ความต่อเนื่องบนปริภูมิทอพอโลยีวางนัยทั่วไปผ่านคลาสพันธุกรรม

นายวิชิตพล ไทยเกื้อ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2559

CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES VIA HEREDITARY CLASSES

Mr. Wichitpon Thaikua

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2016

Thesis Title	CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES VIA HEREDITARY CLASSES
By	Mr. Wichitpon Thaikua
Field of Study	Mathematics
Thesis Advisor	Pongdate Montagantirud, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

. . . . Dean of the Faculty of Science

(Associate Professor Polkit Sangvanich, Ph.D.)

THESIS COMMITTEE

. . . . Chairman

(Associate Professor Wicharn Lewkeeratiyutkul, Ph.D.)

. . . . Thesis Advisor

(Pongdate Montagantirud, Ph.D.)

. . . . Examiner

(Associate Professor Phichet Chaoha, Ph.D.)

. . . . External Examiner

(Assistant Professor Gun Sunyeekhan, Ph.D.)

วิชิตพล ไทยเกื้อ: ความต่อเนื่องบนปริภูมิทอพอโลยีวางนัยทั่วไปผ่านคลาสพันธุกรรม (CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES VIA HEREDITARY CLASSES) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ดร.พงษ์เดช มนทกานติรัตน์, 39 หน้า.

ทอพอโลยีวางนัยทั่วไปเป็นกลุ่มที่ประกอบไปด้วยเซตว่างและยูเนียนของสมาชิกในกลุ่มต้องอยู่ในกลุ่ม เราใช้แนวคิดของคลาสพันธุกรรมในการขยายทอพอโลยีวางนัยทั่วไปสู่ทอพอโลยีวางนัยทั่วไปที่มีสมาชิกเท่ากับหรือมากกว่าเดิม เราเรียกทอพอโลยีวางนัยทั่วไปที่เกิดจากคลาสพันธุกรรมว่า ทอพอโลยีวางนัยทั่วไปผ่านคลาสพันธุกรรม ในวิทยานิพนธ์นี้ เราศึกษาความต่อเนื่องบนปริภูมิทอพอโลยีวางนัยทั่วไปผ่านคลาสพันธุกรรม เราได้ผลลัพธ์ของความต่อเนื่องบนปริภูมิทอพอโลยีวางนัยทั่วไปผ่านคลาสพันธุกรรมในสถานการณ์ที่แตกต่างกัน

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์	ลายมือชื่อนิสิต
สาขาวิชา คณิตศาสตร์	ลายมือชื่อ อ.ที่ปรึกษาหลัก
ปีการศึกษา 2559	

5872053223 : MAJOR MATHEMATICS

KEYWORDS : GENERALIZED TOPOLOGICAL SPACES, HEREDITARY CLASSES, $(\mu, \nu)-$CONTINUOUS FUNCTIONS

WICHITPON THAIKUA : CONTINUITY ON GENERALIZED TOPOLOGICAL SPACES VIA HEREDITARY CLASSES

ADVISOR : PONGDATE MONTAGANTIRUD, Ph.D., 39 pp.

A generalized topology is a collection of subsets of a given nonempty set containing the empty set and arbitrary unions of the elements in this collection. By using a concept of hereditary classes, a generalized topology can be extended to the new one, called a generalized topology via a hereditary class. In this thesis, we study on continuity on generalized topological spaces via hereditary classes. The continuity can be described in various situations.

Department: Mathematics and Computer Science Student's Signature: ....

Field of Study: ....Mathematics...... Advisor's Signature: ....

Academic Year: ....2016.........

ACKNOWLEDGEMENTS

I am greatly indebted to Dr.Pongdate Montagantirud, my thesis advisor for his suggestions and helpful advice in preparing and writing this thesis. I would like to express my special thanks to Associate Professor Dr.Wicharn Lewkeeratiyutkul, Associate Professor Dr.Phichet Chaoha and Assistant Professor Dr.Gun Sunyeekhan, my thesis committee, for their suggestions and comments. I feel very thankful to all my teachers who have taught me all along, especially the teacher who teach in the subject about topology.

In particular, I wish to express my thankfulness to Chulalongkorn university for financial supports. I wish to thank my friends and the others in the same departments for giving me good experiences at Chulalongkorn university.

Finally, I would like to express my deep gratitude to my beloved family for their love and encouragement throughout my graduate study.

CONTENTS

CHAPTER I

INTRODUCTION

Topology has played an important role in other branches of mathematics such as methematical analysis, complex analysis, functional analysis, etc. In the point-set topology, topologists research about the class of open sets or the class of nearly open sets and their properties. In 1997 [1], Császár defined the generalized open sets which are called γ-open sets. These cover every class of nearly open sets. Various properties are studied such as an arbitrary union of γ-open sets is γ-open. In 2002 [2], he used properties of γ-open sets to define a generalized topology. The definition of a generalized topology is a topology ignored the following conditions : the whole set belongs to topology and a finite intersection of open sets is open. Moreover, he determined the properties on generalized topologies comparing to the topological properties. In 2007 [4], he gave the concept of an extension of generalized topological spaces by using hereditary classes (a collection $\mathcal{H}$ of subsets of the space such that every subset of elements in $\mathcal{H}$ must belong to $\mathcal{H}$). The concept of hereditary class was introduced in 1990 by Hamlett and Janković [5]. They used this concept to generalize the set of all ω-accumulation points on a set A (a point such that every its neighborhood contains infinite elements of A), the set of all condensation points on a set A (a point such that every its neighborhood contains uncountable elements of A) and also the closure of A. They introduced the concept of an extension of topology by using an ideal. This topology contains the above sets as closed sets. In this thesis, we study the continuity on generalized topological spaces via hereditary classes. The organization of this thesis is described as follows.

The definition of a topology and its properties are described in Chapter 2. We also introduce a concept of nearly open sets and give some examples.

Chapter 3 is entirely devoted to generalized topological spaces and hereditary classes. The main arguments are based on [1],[2],[3],[4]. The first section gives the definition of generalized open sets and their properties. The generalized topological spaces are introdced in the next section. The relation between a generalized topological space and a collection of γ-open sets are described in this section. Moreover, Császár gave the definition of hereditary classes and the concept of generalized topological spaces via hereditary classes and some properties. These will be described in the last section.

The main results of this thesis are obtained in Chapter 4. In the first theorem, we prove a generalization of pasting lemma and try to cut some conditions on this lemma by using hereditary classes. The next theorems show that the continuity between two generalized topological spaces can be preserved on generalized topological spaces via hereditary classes in various situations. By applying these theorems, we obtain the associated ones concerning open maps. Moreover, some hereditary class on a subspace of generalized topologies via hereditary classes is given. Finally, we prove that in some conditions one can construct a hereditary class that makes a given function continuous on the generalized topological space via this hereditary class.

CHAPTER II

THE DEVELOPEMENT OF TOPOLOGY AND SOME PROPERTIES

2.1 A developement from geometry to topology

For the first period of studying mathematics, mathematicians studied and researched about algebra and geometry. Algebra is the study on operations of the numbers , solving equations, etc. Geometry is the study of the shape, the area, the volume of object, etc. In geometry, we have a very important theorem that is Pythagoras's theorem, which describes about the lenght of the opposite side of a right angle. Pythagoras's theorem is defined by the following statement.

Theorem 2.1. *Let $\triangle ABC$ be a right triangle such that the angle B is a right angle. Assume that the lenght of the line segment $\overline{AB}$ is equal to a units and the lenght of the line segment $\overline{BC}$ is equal to b units. Then the lenght of the line segment $\overline{AC}$ is equal to $\sqrt{a^2+b^2}$.*

From the concept of Pythagoras's theorem, if we assign each vertex of a triangle a point on $\mathbb{R}^2$, the set of all 2-tuples of real numbers. then we can defind the distance between two points in $\mathbb{R}^2$ as follows.

Definition 2.2. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be two points in $\mathbb{R}^2$. the distance between x and y is defined by $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

After that, mathematicians generalized this concept and defined the distance between two points in $\mathbb{R}^n$, the set of all n-tuples of real numbers.

Definition 2.3. Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ be two points in $\mathbb{R}^n$. The distance between x and y is defined to be

$$\| x - y \|_n = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2},$$

and is called the Euclidean norm or the usual norm.

Next, they tried to find another distance on $\mathbb{R}^2$ other than the Euclidean norm which preserves some properties of the latter. This gives rise to the concept of a metric on $\mathbb{R}^n$. In general, a metric on a nonempty set X is defined as follows.

Definition 2.4. Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}$ is called a **metric** on X if the following statements are true :

1. $d(x, y) \geq 0$ when $x, y \in X$.
2. $d(x, y) = d(y, x)$ when $x, y \in X$.
3. $d(x, y) = 0$ if and only if $x = y$ when $x, y \in X$.
4. $d(x, z) \leq d(x, y) + d(y, z)$ when $x, y, z \in X$.

A set X equipped with a metric d is called a metric space, denoted by (X, d).

Example 2.5. The following are examples of metric spaces.

1. $(\mathbb{R}, d_1)$, where $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $d_1(x, y) = | \ x - y \ |$ when $x, y \in \mathbb{R}$.

2. $(\mathbb{R}^n, d_n)$, where $d_n : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$d_n(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2},$$

when $x, y \in \mathbb{R}^n$. Note that if $n = 1$, then $\| x - y \|_1 = | \ x - y \ |$.

3. $(\mathbb{R}^n, d_0)$, where $d_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined by

$$d_0(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|,$$

when $x, y \in \mathbb{R}^n$.

Since a metric describes the distance between two points on a nonempty set X, it can describe some behavior on X. For example, a sequence on X converges to a point x in X if the sequence's terms are eventually close to x. A function between two metric spaces is continuous if it maps nearby points to nearby points. These concepts are explained as follows.

