISOR : ASST. PROF. SUJIN KHOMRUTAI, Ph.D., 58 pp.

In this thesis, we investigate some behaviors of the rational difference equations

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + B_2 x_{n-2} + \cdots + B_k x_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

where $k \in \mathbb{N}$, the parameters $A, B_0, B_1, B_2, \ldots, B_k$, and the initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0$ are nonnegative real numbers. We classify all periodic solutions with prime period-two and prime period-three. Then, we also present the existence of periodic solutions with prime period-p, where p is a prime number. Moreover, we prove the convergence of solutions to a period-$(k+1)$ solution. Finally, the local attractivity of solutions about some periodic with prime period-two and prime period-three is established.

Department : ...Mathematics and.....	Student's Signature : ....
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Field of Study : .....Mathematics......	
Academic Year : ....2016............	

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to Assistant Professor Dr.Sujin Khomrutai, my thesis advisor, for his suggestions and helpful advice thoughout the process of researching and writing this thesis. I would like to give special thanks to Assistant Professor Dr.Keng Wiboonton, Assistant Professor Dr.Ratinan Boonklurb and Assistant Professor Dr.Tawikan Treeyaprasert, my thesis committee, for their very valuable comments. I feel very thankful to all my teachers who have taught me for my knowledge and skill.

I would like to express my thankfulness to the Human Resource Development in Science Project (Science Achievement Scholarship of Thailand, SAST) for financial supports. I am graceful to my friends who have supported and shared good experiences together as well.

Finally, I must express my very profound gratitude to my family for their unfailing support and continuous encouragement throughout my years of study. This accomplishment would not have been possible without them. Thank you.

CONTENTS

CHAPTER I

INTRODUCTION

Difference equations have been used to explain real dynamical systems that evolve discretely, in which the independent variables are integers. They are extremely useful in biology, physiology, physics, economics, and engineering, etc. (see for example, [8, 10]). In fact, almost all physical systems, biological processes and real-world applications can be modeled by nonlinear difference equations. Nonlinear difference equations may behave very complicated and there have not yet been any standard methods for studying them. Therefore, many researchers have been concentrated on rational difference equations because they believe that results about such equations offer prototypes towards the development of the basic global theory of nonlinear difference equations [17, 20].

In recent years, many types of rational difference equations have been investigated. There are a great interest in studying some behaviors of rational difference equations such as asymptotic stability, boundedness nature and periodicity character. For example, Kulenović et al. [18] investigated the global asymptotic stability of the positive equilibrium of the equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{A + x_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where the parameters and initial conditions x_{-1} and x_0 are nonnegative. DeVault et al. [9] studied the rational difference equation

$$x_{n+1} = \frac{p + x_{n-k}}{q x_n + x_{n-k}}, \quad n = 0, 1, 2, \ldots,$$

with the positive parameters and positive initial conditions and obtained the existence of period-two solutions and some sufficient conditions for global asymptotic stability of positive equilibrium. Kulenović and Ladas [17] studied the second order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

with nonnegative parameters and nonnegative initial conditions. They investigated the global stability, the boundedness and the periodicity of solutions of all special cases of such equation. Camouzis and Ladas [5] presented several results, open problems and conjectures on period-three solutions of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, 2, \ldots,$$

with nonnegative parameters and nonnegative initial conditions. They also obtained the period-three trichotomy behavior of some special cases of the above equation. For other related works, see [1, 4, 6, 7, 11–15, 19–21].

Our aim in this work is to investigate some behaviors of solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + B_2 x_{n-2} + \cdots + B_k x_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where the parameters $A, B_0, B_1, B_2, \ldots, B_k$ and the initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0$ are nonnegative real numbers such that the denominator in (1.1) is always positive.

The behavior of solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{A + Bx_n + x_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

was studied by Kulenović and Ladas in [17]. They established that when $A < 1$, (1.2) has the unique positive equilibrium

$$\bar{x} = \frac{1 - A}{1 + B}$$

and then the positive equilibrium $\bar{x}$ is locally asymptotically stable when $B < 1$ and is unstable when $B > 1$. Furthermore, they examined the existence of prime period-two solutions of (1.2) and showed that when $A < 1$ and $B > 1$, the prime period-two solution

$$\ldots, 0, 1 - A, 0, 1 - A, \ldots$$

of (1.2) is locally asymptotically stable.

In [5], Camouzis and Ladas investigated the behaviors of solutions of

$$x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1} + x_{n-2}}, \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

They proved that when $A \in [0, 1)$, (1.3) has the positive equilibrium point

$$\bar{x} = \frac{1 - A}{B + C + 1},$$

which is locally asymptotically stable provided

$$B < \frac{2 - A + \sqrt{A^2 + 8}}{2} \quad \text{and} \quad C < \frac{1 - 2B + \sqrt{5 + 4A + 4B(1 - A)}}{2}.$$

Furthermore, when $A, B + C \in [0, 1)$, every positive solution of (1.3) converges to the positive equilibrium $\bar{x}$ of (1.3). They also classified that all possible prime period-three solution of (1.3) are of the form

$$\ldots, 0, 0, \phi, 0, 0, \phi, \ldots$$

with $\phi \in (0, \infty)$ or

$$\ldots, 0, \phi, \psi, 0, \phi, \psi, \ldots$$

with $\phi, \psi \in (0, \infty)$ or

$$\ldots, \phi, \psi, \omega, \phi, \psi, \omega, \ldots$$

with $\phi, \psi, \omega \in (0, \infty)$. Moreover, they obtained that when $A \in [0, 1)$ and $B = C = 1$, every solution of (1.3) converges to a period-three solution.

We notice that (1.1) is a generalization of (1.2) and (1.3). For arbitrary positive integer k, it follows from the result in [6] that every positive solution of (1.1) converges to its positive equilibrium point

$$\bar{x} = \frac{1-A}{B_0 + B_1 + B_2 + \cdots + B_k},$$

provided $A \in [0,1)$ and $B_0 + B_1 + B_2 + \cdots + B_{k-1} < B_k$. Although the stability of equilibrium point of (1.1) has been investigated, however, several important problems remain open such as the periodicity and the local stability of periodic solutions of such equation. Therefore, in this thesis, we intent to investigate these behaviors. The results in this work are organized as follows.

Chapter II contains some basic definitions and results about difference equations which will be useful thoughtout this thesis.

Chapter III, the periodic character of (1.1) is investiged. We establish the existence of periodic solutions with prime period-two, prime period-three and extend to prime period-p, where p is a prime number. We use the idea of the proof of Theorem 3 in [5] to show that every solution of (1.1) converges to a period-$(k+1)$ solution under certain conditions.

Chapter IV, the local asymptotic stability of some periodic solutions of (1.1) is examined by using the linearized stability analysis.

Chapter V, we give some numerical examples to illustrate the results that we obtained. Also, we use MATLAB R2017a to draw graphs that represent the behavior of their solutions.

CHAPTER II

PRELIMINARIES

In this chapter, we introduce the definitions of some behaviors of solutions of difference equations. We also present some known results that will be useful for studying our work.

2.1 Stability and Linearized Stability Analysis

Definition 2.1 ([6, 12])**.** A **difference equation of order** $k+1$ is an equation of the form

$$x_{n+1} = f(x_n, x_{n-1}, x_{n-2}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

where f is a function which maps some set I^{k+1} into I and I is an interval of real numbers.

Definition 2.2 ([6, 13])**.** A **solution** of (2.1) is a sequence $\{x_n\}_{n=-k}^{\infty}$ that satisfies (2.1) for all $n \geq 0$. If we give initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0 \in I$, then there exists a unique solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1).

Definition 2.3 ([12, 13])**.** An **equilibrium point** of (2.1) is a point $\bar{x} \in I$ that satisfies the condition

$$\bar{x} = f(\bar{x}, \bar{x}, \bar{x}, \ldots, \bar{x}).$$

That is, the constant sequence $\{x_n\}_{n=-k}^{\infty}$ with $x_n = \bar{x}$ for all $n \geq -k$ is a solution of (2.1), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2.4 ([13])**.** Let $\bar{x}$ be an equilibrium point of (2.1).

(i) $\bar{x}$ is called **locally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of (2.1) with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \delta,$$

then

$$|x_n - \bar{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) $\bar{x}$ is called **locally asymptotically stable** if it is locally stable and if there exists $\gamma > 0$ such that $\{x_n\}_{n=-k}^{\infty}$ is a solution of (2.1) with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + \cdots + |x_0 - \bar{x}| < \gamma,$$

then

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iii) $\bar{x}$ is called a **global attractor** if for every solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1), we have

$$\lim_{n \to \infty} x_n = \bar{x}.$$

(iv) $\bar{x}$ is called **globally asymptotically stable** if it is locally stable and a global attractor.

(v) $\bar{x}$ is called **unstable** if it is not locally stable.

In general, we can rewrite (2.1) in vector form as

$$X_{n+1} = T(X_n), \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

where

$$X_n = \begin{bmatrix} x_n^{(0)} \\ x_n^{(1)} \\ x_n^{(2)} \\ \vdots \\ x_n^{(k-1)} \\ x_n^{(k)} \end{bmatrix}, \quad x_n^{(j)} = x_{n-k+j} \quad (j \in \{0,1,2,\ldots,k\})$$

and

$$T(X_n) = \begin{bmatrix} x_n^{(1)} \\ x_n^{(2)} \\ x_n^{(3)} \\ \vdots \\ x_n^{(k)} \\ f(x_n^{(0)}, x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(k)}) \end{bmatrix}.$$

Let $\{X_n\}_{n=0}^{\infty}$ be a unique solution of (2.2) which is determined by the initial condition $X_0 = (x_{-k}, x_{-k+1}, x_{-k+2}, \ldots, x_0) \in I^{k+1}$. Previously, we defined an equilibrium point of (2.1). Similarly, we can also define an equilibrium point of (2.2) to be a vector $\bar{X} \in I^{k+1}$ such that $\bar{X} = T(\bar{X})$ (see [6,15]). Next, we present the stability definitions for the difference equation (2.2). Let $\|\cdot\|$ denote the Euclidean norm.

Definition 2.5 ([15])**.** Let $\bar{X}$ be an equilibrium point of (2.2).

(i) $\bar{X}$ is called **locally stable** if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|X_0 - \bar{X}\| < \delta$ implying

$$\|X_n - \bar{X}\| < \epsilon \quad \text{for all} \quad n \geq 0.$$

(ii) $\bar{X}$ is called **locally asymptotically stable** if it is locally stable and there exists $\gamma > 0$ such that $\|X_0 - \bar{X}\| < \gamma$ implying

$$\lim_{n\to\infty} \|X_n - \bar{X}\| = 0.$$

(iii) $\bar{X}$ is called **globally asymptotically stable** if it is locally asymptotically stable and for every X_0,

$$\lim_{n\to\infty} \|X_n - \bar{X}\| = 0.$$

(iv) $\bar{X}$ is called **unstable** if it is not locally stable.

Suppose that the function f is a continuously differentiable in some open neighborhood of an equilibrium point $\bar{x}$. Let

$$p_i = \frac{\partial}{\partial u_i} f(\bar{x}, \bar{x}, \bar{x}, \ldots, \bar{x}) \quad \text{for all} \quad i \in \{0, 1, 2, \ldots, k\}$$

denote the partial derivative of $f(u_0, u_1, u_2, \ldots, u_k)$ with respect to u_i evaluated at the equilibrium point $\bar{x}$ of (2.1). Then, the linearized equation of (2.1) about the equilibrium point $\bar{x}$ is the linear difference equation of the form

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + p_2 y_{n-2} + \cdots + p_k y_{n-k}, \quad n = 0, 1, 2, \ldots, \tag{2.3}$$

and the equation

$$\lambda^{k+1} - p_0 \lambda^k - p_1 \lambda^{k-1} - \cdots - p_{k-1}\lambda - p_k = 0 \tag{2.4}$$

is called the characteristic equation of (2.3) about $\bar{x}$ (see [6]). Likewise, we assume that the function T is a continuously differentiable in some open neighborhood of an equilibrium point $\bar{X}$. Then, the linearized equation of (2.2) about the equilibrium point $\bar{X}$ is the equation of the form

$$Y_{n+1} = J_T(\bar{X}) Y_n, \tag{2.5}$$

where $J_T(\bar{X})$ is the Jacobian matrix of T at the equilibrium point $\bar{X}$ and the characteristic equation of (2.5) about $\bar{X}$ is

$$\det(J_T(\bar{X}) - \lambda I_{k+1}) = 0. \tag{2.6}$$

The following theorems will be useful for studying the local stability character of an equilibrium point $\bar{X}$ of (2.2).

Theorem 2.6 ([15])**.** *Let $\bar{X}$ be an equilibrium point of (2.2) and assume that T is a continuously differentiable in I^{k+1}. Then, the following statements are true:*

(i) If all eigenvalues of the Jacobian matrix $J_T(\bar{X})$ lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\bar{X}$ of (2.2) is locally asymptotically stable.

(ii) If at least one eigenvalue of $J_T(\bar{X})$ has absolute value greater than one, then the equilibrium point $\bar{X}$ of (2.2) is unstable.

Theorem 2.7 ([13])**.** *Assume that $p_0, p_1, p_2, \ldots, p_k$ are real numbers such that*

$$|p_0| + |p_1| + |p_2| + \cdots + |p_k| < 1.$$

Then, all roots of (2.4) lie inside the open unit disk $|\lambda| < 1$.

2.2 Periodicity

Definition 2.8 ([13])**.** A solution $\{x_n\}_{n=-k}^{\infty}$ of (2.1) is called **periodic with period** p (or a period-p solution) if there exists an integer $p \geq 1$ such that

$$x_{n+p} = x_n \quad \text{for all} \;\; n \geq -k. \tag{2.7}$$

A solution is called **periodic with prime period** p if p is the smallest positive integer for which (2.7) holds. In this case, a *p-tuple*

$$(x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+p})$$

of any p consecutive values of the solution is called a p**-cycle** of (2.1).

Finally, to study the convergence to periodic solutions, we have the following lemma which was proved in [13].

