ไอดีลฟี-เอ็น-แอบซอร์บิงและไอดีลฟี-วางนัยทั่วไป-เอ็น-แอบซอร์บิงของกึ่งริงสลับที่

นางสาวภัทราวรรณ เพชรแก้ว

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ϕ-n-ABSORBING IDEALS AND ϕ-GENERALIZED-n-ABSORBING IDEALS OF COMMUTATIVE SEMIRINGS

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A Dissertation Submitted in Partial Fulfillment of the Requirements
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ภัทราวรรณ เพชรแก้ว : ไอดีลฟี-เอ็น-แอบซอร์บิงและไอดีลฟี-วางนัยทั่วไป-เอ็น-แอบซอร์บิงของกึ่งริงสลับที่ (ϕ-n-ABSORBING IDEALS AND ϕ-GENERALIZED-n-ABSORBING IDEALS OF COMMUTATIVE SEMIRINGS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : รศ. ดร. อมร วาสนาวิจิตร์, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : ผศ.ดร.ศจี เพียรสกุล, 111 หน้า.

ในดุษฎีนิพนธ์เล่มนี้ เราแนะนำแนวคิดของไอดีลฟี-ปฐมภูมิ, ไอดีลฟี-เอ็น-แอบซอร์บิงและไอดีลฟี-วางนัยทั่วไป-เอ็น-แอบซอร์บิงของกึ่งริงสลับที่ R ที่มีเอกลักษณ์ที่ไม่เป็นศูนย์ เมื่อฟีคือฟังก์ชันจากเซตของไอดีลทั้งหมดของ R ไปยังเซตของไอดีลทั้งหมดของ R หรือเซตว่าง แนวคิดเหล่านี้ถูกขยายมาจากไอดีลปฐมภูมิ, ไอดีลเอ็น-แอบซอร์บิงและไอดีลวางนัยทั่วไปเอ็น-แอบซอร์บิงของริงสลับที่ที่มีเอกลักษณ์ที่ไม่เป็นศูนย์ ตามลำดับ เราศึกษาไอดีลฟี-ปฐมภูมิ, ไอดีลฟี-เอ็น-แอบซอร์บิงและไอดีลฟี-วางนัยทั่วไป-เอ็น-แอบซอร์บิงในโครงสร้างกึ่งริงสามประเภท กล่าวคือ กึ่งริง (โดยทั่วไป), กึ่งริงผลหารและกึ่งริงของเศษส่วน ในมุมมองที่ต่างกัน ยิ่งไปกว่านั้น เราตรวจสอบความสัมพันธ์ระหว่างไอดีลฟี-ปฐมภูมิ, ไอดีลฟี-เอ็น-แอบซอร์บิงและไอดีลฟี-วางนัยทั่วไป-เอ็น-แอบซอร์บิง และสามารถสรุปได้ว่าไอดีลฟี-ปฐมภูมิไม่เป็นไอดีลฟี-เอ็น-แอบซอร์บิงและไอดีลฟี-เอ็น-แอบซอร์บิงไม่เป็นไอดีลฟี-ปฐมภูมิ แต่อย่างไรก็ตาม ทั้งคู่ต่างเป็นไอดีลฟี-วางนัยทั่วไป-เอ็น-แอบซอร์บิง

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PATTARAWAN PETCHKAEW : ϕ-n-ABSORBING IDEALS AND ϕ-GENERALIZED-n-ABSORBING IDEALS OF COMMUTATIVE SEMIRINGS. ADVISOR : ASSOC. PROF. AMORN WASANAWICHIT, Ph.D., CO-ADVISOR : ASST. PROF. SAJEE PIANSKOOL, Ph.D., 111pp.

In this dissertation, we introduce the concepts of ϕ-primary ideals, ϕ-n-absorbing ideals and ϕ-generalized-n-absorbing ideals of a commutative semiring R with nonzero identity where ϕ is a function from the set of ideals of R into the set of ideals of R or the empty set. These are extended from primary ideals, n-absorbing ideals and generalized n-absorbing ideals of commutative rings with nonzero identity, respectively. We investigate ϕ-primary ideals, ϕ-n-absorbing ideals and ϕ-generalized-n-absorbing ideals in three types of semirings structures; namely, semirings (in general), quotient semirings and semirings of fractions in various points of view. In addition, we examine relationships among ϕ-primary ideals, ϕ-n-absorbing ideals and ϕ-generalized-n-absorbing ideals and can conclude that ϕ-primary ideals and ϕ-n-absorbing ideals do not imply each other; nevertheless, both imply ϕ-generalized-n-absorbing ideals.

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CONTENTS

CHAPTER I
INTRODUCTION

Research on semirings has been studied in many ways such as prime and semiprime ideals, quotient semirings, additive-regular semirings, etc. One of those that we are interested in is prime ideals. It is natural that almost research of prime ideals of semirings are extended from results of prime ideals of rings.

One knows that prime ideals play an important role in rings. Recall that a proper ideal I of a commutative ring R with nonzero identity is said to be a ***prime ideal*** if whenever $a, b \in R$ with $ab \in I$, either $a \in I$ or $b \in I$. In 2003, D. D. Anderson and E. Smith [4] generalized the concept of prime ideals to weakly prime ideals of a ring. They defined a ***weakly prime ideal*** I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - \{0\}$, either $a \in I$ or $b \in I$. After that, in 2005, S. M. Bhatwadekar and P. K. Sharma [11] generalized the concept of weakly prime ideals to almost prime ideals of a ring. They defined an ***almost prime ideal*** I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - I^2$, either $a \in I$ or $b \in I$.

In 2008, D. D. Anderson and M. Batanieh [3] generalized the concept of prime ideals, weakly prime ideals and almost prime ideals to ϕ-prime ideals of a commutative ring R with nonzero identity where $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ is a function in which $\mathscr{I}(R)$ is the set of ideals in such ring. They defined a ***ϕ-prime ideal*** I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - \phi(I)$, either $a \in I$ or $b \in I$.

We can see that the direction of this extension of prime ideals of commutative rings with nonzero identity starting with changing the condition that $ab \in I$ to $ab \in I - \{0\}$ which is called weakly prime ideals. Later, the condition that

$ab \in I - \{0\}$ of weakly prime ideals was changed to $ab \in I - I^2$ which is called almost prime ideals. This is one of natural ways to generalize prime ideals by subtracting some ideals from the ideal I. This led to the extension of prime ideals by changing the condition that $ab \in I$ to $ab \in I - \phi(I)$ where ϕ is a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ which can be defined in several ways, e.g., $\phi(J) = \varnothing$ for all ideals J, $\phi(J) = \{0\}$ for all ideals J, $\phi(J) = J^2$ for all ideals J, $\phi(J) = J^n$ where $n \in \mathbb{N}$ for all ideals J, etc. This makes the definition of ϕ-prime ideals both support old definitions and extend them. This is very interesting and becomes an inspiration of doing this research.