Definition 2.6. Let (X, d) be a metric space. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X **converges** to a point $x_0 \in X$ if for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $n \geq N_\epsilon$ implies $d(x_n, x_0) < \epsilon$.

Definition 2.7. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. We say that f is **continuous at a point** $x_0 \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$ if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$.

Definition 2.8. Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is **continuous on** X if f is continuous at every points in X.

Theorem 2.9. *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Then f is continuous at $x_0 \in X$ if and only if every sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x_0 implies $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x_0)$.*

To generalize the concept of metric spaces, we need to introduce the definition of an d-open ball.

Definition 2.10. Let (X, d) be a metric space. Given an $\epsilon > 0$ and $x \in X$, a set $\{y \in X \mid d(x, y) < \epsilon)\}$ is said to be a d**-open ball** centered at x with radius ϵ, denoted by $B_d(x\ ;\ \epsilon)$.

We observe that the definition of a convergent sequence and a continuous function can be rewritten by replacing $d(x, x_0) < \delta$ by $x \in B_d(x_0\ ;\ \delta)$. Open balls play an important role in the concept of metric spaces. Open balls are used to define openness of a subset of a metric space and neighborhoods in a metric space.

Definition 2.11. Let (X, d) be a metric space and $O \subset X$. Then O is said to be d**-open** if for any $x \in O$ there is an $\epsilon > 0$ such that $B_d(x\ ;\ \epsilon) \subset O$.

Definition 2.12. Let (X, d) be a metric space and $x \in V \subset X$. Then V is said to be a **neighborhood of** x if there exists a d-open set O such that $x \in O \subset V$.

Remark 2.13. Every d-open set is a neighbornood of each of its points.

If there is no ambiguity, then d-open is said to be open.

Proposition 2.14. *For any metric space, the following statements are true.*

1. *The empty set and the whole set are open.*
2. *An arbitrary union of open sets is open.*
3. *A finite intersection of open sets is open.*

Remark 2.15. For any metric space, any open ball is an open set.

We look back to the definition of continuous functions. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. We say that f is continuous at $x_0 \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$, if $d_X(x, x_0) < \delta$, then $d_Y(f(x), f(x_0)) < \epsilon$. We can rewrite the definition of continuous functions as below.

" f is continuous at $x_0 \in X$ if for any $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$, if $x \in B_{d_X}(x_0\ ;\ \delta)$, then $f(x) \in B_{d_Y}(f(x_0)\ ;\ \epsilon)$. "

Moreover, we can show that if f is continuous on X, then $f^{-1}(B_{d_Y}(f(x_0)\ ;\ \epsilon))$ is d_X-open for all $x_0 \in X$ and $\epsilon > 0$. Applying this fact shows that f is continuous on X if and only if the preimage of a d_Y-open is d_X-open. By ignoring the metric, we can use the properties of open sets to define a topological space which is a generalization of a metric space. These will be described and also use the above to define ccontinuity of functions in the next section.

2.2 Introduction to topology

Definition 2.16. Let X be a nonempty set. The collection τ of subsets of X is said to be a **topology** on X when τ satisfies the following conditions.

1. The empty set and the whole set belong to τ.
2. Any arbitary union of elements in τ belongs to τ.
3. A finite intersection of elements in τ belongs to τ.

An element in τ is called an open set (in X) and the pair (X, τ) is called a topological space.

Example 2.17. The following are examples of topologies.

1. Let (X, d) be a metric space. The collection of all open sets is a topology on X, denoted by τ_d. The topology τ_d is called the topology generated by d.

2. Let X be a nonempty set. Define $\tau = \{\varnothing, X\}$. Then τ is a topology and this topology τ is called the indiscrete topology or the trivial topology on X and denoted by $\tau_{indiscrete}$.

3. Let X be a nonempty set. Define $\tau = \mathcal{P}(X)$. Then τ is a topology and this topology τ is called the discrete topology on X and denoted by $\tau_{discrete}$.

4. Let $X = \{0, 1\}$. Define $\tau = \{\varnothing, \{0\}, X\}$. Then τ is a topology on X. This topology is called Sierpiński topology. Note that Sierpiński topology is the smallest topology which is neither discrete nor indicrete.

5. Let X be a nonempty set. Define $\tau = \{A \subset X \mid A^c \text{ is a finite set }\} \cup \{\varnothing\}$. Then τ is a topology on X. This topology is called the cofinite topology, denoted by $\tau_{cofinite}$.

Definition 2.18. Let (X, τ) be a topological space and $F \subset X$. Then F is said to be closed (in X) if $X - F$ is open.

Example 2.19. The following are examples of closed sets.

1. For an indiscrete topological space, only $\varnothing$ and X are closed.
2. For a discrete topological space, every subset of X is closed.
3. From Example 1.17.4., $\varnothing$, X and $\{1\}$ are closed.
4. From Example 1.17.5., finite sets are closed in X.
5. For any topological space X, $\varnothing$ and X are both open and closed.

Example 2.20. Let $X = \{a, b, c\}$. Define a topology $\tau = \{\varnothing, \{a\}, \{a, b, c\}\}$ on X. Then $\{b\}$ is neither open nor closed.

Next, we will define a natural topology on a subset of a topological space.

Definition 2.21. Let (X, τ) be a topological space and A a subset of X. The relative topology τ_A on A is defined by

$$\tau_A = \{O \cap A \mid \text{for all } O \in \tau\}.$$

The pair (A, τ_A) is called a subspace of (X, τ).

Example 2.22. The following are examples of subspace of X

1. Let $(\mathbb{R}, \tau_{d_1})$ be a topological space. Let $A = (0, 1]$. Then τ_A is the collection of all open intervals such that contained $(0, 1)$ unions the empty set and a set A. So (A, τ_A) is a subspace of X.

2. For an indiscrete topological space, every subset A of X has the relative topology on A to be $\tau_A = \{\varnothing, A\}$. So (A, τ_A) is a subspace of X.

3. For a discrete topological space, every subset A of X has the relative topology on A to be $\tau_A = \mathcal{P}(A)$. So (A, τ_A) is a subspace of X.

4. From Example 1.20, given $A = \{b\}$ and so $\tau_A = \{\varnothing, \{b\}\}$ is the relative topology on A. So (A, τ_A) is a subspace of X.

From the above examples, we see that $\{b\}$ is not open in X but open in A.

Remark 2.23. For any topological space X and $A \subset$ X. If A is open in X, then A open set in A must be open in X.

Let (X, τ) be a topological space and $A \subset X$. We want to find the largest open set that is contained in A. This leads to the definition of neighborhoods and interior points as follows.

Definition 2.24. Let (X, τ) be a topological space and $x \in V \subset X$. Then V is said to be a **neighborhood of x** if there is an open set O such that $x \in O \subset V$.

Definition 2.25. Let (X, τ) be a topological space and $A \subset X$. A point $x \in X$ is said to be an **interior point** of A if there is a neighborhood V of x such that $x \in V \subset A$.

Remark 2.26. A subset O is an open set in X if and only if O is a neighborhood of each of its points.

Theorem 2.27. *Let (X, τ) be a topological space and $A \subset X$. The set of all interior points of A is the largest open set contained in A.*

Remark 2.28. The set of all interior points of A is called the interior of A, denoted by A°.

The next theorem shows some properties of the interior.

Theorem 2.29. *Let (X, τ) be a topological space and $A, B \subset X$. The following statements are true:*

1. *$A \subset B$ implies $A^\circ \subset B^\circ$.*
2. *A is open if and only if $A^\circ = A$.*
3. *$A^\circ \cup B^\circ \subset (A \cup B)^\circ$.*

Now we will use the concept of interior sets to find the smallest closed set containing a subset A of X. First, we notice that $A^\circ = \bigcup \{G \subset A \mid G \text{ is open }\}$.

Then $X-(A^\circ)=\bigcap\{X-A\subset F \mid F \text{ is closed }\}$. Therefore, we obtian the smallest closed set containing $X-A$. The following is the definition of the smallest closed set containing A.

Definition 2.30. Let (X,τ) be a topological space and $A\subset X$. The closure of A is defined to be $X-(X-A)^\circ$ which is the smallest closed set containing in A, denoted by $\overline{A}$.

The next theorem shows some properties of the closure.

Theorem 2.31. *Let (X,τ) be a topological space and $A,B\subset X$. The following statements are true.*

1. *$A\subset B$ implies $\overline{A}\subset\overline{B}$.*
2. *A is closed if and only if $\overline{A}=A$,*
3. *$\overline{A}\cup\overline{B}=\overline{(A\cup B)}$.*

Example 2.32. Let $(\mathbb{R},\tau_{d_1})$ be a topological space. Recall that τ_{d_1} is the set of all d_1-open sets. Since $d_1(x,y)=\mid x-y\mid$ when $x,y\in\mathbb{R}$, a d_1-open set is generated by open intevals on $\mathbb{R}$. Let $A=(0,1)$. Then $A^\circ=(0,1)$ and $\overline{A}=[0,1]$.

Next, we will describe equivalent definitions of the closure.

Definition 2.33. Let (X,τ) be a topological space and $A\subset X$. A point $x\in X$ is said to be a limit point of A if every neighborhood of x meets $A-\{x\}$.

Remark 2.34. The set of all limit points of A is called the derived set of A, denoted by A'.

Theorem 2.35. *Let (X,τ) be a topological space and $A\subset X$. The following statements are true.*

1. *$\overline{A}=A\cup A'$.*
2. *$\overline{A}$ is the set of all points x in X such that every neighborhood of x meets A.*

Example 2.36. From Example 2.32, we can see that 0 , 1 are the limit points of A that do not belong to A. By Theorem 2.35, $\overline{A} = A \cup A' = [0, 1]$.

By applying the interior and the closure, we can define particular functions from the power set of X to itself as follows.

Definition 2.37. Let (X, τ) be a topological space. Define the function $\iota : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\iota(A) = A^\circ$ when $A \in \mathcal{P}(X)$. Then ι is called the interior operator on X.