Lemma 2.9 ([13])**.** *Let* $\{x_n\}_{n=-k}^{\infty}$ *be a solution of (2.1) and let* $p \geq 1$ *be a positive integer. Suppose that there exist real numbers* $l_0, l_1, l_2, \ldots, l_{p-1} \in I$ *such that*

$$\lim_{n\to\infty} x_{pn+j} = l_j \quad \textit{for all} \ \ j \in \{0, 1, 2, \ldots, p-1\}.$$

Finally, let $\{y_n\}_{n=-k}^{\infty}$ *be the period-p sequence of real numbers in* I *such that for every integer* j *with* $0 \leq j \leq p-1$*, we have*

$$y_{pn+j} = l_j \quad \textit{for all} \ \ n \in \{0, 1, 2, \ldots\}.$$

Then, the following statements are true:

(i) $\{y_n\}_{n=-k}^{\infty}$ *is a period-p solution of (2.1).*

(ii) $\lim_{n\to\infty} x_{pn+j} = y_j$ *for every* $j \geq -k$*.*

CHAPTER III

PERIODICITY OF SOLUTIONS

In this chapter, we investigate the periodic character of (1.1). First, we establish the existence of periodic solutions with prime period- two, prime period-three and extend to prime period-p, where p is a prime number. Finally, we also prove the convergence of solution to a (not necessarily prime) period-$(k+1)$ solution.

3.1 Period-Two Solutions of (1.1)

We now examine the existence of prime period-two solutions of (1.1). It is easy to verify that the only positive prime period-one solution of (1.1) when $0 \leq A < 1$ is

$$x_n = \frac{1-A}{B_0 + B_1 + B_2 + \cdots + B_k} \quad \text{for all} \quad n \geq -k.$$

In this case, the solution is called an equilibrium solution of (1.1). Let us define the following notations which are used throughout this section:

$$b_0 := \sum_{\substack{2|i \\ 0 \leq i \leq k}} B_i \quad \text{and} \quad b_1 \; := \sum_{\substack{2|(i-1) \\ 0 \leq i \leq k}} B_i. \tag{3.1}$$

Theorem 3.1. *Let k be an even integer. Then, (1.1) has no positive solution of prime period-two.*

Proof. Suppose that there exists a distinctive positive solution

$$\ldots, \phi, \psi, \phi, \psi, \phi, \psi, \ldots$$

of prime period-two of (1.1). Since k is even, $x_n = x_{n-k}$. It follows from (1.1) that

$$\begin{aligned}\phi &= \frac{\psi}{A + B_0\psi + B_1\phi + B_2\psi + B_3\phi + \cdots + B_k\psi} \\ &= \frac{\psi}{A + (B_0 + B_2 + B_4 + \cdots + B_k)\psi + (B_1 + B_3 + B_5 + \cdots + B_{k-1})\phi} \\ &= \frac{\psi}{A + b_0\psi + b_1\phi}\end{aligned}$$

and

$$\begin{aligned}\psi &= \frac{\phi}{A + B_0\phi + B_1\psi + B_2\phi + B_3\psi + \cdots + B_k\phi} \\ &= \frac{\phi}{A + (B_0 + B_2 + B_4 + \cdots + B_k)\phi + (B_1 + B_3 + B_5 + \cdots + B_{k-1})\psi} \\ &= \frac{\phi}{A + b_0\phi + b_1\psi}.\end{aligned}$$

Then, we get

$$A\phi + b_0\phi\psi + b_1\phi^2 = \psi \tag{3.2}$$

and

$$A\psi + b_0\phi\psi + b_1\psi^2 = \phi. \tag{3.3}$$

Subtracting (3.2) by (3.3) gives

$$A(\phi - \psi) + b_1(\phi^2 - \psi^2) = -(\phi - \psi).$$

Since $\phi \neq \psi$, it follows that

$$A + b_1(\phi + \psi) = -1.$$

Notice that $A + b_1(\phi + \psi)$ is negative. Because A, b_1, ϕ and ψ are all nonnegative, we have a contradiction. Therefore, the proof is complete. $\square$

The following result states the necessary and sufficient conditions that (1.1), when k is odd, has a prime period-two solution.

Theorem 3.2. *Let k be an odd integer. Assume that $b_1 > 0$. (1.1) has a prime period-two solution if and only if*

$$0 < A < 1 \quad or \quad (A = 0 \quad and \quad b_0 > 0). \tag{3.4}$$

Proof. Suppose that there exists a nonnegative prime period-two solution

$$\ldots, \phi, \psi, \phi, \psi, \phi, \psi, \ldots$$

of (1.1). We will show that (3.4) holds. Since k is odd, it follows from (1.1) that

$$\begin{aligned}\phi &= \frac{\phi}{A + B_0\psi + B_1\phi + B_2\psi + B_3\phi + \cdots + B_k\phi} \\ &= \frac{\phi}{A + (B_0 + B_2 + B_4 + \cdots + B_{k-1})\psi + (B_1 + B_3 + B_5 + \cdots + B_k)\phi} \\ &= \frac{\phi}{A + b_0\psi + b_1\phi} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\psi &= \frac{\psi}{A + B_0\phi + B_1\psi + B_2\phi + B_3\psi + \cdots + B_k\psi} \\ &= \frac{\psi}{A + (B_0 + B_2 + B_4 + \cdots + B_{k-1})\phi + (B_1 + B_3 + B_5 + \cdots + B_k)\psi} \\ &= \frac{\psi}{A + b_0\phi + b_1\psi}. \end{aligned} \tag{3.6}$$

Then,

$$A\phi + b_0\phi\psi + b_1\phi^2 = \phi \tag{3.7}$$

and

$$A\psi + b_0\phi\psi + b_1\psi^2 = \psi. \tag{3.8}$$

Subtracting (3.7) by (3.8) gives

$$A(\phi - \psi) + b_1(\phi^2 - \psi^2) = \phi - \psi.$$

Since $\phi \neq \psi$, it follows that

$$A + b_1(\phi + \psi) = 1.$$

Thus, we obtain

$$\phi + \psi = \frac{1 - A}{b_1}. \tag{3.9}$$

Since $\phi + \psi$ is positive, it implies that $0 \leq A < 1$. Now we suppose that $A \notin (0, 1)$. Then, $A = 0$ and by adding (3.7) and (3.8), we have

$$2b_0\phi\psi + b_1(\phi^2 + \psi^2) = \phi + \psi.$$

The relation $\phi^2 + \psi^2 = (\phi + \psi)^2 - 2\phi\psi$ implies that

$$2b_0\phi\psi + b_1(\phi + \psi)^2 - 2b_1\phi\psi = \phi + \psi,$$

and thus,

$$2\left(b_0 - b_1\right)\phi\psi + b_1(\phi + \psi)^2 = \phi + \psi.$$

Then, we see that

$$\begin{aligned} 2\left(b_0 - b_1\right)\phi\psi &= (\phi + \psi) - b_1(\phi + \psi)^2 \\ &= (\phi + \psi)\left[1 - b_1(\phi + \psi)\right]. \end{aligned}$$

It follows from (3.9) that

$$1 - b_1(\phi + \psi) = 1 - b_1\left(\frac{1}{b_1}\right) = 0.$$

Therefore, we obtain

$$2\left(b_0 - b_1\right)\phi\psi = 0.$$

If $\phi\psi \neq 0$, then $b_0 = b_1$, which implies that $b_0 > 0$. If $\phi = 0$, then it follows from (3.5) and (3.6) that

$$0 = \frac{0}{b_0\psi} \quad \text{and} \quad \psi = \frac{1}{b_1}.$$

Thus, b_0 must be positive. Similarly, if $\psi = 0$, then $b_0 > 0$. Hence, (3.4) is true.

Conversely, we assume that (3.4) holds. Then, there exist two distinct nonnegative real numbers ϕ and ψ such that

$$\phi = 0 \quad \text{and} \quad \psi = \frac{1-A}{b_1}.$$

Suppose that

$$x_{-k} = 0, \quad x_{-k+1} = \psi, \quad x_{-k+2} = 0, \quad x_{-k+3} = \psi, \quad \ldots, \quad x_{-1} = 0 \quad \text{and} \quad x_0 = \psi.$$

From (1.1), we have

$$\begin{aligned} x_1 &= \frac{x_{-k}}{A + B_0x_0 + B_1x_{-1} + B_2x_{-2} + \cdots + B_kx_{-k}} \\ &= \frac{0}{A + (B_0 + B_2 + B_4 + \cdots + B_{k-1})\psi} \\ &= \frac{0}{A + b_0\psi}. \end{aligned}$$

It follows from (3.4) that $A + b_0\psi > 0$ and then $x_1 = 0$. Thus, we see that

$$\begin{aligned} x_2 &= \frac{x_{-k+1}}{A + B_0x_1 + B_1x_0 + B_2x_{-1} + \cdots + B_kx_{-k+1}} \\ &= \frac{\psi}{A + (B_1 + B_3 + B_5 + \cdots + B_k)\psi} \\ &= \frac{\psi}{A + b_1\psi} \\ &= \frac{\psi}{A + (1-A)}, \end{aligned}$$

which implies that $x_2 = \psi$. Using the mathematical induction, we have

$$x_{2n-1} = 0 \quad \text{and} \quad x_{2n} = \psi \quad \text{ for all } \; n \geq \frac{1-k}{2}.$$

Therefore, the difference equation (1.1) has a prime period-two solution

$$\ldots, 0, \psi, 0, \psi, 0, \psi, \ldots$$

Consequently, the proof is now complete. $\square$

3.2 Period-Three Solutions of (1.1)

In this section, the existence of prime period-three solutions of (1.1) will be investigated. We now give the necessary and sufficient conditions that (1.1) has a periodic solution with prime period-three. The following notation is used throughout this section:

$$b_i := \sum_{\substack{3|(m-i)\\ 0\leq m\leq k}} B_m \tag{3.10}$$

for each $i \in \{0, 1, 2\}$.

Theorem 3.3. *Let k be a positive interger such that $3 \mid (k+1)$ and $b_2 > 0$. (1.1) has positive prime period-three solutions of the form*

$$\ldots, 0, 0, \gamma, 0, 0, \gamma, \ldots \tag{3.11}$$

with $\gamma \in (0, \infty)$ if and only if

$$0 < A < 1 \quad or \quad (A = 0 \quad and \quad b_0, b_1 > 0). \tag{3.12}$$

Furthermore, when (3.12) holds, (1.1) has a unique positive prime period-three solution with

$$\gamma = \frac{1-A}{b_2}.$$

Proof. Assume that there exists a prime period-three solution

$$\ldots, \phi, \psi, \gamma, \phi, \psi, \gamma, \ldots.$$

Since $3 \mid (k+1)$, it follows from (1.1) that

$$\phi = \frac{\phi}{A + b_0\gamma + b_1\psi + b_2\phi}, \tag{3.13}$$

$$\psi = \frac{\psi}{A + b_0\phi + b_1\gamma + b_2\psi}, \tag{3.14}$$

$$\gamma = \frac{\gamma}{A + b_0\psi + b_1\phi + b_2\gamma}. \tag{3.15}$$

Suppose that $\phi = \psi = 0$ and $\gamma > 0$. From (3.13) and (3.14), we have

$$0 = \frac{0}{A + b_0\gamma} \quad \text{and} \quad 0 = \frac{0}{A + b_1\gamma},$$

which means $A > 0$ or $(A = 0$ and $b_0, b_1 > 0)$. Then, from (3.15), we see that

$$\gamma = \frac{1 - A}{b_2},$$

which implies $0 < A < 1$. Therefore, (3.12) is true.

Conversely, we assume that (3.12) holds. Then, there exists a positive real number γ such that

$$\gamma = \frac{1 - A}{b_2}.$$

Suppose that

$$x_{-k} = 0, \quad x_{-k+1} = 0, \quad x_{-k+2} = \gamma, \quad \ldots, \quad x_{-2} = 0, \quad x_{-1} = 0 \quad \text{and} \quad x_0 = \gamma.$$

From (1.1), we have

$$\begin{aligned} x_1 &= \frac{x_{-k}}{A + B_0x_0 + B_1x_{-1} + B_2x_{-2} + \cdots + B_kx_{-k}} \\ &= \frac{0}{A + b_0\gamma + b_1(0) + b_2(0)} \\ &= \frac{0}{A + b_0\gamma}. \end{aligned}$$

By (3.12), we have $A + b_0\gamma > 0$. This implies that $x_1 = 0$. Then,

$$\begin{aligned} x_2 &= \frac{x_{-k+1}}{A + B_0x_1 + B_1x_0 + B_2x_{-1} + \cdots + B_kx_{-k+1}} \\ &= \frac{0}{A + b_0(0) + b_1\gamma + b_2(0)} \\ &= \frac{0}{A + b_1\gamma}. \end{aligned}$$

From (3.12), we get $A + b_1\gamma > 0$. It implies that $x_2 = 0$. Then,

$$\begin{aligned} x_3 &= \frac{x_{-k+2}}{A + B_0x_2 + B_1x_1 + B_2x_0 + \cdots + B_kx_{-k+2}} \\ &= \frac{\gamma}{A + b_0(0) + b_1(0) + b_2\gamma} \\ &= \frac{\gamma}{A + b_2\gamma} \\ &= \frac{\gamma}{A + (1 - A)}. \end{aligned}$$

Thus, $x_3 = \gamma$. Using the mathematical induction, we have

$$x_{3n-2} = 0, \quad x_{3n-1} = 0 \quad \text{and} \quad x_{3n} = \gamma \quad \text{for all} \quad n \geq \frac{2-k}{3}.$$

Hence, (1.1) has prime period-three solutions of the form (3.11). □

Theorem 3.4. *Let k be a positive integer such that $3 \mid (k+1)$. Assume that $A + b_0 + b_1 > 0$ and $b_2 > 0$. (1.1) has positive prime period-three solutions of the form*

$$\ldots, 0, \psi, \gamma, 0, \psi, \gamma, \ldots \tag{3.16}$$

with $\psi, \gamma \in (0, \infty)$ if and only if one of the following statements holds:

(i) $0 \leq A < 1$ *and* $b_0, b_1 \in (b_2, \infty)$.

(ii) $0 \leq A < 1$ *and* $b_0, b_1 \in [0, b_2)$.