Many concepts of rings are extended to those of semirings so are the concepts of prime ideals and weakly prime ideals. J. S. Golan [17] introduced the concept of prime ideals of a semiring in 1999. He defined a ***prime ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I$, then $a \in I$ or $b \in I$. After that, V. Gupta and J. N. Chaudhari [20] introduced the notion of weakly prime ideals of a semiring in 2008. They defined a ***weakly prime ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - \{0\}$, then $a \in I$ or $b \in I$. This brought us to extend the concepts of prime ideals, weakly prime ideals of semirings and ϕ-prime ideals of rings to ϕ-prime ideals of semirings. In the same fashion as the idea of ϕ-prime ideals of rings, for a semiring R, we define ϕ to be a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ where $\mathscr{I}(R)$ is the set of ideals of the semiring R and define a ***ϕ-prime ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - \phi(I)$, either $a \in I$ or $b \in I$.

The inspiration of the next target of this research arose from the following. In 2007, A. Badawi [10] introduced the notion of 2-absorbing ideals of a ring. He defined a ***2-absorbing ideal*** I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, either $ab \in I$ or $ac \in I$ or $bc \in I$. After that, in 2011, D. F. Anderson and A. Badawi [2] generalized this to n-absorbing ideals (with integer $n \geq 2$) of a ring. They defined an ***n-absorbing ideal*** I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1x_2\cdots x_{n+1} \in I$, then $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Next, in 2012, M. Ebrahimpour and R. Nekooei [16] gave the definition of $(n-1, n)$-ϕ-prime ideals (with integer $n \geq 2$) of a ring. They defined an ***$(n-1, n)$-ϕ-prime ideal*** I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $x_1, x_2, \ldots, x_n \in R$ with $x_1x_2\cdots x_n \in I - \phi(I)$, then $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n \in I$ for some $i \in \{1, 2, \ldots, n\}$. Obviously, $(n-1, n)$-ϕ-prime ideal is just a ϕ-$(n-1)$-absorbing ideal.

In our work, we also extend n-absorbing ideals and $(n-1, n)$-ϕ-prime ideals of a ring to n-absorbing ideals and ϕ-n-absorbing ideals of a semiring. We define an ***n-absorbing ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1x_2\cdots x_{n+1} \in I$, then $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Besides, we define a ***ϕ-n-absorbing ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1x_2\cdots x_{n+1} \in I - \phi(I)$, then $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Moreover, we obtain that our first results of ϕ-prime ideals of semirings are the specific case of the results of ϕ-n-absorbing ideals of semirings.

Afterwards, we would like to generalize the concept of ϕ-prime ideals in other ways. We found that, in 2012, A. Y. Darani [15] generalized the idea of ϕ-prime ideals to ϕ-primary ideals of a ring. He defined a ***ϕ-primary ideal*** I of a commutative ring R with nonzero identity to be a propper ideal and if whenever $a, b \in R$ with $ab \in I - \phi(I)$, either $a \in I$ or $b^n \in I$ for some positive integer n.

By the same idea as our first results, our interest is also to extend the concept of ϕ-primary ideals of rings to ϕ-primary ideals of semirings. Primary ideals of a semiring have been introduced and studied by S. E. Atani and M. S. Kohan in 2010 [9]. They defined a ***primary ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I$, then $a \in I$ or $b^n \in I$ for some positive integer n. Subsequently, in 2011, J. N. Chaudhari and B. R. Bonde [12] generalized the notion of primary ideals of semirings to weakly primary ideals of semirings. They defined a ***weakly primary ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b^n \in I$ for some positive integer n.

In this dissertation, we also aim to extend the concepts of primary ideals, weakly primary ideals of semirings and ϕ-primary ideals of rings to ϕ-primary ideals of semirings. We define a ***ϕ-primary ideal*** I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - \phi(I)$, either $a \in I$ or $b^n \in I$ for some positive integer n. In addition, we obtain that the concepts of ϕ-primary ideals and ϕ-n-absorbing ideals do not imply each other.

Finally, our last target is set according to the following idea. In 2015, S. Chinwarakorn and S. Pianskool [14] defined a new type of ideals which is still a generalization of primary ideals and n-absorbing ideals of a ring. They defined a ***generalized n-absorbing ideal*** (simply ***Gn-absorbing ideal***) I of a commutative ring R with nonzero identity to be a proper ideal and if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1 x_2 \cdots x_{n+1} \in I$, then $(x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})^\alpha \in I$ for some positive integer α and for some $i \in \{1, 2, \ldots, n+1\}$.

For the final part of this dissertation, we extend the idea of generalized n-absorbing ideals of a ring to ϕ-generalized-n-absorbing ideals of a semiring. We define a ***ϕ-generalized-n-absorbing ideal*** (simply ***ϕ-Gn-absorbing ideal***) I of a commutative semiring R with nonzero identity to be a proper ideal and if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1 x_2 \cdots x_{n+1} \in I - \phi(I)$, then $(x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1})^\alpha \in I$ for some positive integer α and for some $i \in \{1, 2, \ldots, n+1\}$.

In this dissertation, we organize our work as follows. Next chapter contains three sections. The first section introduces basic definitions, notation, examples, elementary properties and some of our results in semirings. The second section contains definitions of Q-ideals (partitioning ideals), quotient semirings, homomorphisms and isomorphisms; in addition, some of our results are given. The last section discusses about semirings of fractions and we obtain some results which are used in other chapters.

In Chapter III, we define almost primary ideals, n-almost primary ideals, ω-primary ideals and ϕ-primary ideals of semirings. Some results in this chapter are analogous to the results given in [15].

In Chapter IV, we give the notion of n-absorbing ideals, weakly n-absorbing ideals, almost n-absorbing ideals, m-almost n-absorbing ideals and ω-n-absorbing ideals of semirings; in addition, we extend these to ϕ-n-absorbing ideals of semirings and investigate them in the same fashion as the results in Chapter III. Moreover, we obtain other results which are not analogous to the results in Chapter III. Besides, we provide some forms of n-absorbing ideals which are not $(n-1)$-absorbing ideals of the semiring $\mathbb{Z}_0^+$.

In Chapter V, we introduce the concepts of generalized n-absorbing ideals, weakly generalized n-absorbing ideals, almost generalized n-absorbing ideals, m-almost generalized n-absorbing ideals and ω-generalized n-absorbing ideals of semirings and extend these to ϕ-generalized-n-absorbing ideals of semirings. Almost results of this chapter are investigated in the same manner as the results of Chapter IV. Moreover, some forms of generalized n-absorbing ideals which are not n-absorbing ideals of the semiring $\mathbb{Z}_0^+$ are obtained.

The contents of Chapter III, Chapter IV and Chapter V are divided into three sections. The first sections are ϕ-primary ideals of semirings, ϕ-n-absorbing ideals of semirings and ϕ-generalized-n-absorbing ideals of semirings, respectively. The second sections of those three chapters concern with decomposable semirings. In the last section, we focus our work on quotient semirings and semirings of fractions.

In the final chapter, Chapter VI, we summarize the main concept of our research and give relationships among each chapter.

CHAPTER II
PRELIMINARIES

In this chapter, we provide some definitions, notations and results which will be used for this dissertation. All contents of this dissertation are investigated in three main types of semiring structures; namely, semirings, quotient semirings and semirings of fractions, so we divide this chapter into three sections. The first section is definitions and fundamental results in semirings. The second section is fundamental results in quotient semirings and the last section is fundamental results in semirings of fractions.