Definition 2.38. Let (X, τ) be a topological space. Define the function $\kappa : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\kappa(A) = \overline{A}$ when $A \in \mathcal{P}(X)$. Then κ is called the closure operator on X.

The interior and the closure operators are well-defined bacause of the closure and the interior must be unique by thier definitions. In [1], the interior and closure operators are studied in the sense of nearly open sets. A various sets of nearly open sets can be defined as follows.

Definition 2.39. Let (X, τ) be a topological space. For each $A \subset X$,

1. A is called semi-open if $A \subset \overline{(A^\circ)}$,
2. A is called preopen if $A \subset (\overline{A})^\circ$,
3. A is called $\alpha-$open if $A \subset (\overline{(A^\circ)})^\circ$,
4. A is called $\beta-$open if $A \subset \overline{((\overline{A})^\circ)}$.

Example 2.40. Let $(\mathbb{R}, \tau_{d_1})$ be a topological space.

1. $A = (0, 1)$ is semi-open, preopen, $\alpha-$open and $\beta-$open.
2. $B = [0, 1]$ is only semi-open and $\beta-$open .
3. $C = \mathbb{Q}$ is only preopen and $\beta-$open.
4. $D = \{0\}$ is neither semi-open, preopen, $\alpha-$open, nor $\beta-$open.
5. $E = [-1, 0) \cup (\mathbb{Q} \cap (0, 1)) \cup (1, 2)$ is only $\beta-$open.

Many researches in topology tried to study the properties on various sets of nearly open sets. The relations among an open set and various sets of nearly open sets can be described by the following mapping.

$$\text{open set} \to \alpha\text{-open set} \to \text{semi-open} \to \beta-\text{open.}$$

$$\text{open set} \to \alpha\text{-open set} \to \text{preopen} \to \beta-\text{open.}$$

From the previous section, we introduce about the continuity between metric spaces. By using the concept of topologies, we will define the definition between topological spaces and give some properties to use in this thesis.

Definition 2.41. Let (X, τ) and (Y, ζ) be topological spaces. A function $f : X \to Y$ is continuous if for each $G \in \zeta$, $f^{-1}(G) \in \tau$.

Theorem 2.42. *Let (X, τ) and (Y, ζ) be topological spaces. Given $f : X \to Y$. The following statement are equivalent.*

1. *f is continuous.*
2. *for each F is closed in Y, then $f^{-1}(F)$ is closed in X.*

Theorem 2.43. *Let (X, τ) and (Y, ζ) be topological spaces. Given $f : X \to Y$ is continuous. The following statement are true.*

1. *For each $A \subset X$, the restriction function $f_{|A}$ on A is continuous.*
2. *If (X, τ) is a discrete space, then f is always continuous.*
3. *If (Y, ζ) is a indiscrete space, then f is always continuous.*

Theorem 2.44. *(Pasting lemma) Let (X, τ) and (Y, ζ) be topological spaces. Let $X = A \cup B$ when A and B both open or closed in X. Let $f : X \to Y$ be a function such that $f_{|A}$ is continuous on A and $f_{|B}$ is continuous on B. Then f is continuous on X.*

CHAPTER III

GENERALIZED TOPOLOGICAL SPACES AND HEREDITARY CLASSES

3.1 Generalized open sets and some properties

In point-set topological research, a large number of papers is interested the class of open sets or the class of nearly open sets. In 1997 [1], Császár introduced a notation of γ-open sets which is a generalization of open and nearly open sets.

Definition 3.1. Let X be a nonempty set. The function $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be **monotonic** if $A \subset B$ implies $\gamma A \subset \gamma B$ for all $A, B \in \mathcal{P}(X)$. The set of all monotonic functions is denoted by $\Gamma(X)$.

Remark 3.2. Let (X, τ) be a topological space. The interior and the closure operators are monotonic which also imply that the compositions between them are monotonic.

Definition 3.3. Let X be a set and γ a monotonic function. Then γ is called **idempotent** if $\gamma^2 A = \gamma\gamma A = \gamma A$, for all $A \in \mathcal{P}(X)$, γ is called **enlarging** if $A \subset \gamma A$ for all $A \in \mathcal{P}(X)$, and γ is called **restricing** if $\gamma A \subset A$ for all $A \in \mathcal{P}(X)$.

Definition 3.4. Let X be a set and γ a monotonic function. For each $A \subset X$, A is called $\gamma-$open if $A \subset \gamma A$.

Remark 3.5. Let (X, τ) be a topological space. If we choose γ is the interior operator, then every $\gamma-$open set is open.

Definition 3.6. Let X be a set and γ a monotonic function. For each $A \subset X$, A is said to be γ**-closed** if $X - A$ is γ-open.

Theorem 3.7. [1] Let X be a set and γ a monotonic function. Any union of $\gamma-$open is $\gamma-$open.

Proof. Let A_α be a $\gamma-$open set for all $\alpha \in \Lambda$. That is $A_\alpha \subset \gamma A_\alpha$ for all $\alpha \in \Lambda$. We know that for each $\beta \in \Lambda$, $A_\beta \subset \bigcup_{\alpha\in\Lambda} A_\alpha$, and then $A_\beta \subset \gamma A_\beta \subset \gamma(\bigcup_{\alpha\in\Lambda} A_\alpha)$ Hence $\bigcup_{\alpha\in\Lambda} A_\alpha \subset \gamma(\bigcup_{\alpha\in\Lambda} A_\alpha)$. □

However, we can show that a finite intersection of γ-open is not necessarily γ-open. Consider the following example

Example 3.8. Let $(\mathbb{R}, \tau_{d_1})$ be a topological space. Consider the sets $A = \mathbb{Q}^c \cup \{0\}$, $B = \mathbb{Q}$, and a monotonic function γ defined by $\gamma(A) = (\overline{A})^\circ$ for all $A \subset \mathbb{R}$. We can see that A and B are γ-open but $A \cap B$ is not γ-open.

Definition 3.9. Let X be a set and γ a monotonic function. For each $A \subset X$, we define $i_\gamma A$ to be the union of all γ-open subsets of A, and $i_\gamma A$ is said to be the γ**-interior** of A.

Remark 3.10. We can regard $i_\gamma A$ as the image of A under the function $i_\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$.

Proposition 3.11. [1] Let X be a set and γ a monotonic function. The following statements are true.

1. $A \subset B \subset X$ implies $i_\gamma A \subset i_\gamma B$.
2. $i_\gamma A \subset A$ when $A \subset X$.
3. $i_\gamma i_\gamma A = i_\gamma A$ when $A \subset X$.

Proof. 1. and 2. They are obvious.

3. Let A be a subset of X. By using (2.) , $i_\gamma i_\gamma A \subset i_\gamma A$. By the definition of i_γ and $i_\gamma A$ is γ-open , $i_\gamma A \subset i_\gamma i_\gamma A$. □

By Proposition 3.11, for each monotonic function γ, i_γ is monotonic and A is i_γ-open if $A = i_\gamma A$.

Theorem 3.12. [1] Let X be a set and γ a monotonic, restricting and idempotent function. Then $\gamma = i_\gamma$, i.e. $\gamma A = i_\gamma A$ for all $A \in \mathcal{P}(X)$.

Proof. Let $A \in \mathcal{P}(X)$. Since γ is idempotent, $\gamma\gamma A = \gamma A$ and so $\gamma A \subset \gamma\gamma A$. Then γA is $\gamma-$open. Next we will claim that γA is the largest $\gamma-$open contained in A. It is obvious that $\gamma A \subset A$ since γ is restricting. Now assume that $B \subset A$ and B is γ-open. Since γ is monotonic, $B \subset \gamma B \subset \gamma A$. Hence we have claim. That is $\gamma A = i_\gamma A$. □

Definition 3.13. Let X be a set and γ a monotonic function. For each $A \subset X$, we define $c_\gamma A$ to be the intersection of all γ-closed supersets of A, and $c_\gamma A$ is said to be the **γ-closure** of A.

Remark 3.14. We can regard $c_\gamma A$ as the image of A under the function $c_\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$.

Proposition 3.15. [1] Let X be a set and γ a monotonic function. For each $A \subset X$, $c_\gamma A = X - i_\gamma(X - A)$ and $i_\gamma A = X - c_\gamma(X - A)$.

Proof. Let $A \subset X$. Then $i_\gamma(X - A) = \bigcup\{M \subset X - A \mid M \text{ is } \gamma\text{-open }\}$. So $X - i_\gamma(X - A) = \bigcap\{A \subset X - M \mid M \text{ is } \gamma\text{-open }\} = \bigcap\{A \subset N \mid N \text{ is } \gamma\text{-closed}\} = c_\gamma A$. The statement $i_\gamma A = X - c_\gamma(X - A)$ can be proved similary as above. □

Proposition 3.16. [1] Let X be a set and γ a monotonic function. The following statements are true.

1. $A \subset B \subset X$ implies $c_\gamma A \subset c_\gamma B$.
2. $A \subset c_\gamma A$ when $A \subset X$.
3. $c_\gamma c_\gamma A = c_\gamma A$ when $A \subset X$.

Proof. 1. and 2. They are obvious.

3. Use (1.) and (2.) and the fact that $c_\gamma c_\gamma A \subset c_\gamma A$. □

Similarly, by the Proposition 3.16, for each monotonic function γ, c_γ is monotonic but every subset is c_γ-open.

Theorem 3.17. [1] Let X be a set and γ a monotonic, enlarging and idempotent function. Then $\gamma = c_\gamma$, i.e. $\gamma A = c_\gamma A$ for all $A \in \mathcal{P}(X)$.