Furthermore, when (i) or (ii) holds, (1.1) has a unique positive prime period-three solution with

$$\psi = \frac{(1-A)(b_2 - b_1)}{b_2^2 - b_0b_1} \quad \text{and} \quad \gamma = \frac{(1-A)(b_2 - b_0)}{b_2^2 - b_0b_1}.$$

Proof. Assume that there exists a positive prime period-three solution

$$\ldots, 0, \psi, \gamma, 0, \psi, \gamma, \ldots.$$

Since $3 \mid (k+1)$, it follows from (1.1) that

$$0 = \frac{0}{A + b_0\gamma + b_1\psi}, \tag{3.17}$$

$$\psi = \frac{\psi}{A + b_1\gamma + b_2\psi}, \tag{3.18}$$

$$\gamma = \frac{\gamma}{A + b_0\psi + b_2\gamma}. \tag{3.19}$$

Then, we have

$$A\psi + b_1\psi\gamma + b_2\psi^2 = \psi$$

and

$$A\gamma + b_0\psi\gamma + b_2\gamma^2 = \gamma.$$

Since $\gamma, \psi > 0$, we obtain that

$$b_1\gamma + b_2\psi = 1 - A \tag{3.20}$$

and

$$b_0\psi + b_2\gamma = 1 - A, \tag{3.21}$$

from which it follows that $0 \le A < 1$. Subtracting (3.20) by (3.21) gives

$$(b_1 - b_2)\gamma = (b_0 - b_2)\psi.$$

Because γ and ψ are both positive, we get either $b_0, b_1 \in (b_2, \infty)$ or $b_0, b_1 \in [0, b_2)$.

Conversely, assume that (i) or (ii) holds. Then, there exist two positive real numbers ψ and γ such that

$$\psi = \frac{(1-A)(b_2 - b_1)}{b_2^2 - b_0 b_1} \quad \text{and} \quad \gamma = \frac{(1-A)(b_2 - b_0)}{b_2^2 - b_0 b_1}.$$

Suppose that

$$x_{-k} = 0, \quad x_{-k+1} = \psi, \quad x_{-k+2} = \gamma, \quad \ldots, \quad x_{-2} = 0, \quad x_{-1} = \psi \quad \text{and} \quad x_0 = \gamma.$$

From (1.1), we have

$$\begin{aligned}
x_1 &= \frac{x_{-k}}{A + B_0x_0 + B_1x_{-1} + B_2x_{-2} + \cdots + B_kx_{-k}} \\
&= \frac{0}{A + b_0\gamma + b_1\psi + b_2(0)} \\
&= \frac{0}{A + b_0\gamma + b_1\psi}.
\end{aligned}$$

Since $A + b_0\gamma + b_1\psi > 0$, $x_1 = 0$. Then,

$$\begin{aligned}
x_2 &= \frac{x_{-k+1}}{A + B_0x_1 + B_1x_0 + B_2x_{-1} + \cdots + B_kx_{-k+1}} \\
&= \frac{\psi}{A + b_0(0) + b_1\gamma + b_2\psi} \\
&= \frac{\psi}{A + b_1\left(\frac{(1-A)(b_2-b_0)}{b_2^2-b_0b_1}\right) + b_2\left(\frac{(1-A)(b_2-b_1)}{b_2^2-b_0b_1}\right)} \\
&= \frac{\psi}{A + (1-A)\left(\frac{b_1b_2-b_0b_1+b_2^2-b_1b_2}{b_2^2-b_0b_1}\right)} \\
&= \frac{\psi}{A + (1-A)}.
\end{aligned}$$

Thus, $x_2 = \psi$. Then, we see that

$$\begin{aligned}
x_3 &= \frac{x_{-k+2}}{A + B_0x_2 + B_1x_1 + B_2x_0 + \cdots + B_kx_{-k+2}} \\
&= \frac{\gamma}{A + b_0\psi + b_1(0) + b_2\gamma} \\
&= \frac{\gamma}{A + b_0\left(\frac{(1-A)(b_2-b_1)}{b_2^2-b_0b_1}\right) + b_2\left(\frac{(1-A)(b_2-b_0)}{b_2^2-b_0b_1}\right)} \\
&= \frac{\gamma}{A + (1-A)\left(\frac{b_0b_2-b_0b_1+b_2^2-b_0b_2}{b_2^2-b_0b_1}\right)} \\
&= \frac{\gamma}{A + (1-A)}.
\end{aligned}$$

Hence, $x_3 = \gamma$. Using the mathematical induction, we have

$$x_{3n-2} = 0, \quad x_{3n-1} = \psi \quad \text{and} \quad x_{3n} = \gamma \quad \text{for all} \quad n \geq \frac{2-k}{3}.$$

Thus, (1.1) has prime period-three solutions of the form (3.16). $\square$

Theorem 3.5. *Let k be a positive integer such that $3 \mid (k+1)$. If $b_i \neq b_j$ for some $i, j \in \{0, 1, 2\}$ with $i \neq j$, then (1.1) has no positive prime period-three solutions of the form*

$$\ldots, \phi, \psi, \gamma, \phi, \psi, \gamma, \ldots \tag{3.22}$$

with $\phi, \psi, \gamma \in (0, \infty)$.

Proof. Assume that there exist $i, j \in \{0, 1, 2\}$ such that $i \neq j$ and $b_i \neq b_j$. Suppose that (1.1) has a prime period-three solution of the form

$$\ldots, \phi, \psi, \gamma, \phi, \psi, \gamma, \ldots,$$

where ϕ, ψ and γ are all positive real numbers. Then, it follows form (1.1) that

$$\begin{aligned} \phi &= \frac{\phi}{A + b_0\gamma + b_1\psi + b_2\phi}, \\ \psi &= \frac{\psi}{A + b_0\phi + b_1\gamma + b_2\psi}, \\ \gamma &= \frac{\gamma}{A + b_0\psi + b_1\phi + b_2\gamma}. \end{aligned}$$

Since $\phi, \psi, \gamma > 0$, we get

$$\begin{aligned} b_0\gamma + b_1\psi + b_2\phi &= 1 - A, \\ b_0\phi + b_1\gamma + b_2\psi &= 1 - A, \\ b_0\psi + b_1\phi + b_2\gamma &= 1 - A, \end{aligned}$$

which is equivalent to the matrix equation

$$\begin{bmatrix} b_2 & b_1 & b_0 \\ b_0 & b_2 & b_1 \\ b_1 & b_0 & b_2 \end{bmatrix} \begin{bmatrix} \phi \\ \psi \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 - A \\ 1 - A \\ 1 - A \end{bmatrix}. \tag{3.23}$$

Let $\mathcal{A}$ be the coefficient matrix of (3.23). We now calculate the determinant of $\mathcal{A}$. Note that

$$\begin{aligned}b_0^3 + b_1^3 + b_2^3 - 3b_0b_1b_2 &= (b_0 + b_1 + b_2)(b_0^2 + b_1^2 + b_2^2 - b_0b_1 - b_1b_2 - b_0b_2)\\ &= \frac{1}{2}(b_0 + b_1 + b_2)[(b_0 - b_1)^2 + (b_1 - b_2)^2 + (b_0 - b_2)^2].\end{aligned}$$

Because $b_i \neq b_j$ for some $i, j \in \{0, 1, 2\}$ with $i \neq j$, $\det \mathcal{A} \neq 0$. By the Cramer's rule, the system of equations (3.23) has a unique solution with

$$\phi = \frac{\det \mathcal{A}_1}{\det \mathcal{A}}, \quad \psi = \frac{\det \mathcal{A}_2}{\det \mathcal{A}} \quad \text{and} \quad \gamma = \frac{\det \mathcal{A}_3}{\det \mathcal{A}},$$

where $\mathcal{A}_i$ is the matrix obtained by replacing the i^{th} column of $\mathcal{A}$ by $(1 - A, 1 - A, 1 - A)^T$. Notice that for any $i \in \{1, 2, 3\}$,

$$\det \mathcal{A}_i = (1 - A)(b_0^2 + b_1^2 + b_2^2 - b_0b_1 - b_1b_2 - b_0b_2).$$

This implies that $\phi = \psi = \gamma$, which is a contradiction. Therefore, (1.1) has no prime period-three solutions of the form (3.22). $\square$

3.3 Period-p Solutions of (1.1)

First, we give the necessary and sufficient conditions that (1.1) has periodic solutions with prime period-p, where p is a prime number. Define the notation that will be useful in this section:

$$b_i := \sum_{\substack{p|(m-i)\\ 0\leq m\leq k}} B_m \tag{3.24}$$

for each $i \in \{0, 1, 2, \ldots, p - 1\}$.

Theorem 3.6. *Let p be a prime number such that $p \mid (k+1)$ and $b_{p-1} > 0$. (1.1) has positive prime period-p solutions of the form*

$$\ldots, 0, \ldots, 0, \gamma_p, 0, \ldots, 0, \gamma_p, \ldots \tag{3.25}$$

with $\gamma_p \in (0, \infty)$ *if and only if*

$$0 < A < 1 \quad or \quad (A = 0 \quad and \quad b_0, b_1, b_2, \dots, b_{p-2} > 0). \tag{3.26}$$

Furthermore, when (3.26) holds, (1.1) has a unique prime positive prime period-p solutions with

$$\gamma = \frac{1 - A}{b_{p-1}}.$$

Proof. Assume that there exists a prime period-p solution

$$\dots, 0, \dots, 0, \gamma_p, 0, \dots, 0, \gamma_p, \dots.$$

of (1.1). Since $p \mid (k + 1)$, it follows from (1.1) that for any $1 \leq i \leq p - 1$,

$$0 = \frac{0}{A + b_{i-1}\gamma_p}$$

and

$$\gamma_p = \frac{\gamma_p}{A + b_{p-1}\gamma_p}.$$

Since $\gamma_p > 0$, $\gamma_p = (1 - A)/b_{p-1}$. From which it implies that (3.26) holds.

Conversely, assume that (3.26) is true. Then, there exists a positive real number γ_p such that

$$\gamma_p = \frac{1 - A}{b_{p-1}}.$$

Notice that for each $0 \leq i \leq p - 2$, $A + b_i\gamma_p > 0$. Suppose that

$$x_{-k} = x_{-k+1} = x_{-k+2} = \dots = x_{-k+p-2} = 0, \quad x_{-k+p-1} = \gamma_p, \quad \dots,$$

$$x_{-p+1} = x_{-p+2} = x_{-p+3} = \dots = x_{-1} = 0, \quad x_0 = \gamma_p.$$

From (1.1), we have

$$\begin{aligned} x_1 &= \frac{x_{-k}}{A + B_0x_0 + B_1x_{-1} + B_2x_{-2} \dots + B_kx_{-k}} \\ &= \frac{0}{A + b_0\gamma_p + b_1(0) + b_2(0) + \dots + b_{p-1}(0)} \\ &= \frac{0}{A + b_0\gamma_p}. \end{aligned}$$

Thus, $x_1 = 0$. Then, we see that

$$\begin{aligned} x_2 &= \frac{x_{1-k}}{A + B_0 x_1 + B_1 x_0 + B_2 x_{-1} \cdots + B_k x_{1-k}} \\ &= \frac{0}{A + b_0(0) + b_1(\gamma_p) + b_2(0) + \cdots + b_{p-1}(0)} \\ &= \frac{0}{A + b_1 \gamma_p}. \end{aligned}$$

Thus, $x_2 = 0$. Then, we obtain that for each $3 \leq i \leq p-1$,

$$\begin{aligned} x_i &= \frac{x_{(i-1)-k}}{A + B_0 x_{i-1} + B_1 x_{i-2} + B_2 x_{i-3} \cdots + B_k x_{(i-1)-k}} \\ &= \frac{0}{A + b_{i-1} \gamma_p}. \end{aligned}$$

Therefore, $x_i = 0$ for all $1 \leq i \leq p-1$ and then we get

$$\begin{aligned} x_p &= \frac{x_{(p-1)-k}}{A + B_0 x_{p-1} + B_1 x_{p-2} + \cdots + B_{p-2} x_1 + B_{p-1} x_0} \\ &= \frac{\gamma_p}{A + b_0(0) + b_1(0) + \cdots + b_{p-2}(0) + b_{p-1}\gamma_p} \\ &= \frac{\gamma_p}{A + b_{p-1}\left(\frac{1-A}{b_{p-1}}\right)}. \end{aligned}$$

This implies that $x_p = \gamma_p$. Using the mathematical induction, we have

$$x_{pn-p+1} = x_{pn-p+2} = x_{pn-p+3} = \cdots = x_{pn-1} = 0$$

and

$$x_{pn} = \gamma_p \quad \text{for all} \quad n \geq \frac{p-1-k}{p}.$$

Hence, (1.1) has positive prime period-p solutions of the form (3.25). The proof is now complete. □

Theorem 3.7. *Let p be a prime number such that $p > 2$ and $p \mid (k+1)$. Assume that $A + b_i > 0$ for $i \in \{0, 1, 2, \ldots, p-2\}$ and $b_{p-1} > 0$. (1.1) has positive prime period-p solutions of the form*

$$\ldots, 0, \ldots, 0, \gamma_j, 0, \ldots, 0, \gamma_p, \ldots \tag{3.27}$$

with $0 < j < p$ *and* $\gamma_j, \gamma_p \in (0, \infty)$ *if and only if one of the following statements holds:*

(i) $0 \leq A < 1$ *and* $b_{j-1}, b_{p-j-1} \in (b_{p-1}, \infty)$.

(ii) $0 \leq A < 1$ *and* $b_{j-1}, b_{p-j-1} \in [0, b_{p-1})$.

Furthermore, when (*i*) *or* (*ii*) *holds, (1.1) has a unique positive prime period-three solution with*

$$\gamma_j = \frac{(1-A)(b_{p-1} - b_{j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}} \quad \textit{and} \quad \gamma_p = \frac{(1-A)(b_{p-1} - b_{p-j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}}.$$

Proof. Assume that there exists a positive prime period-three solution

$$\ldots, 0, \ldots, 0, \gamma_j, 0, \ldots, 0, \gamma_p, \ldots,$$

where $\gamma_j, \gamma_p \in (0, \infty)$. Since $p \mid (k+1)$, it follows from (1.1) that

$$\gamma_j = \frac{\gamma_j}{A + b_{j-1}\gamma_p + b_{p-1}\gamma_j}$$

and

$$\gamma_p = \frac{\gamma_p}{A + b_{p-j-1}\gamma_j + b_{p-1}\gamma_p}.$$

Since $\gamma_j, \gamma_p > 0$, we obtain that

$$b_{j-1}\gamma_p + b_{p-1}\gamma_j = 1 - A \tag{3.28}$$

and

$$b_{p-j-1}\gamma_j + b_{p-1}\gamma_p = 1 - A, \tag{3.29}$$

from which it follows that $0 \leq A < 1$. Subtracting (3.28) by (3.29) gives

$$(b_{j-1} - b_{p-1})\gamma_p = (b_{p-j-1} - b_{p-1})\gamma_j.$$

Because γ_j and γ_p are both positive, we get either $b_{j-1}, b_{p-j-1} \in (b_{p-1}, \infty)$ or $b_{j-1}, b_{p-j-1} \in [0, b_{p-1})$. Therefore, the statement (i) or (ii) holds.