Throughout this work, let $\mathbb{Z}$ denote the set of integers, $\mathbb{N}$ the set of natural numbers (positive integers), $\mathbb{Z}_0^+$ the set of nonnegative integers, $\mathbb{Q}_0^+$ the set of nonnegative rational numbers, $\mathbb{R}_0^+$ the set of nonnegative real numbers and $\mathbb{Z}_n = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\}$ where $n \in \mathbb{N}$.

2.1 Definitions and Fundamental Results in Semirings

First of all, the definition of semirings along with some results based on [17] by J. S. Golan are presented.

Definition 2.1.1. [17] A ***semiring*** R is defined as an algebraic system $(R, +, \cdot)$ on which the operations of addition $+$ and multiplication $\cdot$ have been defined such that the following conditions are satisfied:

(1) $(R, +)$ is a commutative monoid with identity element 0, called the zero;

(2) $(R, \cdot)$ is a semigroup (we write ab instead of $a \cdot b$ for all $a, b \in R$);

(3) the multiplication distributes over the addition; and

(4) $r0 = 0 = 0r$ for all $r \in R$.

Let R be a semiring. Then R is said to be ***commutative*** if $ab = ba$ for all $a, b \in R$. If $1 \in R$ and $a1 = a = 1a$ for all $a \in R$, then 1 is called the ***identity*** of the semiring R. If R contains an identity $1 \neq 0$, then R is called a ***semiring with nonzero identity***. Moreover, $1 \in R$ stands for the identity of the semiring R.

By the definition of semirings, it is easy to see that semirings are generalizations of rings. Therefore, every ring is a semiring. However, the converse of this statement is not true. For example, $\mathbb{Z}_0^+$ under usual addition and usual multiplication is a semiring but is not a ring. Moreover, from now on, if we give examples of semirings by omitting their binary operations, it means that their operations are usual addition and usual multiplication.

Example 2.1.2. [17] (1) $\mathbb{Z}_0^+, \mathbb{Q}_0^+$ and $\mathbb{R}_0^+$ are commutative semirings with nonzero identity which is the number 1.

(2) $\mathbb{Z}_0^+[t]$, the set of polynomials in t over the semiring $\mathbb{Z}_0^+$, is a commutative semiring with nonzero identity which is the constant polynomial 1 under usual addition and usual multiplication of polynomials.

(3) Let $\mathbb{B} = \{0, 1\}$. Then $\mathbb{B}$ forms a semiring under operations $+$ and $\cdot$ given as follows:

$+$	0	1
0	0	1
1	1	1

and

$\cdot$	0	1
0	0	0
1	0	1

Then $\mathbb{B}$ is a commutative semiring with nonzero identity which is the element 1. In fact, this semiring $\mathbb{B}$ is called the ***Boolean semiring***. Moreover, $\mathbb{B}[t]$, the set of polynomials in t over the semiring $\mathbb{B}$, is a commutative semiring with nonzero identity which is the constant polynomial 1.

(4) Let $R = \{0, 1, u\}$. Then R forms a semiring under operations $+$ and $\cdot$ given as follows:

$+$	0	1	u
0	0	1	u
1	1	1	u
u	u	u	u

and

$\cdot$	0	1	u
0	0	0	0
1	0	1	u
u	0	u	u

Then R is a commutative semiring with nonzero identity which is the element 1.

(5) Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}_0^+ \right\}$. Then R is a noncommutative semiring with nonzero identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ under usual addition and usual multiplication of matrices.

In this research, all considered semirings are assumed to be commutative semirings with nonzero identity. Moreover, all referred rings in this dissertation are commutative rings with nonzero identity. Thus we simply write "semiring"or "ring" in stead of "commutative semiring with nonzero identity"or "commutative ring with nonzero identity", respectively.

Definition 2.1.3. [17] A nonempty subset I of a semiring R is called an ***ideal*** of R if it satisfies the following conditions:

(1) if $a, b \in I$, then $a + b \in I$; and

(2) if $a \in I$ and $r \in R$, then $ar \in I$.

If a is an element in a semiring R, then $aR = \{ar \mid r \in R\}$ is an ideal of R, called a ***principal ideal***. From the definition of ideals of semirings, if we consider the semiring $\mathbb{Z}_0^+$, then ideals of $\mathbb{Z}_0^+$ may not be in the form $m\mathbb{Z}_0^+$ where $m \in \mathbb{Z}_0^+$. Examples of ideals of $\mathbb{Z}_0^+$ are $m\mathbb{Z}_0^+$ for all $m \in \mathbb{Z}_0^+$, $\mathbb{Z}_0^+ - \{1\}$ and $\{0, 3\} \cup \{5, 6, 7, \ldots\}$.

Notation 2.1.4. [17] Let I and J be ideals of a semiring R and m a positive integer. Let

$$I + J = \left\{ a + b \,\middle|\, a \in I \text{ and } b \in J \right\},$$

$$IJ = \left\{ \sum_{i=1}^{n} a_i b_i \,\middle|\, a_i \in I \text{ and } b_i \in J \text{ for all } i \in \{1, 2, \ldots, n\} \right\} \text{ and}$$

$$I^m = \left\{ \sum_{i=1}^{n} a_{i1}a_{i2}\cdots a_{im} \,\middle|\, n \in \mathbb{N} \text{ and } a_{i1}, a_{i2}, \ldots, a_{im} \in I \text{ for all } i \in \{1, 2, \ldots, n\} \right\}.$$

Proposition 2.1.5. [17] *Let I and J be ideals of a semiring R and m a positive integer. Then the following statements hold.*

(1) $I + J$ is an ideal of R containing both I and J.

(2) IJ is an ideal of R contained in I and J.

(3) I^m is an ideal of R contained in I; in addition, if $n_1, n_2 \in \mathbb{N}$ are such that $n_1 \geq n_2$, then $I^{n_1} \subseteq I^{n_2}$.

Notation 2.1.6. [17] Let A be a nonempty subset of a semiring R. Let

$$\langle A \rangle = \left\{ \sum_{i=1}^{n} a_i r_i \,\middle|\, n \in \mathbb{N}, a_i \in A \text{ and } r_i \in R \text{ for all } i \in \{1, 2, \ldots, n\} \right\}.$$

Proposition 2.1.7. [17] *Let A be a nonempty subset of a semiring R. Then $\langle A \rangle$ is the smallest ideal of R containing A.*

For a nonempty subset A of a semiring R, the ideal $\langle A \rangle$ is said to be the ***ideal generated by A***. If $A = \{a\}$, then $\langle A \rangle = \langle a \rangle = aR$, see [17].

Proposition 2.1.8. [17] *If I and J are ideals of a semiring R, then $I + J$ is the unique minimal member of the family of all ideals of R containing both I and J and $I \cap J$ is the unique maximal member of the family of all ideals of R contained in I and J.*

Definition 2.1.9. [17] An ideal I of a semiring R is called a ***k-ideal*** (***subtractive ideal***) of R if whenever $x, y \in R$ and $x, x + y \in I$, then $y \in I$.