Proof. Apply Theorem 3.12 , Proposition 3.15, and Proposition 3.16. □

3.2 Generalized topological spaces

In 2002 [2], Császár observed that the collection of γ-open sets has some properties similar to the one of the classical open sets. That is, for each monotonic function γ, the empty set is γ-open and arbitrary union of γ- open sets is γ-open. So he introduced the definition of a generalized topological space as follows.

Definition 3.18. Let X be a nonempty set. A collection μ of subsets of X is said to be a **generalized topology** on X if it satisfies the following statements.

1. The empty set is always in μ;
2. An arbitrary union of elements in μ is in μ.

The pair (X,μ) is called a generalized topological space, and an element in μ is called a **μ-open** set. It is easy to see that every topology is a generalized topology.

Definition 3.19. Let (X,μ) be a generalized topological space. For each $A \subset X$, A is called **μ-closed** if $X - A$ is μ-open.

Definition 3.20. Let (X, μ) be a generalized topological space. Then μ is said to be a quasi-topology on X if it satisfies the following property.

$$A, B \in \mu \text{ implies } A \cap B \in \mu.$$

The pair (X, μ) is called a quasi-topological space.

The concept of subspaces of generalized topological spaces can be given by the following definition.

Definition 3.21. Let (X, μ) be a generalized topological space and A a subset of X. The relative generalized topology μ_A on A is defined by

$$\mu_A = \{M \cap A \mid \text{for all } M \in \mu\}.$$

The pair (A, μ_A) is called a subspace of (X, τ).

From the definition of γ-open sets, we can see that each monotonic function can produce the generalized topological space. In the other way around, the following theorem shows that for each generalized topological space, there is a monotonic function γ such that every element in the generalized topology is γ-open.

Theorem 3.22. [2] Let (X, μ) be a generalized topological space. Then there is a monotonic function $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ such that μ is the collection of all γ-open sets.

Proof. Define the function $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\gamma A = \bigcup\{G \subset A \mid G \in \mu\}$ when $A \in \mathcal{P}(X)$. It is clear that γ is well-defined , monotonic and $\gamma A \in \mu$. Note that $\gamma A \subset A$. If $A \in \mu$, then $A \subset \gamma A$ and hence A is γ-open. Conversely, if A is γ-open, then $A \subset \gamma A \subset A$ and hence, $A = \gamma A$ which is in μ. □

By the theorem above, we see that γA is the union of all open sets in μ contained in A. Then we will define the μ-interior and the μ-closure of A as follows.

Definition 3.23. Let (X,μ) be a generalized topological space. For each $A \subset X$, we define $i_\mu A$ to be the union of all μ-open subsets of A, and $i_\mu A$ is said to be the **μ-interior** of A.

Definition 3.24. Let (X,μ) be a generalized topological space. For each $A \subset X$, we define $c_\mu A$ to be the intersection of all μ-closed supersets of A, and $c_\mu A$ is said to be the **μ-closure** of A.

Similarly, we can regard i_μ and c_μ as functions from $\mathcal{P}(X)$ to itself and we obtain the following propositions as well.

Proposition 3.25. [3] Let (X, μ) be a generalized topological space. The following statements are true.

1. $A \subset B \subset X$ implies $i_\mu A \subset i_\mu B$.
2. $i_\mu A \subset A$ when $A \subset X$.
3. $i_\mu i_\mu A = i_\mu A$ when $A \subset X$.

Proposition 3.26. [3] Let (X, μ) be a generalized topological space. For each $A \subset X$, $c_\mu A = X - i_\mu(X - A)$ and $i_\mu A = X - c_\mu(X - A)$.

Proposition 3.27. [3] Let (X, μ) be a generalized topological space. The following statements are true.

1. $A \subset B \subset X$ implies $c_\mu A \subset c_\mu B$.
2. $A \subset c_\mu A$ when $A \subset X$.
3. $c_\mu c_\mu A = c_\mu A$ when $A \subset X$.

3.3 Generalized topological spaces via hereditary classes

In 1990 [5], Hamlett and Janković studied on how to construct new topologies from an old one. This can be described as follows. Let (X, τ) be a topological space and $A \subset X$. The closure of A in X is the set of all points in X such that every neighborhood of x meets A, i.e.

$$x \in \bar{A} \longleftrightarrow A \cap O \neq \varnothing \text{ for all } O \in \tau \text{ containing } x.$$

We observe that the condition $A \cap O \neq \varnothing$ can be rewritten as $A \cap O \notin \{\varnothing\}$. Then we can generalize the concept of the closure by replacing $\{\varnothing\}$ by a collection of subsets of X. To obtain a new topology, note for now that not every collection of subsets of X can be replaced. This leads to a concept of a hereditary class and an ideal.

Definition 3.28. [5] Let X be a nonempty set. A collection $\mathcal{H}_X$ of subsets of X is said to be a **hereditary class** on X if for each $A, B \in \mathcal{P}(X)$,

$$A \subset B \text{ and } B \in \mathcal{H}_X \text{ imply } A \in \mathcal{H}_X.$$

If we add the property :

$$\text{for each } A, B \in \mathcal{P}(X),\ A, B \in \mathcal{H}_X \text{ implies } A \cup B \in \mathcal{H}_X,$$

then a hereditary class on X is said to be an **ideal**, denoted by $\mathcal{I}_X$.

Hamlett and Janković used ideals to construct new topologies. In 2007 [4], Császár used this concept to construct generalized topologies via hereditary classes. Note that the construction on generalized topology only requires a hereditary class. The condition for being an ideal is not necessary. Before we see how to construct the new generalzied topology, we need a significant concept on subsets of the space. Moreover, some properties will also be described. Throughout this thesis, $(X, \mu, \mathcal{H})$ denotes a generalized topological space (X, μ) together with a hereditary class $\mathcal{H}$.

Definition 3.29. Let $(X, \mu, \mathcal{H})$ be a generalized topological space. For each $A \subset X$, we define

$$A^*_{\mathcal{H}} = \{x \in X \mid M \cap A \notin \mathcal{H} \text{ when } M \in \mu \text{ containing } x\}.$$

If there is no ambiguity, then $A^*_{\mathcal{H}}$ can be denoted by A^*.

Remark 3.30. 1. If we choose a hereditary class $\mathcal{H} = \{\varnothing\}$, then $A^*_{\mathcal{H}} = c_\mu A$.

2. $A^*_{\mathcal{H}}$ depends on a hereditary class on X as seen in the following example.

Example 3.31. Let $X = \{a, b, c, d\}$. Define $\mu = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$. It is easy to see that μ is a generalized topology on X. Let $A = \{a, c\}$. Assume that $\mathcal{H}_1 = \{\varnothing, \{a\}\}$ and $\mathcal{H}_2 = \{\varnothing, \{b\}\}$ are hereditary classes on X. So $A^*_{\mathcal{H}_1} = \{b, c\}$ and $A^*_{\mathcal{H}_2} = \{a, b, c\}$. Note that $A \cap \{a\} \in \mathcal{H}_1$ implies $a \notin A^*_{\mathcal{H}_1}$, and $a \in A^*_{\mathcal{H}_2}$ because of $A \cap \{a\} \notin \mathcal{H}_2$ and $A \cap \{a, b, c\} \notin \mathcal{H}_2$.

Next, we describe some properties of A^* and we observe that in a topological space, $\overline{A}$ also satisfies these properties.

Proposition 3.32. [4] Let $(X, \mu, \mathcal{H})$ be a generalized topological space and $A, B \subset X$. The following statements are true :

1. $A \subset B$ implies $A^* \subset B^*$.
2. $A^* \subset c_\mu A$.
3. If $M \in \mu$ and $M \cap A \in \mathcal{H}$, then $M \cap A^* = \varnothing$.
4. A^* is μ-closed.
5. A is μ-closed implies $A^* \subset A$.
6. $(A^*)^* \subset A^*$ when $A \subset X$.
7. $X = X^*$ if and only if $\mu \cap \mathcal{H} = \{\varnothing\}$.

Proof. 1. Suppose that $x \notin B^*$. Then there is a set $M \in \mu$ such that $M \cap B \in \mathcal{H}$. Since $A \subset B$, then $M \cap A \in \mathcal{H}$ by the hereditary property . Therefore $x \notin A^*$.

2. Let $x \notin c_\mu A$. Then there is a set $M \in \mu$ such that $M \cap A = \varnothing$. So $M \cap A \in \mathcal{H}$. So $x \notin A^*$.

3. Assume that $M \in \mu$ and $M \cap A \in \mathcal{H}$. Suppose that $M \cap A^* \neq \varnothing$. There is $x \in M \cap A^*$. Then $M \cap A \notin \mathcal{H}$ which is a contradiction. So $M \cap A^* = \varnothing$.

4. Let $x \in X - A^*$. Then $x \notin A^*$ and there is a set $M \in \mu$ such that

$M \cap A \in \mathcal{H}$. By (3.), $M \cap A^* = \varnothing$. Then $x \notin c_\mu A^*$. So $c_\mu A^* \subset A^* \subset c_\mu A^*$ and $c_\mu A^* = A^*$. Hence A^* is a μ-closed set.

5. Assume that A is μ-closed. We obtain that $A^* \subset c_\mu A = A$.

6. Since A^* is μ-closed and using (5.), $(A^*)^* \subset A^*$.

7. Assume that $X = X^*$. Let $M \in \mu$. If $M \neq \varnothing$ then there is $x \in M$ such that $M \cap X \notin \mathcal{H}$. So $M \notin \mathcal{H}$. Hence $\mu \cap \mathcal{H} = \{\varnothing\}$. Conversely, assume that $\mu \cap \mathcal{H} = \{\varnothing\}$. Let $x \in X$. Then $M \cap X = M \notin \mathcal{H}$ when $x \in M \in \mu$. so $x \in X^*$. So we get $X \subset X^* \subset X$. Hence $X^* = X$. □

By using the concept of Proposition 3.32 (7.), we obtain the following definition.