Conversely, let $j \in \{1, 2, 3, \ldots, p-1\}$. Assume that (i) or (ii) holds. Then, there exist positive real numbers γ_j and γ_p such that

$$\gamma_j = \frac{(1-A)(b_{p-1} - b_{j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}} \quad \text{and} \quad \gamma_p = \frac{(1-A)(b_{p-1} - b_{p-j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}}.$$

Suppose that

$$x_{-k} = x_{-k+1} = x_{-k+2} = \cdots = x_{-k+j-2} = 0,\ x_{-k+j-1} = \gamma_j,$$

$$x_{-k+j} = x_{-k+j+1} = x_{-k+j+2} = \cdots = x_{-k+p-2} = 0,\ x_{-k+p-1} = \gamma_p, \quad \ldots,$$

$$x_{-p+1} = x_{-p+2} = x_{-p+3} = \cdots = x_{-p+j-1} = 0,\ x_{-p+j} = \gamma_j,$$

$$x_{-p+j+1} = x_{-p+j+2} = x_{-p+j+3} = \cdots = x_{-1} = 0,\ x_0 = \gamma_p.$$

From (1.1), for each $1 \le i \le j-1$, we have

$$x_i = \frac{0}{A + b_{i-1}\gamma_p + b_{p-j+i-1}\gamma_j}.$$

Note that $A + b_{i-1}\gamma_p + b_{p-j+i-1}\gamma_j > 0$ for all $1 \le i \le j-1$. This implies that $x_i = 0$ for all $1 \le i \le j-1$. Then, we see that

$$\begin{aligned} x_j &= \frac{x_{-k+j-1}}{A + b_{j-1}x_0 + b_{p-1}x_{-p+j}} \\ &= \frac{\gamma_j}{A + b_{j-1}\gamma_p + b_{p-1}\gamma_j} \\ &= \frac{\gamma_j}{A + b_{j-1}\left(\frac{(1-A)(b_{p-1}-b_{p-j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}}\right) + b_{p-1}\left(\frac{(1-A)(b_{p-1}-b_{j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}}\right)} \\ &= \frac{\gamma_j}{A + (1-A)}. \end{aligned}$$

This implies that $x_j = \gamma_j$. Next, for each $j+1 \le i \le p-1$, we have

$$\begin{aligned} x_i &= \frac{x_{-k+i-1}}{A + b_{i-1}x_0 + b_{-j+i-1}x_j} \\ &= \frac{0}{A + b_{i-1}\gamma_p + b_{-j+i-1}\gamma_j}. \end{aligned}$$

Note that $A + b_{i-1}\gamma_p + b_{-j+i-1}\gamma_j > 0$ for all $j+1 \leq i \leq p-1$. Thus, $x_i = 0$ for all $j+1 \leq i \leq p-1$. Then, we get

$$\begin{aligned}
x_p &= \frac{x_{-k+p-1}}{A + b_{p-1}x_0 + b_{p-j-1}x_j} \\
&= \frac{\gamma_p}{A + b_{p-1}\gamma_p + b_{p-j-1}\gamma_j} \\
&= \frac{\gamma_p}{A + b_{p-1}\left(\frac{(1-A)(b_{p-1}-b_{p-j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}}\right) + b_{p-j-1}\left(\frac{(1-A)(b_{p-1}-b_{j-1})}{b_{p-1}^2 - b_{p-j-1}b_{j-1}}\right)} \\
&= \frac{\gamma_p}{A + (1-A)}.
\end{aligned}$$

Hence, $x_p = \gamma_p$. Using the mathematical induction, we have

$$\begin{gathered}
x_{pn-p+1} = x_{pn-p+2} = \cdots = x_{pn-p+j-1} = x_{pn-p+j+1} = \cdots = x_{pn-1} = 0, \\
x_{pn-p+j} = \gamma_j \quad \text{and} \quad x_{pn} = \gamma_p \quad \text{for all} \quad n \geq \frac{p-1-k}{p}.
\end{gathered}$$

Therefore, (1.1) has positive prime period-p solutions of the form (3.27). The proof is now complete. $\square$

Here, we give the following lemma to study the existence of p-cycle, namely, $(0, \ldots, 0, \gamma_j, 0, \ldots, 0, \gamma_l, 0, \ldots, 0, \gamma_p)$, where γ_j, γ_l and γ_p are all positive real numbers.

Lemma 3.8. *Let $\mathcal{A}$ be the matrix*

$$\begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(2)} & a_{13}^{(3)} \\
a_{21}^{(3)} & a_{22}^{(1)} & a_{23}^{(2)} \\
a_{31}^{(2)} & a_{32}^{(3)} & a_{33}^{(1)}
\end{bmatrix}$$

and for any $1 \leq i, j \leq 3$,

$$\begin{aligned}
&a_{ij_1}^{(1)} \geq a_{ij_2}^{(2)} \geq a_{ij_3}^{(3)} \geq 0, \\
&a_{i_1 j}^{(1)} \geq a_{i_2 j}^{(2)} \geq a_{i_3 j}^{(3)} \geq 0,
\end{aligned} \tag{3.30}$$

where $i_1, i_2, i_3, j_1, j_2, j_3 \in \{1, 2, 3\}$.

(i) If

$$\min\{a_{ij}^{(1)} \mid i,j \in \{1,2,3\}\} > \max\{a_{ij}^{(2)} \mid i,j \in \{1,2,3\}\} \tag{3.31}$$

or

$$\min\{a_{ij}^{(2)} \mid i,j \in \{1,2,3\}\} > \max\{a_{ij}^{(3)} \mid i,j \in \{1,2,3\}\}, \tag{3.32}$$

then $\det \mathcal{A} > 0$.

(ii) Let $l \in \{1,2,3\}$*. If* $a_{i_1 l}^{(1)} = a_{i_2 l}^{(2)} = a_{i_3 l}^{(3)} = c > 0$ *and*

$$\min\{a_{ij}^{(1)} \mid j \in \{1,2,3\} \setminus \{l\}\} > \max\{a_{ij}^{(2)} \mid j \in \{1,2,3\} \setminus \{l\}\} \tag{3.33}$$

or

$$\min\{a_{ij}^{(2)} \mid j \in \{1,2,3\} \setminus \{l\}\} > \max\{a_{ij}^{(3)} \mid j \in \{1,2,3\} \setminus \{l\}\}, \tag{3.34}$$

then $\det \mathcal{A} > 0$.

Proof. (i) Assume that (3.31) holds. We now consider

$$\begin{aligned}
\det \mathcal{A} &= a_{11}^{(1)}\left(a_{22}^{(1)}a_{33}^{(1)} - a_{32}^{(3)}a_{23}^{(2)}\right) + a_{12}^{(2)}\left(a_{23}^{(2)}a_{31}^{(2)} - a_{33}^{(1)}a_{21}^{(3)}\right) + a_{13}^{(3)}\left(a_{32}^{(3)}a_{21}^{(3)} - a_{31}^{(2)}a_{22}^{(1)}\right)\\
&> a_{12}^{(2)}\left(a_{22}^{(1)}a_{33}^{(1)} - a_{32}^{(3)}a_{23}^{(2)}\right) + a_{12}^{(2)}\left(a_{23}^{(2)}a_{31}^{(2)} - a_{33}^{(1)}a_{21}^{(3)}\right) + a_{12}^{(2)}\left(a_{32}^{(3)}a_{21}^{(3)} - a_{31}^{(2)}a_{22}^{(1)}\right)\\
&= a_{12}^{(2)}\left[a_{22}^{(1)}\left(a_{33}^{(1)} - a_{31}^{(2)}\right) + a_{23}^{(2)}\left(a_{31}^{(2)} - a_{32}^{(3)}\right) + a_{21}^{(3)}\left(a_{32}^{(3)} - a_{33}^{(1)}\right)\right]\\
&> a_{12}^{(2)}\left[a_{23}^{(2)}\left(a_{33}^{(1)} - a_{31}^{(2)}\right) + a_{23}^{(2)}\left(a_{31}^{(2)} - a_{32}^{(3)}\right) + a_{23}^{(2)}\left(a_{32}^{(3)} - a_{33}^{(1)}\right)\right]\\
&= a_{12}^{(2)}a_{23}^{(2)}(0).
\end{aligned}$$

This implies that $\det \mathcal{A} > 0$. Similarly, if (3.32) holds, then $\det \mathcal{A} > 0$.

(ii) Let $l \in \{1,2,3\}$. Assume that $a_{i_1 l}^{(1)} = a_{i_2 l}^{(2)} = a_{i_3 l}^{(3)} = c > 0$ and (3.33) holds.

Case 1. $l = 1$,

$$\det \mathcal{A} = \det \begin{bmatrix} c & a_{12}^{(2)} & a_{13}^{(3)} \\ c & a_{22}^{(1)} & a_{23}^{(2)} \\ c & a_{32}^{(3)} & a_{33}^{(1)} \end{bmatrix}$$

$$\begin{aligned}
&= c\left[a_{12}^{(2)}\left(a_{23}^{(2)} - a_{33}^{(1)}\right) + a_{22}^{(1)}\left(a_{33}^{(1)} - a_{13}^{(3)}\right) + a_{32}^{(3)}\left(a_{13}^{(3)} - a_{23}^{(2)}\right)\right] \\
&> c\left[a_{12}^{(2)}\left(a_{23}^{(2)} - a_{33}^{(1)}\right) + a_{12}^{(2)}\left(a_{33}^{(1)} - a_{13}^{(3)}\right) + a_{12}^{(2)}\left(a_{13}^{(3)} - a_{23}^{(2)}\right)\right] \\
&= c a_{12}^{(2)}(0).
\end{aligned}$$

This implies that $\det \mathcal{A} > 0$.

Case 2. $l = 2$,

$$\det \mathcal{A} = \det \begin{bmatrix} a_{11}^{(1)} & c & a_{13}^{(3)} \\ a_{21}^{(3)} & c & a_{23}^{(2)} \\ a_{31}^{(2)} & c & a_{33}^{(1)} \end{bmatrix}$$

$$\begin{aligned}
&= c\left[a_{11}^{(1)}\left(a_{33}^{(1)} - a_{23}^{(2)}\right) + a_{21}^{(3)}\left(a_{13}^{(3)} - a_{33}^{(1)}\right) + a_{31}^{(2)}\left(a_{23}^{(2)} - a_{13}^{(3)}\right)\right] \\
&> c\left[a_{31}^{(2)}\left(a_{33}^{(1)} - a_{23}^{(2)}\right) + a_{31}^{(2)}\left(a_{13}^{(3)} - a_{33}^{(1)}\right) + a_{31}^{(2)}\left(a_{23}^{(2)} - a_{13}^{(3)}\right)\right] \\
&= c a_{31}^{(2)}(0).
\end{aligned}$$

Then, we get $\det \mathcal{A} > 0$.

Case 3. $l = 3$,

$$\det \mathcal{A} = \det \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(2)} & c \\ a_{21}^{(3)} & a_{22}^{(1)} & c \\ a_{31}^{(2)} & a_{32}^{(3)} & c \end{bmatrix}$$

$$\begin{aligned}
&= c\left[a_{11}^{(1)}\left(a_{22}^{(1)} - a_{32}^{(3)}\right) + a_{21}^{(3)}\left(a_{32}^{(3)} - a_{12}^{(2)}\right) + a_{31}^{(2)}\left(a_{12}^{(2)} - a_{22}^{(1)}\right)\right] \\
&> c\left[a_{31}^{(2)}\left(a_{22}^{(1)} - a_{32}^{(3)}\right) + a_{31}^{(2)}\left(a_{32}^{(3)} - a_{12}^{(2)}\right) + a_{31}^{(2)}\left(a_{12}^{(2)} - a_{22}^{(1)}\right)\right] \\
&= c a_{31}^{(2)}(0).
\end{aligned}$$

Then, we obtain that $\det \mathcal{A} > 0$. Similarly, if $a^{(1)}_{i_1 l} = a^{(2)}_{i_2 l} = a^{(3)}_{i_3 l} = c > 0$ and (3.34) holds, then $\det \mathcal{A} > 0$. $\square$

By interchanging the pair of columns or rows of the matrix in Lemma 3.8, we have the following corollary.

Corollary 3.9. *Let $\mathcal{A}$ be the matrix*

$$\begin{bmatrix} a^{(1)}_{11} & a^{(3)}_{12} & a^{(2)}_{13} \\ a^{(2)}_{21} & a^{(1)}_{22} & a^{(3)}_{23} \\ a^{(3)}_{31} & a^{(2)}_{32} & a^{(1)}_{33} \end{bmatrix}, \quad \begin{bmatrix} a^{(3)}_{11} & a^{(1)}_{12} & a^{(2)}_{13} \\ a^{(2)}_{21} & a^{(3)}_{22} & a^{(1)}_{23} \\ a^{(1)}_{31} & a^{(2)}_{32} & a^{(3)}_{33} \end{bmatrix} \quad or \quad \begin{bmatrix} a^{(3)}_{11} & a^{(2)}_{12} & a^{(1)}_{13} \\ a^{(1)}_{21} & a^{(3)}_{22} & a^{(2)}_{23} \\ a^{(2)}_{31} & a^{(1)}_{32} & a^{(3)}_{33} \end{bmatrix}$$

such that (3.30) holds.

(i) If the matrix $\mathcal{A}$ satisfies (3.31) or (3.32), then $\det \mathcal{A} > 0$.

(ii) Let $l \in \{1, 2, 3\}$. If $a^{(1)}_{i_1 l} = a^{(2)}_{i_2 l} = a^{(3)}_{i_3 l} = c > 0$ and $\mathcal{A}$ satisfies (3.33) or (3.34), then $\det \mathcal{A} > 0$.