Certainly, k-ideals are ideals but the converse is not true. For example, the ideal $\mathbb{Z}_0^+ - \{1\}$ of the semiring $\mathbb{Z}_0^+$ is not a k-ideal because $2, 2 + 1 \in \mathbb{Z}_0^+ - \{1\}$ but $1 \notin \mathbb{Z}_0^+ - \{1\}$. Moreover, k-ideals play a very important role in this dissertation because several of our main results need the property of k-ideals. In the following, we provide examples of k-ideals as well as examples of ideals which are not k-ideals.

Example 2.1.10. [17] (1) Consider the semiring $\mathbb{Z}_0^+$ and the ideal $I = 2\mathbb{Z}_0^+$ of $\mathbb{Z}_0^+$. To show that I is a k-ideal, let $a, b \in \mathbb{Z}_0^+$ be such that $a, a+b \in I$. Then $a = 2n$ and $a + b = 2m$ for some $n, m \in \mathbb{Z}_0^+$. Thus $2n + b = 2m$. Next, we see $2n, b, 2m$ as elements of $\mathbb{Z}$, and hence we obtain that $b = 2(m - n)$. Since $b \in \mathbb{Z}_0^+$, we have $m - n \in \mathbb{Z}_0^+$. Then $b \in 2\mathbb{Z}_0^+ = I$. Therefore, I is a k-ideal of $\mathbb{Z}_0^+$.

(2) $m\mathbb{Z}_0^+$ is a k-ideal of the semiring $\mathbb{Z}_0^+$ for any $m \in \mathbb{Z}_0^+$.

(3) Consider the semiring $R = \{0, 1, u\}$ given in Example 2.1.2 (4). Let $I = \{0, u\}$. Then I is an ideal of R which is not a k-ideal because $u, u + 1 = u \in I$ but $1 \notin I$.

(4) Consider the semiring $\mathbb{Z}_0^+[t]$. Let I be the ideal of $\mathbb{Z}_0^+[t]$ generated by $t + 1$, that is $I = \{(t+1)f(t) \mid f(t) \in \mathbb{Z}_0^+[t]\}$. Then $(t+1)^3 \in I$. Since $(t+1)3t+(t^3+1) = (t + 1)^3 \in I$ and $(t + 1)3t \in I$ but $t^3 + 1 \notin I$, the ideal I is not a k-ideal.

By Proposition 2.1.8, we know that any sum of ideals is an ideal but this statement is not true for k-ideals. For example, $2\mathbb{Z}_0^+$ and $3\mathbb{Z}_0^+$ are k-ideals of the semiring $\mathbb{Z}_0^+$ but $2\mathbb{Z}_0^+ + 3\mathbb{Z}_0^+ = \mathbb{Z}_0^+ - \{1\}$ is not a k-ideal. Moreover, Proposition 2.1.8 also shows that the intersection of ideals is an ideal and this statement holds for k-ideals as we shown in the next result.

Proposition 2.1.11. *Let R be a semiring. If I and J are k-ideals of R, then $I \cap J$ is a k-ideal of R.*

Proof. Assume that I and J are k-ideals of R. Then $I \cap J$ is an ideal of R. Let $a, b \in R$ be such that $a, a + b \in I \cap J$. Since I is a k-ideal and $a, a + b \in I$, we obtain $b \in I$. Similarly, $b \in J$. Hence $b \in I \cap J$. Therefore, $I \cap J$ is a k-ideal of R. □

An element a of a semiring R is said to be ***multiplicatively regular*** if there exists an element b of R satisfying $aba = a$. A semiring R is called a ***multiplicatively regular semiring*** if each element of R is multiplicatively regular, see [17].

From Proposition 2.1.11, it is suspected that if I and J are k-ideals of a semiring R, then IJ is a k-ideal of R or not. In 1999, J. S. Golan shown that if R is a multiplicatively regular semiring, then $IJ = J \cap J$ for all ideals I and J of R.

Hence, we can conclude that if I and J are k-ideals of a multiplicatively regular semiring R, then IJ is a k-ideal of R by Proposition 2.1.11.

In rings, we know that any union of ideals of rings need not be an ideal but if a union of ideals is an ideal, then it must be equal to one of them. However, this statement is not true in general if we consider in semirings.

Example 2.1.12. Consider the semiring $\mathbb{Z}_0^+$. Let $A = \mathbb{Z}_0^+ - \{1, 2, 5\}$ and $B = 5\mathbb{Z}_0^+$. Then A and B are ideals of $\mathbb{Z}_0^+$ such that A is not a k-ideal because $3, 3 + 1 \in A$ but $1 \notin A$. Let $I = A \cup B = \mathbb{Z}_0^+ - \{1, 2\}$. Hence I is an ideal of $\mathbb{Z}_0^+$ but $I \neq A$ and $I \neq B$.

In the next proposition, we show that, for semirings, if the union of k-ideals is an ideal, then it is equal to one of them.

Proposition 2.1.13. *Let R be a semiring and A,B k-ideals of R. If $I = A \cup B$ is an ideal of R, then $I = A$ or $I = B$ (certainly, I must be a k-ideal of R).*

Proof. Let $I = A \cup B$ be an ideal of R. Suppose that $I \neq A$ and $I \neq B$. Then $B \not\subseteq A$ and $A \not\subseteq B$. Thus there exist $a \in A - B$ and $b \in B - A$. Since I is an ideal, $a + b \in I = A \cup B$. Hence $a + b \in A$ or $a + b \in B$. Without loss of generality, suppose that $a+b \in A$. Then $b \in A$ because A is a k-ideal. This is a contradiction. Therefore, $I = A$ or $I = B$ and then I is a k-ideal of R. □

Notation 2.1.14. [17] Let R be a semiring, I an ideal of R and $a \in R$. Let

$$(I : a) = \{x \in R \,|\, xa \in I\}.$$

Example 2.1.15. Consider the ideal $6\mathbb{Z}_0^+$ of the semiring $\mathbb{Z}_0^+$. Then $(6\mathbb{Z}_0^+ : 2) = \{x \in \mathbb{Z}_0^+ \,|\, 2x \in 6\mathbb{Z}_0^+\} = 3\mathbb{Z}_0^+$.

Proposition 2.1.16. *Let R be a semiring and $a \in R$. Then the following statements hold.*

(1) If I is an ideal of R, then $(I : a)$ is an ideal of R and $I \subseteq (I : a)$.

(2) If I is a k-ideal of R, then $(I : a)$ is a k-ideal of R.

(3) If I and J are ideals of R such that $I \subseteq J$, then $(I : a) \subseteq (J : a)$.

(4) $\langle a \rangle \subseteq (\langle a \rangle^2 : a)$.

Proof. (1) Let I be an ideal of R. Then I is a nonempty subset of R. Let $b \in I$. Then $ba \in I$ because I is an ideal. Thus $b \in (I : a)$, i.e., $(I : a)$ is a nonempty subset of R. In addition, it is obvious that $I \subseteq (I : a)$. Next, let $x, y \in (I : a)$ and $r \in R$. Then $xa \in I$ and $ya \in I$. Since I is an ideal, we obtain $(x + y)a = xa + ya \in I$ and $rxa \in I$. Then $x + y \in (I : a)$ and $rx \in (I : a)$. Therefore, $(I : a)$ is an ideal of R.