Definition 3.33. Let $(X, \mu, \mathcal{H})$ be a generalized topological space. A hereditary class $\mathcal{H}$ is said to be $\mu-$codense if $\mathcal{H} \cap \mu = \{\varnothing\}$.

Theorem 3.34. [6] Let $(X, \mu, \mathcal{I})$ be a quasi-topological space together with an ideal $\mathcal{I}$. For each $A, B \subset X$, $A^* \cup B^* = (A \cup B)^*$.

Proof. Let $A, B \subset X$. From Proposition 3.32 (1.), we get $A^* \cup B^* \subset (A \cup B)^*$. Suppose that $x \notin A^* \cup B^*$. Then there exist $M, N \in \mu$ contaning x such that $M \cap A \in \mathcal{I}$ and $N \cap B \in \mathcal{I}$. So we can see that $M \cap N \cap A \in \mathcal{I}$ and $M \cap N \cap B \in \mathcal{I}$ bacause of the hereditary property. Since $\mathcal{I}$ is an ideal, $(M \cap N \cap A) \cup (M \cap N \cap B) \in \mathcal{I}$. We get $(M \cap N) \cap (A \cup B) \in \mathcal{I}$. Therefore $x \notin (A \cup B)^*$ because of $M \cap N \in \mu$. Hence $A^* \cup B^* = (A \cup B)^*$. □

In [6], Renukadevi and Vimladevi show the following example such that the above theorem is not true if $\mathcal{H}$ is a hereditary class not an ideal.

Example 3.35. Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, \{a\}, \{d\}, \{a, c\}, \{a, d\}, \{a, c, d\}\}$ a quasi-topology on X. Define a hereditary class $\mathcal{H}$ on X by $\mathcal{H} = \{\varnothing, \{a\}, \{c\}\}$.

Let $A = \{a\}$ and $B = \{b, c\}$. Then $A^* = \{b\}$ and $B^* = \{b\}$. So $A^* \cup B^* = \{b\}$. Consider $A \cup B = \{a, b, c\}$. We have $(A \cup B)^* = \{b, c\}$. Hence $A^* \cup B^* \neq (A \cup B)^*$.

The following example shows that the above theorem is not true if μ is not a quasi-topology.

Example 3.36. Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a, b\}, \{a, c\}, X\}$ a generalized topology on X which is not a quasi-topology. Define a hereditary class $\mathcal{H}$ on X by $\mathcal{H} = \{\varnothing, \{a\}\}$. Let $A = \{b\}$ and $B = \{c\}$. Then $A^* = \{b\}$ and $B^* = \{c\}$. So $A^* \cup B^* = \{b, c\}$. Consider $A \cup B = \{b, c\}$. We have $(A \cup B)^* = X$. Hence $A^* \cup B^* \neq (A \cup B)^*$.

In some situations, seeing Example 3.31 and 3.35, A^* does not contain A. That is A^* cannot give the sense of the closure of A anymore. To generalize the concept of the closure of A, we need the following definition.

Definition 3.37. Let $(X, \mu, \mathcal{H})$ be a generalized topological space. For each $A \subset X$,

$$c^*_{\mu,\mathcal{H}}A = A \cup A^*.$$

Similarly, if there is no ambiguity, then $c^*_{\mu,\mathcal{H}}A$ can be denoted by c^*A.

Lemma 3.38. [4] Let $(X,\mu,\mathcal{H})$ be a generalized topological space. The following statements are true:

1. $A \subset B \subset X$ implies $c^*A \subset c^*B$.
2. For each $A \subset X$, $(A \cup A^*)^* \subset A^*$.
3. Let $\beta : \mathcal{P}(X) \to \mathcal{P}(X)$ be a monotonic function such that $\beta(A \cup \beta A) \subset A \cup \beta A$ for all $A \subset X$. Then $c^{'}A = A \cup \beta A$ defines an enlarging, monotone and idempotent function.
4. c^* is enlarging, monotone and idempotent.

Proof. 1. Let $A \subset B \subset X$. By using Proposition 3.32 (1.) , $c^*A = A \cup A^* \subset B \cup B^* = c^*B$.

2. Let $A \subset X$ and $x \notin A^*$. Then there is a set $M \in \mu$ containing x such that $M \cap A \in \mathcal{H}$. By Proposition 3.32 (3.) , $M \cap A^* = \varnothing$. So $M \cap (A \cup A^*) = M \cap A \in \mathcal{H}$. Therefore $x \notin (A \cup A^*)^*$.

3. It is obvious that $c^{'}$ is enlarging and monotonic. Moreover,

$$c^{'}c^{'}A = c^{'}(A \cup \beta A) = (A \cup \beta A) \cup \beta(A \cup \beta A) \subset A \cup \beta A = c^{'}A \subset c^{'}c^{'}A.$$

So $c^{'}c^{'}A = c^{'}A$.

4. Use (1.) , (2.) and (3.) when we consider a monotonic $\beta : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\beta A = A^*$ for all $A \subset X$. □

Remark 3.39. By the above lemma, we have $(c^*A)^* \subset A^*$. Since $A \subset c^*A$, by using (1.) , $A^* \subset (c^*A)^*$. Hence $(c^*A)^* = A^*$.

By applying Lemma 3.38 and Theorem 3.17, there is a generalized topological space μ^* such that c^*A is the intersection of all μ^*-closed supersets of A , i.e. $M \in \mu^*$ if and only $c^*(X - M) = X - M$. Hence, we obtain the new generalized topological space.

Definition 3.40. Let $(X, \mu, \mathcal{H})$ be a generalized topological space. We define a generalized topology on X via a hereditary class $\mathcal{H}$ by

$$\mu^*_{\mathcal{H}} = \{M \subset X \mid c^*(X - M) = (X - M)\}.$$

An element in $\mu^*_{\mathcal{H}}$ is said to be $\mu^*_{\mathcal{H}}$-open.

Again, if there is no ambiguity, then $\mu^*_{\mathcal{H}}$ can be denoted by μ^*.

Example 3.41. By using Example 3.31, we can show that $\mu^*_{\mathcal{H}_1} = \{\varnothing, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mu^*_{\mathcal{H}_2} = \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Remark 3.42. If $\mathcal{H} = \{\varnothing\}$, then $\mu^* = \mu$.

Definition 3.43. Let $(X, \mu, \mathcal{H})$ be a generalized topological space. For each $F \subset X$, F is said to be μ^*- closed if and only if $X - F$ is μ^* open.

The following are some properties about the generalized topology μ^*.

Proposition 3.44. [4] Let $(X, \mu, \mathcal{H})$ be a generalized topological space. The following statements are true:

1. F is μ^*-closed if and only if $F^* \subset F$.

2. $\mu \subset \mu^*$.

Proof. 1. F is μ^*-closed if and only if $F = c^*F$ if and only if $F^* \subset F$.

2. Let $M \in \mu$. Then $X - M$ is μ-closed. By Proposition 3.32, $(X-M)^* \subset (X-M)$. By using (1.), $X-M$ is μ^*-closed. Hence M is μ^*-open. □

Proposition 3.45. [4] Let $(X,\mu,\mathcal{H})$ be a generalized topological space. The following statements are equivalent:

1. $A \subset A^*$.
2. $A^* = c^*A$.
3. $cA \subset A^*$.
4. $A^* = c_\mu A$.

Proof. (1.) $\leftrightarrow$ (2.) Use the definition of c^*.

(1.) $\rightarrow$ (3.) By using Proposition 3.32 (4.), A^* is μ-closed.

(3.) $\rightarrow$ (4.) By using Proposition 3.32 (2.), $A^* \subset cA$.

(4.) $\rightarrow$ (1.) Use the fact that $A \subset c^*$.

□

From Example 3.31, if a generalized topology contains a large number of elements, then it is complicated to determine the generalized topology via the hereditary class. However, there is an easier way to determine a generalized topology via a hereditary class. This needs the concept of a base for a generalized topology.

Definition 3.46. Let (X, μ) be a generalized topological space. The collection $\mathcal{B}$ is a **base** for μ if and only if every $M \in \mu$ is a union of elements of $\mathcal{B}$.

Theorem 3.47. [4] Let $(X, \mu, \mathcal{H})$ be a generalized topological space. The set $\{ M - H \mid \text{for all } M \in \mu \text{ and } H \in \mathcal{H} \}$ constitutes a base for μ^*.

Proof. Let $\mathcal{B} = \{ M - H \mid \text{for all } M \in \mu \text{ and } H \in \mathcal{H} \}$. For each $M \in \mu$ and $H \in \mathcal{H}$, by using Proposition 3.44 (1.), we can show that $F := X - (M - H)$ is μ^*-closed. Then $\mathcal{B} \subset \mu^*$. Next, we will show that each element of μ^* can be written as a union of elements of $\mathcal{B}$. Let $A \in \mu^*$. Then $C := X - A$ is μ^*- closed and $C^* \subset C$ by Proposition 3.44 (1.). Thus $x \in A$ implies $x \notin C^*$, and there exists $M_x \in \mu$ such that $x \in M_x$ and $H_x := M_x \cap C \in \mathcal{H}$. Therefore for each $x \in A$, $x \in M_x - H_x \subset X - C = A$, i.e. $A = \bigcup_{x \in A} (M_x - H_x)$. □

CHAPTER IV

CONTINUITY ON GENERALIZED TOPOLOGICAL SPCAES VIA HEREDITARY CLASSES

In this chapter, we will prove the main results of this thesis. First of all, the definition of continuous functions between generalized topological spaces was introduced by [2] as follows.

Definition 4.1. Let (X, μ) and (Y, ν) be generalized topological spaces. A function $f : X \to Y$ is said to be (μ, ν)**-continuous** if for each $N \in \nu$, $f^{-1}(N) \in \mu$.