Theorem 3.10. *Let p be a prime number such that $p > 3$ and $p \mid (k+1)$. Assume that $b_{p-1} > 0$. (1.1) has positive prime period-p solutions of the form*

$$\ldots, 0, \ldots, 0, \gamma_j, 0, \ldots, 0, \gamma_l, 0, \ldots, 0, \gamma_p, \ldots \tag{3.35}$$

with $0 < j < l < p$ and $\gamma_j, \gamma_l, \gamma_p \in (0, \infty)$ if one of the following statements holds:

(i) $0 \le A < 1$, $b_0 \le b_1 \le b_2 \le \cdots \le b_{p-2}$ *and* $b_{p-1} < b_i$ *for all* $0 \le i \le p-2$.

(ii) $0 \le A < 1$, $b_0 \le b_1 \le b_2 \le \cdots \le b_{p-2}$ *and* $b_{p-1} > b_i$ *for all* $0 \le i \le p-2$.

(iii) $0 \le A < 1$, $b_0 \ge b_1 \ge b_2 \ge \cdots \ge b_{p-2}$ *and* $b_{p-1} < b_i$ *for all* $0 \le i \le p-2$.

(iv) $0 \le A < 1$, $b_0 \ge b_1 \ge b_2 \ge \cdots \ge b_{p-2}$ *and* $b_{p-1} > b_i$ *for all* $0 \le i \le p-2$.

Proof. We find the positive prime period-p solutions of the form (3.35). It follows from (1.1) that

$$\gamma_j = \frac{\gamma_j}{A + b_{j-1}\gamma_p + b_{p+j-l-1}\gamma_l + b_{p-1}\gamma_j},$$
$$\gamma_l = \frac{\gamma_l}{A + b_{l-j-1}\gamma_j + b_{l-1}\gamma_p + b_{p-1}\gamma_l}$$

and

$$\gamma_p = \frac{\gamma_p}{A + b_{p-l-1}\gamma_l + b_{p-j-1}\gamma_j + b_{p-1}\gamma_p}.$$

Then,

$$b_{j-1}\gamma_p + b_{p+j-l-1}\gamma_l + b_{p-1}\gamma_j = 1 - A,$$
$$b_{l-j-1}\gamma_j + b_{l-1}\gamma_p + b_{p-1}\gamma_l = 1 - A$$

and

$$b_{p-l-1}\gamma_l + b_{p-j-1}\gamma_j + b_{p-1}\gamma_p = 1 - A.$$

The above system is equivalent to the matrix equation

$$\begin{bmatrix} b_{p-1} & b_{p+j-l-1} & b_{j-1} \\ b_{l-j-1} & b_{p-1} & b_{l-1} \\ b_{p-j-1} & b_{p-l-1} & b_{p-1} \end{bmatrix} \begin{bmatrix} \gamma_j \\ \gamma_l \\ \gamma_p \end{bmatrix} = \begin{bmatrix} 1-A \\ 1-A \\ 1-A \end{bmatrix}.$$

To solve this equation by using the Cramer's rule, let $\mathcal{A}$ denote the coefficient matrix and $\mathcal{A}_i$ the matrix obtained by replacing the i^{th} column of $\mathcal{A}$ by $(1-A, 1-A, 1-A)^T$. Assume that (i) holds. Then, we can rewrite the matrix $\mathcal{A}$ as

$$\begin{bmatrix} b^{(3)}_{p-1} & b^{(1)}_{p+j-l-1} & b^{(2)}_{j-1} \\ b^{(2)}_{l-j-1} & b^{(3)}_{p-1} & b^{(1)}_{l-1} \\ b^{(1)}_{p-j-1} & b^{(2)}_{p-l-1} & b^{(3)}_{p-1} \end{bmatrix}.$$

From (i), we have

$$\min\{b_{j-1}^{(2)}, b_{l-j-1}^{(2)}, b_{p-l-1}^{(2)}\} > b_{p-1}^{(3)} = \max\{b_{p-1}^{(3)}\}.$$

Then, it follows from Corollary 3.9(i) that $\det \mathcal{A} > 0$. Thus, the unique solution to this equation is given by

$$\gamma_j = \frac{\det \mathcal{A}_1}{\det \mathcal{A}}, \ \gamma_l = \frac{\det \mathcal{A}_2}{\det \mathcal{A}} \ \text{ and } \ \gamma_p = \frac{\det \mathcal{A}_3}{\det \mathcal{A}}.$$

Notice that

$$\min\{b_{j-1}^{(2)}, b_{p-l-1}^{(2)}\}, \ \min\{b_{j-1}^{(2)}, b_{l-j-1}^{(2)}\}, \ \min\{b_{l-j-1}^{(2)}, b_{p-l-1}^{(2)}\} > \max\{b_{p-1}^{(3)}\}.$$

By Corollary 3.9(ii), we obtain that $\det \mathcal{A}_1, \det \mathcal{A}_2, \det \mathcal{A}_3 > 0$. Therefore, γ_j, γ_l and γ_p are all positive real numbers. Similarly, by using Lemma 3.8 and Corollary 3.9, we can show that (1.1) has a unique positive solution when $(ii), (iii)$ or (iv) holds. The proof is now complete. $\square$

Next, we present some definitions and results which will be useful to study the non-existence of p-cycle, $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p)$ of (1.1), where γ_i's are all positive real numbers.

Definition 3.11 ([2])**.** For a given vector $A := (a_0, a_1, a_2, \ldots, a_{n-1})$ in $\mathbb{R}^n$, let $C(A)$ denote the circulant matrix of A,

$$C(A) := \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}.$$

Theorem 3.12 ([16])**.** *If $\{v_j\}_{0 \leq j \leq n-1}$ is a weakly monotone sequence (that is, a nondecreasing or nonincreasing sequence) of nonnegative or nonpositive real* *numbers, then the circulant matrix $C(V)$ with the vector $V = (v_0, v_1, v_2, \ldots, v_{n-1})$ is singular if and only if for some integer $d \mid n$ and $d \geq 2$, V consists of n/d consecutive constant blocks of length d. In particular, if the sequence $\{v_j\}_{0\leq j\leq n-1}$ is strictly monotone of nonnegative or nonpositive, then V is non-singular.*

Definition 3.13 ([3])**.** Let $S = [s_{ij}]$ denote the n-by-n shift operator where $s_{ij} = 1$ if $i - j \equiv 1 \pmod{n}$, $s_{ij} = 0$ otherwise.

The following properties of S are given in [3], which are $S^n = I_n, S^{-1} = S^T$ and MS^k shifts the columns of matrix M over k steps to the left. Moreover, it is easy to see that $M(S^T)^k$ shifts the columns of M over k steps to the right. Similarly, S^kM and $(S^T)^kM$ are the matrices that shift the rows of matrix M over k steps up and down, respectively.

Lemma 3.14. *Let $V = (v_0, v_1, v_2, \ldots, v_{n-1})$ be a vector over $\mathbb{R}^n$ and $C(V)$ a circulant matrix of V. Let $C(V)_i$ denote a matrix formed by replacing the i^{th} column of $C(V)$ by $(c, c, c, \ldots, c)^T \in \mathbb{R}^n$. Then, we have the following results:*

(i) $C(V)_i = S^TC(V)_{i+1}S = SC(V)_{i+1}S^T$ *for all* $i \in \{1, 2, 3, \ldots, n-1\}$.

(ii) $\det C(V)_i = \det C(V)_{i+1}$ *for all* $i \in \{1, 2, 3, \ldots, n-1\}$.

Proof. (i) For any $i \in \{1, 2, 3, \ldots, n-1\}$, we consider

$$S^TC(V)_{i+1}S = S^T \begin{bmatrix} v_0 & v_1 & \cdots & v_{i-1} & c & v_{i+1} & \cdots & v_{n-1} \\ v_{n-1} & v_0 & \cdots & v_{i-2} & c & v_i & \cdots & v_{n-2} \\ v_{n-2} & v_{n-1} & \cdots & v_{i-3} & c & v_{i-1} & \cdots & v_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ v_2 & v_3 & \cdots & v_{i+1} & c & v_{i+3} & \cdots & v_1 \\ v_1 & v_2 & \cdots & v_i & c & v_{i+2} & \cdots & v_0 \end{bmatrix} S.$$

By using the properties of S, $S^T C(V)_{i+1}$ is the matrix that shifts each rows of $C(V)_{i+1}$ one step up, then we get

$$
\begin{aligned}
S^T C(V)_{i+1} S &= \begin{bmatrix} v_{n-1} & v_0 & \cdots & v_{i-2} & c & v_i & \cdots & v_{n-2} \\ v_{n-2} & v_{n-1} & \cdots & v_{i-3} & c & v_{i-1} & \cdots & v_{n-3} \\ v_{n-3} & v_{n-2} & \cdots & v_{i-4} & c & v_{i-2} & \cdots & v_{n-4} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ v_1 & v_2 & \cdots & v_i & c & v_{i+2} & \cdots & v_0 \\ v_0 & v_1 & \cdots & v_{i-1} & c & v_{i+1} & \cdots & v_{n-1} \end{bmatrix} S \\
&= \begin{bmatrix} v_0 & v_1 & \cdots & v_{i-2} & c & v_i & \cdots & v_{n-1} \\ v_{n-1} & v_0 & \cdots & v_{i-3} & c & v_{i-1} & \cdots & v_{n-2} \\ v_{n-2} & v_{n-1} & \cdots & v_{i-4} & c & v_{i-2} & \cdots & v_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ v_2 & v_3 & \cdots & v_i & c & v_{i+2} & \cdots & v_1 \\ v_1 & v_2 & \cdots & v_{i-1} & c & v_{i+1} & \cdots & v_0 \end{bmatrix} \\
&= C(V)_i.
\end{aligned}
$$

Similarly, we can show that $SC(V)_{i+1}S^T = C(V)_i$.

(ii) We consider

$$
\det S = \det \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = (-1)^{1+n} \det I_{n-1} = (-1)^{1+n}.
$$

Note that $\det S = \det S^T$. Then, it follows from (ii) that for $i \in \{1, 2, 3, \ldots, n-1\}$,

$$\begin{aligned}
\det C(V)_i &= \det\left(S^T C(V)_{i+1} S\right) \\
&= \left(\det S^T\right)\left(\det C(V)_{i+1}\right)\left(\det S\right) \\
&= (-1)^{1+n} \det C(V)_{i+1} (-1)^{1+n} \\
&= \det C(V)_{i+1}.
\end{aligned}$$

Thus, we obtain that $\det C(V)_i = \det C(V)_{i+1}$ for all $i \in \{1, 2, 3, \ldots, n-1\}$. $\square$

The following theorem gives the sufficient condition that (1.1) has no p-cycle, $(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p)$, where γ_i's are all positive real numbers.

Theorem 3.15. *Let p be a prime number such that $p \mid (k+1)$. Let $\{b_i\}_{0 \le i \le p-1}$ be a weakly monotone sequence of nonnegative real numbers such that $b_i \neq b_j$ for some $i, j \in \{0, 1, 2, \ldots, p-1\}$ with $i \neq j$. Then, (1.1) has no prime period-p solution of the form*

$$\ldots, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p, \ldots,$$

where γ_i's are all positive real numbers.

Proof. Suppose there exists a prime period-p solution of the form

$$\ldots, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p, \ldots,$$

where γ_i's are all positive real numbers. Since $p \mid (k+1)$, it follows from (1.1) that

$$\begin{aligned}
\gamma_1 &= \frac{\gamma_1}{A + b_0\gamma_p + b_1\gamma_{p-1} + b_2\gamma_{p-2} + \cdots + b_{p-2}\gamma_2 + b_{p-1}\gamma_1}, \\
\gamma_2 &= \frac{\gamma_2}{A + b_0\gamma_1 + b_1\gamma_p + b_2\gamma_{p-1} + \cdots + b_{p-2}\gamma_3 + b_{p-1}\gamma_2}, \\
\gamma_3 &= \frac{\gamma_3}{A + b_0\gamma_2 + b_1\gamma_1 + b_2\gamma_p + \cdots + b_{p-2}\gamma_4 + b_{p-1}\gamma_3}, \\
&\vdots \\
\gamma_{p-1} &= \frac{\gamma_{p-1}}{A + b_0\gamma_{p-2} + b_1\gamma_{p-3} + b_2\gamma_{p-4} + \cdots + b_{p-2}\gamma_p + b_{p-1}\gamma_{p-1}}
\end{aligned}$$

and

$$\gamma_p = \frac{\gamma_p}{A + b_0\gamma_{p-1} + b_1\gamma_{p-2} + b_2\gamma_{p-3} + \cdots + b_{p-2}\gamma_1 + b_{p-1}\gamma_p}.$$

Since $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p \neq 0$, we get

$$\begin{aligned}
b_0\gamma_p + b_1\gamma_{p-1} + b_2\gamma_{p-2} + \cdots + b_{p-2}\gamma_2 + b_{p-1}\gamma_1 &= 1 - A,\\
b_0\gamma_1 + b_1\gamma_p + b_2\gamma_{p-1} + \cdots + b_{p-2}\gamma_3 + b_{p-1}\gamma_2 &= 1 - A,\\
b_0\gamma_2 + b_1\gamma_1 + b_2\gamma_p + \cdots + b_{p-2}\gamma_4 + b_{p-1}\gamma_3 &= 1 - A,\\
&\vdots\\
b_0\gamma_{p-2} + b_1\gamma_{p-3} + b_2\gamma_{p-4} + \cdots + b_{p-2}\gamma_p + b_{p-1}\gamma_{p-1} &= 1 - A,\\
b_0\gamma_{p-1} + b_1\gamma_{p-2} + b_2\gamma_{p-3} + \cdots + b_{p-2}\gamma_1 + b_{p-1}\gamma_p &= 1 - A,
\end{aligned}$$

which is equivalent to the matrix equation

$$\begin{bmatrix}
b_{p-1} & b_{p-2} & b_{p-3} & \cdots & b_1 & b_0\\
b_0 & b_{p-1} & b_{p-2} & \cdots & b_2 & b_1\\
b_1 & b_0 & b_{p-1} & \cdots & b_3 & b_2\\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
b_{p-3} & b_{p-4} & b_{p-5} & \cdots & b_{p-1} & b_{p-2}\\
b_{p-2} & b_{p-3} & b_{p-4} & \cdots & b_0 & b_{p-1}
\end{bmatrix}
\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \vdots\\ \gamma_{p-1}\\ \gamma_p\end{bmatrix}
=
\begin{bmatrix}1-A\\ 1-A\\ 1-A\\ \vdots\\ 1-A\\ 1-A\end{bmatrix}.$$

Let $C(B)$ denote a circulant matrix of the vector $B = (b_{p-1}, b_{p-2}, b_{p-3}, \ldots, b_0)$. Since $b_i \neq b_j$ for some $i, j \in \{0, 1, 2, \ldots, p-1\}$ with $i \neq j$ and p is a prime number, B contains no p/d consecutive constant blocks of length d with $d \mid p$ and $d \geq 2$. It follows from Theorem 3.12 that determinant of $C(B)$ is non-zero. In fact, $\det C(B) > 0$ according to Theorem 1 of Beckenbach and Bellman [2]. By using the Cramer's rule, the unique solution to this system is given by

$$\gamma_i = \frac{\det C(B)_i}{\det C(B)},$$

where $C(B)_i$ is a matrix obtained by replacing the i^{th} column of $C(B)$ by the vector $(1-A, 1-A, 1-A, \ldots, 1-A)^T$. By Lemma 3.14($ii$), we obtain that $\det C(B)_i$'s are all equal. It implies that $\gamma_1 = \gamma_2 = \gamma_3 = \cdots = \gamma_p$, which is a contradiction. $\square$

Remark 1. The case when $\{b_i\}_{0\le i\le p-1}$ is a weakly monotone sequence of non-negative real numbers such that $b_i \neq b_j$ for some $i, j \in \{0, 1, 2, \ldots, p-1\}$ with $i \neq j$, it follows from Theorem 3.15 that a p-cycle of (1.1) must contain at least one zero.