(2) Let I be a k-ideal of R. Then $(I : a)$ is an ideal of R by (1). Next, let $x, y \in R$ be such that $x, x + y \in (I : a)$. Thus $xa \in I$ and $xa + ya = (x + y)a \in I$. Since I is a k-ideal and $xa, xa + ya \in I$, it follows that $ya \in I$. Hence $y \in (I : a)$. Therefore, I is a k-ideal of R.

(3) Assume that I and J are ideals of R such that $I \subseteq J$. Let $x \in (I : a)$. Then $xa \in I$ so that $xa \in J$. Hence $x \in (J : a)$. Therefore, $(I : a) \subseteq (J : a)$.

(4) Let $x \in \langle a \rangle$. Then $x = ra$ for some $r \in R$. Hence $xa = ra^2$, that is $xa \in \langle a \rangle^2$. Thus $x \in (\langle a \rangle^2 : a)$. Therefore, $\langle a \rangle \subseteq (\langle a \rangle^2 : a)$. □

The reverse inclusion in the statement (4) of Proposition 2.1.16 is not true as shown in the following example.

Example 2.1.17. Consider the semiring $R = \{0, 1, u\}$ given in Example 2.1.2 (4). Since $1 \in R$, we obtain $\langle u \rangle^2 = Ru^2$. Then $\langle u \rangle^2 = Ru^2 = Ru = \langle u \rangle = \{0, u\}$. Since $1u = u \in \{0, u\} = \langle u \rangle^2$, we gain $1 \in (\langle u \rangle^2 : u)$. Hence $(\langle u \rangle^2 : u) \nsubseteq \langle u \rangle$ because $1 \notin \langle u \rangle$.

Definition 2.1.18. [17] Let R be a semiring. A proper ideal I of R is said to be a ***prime ideal*** if whenever $a, b \in R$ and $ab \in I$, then $a \in I$ or $b \in I$.

There are many researchers interested in prime ideals of both rings and semirings. Moreover, it is well-known that all ideals of the ring $\mathbb{Z}$ are in form $m\mathbb{Z}$ where $m \in \mathbb{Z}$ and its prime ideals are $\{0\}$ and $\langle p \rangle$ where p is a prime number. Since $n\mathbb{Z}_0^+$ where $n \in \mathbb{Z}_0^+$ are one type of ideals of the semiring $\mathbb{Z}_0^+$, it is interesting to know all prime ideals of $\mathbb{Z}_0^+$. This is done by V. Gupta and J. N. Chaudhari given in the following example.

Example 2.1.19. [21] In the semiring $\mathbb{Z}_0^+$, all the prime ideals of $\mathbb{Z}_0^+$ are $\{0\}$, $\langle p \rangle = p\mathbb{Z}_0^+$ for some prime number p and $\langle 2, 3 \rangle = \mathbb{Z}_0^+ - \{1\}$.

As above example, of course, it is easy to find ideals of the semiring $\mathbb{Z}_0^+$ which are not prime ideals. In the following, we provide an example of an ideal of other semiring which is not a prime ideal.

Example 2.1.20. We know that the ideal $I = \mathbb{Z}_0^+ - \{1\}$ of the semiring $\mathbb{Z}_0^+$ is a prime ideal. However, if t is an indeterminate, then $I[t]$ is an ideal of the semiring $\mathbb{Z}_0^+[t]$ which is not a prime ideal because $(1+2t+3t^2)(3+t) = 3+7t+11t^2+3t^3 \in I[t]$ but $1 + 2t + 3t^2, 3 + t \notin I[t]$.

We know what prime ideals of the semiring $\mathbb{Z}_0^+$ are and Example 2.1.20 shows that there is a prime ideal I of the semiring $\mathbb{Z}_0^+$ such that $I[t]$ is not a prime ideal of the semiring $\mathbb{Z}_0^+[t]$ where t is an indeterminate. This makes us wonder what prime ideals of the semiring $\mathbb{Z}_0^+[t]$ are and the answer is provided as follows.

Proposition 2.1.21. [17] *Let R be a semiring, I an ideal of R and t an indeterminate over R. Then $I[t]$ is a prime ideal of $R[t]$ if and only if I is a prime k-ideal.*

Example 2.1.22. Consider the semiring $\mathbb{Z}_0^+$. Then $11\mathbb{Z}_0^+$ is a prime k-ideal of $\mathbb{Z}_0^+$ from Example 2.1.10 (2) and Example 2.1.19. Hence $11\mathbb{Z}_0^+[t]$ is a prime ideal of the semiring $\mathbb{Z}_0^+[t]$ where t is an indeterminate.

Definition 2.1.23. [17] Let R be a semiring. The ***radical of an ideal I of R***, denoted by $\sqrt{I}$, is defined to be the set of all $a \in R$ for which $a^n \in I$ for some positive integer n.

For an ideal I of a semiring R, one can show that $\sqrt{I}$ is an ideal of R containing I, see [17]. Moreover, if we consider the semiring $\mathbb{Z}_0^+$, then, for examples,

$$\sqrt{2\mathbb{Z}_0^+} = \{r \in \mathbb{Z}_0^+ \mid r^n \in 2\mathbb{Z}_0^+ \text{ for some } n \in \mathbb{N}\} = 2\mathbb{Z}_0^+,$$
$$\sqrt{\mathbb{Z}_0^+ - \{1\}} = \{r \in \mathbb{Z}_0^+ \mid r^n \in \mathbb{Z}_0^+ - \{1\} \text{ for some } n \in \mathbb{N}\} = \mathbb{Z}_0^+ - \{1\},$$
$$\sqrt{9\mathbb{Z}_0^+} = \{r \in \mathbb{Z}_0^+ \mid r^n \in 9\mathbb{Z}_0^+ \text{ for some } n \in \mathbb{N}\} = 3\mathbb{Z}_0^+.$$

Proposition 2.1.24. *Let R be a semiring and I an ideal of R. Then the following statements hold.*

(1) $\sqrt{I} = \sqrt{\sqrt{I}}$.

(2) For all $n \in \mathbb{N}$, $\sqrt{I} = \sqrt{I^n}$.

Proof. (1) Since $\sqrt{I}$ is an ideal of R, we obtain $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. Thus it remains to show that $\sqrt{\sqrt{I}} \subseteq \sqrt{I}$. Let $a \in \sqrt{\sqrt{I}}$. Thus $a^m \in \sqrt{I}$ for some $m \in \mathbb{N}$. Then $(a^m)^l \in I$ for some $l \in \mathbb{N}$, i.e., $a^{ml} \in I$. Hence $a \in \sqrt{I}$. Therefore, $\sqrt{I} = \sqrt{\sqrt{I}}$.

(2) Let $n \in \mathbb{N}$. Since $I^n \subseteq I$, we have $\sqrt{I^n} \subseteq \sqrt{I}$. To show that $\sqrt{I} \subseteq \sqrt{I^n}$, let $x \in \sqrt{I}$. Then there exists $m \in \mathbb{N}$ such that $x^m \in I$. Thus $\underbrace{x^m x^m \cdots x^m}_{n \text{ copies}} \in I^n$. Hence $x^{mn} \in I^n$, and so $x \in \sqrt{I^n}$. Then $\sqrt{I} \subseteq \sqrt{I^n}$. Therefore, $\sqrt{I} = \sqrt{I^n}$ for all $n \in \mathbb{N}$. □

The following proposition is a tool that helps us to find the radicals of the principal ideals of the semiring $\mathbb{Z}_0^+$ more easily.