Theorem 4.2. *(Pasting lemma on quasi-topological spaces.) Let (X, μ) be a quasi-topological space and (Y, ν) a generalized topological space. Let $X = A \cup B$ when A and B both μ-closed or μ-open. Let $f : X \to Y$ be a function such that $f_{|A}$ is $(\mu_A, \nu)-$continuous and $f_{|B}$ is $(\mu_B, \nu)-$continuous. Then f is $(\mu, \nu)-$ continuous.*

Proof. First, consider the case that A and B are both μ-open. Let $N \in \nu$. Then $f_{|A}^{-1}(N) \in \mu_A$ and $f_{|B}^{-1}(N) \in \mu_B$. That is $f^{-1}(N) \cap A \in \mu_A$ and $g^{-1}(N) \cap B \in \mu_B$. Since A and B are μ-open, $f^{-1}(N) \cap A$ and $g^{-1}(N) \cap B$ are in μ. So $f^{-1}(N) = f^{-1}(N) \cap X = f^{-1}(N) \cap (A \cup B) = (f^{-1}(N) \cap A) \cup (f^{-1}(N) \cap B) \in \mu$. Hence $f : X \to Y$ is $(\mu, \nu)-$ continuous. Secondary, if A and B are both μ-closed, then we can show that f is still (μ, ν)-continuous by using the similar argument and the fact that f is continuous if the preimage of a closed set is closed. $\square$

Theorem 4.3. *(Pasting lemma on quasi-topological spaces via codense ideals.) Let (X, μ) be a quasi-topological space, (Y, ν) a generalized topological space and $\mathcal{H}_X$ a $\mu-$codense ideal. Let $X = A \cup B$ and $f : X \to Y$ a function such that $f_{|A^*}$ is $(\mu_{A^*}, \nu)-$continuous and $f_{|B^*}$ is $(\mu_{B^*}, \nu)-$continuous. Then f is $(\mu, \nu)-$ continuous.*

Proof. Let $N \in \nu$. Then $Y - N$ is ν-closed. That is $f^{-1}(Y-N) \cap A^* = f_{|A^*}^{-1}(Y-N)$ is μ_{A^*}-closed and $f^{-1}(Y - N) \cap B^* = f_{|B^*}^{-1}(Y - N)$ is μ_{B^*}-closed. Since A^* and B^* are μ-closed, $f^{-1}(Y - N) \cap A^*$ and $f^{-1}(Y - N) \cap B^*$ are μ-closed. Consider $f^{-1}(Y-N) \cap (A^* \cup B^*) = f^{-1}(Y-N) \cap A^*) \cup f^{-1}(Y-N) \cap B^*)$ which is μ-closed. By Theorem 3.34, we get $f^{-1}(Y - N) \cap (A^* \cup B^*) = f^{-1}(Y - N) \cap (A \cup B)^*$. By Proposition 3.32 (7.), $f^{-1}(Y-N) \cap (A \cup B)^* = f^{-1}(Y-N) \cap (X)^* = f^{-1}(Y-N) \cap X$. So $f^{-1}(Y - N)$ is μ-closed. Then $f^{-1}(N) \in \mu$. Hence f is (μ, ν)-continuous. □

Form Proposition 3.44, we know that $\mu \subset \mu^*$. So we can easily prove the following theorem.

Theorem 4.4. *Let (X, μ) and (Y, ν) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that f is a (μ, ν)-continuous function from X to Y. Then f is $(\mu^*, \nu)-$continuous function.*

Proof. Let $N \in \nu$. Since f is $(\mu, \nu)-$continuous, $f^{-1}(N) \in \mu$. From Theorem 3.44 (2.), $f^{-1}(N) \in \mu^*$. Hence f is $(\mu^*, \nu)-$continuous. □

From the above theorem, if we replace the generalized topological space (Y, ν) by the generalized topological space $(Y, \nu^*_{\mathcal{H}_Y})$ via some hereditary class $\mathcal{H}_Y$, then we come up with a question whether f is (μ^*, ν^*)-continuous or not. Now, let us consider the following example.

Example 4.5. Given $X = [0, 1]$, $Y = [1, 2]$. Let μ be the usual topology on X and ν the indiscrete topology on Y. Define $f : (X, \mu) \to (Y, \nu)$ by $f(x) = x + 1$.

Then f is a $(\mu, \nu)-$continuous bijection from X onto Y. Define hereditary classes on X and Y by

$$\mathcal{H}_X = \{\varnothing\} \cup \{\{x\} \in \mathcal{P}(X) \mid x \in X\} \text{ and } \mathcal{H}_Y = \{\varnothing\} \cup \{A \subset Y \mid A \subset (1, 2]\},$$

respectively. From Theorem 3.47 , we know

$$\mu^*_{\mathcal{H}_X} = \{\bigcup_{\alpha\in\Lambda}(O_\alpha - H_\alpha) \mid \text{for all } O_\alpha \in \mu \text{ and } H_\alpha \in \mathcal{H}_X\} \text{ and,}$$
$$\nu^*_{\mathcal{H}_Y} = \{\bigcup_{\alpha\in\Lambda}(O_\alpha - H_\alpha) \mid \text{for all } O_\alpha \in \nu \text{ and } H_\alpha \in \mathcal{H}_Y\}.$$

We can conclude that $\{1\} \in \nu^*_{\mathcal{H}_Y}$ because $\{1\} = [1, 2] - (1, 2]$, $[1, 2] \in \nu$, and $(1, 2] \in \mathcal{H}_Y$. However $f^{-1}(\{1\}) = \{0\} \notin \mu^*_{\mathcal{H}_X}$ because the elements in $\mu^*_{\mathcal{H}_X}$ cannot be written in a form of singleton sets, i.e. the elements in $\{M - H \mid M \in \mu$ and $H \in \mathcal{H}_X\}$ are disjoint unions of intervals. For examples, if $M_1 = (a, b)$ when $a, b \in (0, 1)$ such that $a < b$ and $H = \{c\}$ when $c \in [0, 1]$, then $M_1 - H = (a, c) \cup (c, b)$ if $c \in (a, b)$, or $M_1 - H = (a, b)$ if $c \notin (a, b)$. If $M_2 = [0, a)$ when $a \in (0, 1)$. So $M_2 - H = [0, c) \cup (c, a)$ if $c \in (0, a)$ or $M_2 - H = [0, a)$ if $c \notin (0, a)$.

This implies that not every hereditary class on Y makes f a (μ^*, ν^*)-continuous function. So we would like to know what conditions give rise to the (μ^*, ν^*)-continuity of a (μ, ν)-continuous function. These various conditions are described as follows.

Theorem 4.6. *Let (X, μ) and $(Y, \nu, \mathcal{H}_Y)$ be generalized topological spaces. If f is a (μ, ν)-continuous injection from X into Y, then there is a hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(A) \mid A \in \mathcal{H}_Y\}.$$

such that f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-continuous.

Proof. First, we prove $\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\}$ is a hereditary class on X. Assume that $C \in \mathcal{H}_X$ and $D \subset C$. Since $C \in \mathcal{H}_X$, there exists $H_1 \in \mathcal{H}_Y$ such that $C = f^{-1}(H_1)$. That is $D \subset f^{-1}(H_1)$ and so $f(D) \subset f(f^{-1}(H_1)) \subset H_1$. Then $f(D) \in \mathcal{H}_Y$ because of $H_1 \in \mathcal{H}_Y$. So $f^{-1}(f(D)) \in \mathcal{H}_X$. Because of f is a injective function, $D = f^{-1}(f(D)) \in \mathcal{H}_X$. Next, to prove that f is $(\mu^*, \nu^*)-$continuous, let $G \in \nu^*$. From Theorem 3.47, ν^* has a base of the form $\{N - H \mid \text{for all } N \in \nu$ and $H \in \mathcal{H}_Y\}$. The set G can be written as $G = \bigcup_{\alpha \in \Lambda} (N_\alpha - H_\alpha)$ when $N_\alpha \in \nu$ and $H_\alpha \in \mathcal{H}_Y$. So

$$f^{-1}(G) = f^{-1}(\bigcup_{\alpha \in \Lambda} (N_\alpha - H_\alpha)) = \bigcup_{\alpha \in \Lambda} f^{-1}(N_\alpha - H_\alpha)).$$

By properties of inverse image, we get $f^{-1}(N_\alpha - H_\alpha) = f^{-1}(N_\alpha) - f^{-1}(H_\alpha)$. Since f is $(\mu, \nu)-$continuous and the first part in this proof, $f^{-1}(N_\alpha) \in \mu$ and $f^{-1}(H_\alpha) \in \mathcal{H}_X$. So $f^{-1}(G)$ is the element in a base for μ^* such that is constructed by a hereditary class $\mathcal{H}_X$ be defined as above. Hence $f^{-1}(G) \in \mu^*$ and f is $(\mu^*, \nu^*)-$countinuous injective function. □

Example 4.7. Consider the previous examples. Define hereditary classes on X and Y by $\mathcal{H}_X = \{\varnothing, \{1\}\}$ and $\mathcal{H}_Y = \{\varnothing, \{2\}\}$, respectively. Then $\nu^*_{\mathcal{H}_Y} = \{\varnothing, [1,2), [1,2]\}$. For each $N \in \nu^*_{\mathcal{H}_Y}$, we can easily check that $f^{-1}(N) \in \mu^*_{\mathcal{H}_X}$. Hence f is $(\mu^*, \nu^*)-$continuous.

The following example shows that the injective property is a necessary condition in the construction of hereditary classes in Theorem 4.6

Example 4.8. Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, $\mu = \{\varnothing, \{a, c\}, \{a, b, c\}\}$ and $\nu = \{\varnothing, \{1, 2\}\}$. Define $g : X \to Y$ by

$$g(a) = 1 \ , \ g(b) = 2 \ , \ g(c) = 2.$$

It is easy to check that g is (μ, ν)-continuous. Let $\mathcal{H}_Y = \{\varnothing, \{2\}\}$ be a hereditary class on Y. Using the construction in Theorem 4.6, we obtain $\mathcal{H}_X = \{\varnothing, \{b, c\}\}$. However, $\mathcal{H}_X$ is not hereditary class on X.