Theorem 3.16. *Let $0 \le A < 1$ and $b_0 = b_1 = b_2 = \cdots = b_{p-1} = C > 0$. Then, (1.1) has infinitely many prime period-p solutions of the form*

$$\ldots, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p, \ldots, \tag{3.36}$$

where γ_i's are all positive real numbers.

Proof. We find the prime period-p solutions of the form (3.36). It follows from (1.1) that

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots + \gamma_p = \frac{1-A}{C}. \tag{3.37}$$

Therefore, the prime period-p solutions of the form (3.36) are given by (3.37) with

$$(\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_p) \neq \left(\frac{1-A}{pC}, \frac{1-A}{pC}, \frac{1-A}{pC}, \ldots, \frac{1-A}{pC}\right).$$

$\square$

3.4 Convergence to Period-$(k+1)$ Solutions

In this section, we give an extension to an open problem 8 in [1]. The convergence of solutions to a (not necessarily prime) period-$(k+1)$ solution is shown. First, we introduce the following lemmas which will be useful in the proof of Theorem 3.19.

Lemma 3.17. *Every solution of (1.1) is bounded if $B_k \neq 0$.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). It follows from (1.1) that

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + B_1 x_{n-1} + B_2 x_{n-2} + \cdots + B_k x_{n-k}} \leq \frac{1}{B_k}.$$

Because $B_k \neq 0$, the sequence $\{x_n\}_{n=-k}^{\infty}$ is bounded by $1/B_k$. $\square$

Lemma 3.18. *Assume that $B_i = C > 0$ for all $0 \leq i \leq k$ and let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). Set*

$$J_n = 1 - A - C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k}).$$

Then, the following statements are true:

(i) $J_{n+1} = \dfrac{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k+1})}{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})} J_n$ *for all* $n \geq 0$.

(ii) $x_{n+1} - x_{n-k} = \dfrac{x_{n-k}}{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})} J_n$ *for all* $n \geq 0$.

Proof. (i) We consider

$$\begin{aligned} J_{n+1} &= 1 - A - Cx_{n+1} - Cx_n - \cdots - Cx_{n-k+1} \\ &= 1 - A - C\left(\frac{x_{n-k}}{A + Cx_n + Cx_{n-1} + \cdots + Cx_{n-k}}\right) - Cx_n - \cdots - Cx_{n-k+1}. \end{aligned}$$

Multiplying the denominator and numerator of the right side of the above equation by $A + Cx_n + Cx_{n-1} + \cdots + Cx_{n-k}$, we obtain

$$\begin{aligned} J_{n+1} = \big(&A + Cx_n + Cx_{n-1} + \cdots + Cx_{n-k} \\ &- A^2 - CAx_n - CAx_{n-1} - \cdots - CAx_{n-k} - Cx_{n-k} \\ &- CAx_n - C^2 x_n x_n - C^2 x_{n-1} x_n - \cdots - C^2 x_{n-k} x_n \\ &- CAx_{n-1} - C^2 x_n x_{n-1} - C^2 x_{n-1} x_{n-1} - \cdots - C^2 x_{n-k} x_{n-1} \\ &\vdots \\ &- CAx_{n-k+1} - C^2 x_n x_{n-k+1} - C^2 x_{n-1} x_{n-k+1} - \cdots - C^2 x_{n-k} x_{n-k+1}\big)/ \\ &\big(A + Cx_n + Cx_{n-1} + \cdots + Cx_{n-k}\big). \end{aligned}$$

This gives

$$J_{n+1} = \frac{1}{(A + Cx_n + Cx_{n-1} + \cdots + Cx_{n-k})}\left[(A + Cx_n + Cx_{n-1} + \cdots + Cx_{n-k+1}) \times \right.$$

$$\left.(1 - A - Cx_n - Cx_{n-1} - \cdots - Cx_{n-k})\right]$$

$$= \frac{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k+1})}{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})} J_n.$$

(ii) We now consider

$$\begin{aligned} x_{n+1} - x_{n-k} &= \frac{x_{n-k}}{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})} - x_{n-k} \\ &= \frac{x_{n-k}\big(1 - A - C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})\big)}{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})} \\ &= \frac{x_{n-k}}{A + C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k})} J_n. \end{aligned}$$

The proof is complete. $\square$

The following theorem states the sufficient conditions that every positive solution of (1.1) converges to a (not necessarily prime) period-$(k+1)$ solution.

Theorem 3.19. *Assume that $B_i = C > 0$ for all $0 \leq i \leq k$. Then, every solution of (1.1) converges to a (not necessarily prime) period-$(k+1)$ solution.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). For each $n \geq 0$, set

$$J_n = 1 - A - C(x_n + x_{n-1} + x_{n-2} + \cdots + x_{n-k}).$$

We have that one and only one of the following three statements is true: $J_0 = 0$ or $J_0 < 0$ or $J_0 > 0$.

If $J_0 = 0$, then by using Lemma 3.18, it is easy to see that $J_n = 0$ for all $n \geq 0$. Then, $x_{n+1} - x_{n-k} = 0$ for all $n \geq 0$. In this case, the solution $\{x_n\}_{n=-k}^{\infty}$ is periodic with period $(k+1)$.

If $J_0 < 0$, then by using Lemma 3.18, $J_n < 0$ and thus, $x_{n+1} - x_{n-k} < 0$ for all $n \geq 0$. In this case, the subsequences $\{x_{(k+1)n+j}\}_{n=0}^{\infty}$ for $0 \leq j \leq k$ of the solution

$\{x_n\}_{n=-k}^{\infty}$ are all decreasing and bounded, so they are convergent. By Lemma 2.9, the solution converges to a period-$(k+1)$ solution of (1.1).

If $J_0 > 0$, then by using Lemma 3.18, $J_n > 0$ and thus, $x_{n+1} - x_{n-k} > 0$ for all $n \geq 0$. In this case, the subsequences $\{x_{(k+1)n+j}\}_{n=0}^{\infty}$ for $0 \leq j \leq k$ of the solution $\{x_n\}_{n=-k}^{\infty}$ are all increasing and bounded, so they are convergent. By Lemma 2.9, the solution converges to a period-$(k+1)$ solution of (1.1). $\square$

Remark 2. The result of Theorem 3.19 means that (1.1), in the case $B_0 = B_1 = B_2 = \cdots = B_k > 0$, admits at least one period-$(k+1)$ solution.

CHAPTER IV

LOCAL STABILITY OF PERIODIC SOLUTIONS

In this chapter, we study the asymptotic stability of some periodic solutions of (1.1). We seperate our work into two sections. In Section 4.1 , the local stability character of a two-cycle is presented and the local stability of a three-cycle of (1.1) is shown in Section 4.2.

4.1 Local Asymptotically Stability of a Two-Cycle

Here, we investigate the local stability character of a two-cycle of (1.1) when k is an odd integer and (3.4) holds. Assume that $b_1 > 0$. Let $\psi = (1 - A)/b_1$. By Theorem 3.2, (1.1) has a prime period-two solution of the form

$$\ldots, 0, \psi, 0, \psi, 0, \psi, \ldots.$$

Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). To study the local stability character of the two-cycle $(0, \psi)$, we rewrite (1.1) in the vector form (2.2). For any $n \geq 0$, set

$$x_n^{(0)} = x_{n-k}, \quad x_n^{(1)} = x_{n-k+1}, \quad x_n^{(2)} = x_{n-k+2}, \quad \ldots, \quad x_n^{(k-1)} = x_{n-1}, \quad x_n^{(k)} = x_n.$$

Then, for any $n \geq 0$, we have

$$\begin{aligned}
x_{n+1}^{(0)} &= x_{n-k+1} = x_n^{(1)}, \\
x_{n+1}^{(1)} &= x_{n-k+2} = x_n^{(2)}, \\
x_{n+1}^{(2)} &= x_{n-k+3} = x_n^{(3)}, \\
&\vdots \\
x_{n+1}^{(k-1)} &= x_n = x_n^{(k)}
\end{aligned}$$

and

$$x_{n+1}^{(k)} = \frac{x_n^{(0)}}{A + B_0 x_n^{(k)} + B_1 x_n^{(k-1)} + B_2 x_n^{(k-2)} + \cdots + B_k x_n^{(0)}}.$$

Let T be the function on $[0,\infty)^{k+1}$ defined for $u_0, u_1, u_2, \ldots, u_k \in [0,\infty)$, by

$$T\left(\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix}\right) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_k \\ \dfrac{u_0}{A + B_0 u_k + B_1 u_{k-1} + \cdots + B_k u_0} \end{bmatrix}. \tag{4.1}$$

Set $U = (u_0, u_1, u_2, \ldots, u_k)^T$. Then, we see that

$$T^2\left(\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix}\right) = \begin{bmatrix} f_0(U) \\ f_1(U) \\ f_2(U) \\ \vdots \\ f_{k-1}(U) \\ f_k(U) \end{bmatrix},$$

where

$$f_i(U) = u_{i+2} \qquad (0 \le i \le k-2),$$
$$f_{k-1}(U) = \frac{u_0}{A + B_0 u_k + B_1 u_{k-1} + \cdots + B_k u_0}$$

and

$$f_k(U) = \frac{u_1}{A + B_0 f_{k-1} + B_1 u_k + B_2 u_{k-1} + \cdots + B_k u_1}.$$

Set $\Psi = (0, \psi, 0, \psi, 0, \psi, \ldots, 0, \psi)^T \in [0, \infty)^{k+1}$. We have

$$T^2\left(\begin{bmatrix} 0 \\ \psi \\ 0 \\ \psi \\ \vdots \\ 0 \\ \psi \end{bmatrix}\right) = \begin{bmatrix} f_0(\Psi) \\ f_1(\Psi) \\ f_2(\Psi) \\ f_3(\Psi) \\ \vdots \\ f_{k-1}(\Psi) \\ f_k(\Psi) \end{bmatrix} = \begin{bmatrix} 0 \\ \psi \\ 0 \\ \psi \\ \vdots \\ 0 \\ \psi \end{bmatrix}.$$

Therefore, Ψ is an equilibrium point of T^2 and the Jacobian matrix of T^2 at this equilibrium point is

$$J_{T^2}(\Psi) = \begin{bmatrix} \frac{\partial f_0}{\partial u_0} & \frac{\partial f_0}{\partial u_1} & \frac{\partial f_0}{\partial u_2} & \cdots & \frac{\partial f_0}{\partial u_k} \\ \frac{\partial f_1}{\partial u_0} & \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \frac{\partial f_2}{\partial u_0} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial u_0} & \frac{\partial f_k}{\partial u_1} & \frac{\partial f_k}{\partial u_2} & \cdots & \frac{\partial f_k}{\partial u_k} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ C & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ D_0 & D_1 & D_2 & D_3 & D_4 & \cdots & D_{k-1} & D_k \end{bmatrix},$$

where

$$C = \frac{1}{A + b_0\psi}, \quad D_0 = \frac{-B_0\psi}{A + b_0\psi}, \quad D_1 = 1 - B_k\psi$$

and

$$D_j = -B_{k-j+1}\psi \qquad (2 \le j \le k).$$

To evaluate the eigenvalues of the matrix $J_{T^2}(\Psi)$, we consider

$$\det(J_{T^2}(\Psi) - \lambda I_{k+1}) = \det \begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ C & 0 & 0 & 0 & 0 & \cdots & -\lambda & 0 \\ D_0 & D_1 & D_2 & D_3 & D_4 & \cdots & D_{k-1} & D_k - \lambda \end{bmatrix}.$$

For finding the above determinant, we use minors and cofactors by expanding along the last row first and continuing this process. To get an idea of what $\det(J_{T^2}(\Psi) - \lambda I_{k+1})$ should be, we find the case when $k = 3$ first:

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ C & 0 & -\lambda & 0 \\ D_0 & D_1 & D_2 & D_3 - \lambda \end{vmatrix} = D_1 \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & 0 & 1 \\ C & -\lambda & 0 \end{vmatrix} + (D_3 - \lambda) \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ C & 0 & -\lambda \end{vmatrix}$$

$$= -D_1 \begin{vmatrix} -\lambda & 1 \\ C & -\lambda \end{vmatrix} - \lambda(D_3 - \lambda) \begin{vmatrix} -\lambda & 1 \\ C & -\lambda \end{vmatrix}.$$

Thus,

$$\begin{aligned} \det(J_{T^2}(\Psi) - \lambda I_4) &= (-D_1 - D_3\lambda + \lambda^2)(\lambda^2 - C) \\ &= \lambda^4 - D_3\lambda^3 - (C + D_1)\lambda^2 + CD_3\lambda + CD_1. \end{aligned}$$

Moreover, if we check the case when $k = 5$, we find

$$\begin{aligned}\det(J_{T^2}(\Psi) - \lambda I_6) &= (-D_1 - D_3\lambda - D_5\lambda^2 + \lambda^3)(\lambda^3 - C) \\ &= \lambda^6 - D_5\lambda^5 - D_3\lambda^4 - (C + D_1)\lambda^3 + CD_5\lambda^2 + CD_3\lambda + CD_1.\end{aligned}$$

Generally, it follows by induction that

$$\begin{aligned}\det(J_{T^2}(\Psi) - \lambda I_{k+1}) = \lambda^{k+1} &- D_k\lambda^k - D_{k-2(1)}\lambda^{k-1} - D_{k-2(2)}\lambda^{k-2} - D_{k-2(3)}\lambda^{k-3} \\ &- \cdots - D_3\lambda^{(k+3)/2} + (C + D_1)\lambda^{(k+1)/2} + CD_k\lambda^{(k-1)/2} \\ &+ CD_{k-2(1)}\lambda^{[k-2(1)-1]/2} + CD_{k-2(2)}\lambda^{[k-2(2)-1]/2} \\ &+ CD_{k-2(3)}\lambda^{[k-2(3)-1]/2} + \cdots + CD_1.\end{aligned}$$

Thus, all eigenvalues of J_{T^2} at the equilibrium point Ψ are all λ's that satisfy

$$\det(J_{T^2}(\Psi) - \lambda I_{k+1}) = 0. \tag{4.2}$$

Theorem 4.1. *Let k be an odd integer, $b_1 > 0$ and $\psi = (1 - A)/b_1$. Assume that (3.4) holds. If*

$$b_0\eta\psi^2 + (A\eta + \eta)\psi + 2 < 0, \tag{4.3}$$

where $\eta = b_1 - 2B_k$ with $\eta < 0$, then the prime period-two solution

$$\ldots, 0, \psi, 0, \psi, 0, \psi, \ldots \tag{4.4}$$

of (1.1) is locally asymptotically stable.