Proposition 2.1.25. *Let m be a positive integer. The radical of the ideal $m\mathbb{Z}_0^+$ of the semiring $\mathbb{Z}_0^+$ is $r\mathbb{Z}_0^+$ where r is the product of all distinct prime factors of m.*

Proof. Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ for some distinct prime numbers $p_1, p_2, \ldots, p_n$ and for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N}$. We would like to show that $\sqrt{m\mathbb{Z}_0^+} = p_1 p_2 \cdots p_n \mathbb{Z}_0^+$. Let $a \in \sqrt{m\mathbb{Z}_0^+}$. Then $a^\alpha \in p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ for some $\alpha \in \mathbb{N}$. Since $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+ \subseteq p_i \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, n\}$, we obtain $a^\alpha \in p_i \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, n\}$. Thus $a \in p_i \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, n\}$ because $p_i \mathbb{Z}_0^+$ are prime ideals for all $i \in \{1, 2, \ldots, n\}$. Hence $a \in p_1 \mathbb{Z}_0^+ \cap p_2 \mathbb{Z}_0^+ \cap \cdots \cap p_n \mathbb{Z}_0^+ \subseteq p_1 \mathbb{Z}_0^+ p_2 \mathbb{Z}_0^+ \cdots p_n \mathbb{Z}_0^+ = p_1 p_2 \cdots p_n \mathbb{Z}_0^+$. Therefore, $\sqrt{m\mathbb{Z}_0^+} \subseteq p_1 p_2 \cdots p_n \mathbb{Z}_0^+$.

Conversely, we show that $p_1 p_2 \cdots p_n \mathbb{Z}_0^+ \subseteq \sqrt{m\mathbb{Z}_0^+}$. Let $x \in p_1 p_2 \cdots p_n \mathbb{Z}_0^+$. Thus

$x = p_1 p_2 \cdots p_n l$ for some $l \in \mathbb{Z}_0^+$. Let $\beta = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then $\beta \in \mathbb{N}$ and

$$\begin{aligned} x^\beta &= (p_1 p_2 \cdots p_n l)^\beta \\ &= (p_1 p_2 \cdots p_n l)^{\alpha_1+\alpha_2+\cdots+\alpha_n} \\ &= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} (p_1^{\alpha_2+\alpha_3+\cdots+\alpha_n} p_2^{\alpha_1+\alpha_3+\cdots+\alpha_n} \cdots p_n^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}} l^{\alpha_1+\alpha_2+\cdots+\alpha_n}) \\ &\in p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+. \end{aligned}$$

Hence $x \in \sqrt{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+} = \sqrt{m\mathbb{Z}_0^+}$. Thus $p_1 p_2 \cdots p_n \mathbb{Z}_0^+ \subseteq \sqrt{m\mathbb{Z}_0^+}$.

Therefore, we can conclude that $\sqrt{m\mathbb{Z}_0^+} = p_1 p_2 \cdots p_n \mathbb{Z}_0^+$. □

Example 2.1.26. Consider the semiring $\mathbb{Z}_0^+$.

(1) The radical of the ideal $85\mathbb{Z}_0^+ = (5 \cdot 17)\mathbb{Z}_0^+$ is $(5 \cdot 17)\mathbb{Z}_0^+ = 85\mathbb{Z}_0^+$.

(2) The radical of the ideal $120\mathbb{Z}_0^+ = (2^3 \cdot 3 \cdot 5)\mathbb{Z}_0^+$ is $(2 \cdot 3 \cdot 5)\mathbb{Z}_0^+ = 30\mathbb{Z}_0^+$.

(3) The radical of the ideal $900\mathbb{Z}_0^+ = (2^2 \cdot 3^2 \cdot 5^2)\mathbb{Z}_0^+$ is $(2 \cdot 3 \cdot 5)\mathbb{Z}_0^+ = 30\mathbb{Z}_0^+$.

A semiring R is said to be ***decomposable*** if it can be written as a product of semirings, i.e., $R = R_1 \times R_2 \times \cdots \times R_m$ for some semirings $R_1, R_2, \ldots, R_m$ where $m \in \mathbb{N}$ with $m \geq 2$. Moreover, the ideals of a decomposable semiring $R_1 \times R_2 \times \cdots \times R_m$ are of the form $I_1 \times I_2 \times \cdots \times I_m$ where I_i is an ideal of R_i for all $i \in \{1, 2, \ldots, m\}$.

The following proposition shows that the radical of an ideal $I = I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$ is equal to a product of the radicals of each component of I.

Proposition 2.1.27. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I_1 \times I_2 \times \cdots \times I_m$ an ideal of R. Then $\sqrt{I_1 \times I_2 \times \cdots \times I_m} = \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$.*

Proof. First, let $(a_1, a_2, \ldots, a_m) \in \sqrt{I_1 \times I_2 \times \cdots \times I_m}$. Then there is $n \in \mathbb{N}$ such that $(a_1, a_2, \ldots, a_m)^n \in I_1 \times I_2 \times \cdots \times I_m$. That is $(a_1^n, a_2^n, \ldots, a_m^n) \in I_1 \times I_2 \times \cdots \times I_m$. Thus $(a_1, a_2, \ldots, a_m) \in \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$. Hence $\sqrt{I_1 \times I_2 \times \cdots \times I_m} \subseteq \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$.

Next, let $(x_1, x_2, \ldots, x_m) \in \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$. There are $n_1, n_2, \ldots, n_m \in \mathbb{N}$ such that $(x_1^{n_1}, x_2^{n_2}, \ldots, x_m^{n_m}) \in I_1 \times I_2 \times \cdots \times I_m$. Thus $(x_1, x_2, \ldots, x_m)^{n_1 n_2 \cdots n_m} = (x_1^{n_1 n_2 \cdots n_m}, x_2^{n_1 n_2 \cdots n_m}, \ldots, x_m^{n_1 n_2 \cdots n_m}) \in I_1 \times I_2 \times \cdots \times I_m$. Then $(x_1, x_2, \ldots, x_m) \in$

$\sqrt{I_1 \times I_2 \times \cdots \times I_m}$. Hence $\sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m} \subseteq \sqrt{I_1 \times I_2 \times \cdots \times I_m}$.

Therefore, $\sqrt{I_1 \times I_2 \times \cdots \times I_m} = \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$. □

Next, we show a relationship between being k-ideals of ideals of decomposable semirings and being k-ideals of each components of those ideals.

Proposition 2.1.28. *Let* $R = R_1 \times R_2 \times \cdots \times R_m$ *be a decomposable semiring and* $I = I_1 \times I_2 \times \cdots \times I_m$ *an ideal of* R*. Then* I *is a* k*-ideal of* R *if and only if* I_i *is a* k*-ideal of* R_i *for all* $i \in \{1, 2, \dots, m\}$*.*

Proof. Assume that I is a k-ideal of R. We prove that I_i is a k-ideal of R_i for all $i \in \{1, 2, \dots, m\}$. Without loss of generality, we show that I_1 is a k-ideal of R_1. Let $x, y \in R_1$ be such that $x, x + y \in I_1$. Then $(x, 0, \dots, 0), (x + y, 0, \dots, 0) \in I$. Hence $(x, 0, \dots, 0), (x, 0, \dots, 0) + (y, 0, \dots, 0) \in I$. Since I is a k-ideal, we obtain $(y, 0, \dots, 0) \in I$. Thus $y \in I_1$ and so I_1 is a k-ideal of R_1.