On the other hand, if we give a hereditary class on X, then we also can construct a hereditary class on Y that makes f (μ^*, ν^*)-continuous.

Theorem 4.9. *Let $(X, \mu, \mathcal{H}_X)$ and (Y, ν) be generalized topological spaces. If f is a (μ, ν)-continuous bijection from X onto Y, then there is a hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(A) \mid A \in \mathcal{H}_X\}.$$

such that f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-continuous.

Proof. To show $\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\}$ is a hereditary class on Y. Assume that $C \in \mathcal{H}_Y$ and $D \subset C$. Since $C \in \mathcal{H}_Y$, there exists $H_1 \in \mathcal{H}_X$ such that $C = f(H_1)$. So $D \subset f(H_1)$. Consider $f^{-1}(D) \subset f^{-1}(f(H_1)) = H_1$ because of f is injective. So $f^{-1}(D) \in \mathcal{H}_X$ and so $f(f^{-1}(D)) \in \mathcal{H}_Y$. By the surjective property of f, $D = f(f^{-1}(D)) \in \mathcal{H}_Y$. Next, we will prove that f is $(\mu^*, \nu^*)-$continuous. Let $G \in \nu^*$. That is G can be wrttien by $G = \bigcup\limits_{\alpha\in\Lambda}(N_\alpha - H_\alpha)$ when $N_\alpha \in \nu$ and $H_\alpha \in \mathcal{H}_Y$. Since $H_\alpha \in \mathcal{H}_Y$, there is $K_\alpha \in \mathcal{H}_X$ such that $f(K_\alpha) = H_\alpha$. Therefore

$$f^{-1}(G) = f^{-1}(\bigcup_{\alpha\in\Lambda}(N_\alpha - H_\alpha)) = \bigcup_{\alpha\in\Lambda} f^{-1}(N_\alpha - H_\alpha)) = \bigcup_{\alpha\in\Lambda} f^{-1}(N_\alpha - f(K_\alpha)).$$

By properties of inverse image, we get $f^{-1}(N_\alpha - f(K_\alpha)) = f^{-1}(N_\alpha) - f^{-1}(f(K_\alpha))$. Since f is surjective, $f^{-1}(f(K_\alpha)) = K_\alpha \in \mathcal{H}_X$ and $f^{-1}(N_\alpha) \in \mu$ by the continuity of f. Therefore $f^{-1}(G)$ is the element in a base for μ^* and so $f^{-1}(G) \in \mu^*$. We can conclude that f is $(\mu^*, \nu^*)-$countinuous bijective function. □

From Theorem 4.6 and Theorem 4.9, we know that if we give any hereditary class on either X or Y under some assumption on the function f, then we always can find the hereditary class so that this construction preserves the continuity of f. Next, we consider the composition function and so it is easy to prove the following theorem.

Theorem 4.10. *Let (X,μ),(Y,ν) and (Z,ω) be generalized topological spaces. Let f be a $(\mu,\nu)-$continuous bijective function from X to Y and g a $(\nu,\mu)-$continuous bijective function from Y to Z. The following statements are true:*

1 If $\mathcal{H}_X$ is a hereditary class on X, we can construct the hereditary classes $\mathcal{H}_Y$ and $\mathcal{H}_Z$ such that $g \circ f$ is $(\mu^*,\omega^*)-$continuous.

2 If $\mathcal{H}_Z$ is a hereditary class on Z, we can construct the hereditary classes $\mathcal{H}_X$ and $\mathcal{H}_Y$ such that $g \circ f$ is $(\mu^*,\omega^*)-$continuous.

3 If $\mathcal{H}_Y$ is a hereditary class on Y, we can construct the hereditary classes $\mathcal{H}_X$ and $\mathcal{H}_Z$ such that $g \circ f$ is $(\mu^*,\omega^*)-$continuous.

Proof. Apply Theorem 4.6 and Theorem 4.9. □

By the theorem above, if we have a hereditary class on Z, then it is enough to assume the injection of f and g. Let us see the following example.

Example 4.11. Let $X = \{a,b,c\}$, $Y = \{1,2,3,4\}$, $Z = \{11,22,33,44,55\}$. Define
$\mu = \{\varnothing,\{a,c\},\{a,b\},\{b,c\},\{a,b,c\}\}$,
$\nu = \{\varnothing,\{1,2\},\{1,3\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\}\}$ and
$\omega = \{\varnothing,\{11,33\}\}$ to be generalized topologies on X, Y, Z, respectively.

Suppose $f : X \to Y$ defined by

$$f(a) = 1\ ,\ f(b) = 3\ ,\ f(c) = 2\ ,$$

and $g : Y \to Z$ defined by

$$g(1) = 11\ ,\ g(2) = 33\ ,\ g(3) = 55\ ,\ g(4) = 22.$$

It is easy to check that f is a (μ, ν)-continuous injection and g is a (ν, ω)-continuous injection. Given a hereditary class $\mathcal{H}_Z = \{\varnothing, \{33\}\}$ on Z, we have $\omega^*_{\mathcal{H}_Z} = \{\varnothing, \{11\}, \{11, 33\}\}$. By Theorem 4.6, we can define $\mathcal{H}_Y = \{\varnothing, \{2\}\}$ and then

$$\nu^*_{\mathcal{H}_Y} = \{\varnothing, \{1\}, \{1, 2\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$

By Theorem 4.6 again , we can define $\mathcal{H}_X = \{\varnothing, \{c\}\}$ and then

$$\mu^*_{\mathcal{H}_X} = \{\varnothing, \{a\}, \{b\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}.$$

Therefore f is (μ^*, ν^*)-continuous and g is (ν^*, ω^*)-continuous. Hence $h := g \circ f$ is (μ^*, ω^*)-continuous.

In a generalized topological space (X, μ), we can define an open map and a homeomorphism which are similar to the ones in a topological space.

Definition 4.12. Let (X, μ) and (Y, ν) be generalized topological spaces. A function $f : X \to Y$ is said to be (μ, ν)**-open** if for each $M \in \mu$, $f(M) \in \nu$.

Definition 4.13. Let (X, μ) and (Y, ν) be generalized topological spaces. A function $f : X \to Y$ is said to be (μ, ν)**-homeomorphism** if f is a (μ, ν)-continuous bijection and f^{-1} is (ν, μ)-continuous.

Theorem 4.14. *Let (X, μ) and (Y, ν) be generalized topological spaces and f a bijection from X onto Y. Then f is $(\mu, \nu)-$open if and only if f^{-1} is $(\nu, \mu)-$continuous.*

Proof. Let $M \in \mu$. Since f is (μ, ν)-open, $f(M) \in \nu$. Then $(f^{-1})^{-1}(M) \in \nu$. Hence f^{-1} is $(\nu, \mu)-$continuous. Conversely, assume that f^{-1} is $(\nu, \mu)-$continuous. Let $M \in \mu$. Since f^{-1} is $(\nu, \mu)-$continuous, $(f^{-1})^{-1}(M) \in \nu$. So $f(M) = (f^{-1})^{-1}(M) \in \nu$. Hence f is (μ, ν)-open. □

Corollary 4.15. *Let* (X, μ) *and* (Y, ν) *be generalized topological spaces and* f *a bijection from* X *onto* Y*. Then* f *is* $(\mu, \nu)-$*homeomorphism if and only if* f *is* $(\mu, \nu)-$*continuous and* (μ, ν)*-open.*

Proof. Apply Theorem 4.14. □

Corollary 4.16. *Let* $(X, \mu, \mathcal{H}_X)$ *and* (Y, ν) *be generalized topological spaces. If* f *is a* (μ, ν)*-open bijection from* X *to* Y*, then there is a hereditary class* $\mathcal{H}_Y$ *on* Y *such that* f *is* $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$*-open.*

Proof. Apply Theorem 4.4 and 4.6 □

Corollary 4.17. *Let* $(X, \mu,)$ *and* $(Y, \nu, \mathcal{H}_Y)$ *be generalized topological spaces. If* f *is a* (μ, ν)*-open bijection from* X *to* Y*, then there is a hereditary class* $\mathcal{H}_X$ *on* X *such that* f *is* $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$*-open.*

Proof. Apply Theorem 4.4 and 4.9. □

Example 4.18. Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$. Define $\mu = \{\varnothing, \{a, c\}, \{a, b\}, \{b, c\}$ $, \{a, b, c\}\}$ and $\nu = \{\varnothing, \{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Suppose $f : X \to Y$ defined by

$$f(a) = 1 \ , \ f(b) = 3 \ , \ f(c) = 2.$$

It is easy to check that f is a (μ, ν)-open bijection. Given a hereditary class $\mathcal{H}_X$ $= \{\varnothing, \{b\}\}$ on X, we have $\mu^*_{\mathcal{H}_X} = \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. By Theorem 4.14, $f^{-1} : Y \to X$ defined by

$$f^{-1}(1) = a \ , \ f^{-1}(2) = c \ , \ f^{-1}(3) = b \ ,$$

and f^{-1} is (ν, μ)-continuous bijection. By Theorem 4.6, we can define $\mathcal{H}_Y =$ $\{\varnothing, \{3\}\}$ and then

$$\nu^*_{\mathcal{H}_Y} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Therefore f^{-1} is $(\nu^*_{\mathcal{H}_Y}, \mu^*_{\mathcal{H}_X})$-continuous. By using Theorem 4.14 again, f is $(\mu^*_{\mathcal{H}_X}, \nu^*_{\mathcal{H}_Y})$-open.