Proof. Assume that (4.3) holds. Then, we have

$$b_0\eta\psi^2 + A\eta\psi + \eta\psi + 2 < 0$$

or

$$(A + b_0\psi)\eta\psi + \eta\psi + 2 < 0.$$

It follows from (3.4) that $A + b_0\psi > 0$. Then,

$$\eta\psi + \frac{\eta\psi}{A + b_0\psi} + \frac{2}{A + b_0\psi} < 0.$$

Then, we see that

$$(B_1 + B_3 + B_5 + \cdots + B_{k-2} - B_k)\,\psi + \frac{(B_1 + B_3 + B_5 + \cdots + B_{k-2} - B_k)\,\psi}{A + b_0\psi} + \frac{2}{A + b_0\psi} < 0,$$

from which it follows that

$$B_1\psi + B_3\psi + B_5\psi + \cdots + B_{k-2}\psi + \left(\frac{1}{A + b_0\psi} + 1 - B_k\psi\right) + \left(\frac{B_1\psi}{A + b_0\psi}\right) + \left(\frac{B_3\psi}{A + b_0\psi}\right) + \left(\frac{B_5\psi}{A + b_0\psi}\right) + \cdots + \left(\frac{B_{k-2}\psi}{A + b_0\psi}\right) + \left(\frac{1 - B_k\psi}{A + b_0\psi}\right) < 1.$$

Since $0 < 1 - A \leq 1$, $B_k(1 - A) \leq B_k \leq b_1$, so

$$B_k\psi = B_k\left(\frac{1 - A}{b_1}\right) \leq 1.$$

Thus, we obtain that

$$|B_1\psi| + |B_3\psi| + |B_5\psi| + \cdots + |B_{k-2}\psi| + \left|\frac{1}{A + b_0\psi} + 1 - B_k\psi\right| + \left|\frac{B_1\psi}{A + b_0\psi}\right| + \left|\frac{B_3\psi}{A + b_0\psi}\right| + \left|\frac{B_5\psi}{A + b_0\psi}\right| + \cdots + \left|\frac{B_{k-2}\psi}{A + b_0\psi}\right| + \left|\frac{1 - B_k\psi}{A + b_0\psi}\right| < 1,$$

that means

$$|D_k| + |D_{k-2}| + |D_{k-4}| + \cdots + |D_3| + |C + D_1| + |CD_k| + |CD_{k-2}| + |CD_{k-4}| + \cdots + |CD_1| < 1.$$

By Theorem 2.7, we conclude that all eigenvalues of (4.2) lie inside the unit disk. Therefore, by Theorem 2.6, the prime period-two solution (4.4) is locally asymptotically stable. $\square$

4.2 Local Asymptotically Stability of a Three-Cycle

In this section, we investigate the local stability character of a three-cycle of (1.1) when k is a positive integer such that $3 \mid (k+1)$ and (3.12) holds. Assume that $b_2 > 0$. Let $\gamma = (1-A)/b_2$. By Theorem 3.3, (1.1) has a prime period-three solution of the form

$$\ldots, 0, 0, \gamma, 0, 0, \gamma, \ldots.$$

Let T be the function on $[0,\infty)^{k+1}$ defined for $U = (u_0, u_1, u_2, \ldots, u_k)^T \in [0,\infty)^{k+1}$ by (4.1). Then, we have

$$T^3\left(\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{k-1} \\ u_k \end{bmatrix}\right) = \begin{bmatrix} f_0(U) \\ f_1(U) \\ f_2(U) \\ \vdots \\ f_{k-1}(U) \\ f_k(U) \end{bmatrix},$$

where

$$\begin{aligned} f_i(U) &= u_{i+3} \qquad (0 \le i \le k-3), \\ f_{k-2}(U) &= \frac{u_0}{A + B_0 u_k + B_1 u_{k-1} + B_2 u_{k-2} + \cdots + B_k u_0}, \\ f_{k-1}(U) &= \frac{u_1}{A + B_0 f_{k-2} + B_1 u_k + B_2 u_{k-1} + \cdots + B_k u_1} \end{aligned}$$

and

$$f_k(U) = \frac{u_2}{A + B_0 f_{k-1} + B_1 f_{k-2} + B_2 u_k + B_3 u_{k-1} + \cdots + B_k u_2}.$$

Set $\Gamma = (0,0,\gamma,0,0,\gamma,\ldots,0,0,\gamma)^T \in [0,\infty)^{k+1}$. We have

$$T^3\left(\begin{bmatrix} 0 \\ 0 \\ \gamma \\ \vdots \\ 0 \\ 0 \\ \gamma \end{bmatrix}\right) = \begin{bmatrix} f_0(\Gamma) \\ f_1(\Gamma) \\ f_2(\Gamma) \\ \vdots \\ f_{k-2}(\Gamma) \\ f_{k-1}(\Gamma) \\ f_k(\Gamma) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \gamma \\ \vdots \\ 0 \\ 0 \\ \gamma \end{bmatrix}.$$

Thus, Γ is an equilibrium point of T^3 and the Jacobian matrix of T^3 at this equilibrium is

$$J_{T^3}(\Gamma) = \begin{bmatrix} \frac{\partial f_0}{\partial u_0} & \frac{\partial f_0}{\partial u_1} & \frac{\partial f_0}{\partial u_2} & \cdots & \frac{\partial f_0}{\partial u_k} \\ \frac{\partial f_1}{\partial u_0} & \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \frac{\partial f_2}{\partial u_0} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial u_0} & \frac{\partial f_k}{\partial u_1} & \frac{\partial f_k}{\partial u_2} & \cdots & \frac{\partial f_k}{\partial u_k} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ C & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & D & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ E_0 & E_1 & E_2 & E_3 & E_4 & E_5 & \cdots & E_{k-1} & E_k \end{bmatrix},$$

where

$$C = \frac{1}{A + b_0\gamma}, \quad D = \frac{1}{A + b_1\gamma}, \quad E_0 = -B_1 C\gamma, \quad E_1 = -B_0 D\gamma, \quad E_2 = 1 - B_k\gamma$$

and

$$E_j = -B_{k-j+2}\gamma \qquad (3 \le j \le k).$$

To evaluate the eigenvalues of the matrix $J_{T^3}(\Gamma)$, we consider $\det(J_{T^3}(\Gamma) - \lambda I_{k+1})$. For finding this determinant, we use minors and cofactors by expanding along the last row first and continuing the process. Then, it follows that

$$\begin{aligned}\det(J_{T^3}(\Gamma) - \lambda I_{k+1}) = (-1)^{k+1} \Bigg[& \lambda^{k+1} - \sum_{i=0}^{(k-5)/3} E_{k-3i}\lambda^{k-i} \\ & - (C + D + E_2)\,\lambda^{(2k+2)/3} \\ & + \sum_{i=0}^{(k-5)/3} (C + D)E_{k-3i}\lambda^{[2k-3i-1]/3} \\ & + (CD + CE_2 + DE_2)\lambda^{(k+1)/3} \\ & - \sum_{i=0}^{(k-5)/3} CDE_{k-3i}\lambda^{[k-3i-2]/3} - CDE_2 \Bigg].\end{aligned}$$

Thus, all eigenvalues of J_{T^3} at the equilibrium point Γ are all λ's that satisfy

$$\det(J_{T^3}(\Gamma) - \lambda I_{k+1}) = 0. \tag{4.5}$$

Theorem 4.2. *Let k be a positive integer such that $3 \mid (k+1)$, $b_2 > 0$ and $\gamma = (1 - A)/b_2$. Let $\delta = A + b_0\gamma$ and $\omega = A + b_1\gamma$. Assume that (3.12) holds. If*

$$(\delta\omega + \delta + \omega + 1)\eta\gamma + 2(\delta + \omega + 1) < 0, \tag{4.6}$$

where $\eta = b_2 - 2B_k$ with $\eta < 0$, then the prime period-three solution

$$\ldots, 0, 0, \gamma, 0, 0, \gamma, \ldots \tag{4.7}$$

of (1.1) is locally asymptotically stable.

Proof. Suppose that inequality (4.6) is true. Then, we obtain that

$$\frac{(\delta\omega+\delta+\omega+1)\eta\gamma+2(\delta+\omega+1)}{\delta\omega}<0.$$

Then,

$$\eta\gamma+\frac{\eta\gamma}{\delta}+\frac{\eta\gamma}{\omega}+\frac{\eta\gamma}{\delta\omega}+\frac{2}{\delta}+\frac{2}{\omega}+\frac{2}{\delta\omega}<0$$

from which it follows that

$$\begin{aligned}(B_2+B_5+B_8+\cdots+B_{k-3}-B_k)\gamma+\frac{(B_2+B_5+B_8+\cdots+B_{k-3}-B_k)\gamma}{\delta}\\ +\frac{(B_2+B_5+B_8+\cdots+B_{k-3}-B_k)\gamma}{\omega}+\frac{(B_2+B_5+B_8+\cdots+B_{k-3}-B_k)\gamma}{\delta\omega}\\ +\frac{2}{\delta}+\frac{2}{\omega}+\frac{2}{\delta\omega}<0.\end{aligned} \tag{4.8}$$

Since $B_k\gamma \leq b_2\gamma$, $B_k\gamma \leq 1-A \leq 1$. It follows by (4.8) that

$$\begin{aligned}|B_2\gamma|+|B_5\gamma|+|B_8\gamma|+\cdots+|B_{k-3}\gamma|+\left|\frac{1}{\delta}+\frac{1}{\omega}+1-B_k\gamma\right|\\ +\left|\frac{B_2\gamma}{\delta}+\frac{B_2\gamma}{\omega}\right|+\left|\frac{B_5\gamma}{\delta}+\frac{B_5\gamma}{\omega}\right|+\left|\frac{B_8\gamma}{\delta}+\frac{B_8\gamma}{\omega}\right|+\cdots+\left|\frac{B_{k-3}\gamma}{\delta}+\frac{B_{k-3}\gamma}{\omega}\right|\\ +\left|\frac{1}{\delta\omega}+\frac{1-B_k\gamma}{\delta}+\frac{1-B_k\gamma}{\omega}\right|+\left|\frac{B_2\gamma}{\delta\omega}\right|+\left|\frac{B_5\gamma}{\delta\omega}\right|+\left|\frac{B_8\gamma}{\delta\omega}\right|+\cdots+\left|\frac{B_{k-3}\gamma}{\delta\omega}\right|<1.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}|E_k|+|E_{k-3}|+|E_{k-6}|+\cdots+|E_5|+|C+D+E_2|+|CE_k+DE_k|\\ +|CE_{k-3}+DE_{k-3}|+|CE_{k-6}+DE_{k-6}|+\cdots+|CE_5+DE_5|\\ +|CD+CE_2+DE_2|+|CDE_k|+|CDE_{k-3}|+|CDE_{k-6}|+\cdots+|CDE_2|<1.\end{aligned}$$

By Theorem 2.7, we conclude that all eigenvalues of (4.5) lie inside the unit disk. Therefore, by Theorem 2.6, the prime period-three solution (4.7) is locally asymptotically stable. □

CHAPTER V

NUMERICAL EXAMPLES

The previous chapters discussed periodicity and local stability of periodic solutions of (1.1). In this chapter, we give some numerical examples to illustrate the results that we obtained. We also use MATLAB R2017a to draw graphs that represent the behavior of solutions to such equations.

Example 5.1. Consider the difference equation

$$x_{n+1} = \frac{x_{n-9}}{0.4 + x_{n-1} + x_{n-3} + x_{n-5} + x_{n-6} + x_{n-7} + 0.2x_{n-9}}, \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

with the initial conditions $x_{-9} = x_{-8} = x_{-7} = x_{-6} = 0,\ x_{-5} = 3,\ x_{-4} = x_{-3} = x_{-2} = x_{-1} = 0$ and $x_0 = 3$. In this example, $k = 9, b_4 = 0.2$ and $A = 0.4$. That means the condition (3.26) of Theorem 3.6 is satisfied. Then, (5.1) has a prime period-5 solution of the form $\ldots, 0, 0, 0, 0, 3, 0, 0, 0, 0, 3, \ldots$ as shown in Figure 5.1.