Conversely, assume that I_i is a k-ideal of R_i for all $i \in \{1, 2, \dots, m\}$. We show that I is a k-ideal of R. Let $(x_1, x_2, \dots, x_m), (y_1, y_2, \dots, y_m) \in R$ be such that $(x_1, x_2, \dots, x_m), (x_1, x_2, \dots, x_m) + (y_1, y_2, \dots, y_m) \in I$. Then $(x_1 + y_1, x_2 + y_2, \dots, x_m + y_m) \in I$. Since I_i is a k-ideal and $x_i, x_i + y_i \in I_i$ for all $i \in \{1, 2, \dots, m\}$, we gain $y_i \in I_i$ for all $i \in \{1, 2, \dots, m\}$. Hence $(y_1, y_2, \dots, y_m) \in I$. Therefore, I is a k-ideal of R. □

2.2 Fundamental Results in Quotient Semirings

In this section, we provide some idea, elementary properties and some of our fundamental results which relate to partitioning ideals and quotient semirings.

There are some results concerning relationships between ϕ-prime ideals (ϕ-primary ideals) of rings in general and ϕ-prime ideals (ϕ-primary ideals) of quotient rings in [3] (in [15]) and then we extend those results to semirings. This made us interested in quotient semirings. First of all, we would like to recall notion of quotient rings.

Let R be a ring and I an ideal of R. Recall that $R/I = \{a + I \mid a \in R\}$ and $\oplus$, $\odot$ are defined on R/I as follows:

$$(a+I) \oplus (b+I) = (a+b)+I \quad \text{and} \quad (a+I) \odot (b+I) = ab+I$$

for all $a, b \in R$. Then $(R/I, \oplus, \odot)$ is a ring and is called the ***quotient ring***.

Nevertheless, for an ideal I of a semiring R, the set $\{a+I \mid a \in R\}$ need not be a partition of R (unlike the set $\{a+I \mid a \in R\}$ where R is a ring) as shown in the following example.

Example 2.2.1. Consider the semiring $\mathbb{Z}_0^+$. Let $I = \{0,3\} \cup \{5,6,7,\ldots\}$. Then I is an ideal of $\mathbb{Z}_0^+$. Thus

$$1+I = \{1,4\} \cup \{6,7,8,9,\ldots\}$$
$$2+I = \{2,5\} \cup \{7,8,9,10,\ldots\}.$$

Hence $1+I \neq 2+I$ and $(1+I) \cap (2+I) \neq \varnothing$. Therefore, $\{a+I \mid a \in \mathbb{Z}_0^+\}$ is not a partition of $\mathbb{Z}_0^+$.

We would like to search for some sets which are partitions of semirings playing the same role as the set $\{a+I \mid a \in R\}$ where I is an ideal of a ring R. Nevertheless, there are some types of ideals that lead to some partitions of semirings.

Definition 2.2.2. [1] An ideal I of a semiring R is called a ***partitioning ideal*** if there exists a subset Q of R such that:

(1) $R = \cup\{q+I \mid q \in Q\}$,

(2) if $q_1, q_2 \in Q$, then $(q_1+I) \cap (q_2+I) \neq \varnothing$ if and only if $q_1 = q_2$.

Therefore, if I is a partitioning ideal of a semiring R, then there exists a subset, say Q, of R such that $\{q+I \mid q \in Q\}$ is a partition of R. We also call I a ***partitioning ideal via the set Q*** or simply call a ***Q-ideal***.

Let $\mathbb{Z}_n^+$ be the nonnegative integers modulo $n \in \mathbb{N}$, that is, $\mathbb{Z}_n^+ = \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ where $\bar{a} = \{a+kn \mid k \in \mathbb{Z}_0^+\}$ for any $a \in \mathbb{Z}_0^+$. Then P. J. Allen showed in [1] that $\mathbb{Z}_n^+$ forms a semiring under addition and multiplication modulo n; in addition, $\mathbb{Z}_n^+$ is also a ring.

Example 2.2.3. (1) Consider the semiring $\mathbb{Z}_0^+$ and its ideal $I = 4\mathbb{Z}_0^+$. Let $Q = \{0, 1, 2, 3\}$. Since

$$0 + I = \{0, 4, 8, 12, \ldots\}, \quad 1 + I = \{1, 5, 9, 13, \ldots\},$$
$$2 + I = \{2, 6, 10, 14, \ldots\}, \; 3 + I = \{3, 7, 11, 15, \ldots\},$$

we obtain $\cup\{q + I \mid q \in Q\} = \mathbb{Z}_0^+$. Next, let $q_1, q_2 \in Q$ be such that $(q_1 + I) \cap (q_2 + I) \neq \varnothing$. From writing explicit elements of the sets $0 + I$, $1 + I$, $2 + I$ and $3 + I$, we can conclude that $q_1 = q_2$. Therefore, I is a Q-ideal of $\mathbb{Z}_0^+$.

(2) Consider the semiring $\mathbb{Z}_6^+$ and its ideal $I = \{\bar{0}, \bar{2}, \bar{4}\}$. Let $Q_1 = \{\bar{0}, \bar{1}\}$, $Q_2 = \{\bar{0}, \bar{3}\}$ and $Q_3 = \{\bar{0}, \bar{5}\}$. Since $\bar{0} + I = \{\bar{0}, \bar{2}, \bar{4}\}$ and $\bar{1} + I = \{\bar{1}, \bar{3}, \bar{5}\} = \bar{3} + I = \bar{5} + I$, we obtain $\cup\{q + I \mid q \in Q_1\} = \cup\{q + I \mid q \in Q_2\} = \cup\{q + I \mid q \in Q_3\} = \mathbb{Z}_6^+$. If $q_1, q_2 \in Q_1$ are distinct, then $(q_1 + I) \cap (q_2 + I) = \varnothing$. Then I is a Q_1-ideal of $\mathbb{Z}_6^+$. Similarly, I is also a Q_2-ideal and a Q_3-ideal of $\mathbb{Z}_6^+$.

Example 2.2.3 (2) shows that it is possible to have several subsets Q of a semiring R which make an ideal I of R be a partitioning ideal via those sets.

Example 2.2.4. [1] (1) Consider the semiring $\mathbb{Z}_0^+$. Let $n \in \mathbb{Z}_0^+$. If $n \in \mathbb{Z}_0^+ - \{0\}$, then $n\mathbb{Z}_0^+$ is a Q-ideal where $Q = \{0, 1, 2, \ldots, n-1\}$. If $n = 0$, then $n\mathbb{Z}_0^+$ is a Q-ideal where $Q = \mathbb{Z}_0^+$. Moreover, the ideal $\mathbb{Z}_0^+ - \{1\}$ is not a partitioning ideal.