Theorem 4.19. *Let $(X, \mu, \mathcal{H}_X)$ and (Y, ν) be generalized topological spaces. If f is a $(\mu, \nu)-$homeomorphism from X to Y, then there is a hereditary class $\mathcal{H}_Y$ on Y that makes f a (μ^*, ν^*)-homeomorphism.*

Proof. Let $f : X \to Y$ be a (μ, ν)-homeomorphism. Then f is a (μ, ν)-open and (μ, ν)-continuous bijection. By Theorem 4.9, we obtain a hereditary class $\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\}$ on Y such that f is (μ^*, ν^*)-continuous. It remains to show that f is (μ^*, ν^*)-open. Let $G \in \mu^*$. Then G can be written by $G = \bigcup_{\alpha\in\Lambda}(M_\alpha - H_\alpha)$ when $M_\alpha \in \mu$ and $H_\alpha \in \mathcal{H}_X$. So $f(G) = f(\bigcup_{\alpha\in\Lambda}(M_\alpha - H_\alpha)) = \bigcup_{\alpha\in\Lambda} f(M_\alpha - H_\alpha)$. Since f is bijective, $\bigcup_{\alpha\in\Lambda} f(M_\alpha - H_\alpha) = \bigcup_{\alpha\in\Lambda}(f(M_\alpha) - f(H_\alpha))$. We see that $f(M_\alpha) \in \nu$ and $f(H_\alpha) \in \mathcal{H}_Y$ bacause f is $(\mu, \nu)-$open. Therefore $f(G) \in \nu^*$ and so f is a $(\mu^*, \nu^*)-$open map. Finally, these imply that f is a (μ^*, ν^*)-homeomorphism. □

Theorem 4.20. *Let (X, μ) and $(Y, \nu, \mathcal{H}_Y)$ be generalized topological spaces. If f is a $(\mu, \nu)-$homeomorphism from X to Y, then there is a hereditary class $\mathcal{H}_X$ on X that makes f a (μ^*, ν^*)-homeomorphism.*

Proof. Consider f^{-1} and apply Theorem 4.19. □

The next quesion about continuous functions will focus on subspaces of a generalized topological spaces. We will show that for each subspace A of X and a hereditary class on X, $(\mu_A)^* = (\mu^*)_A$ for some hereditary class on A.

Definition 4.21. Let (X, μ) be a nonempty set, $\mathcal{H}$ a hereditary class on X, and $A \subset X$. The relative hereditary class $\mathcal{H}_A$ on A is defined by

$$\mathcal{H}_A = \{H \cap A \mid \text{for all } H \in \mathcal{H}\}$$

Remark 4.22. We will show that $\mathcal{H}_A$ is a hereditary class on A. Let $C \in \mathcal{H}_A$ and $D \subset C$. Since $C \in \mathcal{H}_A$, there exists $H \in \mathcal{H}$ such that $C = H \cap A$. We see that $D \subset H \cap A \subset H$ and so $D \in \mathcal{H}$. Hence $D = D \cap A \in \mathcal{H}_A$.

Theorem 4.23. *Let $(X, \mu, \mathcal{H})$ be a generalized topological space and A a subset of X. For the reltive hereditary class $\mathcal{H}_A$ on A, $(\mu_A)^* = (\mu^*)_A$.*

Proof. Let $V \in (\mu^*)_A$. There exists $G \in \mu^*$ such that $V = G \cap A$. Since $G \in \mu^*$, $G = \bigcup_{\alpha\in\Lambda}(M_\alpha - H_\alpha)$ when $M_\alpha \in \mu$ and $H_\alpha \in \mathcal{H}$. Then

$$V = (\bigcup_{\alpha\in\Lambda}(M_\alpha - H_\alpha)) \cap A = (\bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c)) \cap A = \bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c \cap A)$$

$$= (\bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c \cap A)) \cup \varnothing = (\bigcup_{\alpha\in\Lambda}(M_\alpha \cap (H_\alpha)^c \cap A)) \cup (\bigcup_{\alpha\in\Lambda}(M_\alpha \cap A^c \cap A))$$

$$= \bigcup_{\alpha\in\Lambda}((M_\alpha \cap (H_\alpha)^c \cap A) \cup (M_\alpha \cap A^c \cap A)) = \bigcup_{\alpha\in\Lambda}((M_\alpha \cap A) \cap ((H_\alpha)^c \cup A^c))$$

$$= \bigcup_{\alpha\in\Lambda}((M_\alpha \cap A) \cap (H_\alpha \cap A)^c) = \bigcup_{\alpha\in\Lambda}((M_\alpha \cap A) - (H_\alpha \cap A)).$$

Since $M_\alpha \cap A \in (\mu)_A$ and $H_\alpha \cap A \in \mathcal{H}_A$, we get $V \in (\mu_A)^*$. Hence $(\mu^*)_A \subset (\mu_A)^*$. Conversely, let $W \in (\mu_A)^*$.Then $W = \bigcup_{\beta\in\Gamma}(U_\beta - K_\beta)$ when $U_\beta \in \mu_A$ and $K_\beta \in \mathcal{H}_A$. Since $U_\beta \in \mu_A$ and $K_\beta \in \mathcal{H}_A$, there exist $M_\beta \in \mu$ and $H_\beta \in \mathcal{H}$ such that $U_\beta = M_\beta \cap A$ and $K_\beta = H_\beta \cap A$. We have $W = \bigcup_{\beta\in\Gamma}((M_\beta \cap A) - (H_\beta \cap A))$. Similarly, $W = \bigcup_{\beta\in\Gamma}(M_\beta \cap A - H_\beta \cap A) = (\bigcup_{\beta\in\Gamma}(M_\beta - H_\beta)) \cap A$. Since $\bigcup_{\beta\in\Gamma}(M_\beta - H_\beta)$ is an element in μ^*, $W \in (\mu^*)_A$. Hence, $(\mu_A)^* \subset (\mu^*)_A$. Therefore $(\mu_A)^* = (\mu^*)_A$. □

Corollary 4.24. *Let $(X, \mu, \mathcal{H})$, (Y, ν) be generalized topological spaces and A a subset of X. For the relative hereditary class $\mathcal{H}_A$ on A and $f : A \to Y$, f is $((\mu_A)^*, \nu)-$continuous if and only if f is $((\mu^*)_{|A}, \nu)-$continuous.*

Proof. Apply Theorem 4.23. □

Up until now, we have seen various situations for having (μ^*, ν^*)-continuity from (μ, ν)-continuity. Next we would like to detemine whether a function can be (μ^*, ν)-continuous without assuming (μ, ν)-continuity. The following examples give the investigation of this statement.

Example 4.25. Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4\}$. Define the generalized topologies $\mu = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{b, c, d\}, X\}$ on X and $\nu = \{\varnothing, \{1\}\}$ on Y. Suppose $f : (X, \mu) \to (Y, \nu)$ defined by

$$f(a) = 4 \ , \ f(b) = 3 \ , \ f(c) = 2 \ , \ f(d) = 1.$$

So f is not $(\mu, \nu)-$continuous. Choose a hereditary class $\mathcal{H}_X = \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$ on X, then

$$\mu^* = \{\varnothing, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}.$$

We can observe that f is $(\mu^*, \nu)-$continuous.

Example 4.26. Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$. Define the generalized topologies $\mu = \{\varnothing, \{a\}, \{d\}, \{a, d\}, \{a, b, d\}\}$ on X and $\nu = \{\varnothing, \{1\}, \{1, 2\}\}$ on Y. Suppose $f : (X, \mu) \to (Y, \nu)$ defined by

$$f(a) = 4 \ , \ f(b) = 3 \ , \ f(c) = 2 \ , \ f(d) = 1.$$

So f is not $(\mu, \nu)-$continuous. Moreover, we can conclude that f is not (μ^*, ν) continuous for all hereditary classes on X because of we know that the element in μ^* has a base in form $\{M - H \mid \text{for all } M \in \mu \text{ and } H \in \mathcal{H}\}$ and so any open set in $\mu^*_{\mathcal{H}}$ cannot contain the element c, for all hereditary class $\mathcal{H}$ on X.

Theorem 4.27. *Let (X, μ) and (Y, ν) be generalized topological spaces and $f : X \to Y$. If $X \in \mu$, then there is always a hereditary class on X such that f is $(\mu^*, \nu)-$continuous.*

Proof. First, we define $\mathcal{H}_f = \{A \subset X - f^{-1}(V) \mid \text{for all } V \in \nu\}$. Claim that $\mathcal{H}_f$ is a hereditary class on X. Let $C \in \mathcal{H}_f$ and $D \subset C$. Since $C \in \mathcal{H}_f$, there exists $V' \in \nu$ such that $C \subset X - f^{-1}(V')$. We get $D \subset X - f^{-1}(V')$ and so $D \in \mathcal{H}_f$. Now suppose that $X \in \mu$. Let $N \in \nu$. Then $X - (X - f^{-1}(N))$ is an element in a base for μ^*. Therefore $f^{-1}(N) \in \mu^*$. □

REFERENCES

[1] Csszr, Á.: Generalized open sets, *Acta Math. Hungar.* **75**(1–2), 65–87 (1997).

[2] Csszr, Á.: Generalized topology, generalized continuity, *Acta Math. Hungar.* **96**(4), 351–357 (2002).

[3] Csszr, Á.: Generalized open sets in generalized topologies, *Acta Math. Hungar.* **108**(1-2), 53–66 (2005).

[4] Csszr, Á.: Modification of generalized topologies via hereditary classes, *Acta Math. Hungar.* **115**(1-2), 29–36 (2007).

[5] Hamlett, T.R., Janković, D.: New topologies from old via ideals, *Amer. Math. Monthly* **97**(4), 295–310 (1990).

[6] Renukadevi, V., Vimladevi, P.: Compatible hereditary classes in GTS, *Acta Math. Hungar.* **147**(2), 259–271 (2015).

VITA

Name : Mr. Wichitpon Thaikua

Date of Birth : 13 December 1992

Place of Birth : Bangkok, Thailand

Education : B.Sc. (Mathematics), Kasetsart University, 2014