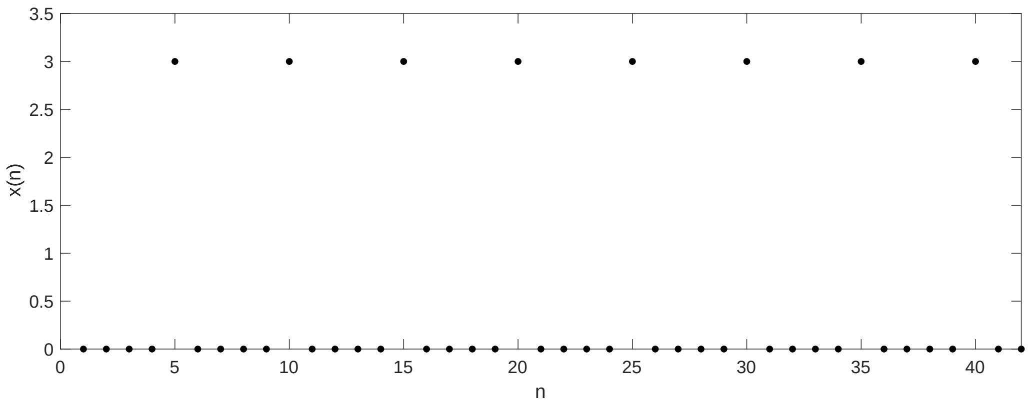

Figure 5.1: The solution of (5.1)

Example 5.2. Consider the difference equation

$$x_{n+1} = \frac{x_{n-9}}{x_n + x_{n-1} + x_{n-3} + 0.25x_{n-6} + x_{n-7} + 1.5x_{n-9}}, \quad n = 0, 1, 2, \ldots, \quad (5.2)$$

with the initial conditions $x_{-9} = x_{-8} = 0,\ x_{-7} = 0.5,\ x_{-6} = 0,\ x_{-5} = 0.25,\ x_{-4} = x_{-3} = 0,\ x_{-2} = 0.5,\ x_{-1} = 0$ and $x_0 = 0.25$. In this example, $k = 9, A = 0, b_{5-1} = b_4 = 1.5, b_{3-1} = b_2 = 1$ and $b_{5-3-1} = b_1 = 1.25$. That means if we set $j = 3$ and $p = 5$, then the condition (ii) of Theorem 3.7 is satisfied. Thus, (5.2) has a prime period-5 solution of the form $\ldots, 0, 0, 0.5, 0, 0.25, 0, 0, 0.5, 0, 0.25, \ldots$ as shown in Figure 5.2.

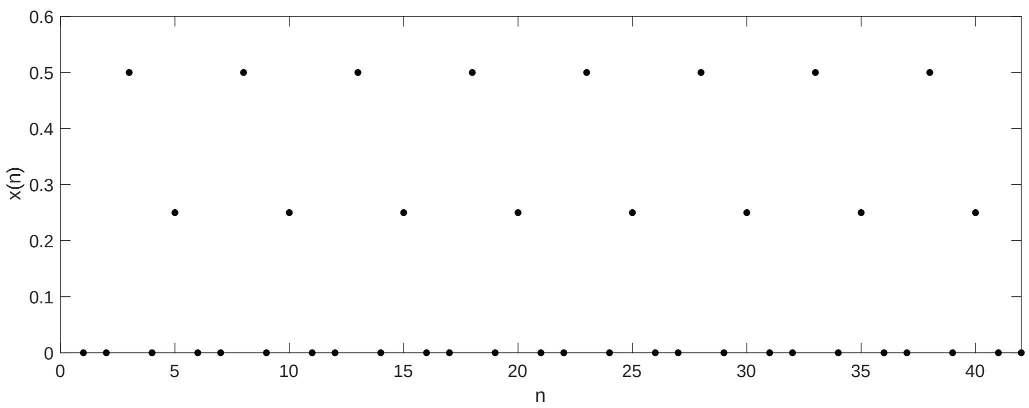

Figure 5.2: The solution of (5.2)

Example 5.3. Consider the difference equation

$$x_{n+1} = \frac{x_{n-3}}{0.5 + 0.1x_n + 0.1x_{n-1} + 0.1x_{n-2} + 0.1x_{n-3}}, \quad n = 0, 1, 2, \ldots, \quad (5.3)$$

with the initial conditions $x_{-3} = 0.1,\ x_{-2} = 1,\ x_{-1} = 0.3$ and $x_0 = 0.6$. Clearly, $B_0 = B_1 = B_2 = B_3 = 0.1$. By Theorem 3.19, the solution of (5.3) converges to a period-4 solution as shown in Figure 5.3.

Example 5.4. Consider the difference equation

$$x_{n+1} = \frac{x_{n-3}}{x_n + 0.1x_{n-1} + 2x_{n-2} + 0.4x_{n-3}}, \quad n = 0, 1, 2, \ldots, \quad (5.4)$$

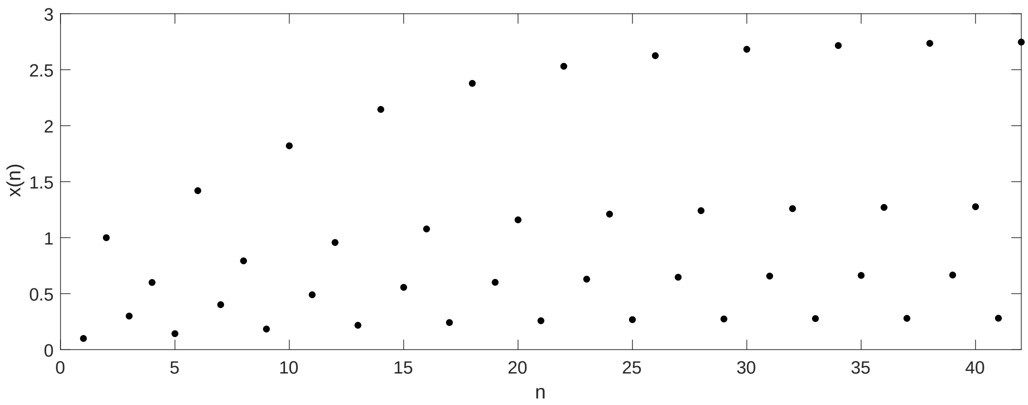

Figure 5.3: The solution of (5.3)

with the initial conditions $x_{-3} = 0.1$, $x_{-2} = 1.5$, $x_{-1} = 0.2$ and $x_0 = 1.7$. In this example, $A = 0, b_1 = 3$ and $b_1 = 0.5$. It is easy to see that the condition (4.3) of Theorem 4.1 is satisfied. Then, the prime period-two solution $\ldots, 0, 2, 0, 2, 0, 2, \ldots$ of (5.4) is locally asymptotically stable as shown in Figure 5.4.

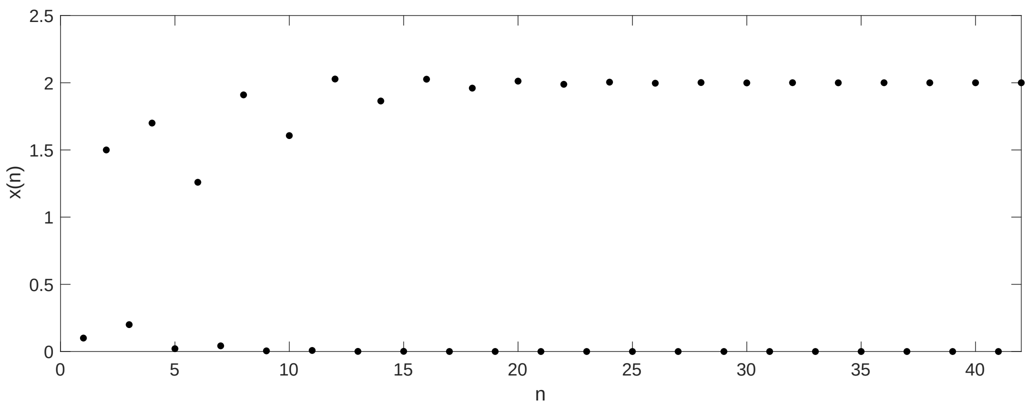

Figure 5.4: The solution of (5.4)

Example 5.5. Consider the difference equation

$$x_{n+1} = \frac{x_{n-3}}{0.2 + x_n + 0.1x_{n-1} + 2x_{n-2} + 0.3x_{n-3}}, \quad n = 0, 1, 2, \ldots, \tag{5.5}$$

with the initial conditions $x_{-3} = 2$, $x_{-2} = 3$, $x_{-1} = 0.5$ and $x_0 = 1$. In this example, $A = 0.2, b_0 = 3$ and $b_1 = 0.4$. It is easy to see that the condition (4.3) of

Theorem 4.1 is satisfied. Then, the prime period-two solution $\ldots, 0, 2, 0, 2, 0, 2, \ldots$ of (5.5) is locally asymptotically stable as shown in Figure 5.5.

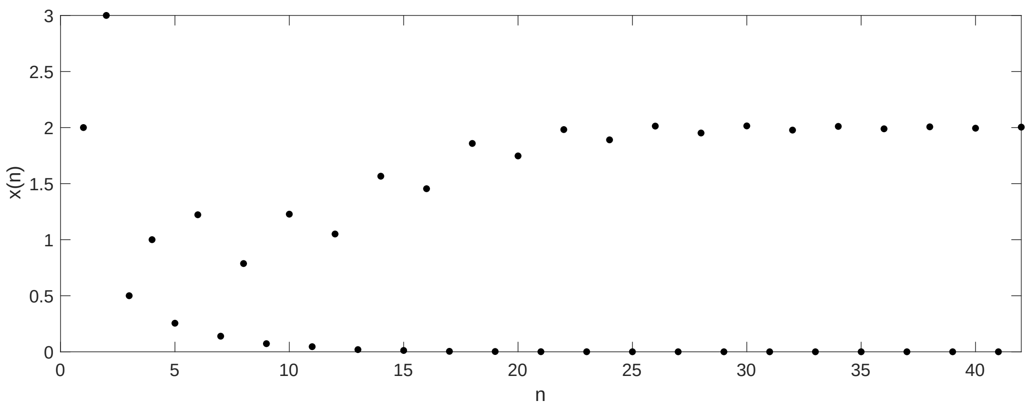

Figure 5.5: The solution of (5.5)

Example 5.6. Consider the difference equation

$$x_{n+1} = \frac{x_{n-5}}{x_n + x_{n-1} + 4x_{n-4} + 0.2x_{n-5}}, \quad n = 0, 1, 2, \ldots, \tag{5.6}$$

with the initial conditions $x_{-5} = 2,\ x_{-4} = 4,\ x_{-3} = 2.5,\ x_{-2} = 3,\ x_{-1} = 6$ and $x_0 = 4$. In this example, $A = 0, b_0 = 1, b_1 = 2$, and $b_2 = 0.2$. It is easy to verify that the condition (4.6) of Theorem 4.2 is satisfied. Then, the prime period-three solution $\ldots, 0, 0, 5, 0, 0, 5, \ldots$ of (5.6) is locally asymptotically stable as shown in Figure 5.6.

Example 5.7. Consider the difference equation

$$x_{n+1} = \frac{x_{n-5}}{0.2 + 1.2x_n + 2x_{n-1} + 0.2x_{n-4} + 0.2x_{n-5}} \quad n = 0, 1, 2, \ldots, \tag{5.7}$$

with the initial conditions $x_{-5} = 2,\ x_{-4} = 4,\ x_{-3} = 6,\ x_{-2} = 3,\ x_{-1} = 2$ and $x_0 = 4$. In this example, $A = 0.2, b_0 = 1.2, b_1 = 2.2$, and $b_2 = 0.2$. It is easy to verify that the condition (4.6) of Theorem 4.2 is satisfied. Then, the prime period-three solution $\ldots, 0, 0, 4, 0, 0, 4, \ldots$ of (5.7) is locally asymptotically stable as shown in Figure 5.7.

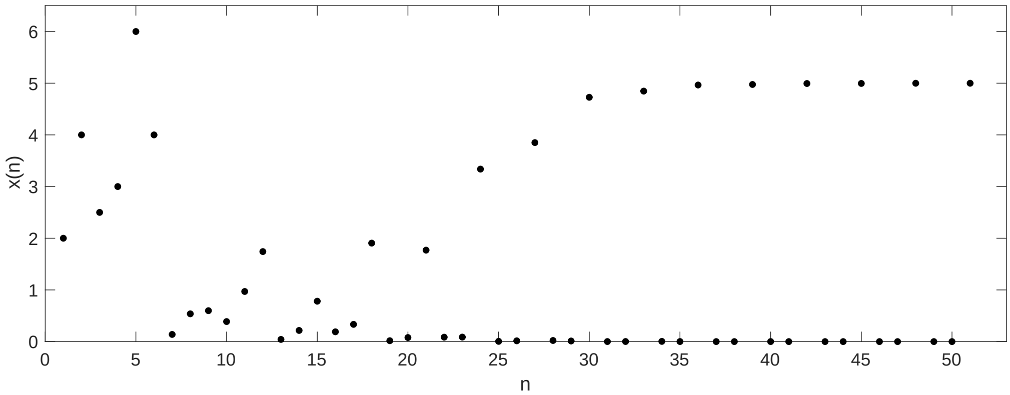

Figure 5.6: The solution of (5.6)

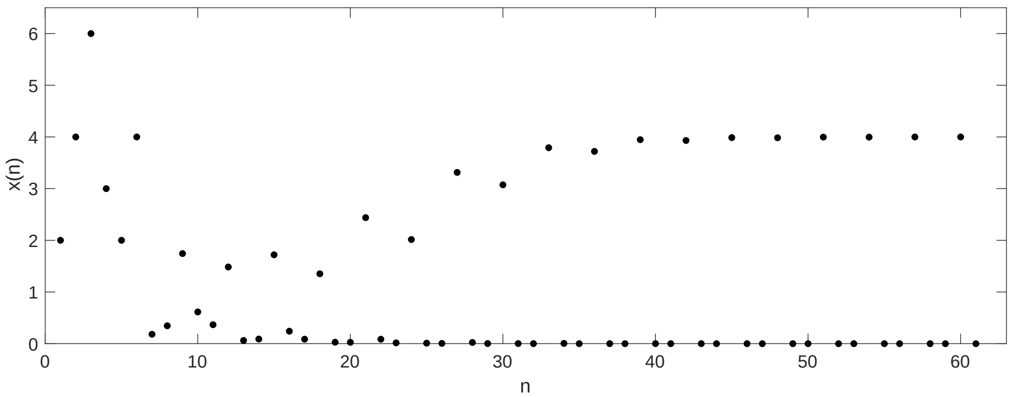

Figure 5.7: The solution of (5.7)

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