(2) Let R be a nonempty well-ordered set and define $a + b = \max\{a, b\}$ and $ab = \min\{a, b\}$ for each $a, b \in R$. Then R together with the two defined operations forms a semiring. If $r \in R$, then the set $I_r = \{x \in R \,|\, x \leq r\}$ is an ideal of R. It is clear from the definition of addition on R that $0 + I_r = I_r$ and $x + I_r = \{x\}$ for each $x > r$. Thus I_r is a Q-ideal where $Q = \{0\} \cup \{x \in R \,|\, x > r\}$.

Proposition 2.2.5. [7] *If I is a partitioning ideal of a semiring, then I is a k-ideal.*

However, the converse of Proposition 2.2.5 is not true. For example, in the semiring $R = (\mathbb{Z}_0^+, \gcd, \mathrm{lcm})$, where gcd is the greatest common divisor and lcm is the least common multiple, the ideal $2\mathbb{Z}_0^+$ is a k-ideal but is not a partitioning ideal, see [7].

Let I be a Q-ideal of a semiring R and $q_1, q_2 \in Q$. By the statement (1) in Definition 2.2.2, there are $q_3, q_4 \in Q$ such that $q_1 + q_2 + I \subseteq q_3 + I$ and $q_1q_2 + I \subseteq q_4 + I$. The uniqueness of q_3 and q_4 is guaranteed by the statement (2) of Definition 2.2.2. To see this, suppose that there exist $q_3', q_4' \in Q$ such that $q_1 + q_2 + I \subseteq q_3' + I$ and $q_1q_2 + I \subseteq q_4' + I$. Thus $(q_3 + I) \cap (q_3' + I) \neq \varnothing$ and $(q_4 + I) \cap (q_4' + I) \neq \varnothing$. Hence $q_3 = q_3'$ and $q_4 = q_4'$. The notion of the uniqueness of q_3 and q_4 leads us to define binary operations on the set $\{q + I \mid q \in Q\}$ in order to form a new semiring.

Let I be a partitioning ideal via the set Q of a semiring R and $R/I = \{q + I \mid q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows:

$$(q_1 + I) \oplus (q_2 + I) = q_3 + I \quad \text{and} \quad (q_1 + I) \odot (q_2 + I) = q_4 + I$$

where $q_3, q_4 \in Q$ are the unique elements such that $q_1 + q_2 + I \subseteq q_3 + I$ and $q_1q_2 + I \subseteq q_4 + I$. This semiring R/I is called the ***quotient semiring of R by I***, see [6].

In addition, for a semiring R and a Q-ideal I of R, since R is a commutative semiring with nonzero identity 1, then R/I is a commutative semiring with nonzero identity $q_1 + I$ where $q_1 \in Q$ such that $1 + I \subseteq q_1 + I$; moreover, its zero element is $q_0 + I$ where $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$.

Example 2.2.6. Let $R = \mathbb{Z}_0^+$ and $I = 6\mathbb{Z}_0^+$. Then I is a Q-ideal where $Q = \{0, 1, 2, 3, 4, 5\}$. Hence $R/I = \{q + 6\mathbb{Z}_0^+ \mid q \in Q\} = \{6\mathbb{Z}_0^+, 1 + 6\mathbb{Z}_0^+, 2 + 6\mathbb{Z}_0^+, 3 + 6\mathbb{Z}_0^+, 4 + 6\mathbb{Z}_0^+, 5 + 6\mathbb{Z}_0^+\}$ is a quotient semiring of R by I which is a commutative semiring with nonzero identity $1 + 6\mathbb{Z}_0^+$. Next, we provide examples of addition $\oplus$ and multiplication $\odot$ of some elements of R/I. We obtain $(1 + 6\mathbb{Z}_0^+) \oplus (2 + 6\mathbb{Z}_0^+) = 3 + 6\mathbb{Z}_0^+$ and $(1 + 6\mathbb{Z}_0^+) \odot (2 + 6\mathbb{Z}_0^+) = 2 + 6\mathbb{Z}_0^+$.

If R is a semiring and I is a partitioning ideal of R via the set Q, then we use the notation R/I_Q instead of the quotient semiring of R by I when we would like to specific that I is a partitioning ideal of R via the set Q.

From Example 2.2.3 (2), there are three quotient semirings of R by I that are

$R/I_{Q_1} = \{q+I \mid q \in Q_1\}$, $R/I_{Q_2} = \{q+I \mid q \in Q_2\}$ and $R/I_{Q_3} = \{q+I \mid q \in Q_3\}$. It is suspected that they are different or not.

Definition 2.2.7. [1] A mapping φ from a semiring R into a semiring R' is called a ***homomorphism*** if $\varphi(a+b) = \varphi(a)+\varphi(b)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in R$. An ***isomorphism*** is a one-to-one and onto homomorphism. Semirings R and R' is said to be ***isomorphic*** (denoted by $R \cong R'$) if there exists an isomorphism from R onto R'.

The following theorem shows that the quotient semirings of R by a partitioning ideal of R via the set Q_1 and via the set Q_2 are isomorphic.

Theorem 2.2.8. [1] *Let I be an ideal of a semiring R. If Q_1 and Q_2 are subsets of R such that I is both a Q_1-ideal and a Q_2-ideal, then*

$$(\{q+I\}_{q\in Q_1}, \oplus_{Q_1}, \odot_{Q_1}) \cong (\{q+I\}_{q\in Q_2}, \oplus_{Q_2}, \odot_{Q_2}).$$

One knows that if R is a ring and I is an ideal of R, then ideals of the quotient ring R/I are in the form J/I where J is an ideal of R and J contains I; however, not all ideals of a semiring R containing a partitioning ideal I can be formed ideals of its quotient semiring R/I. J. N. Chuadhari and D. R. Bonde introduced, in 2014, another kind of ideals of semirings that lead to ideals of its quotient semirings. Moreover, these ideals are a generalization of k-ideals.

Definition 2.2.9. [13] Let I be an ideal of a semiring R. An ideal P of R containing I is said to be a ***subtractive extension of*** I if whenever $x, y \in R$ and $x \in I, x+y \in P$, then $y \in P$.

Note that, every k-ideal of a semiring R containing an ideal I of R is a subtractive extension of I; nevertheless, the converse of this statement is not true as shown in the following.

Example 2.2.10. Let $I = 4\mathbb{Z}_0^+ \times \{0\}$ and $P = 2\mathbb{Z}_0^+ \times (\mathbb{Z}_0^+ - \{1\})$. Then I and P are ideals of the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ such that $I \subseteq P$. Since $(4, 2), (4, 2) + (2, 1) = (6, 3) \in P$ but $(2, 1) \notin P$, it follows that P is not a k-ideal of R. Let $x \in I$ and

$x + y \in P$. Thus $x = (4n, 0)$ for some $n \in \mathbb{Z}_0^+$ and $x + y = (2m, l)$ for some $m \in \mathbb{Z}_0^+$ and for some $l \in \mathbb{Z}_0^+ - \{1\}$. Let $y = (a, b)$ for some $a, b \in \mathbb{Z}_0^+$. Then $(2m, l) = x + y = (4n, 0) + (a, b) = (4n + a, b)$. Hence $4n + a = 2m$ and $b = l$, and so we obtain $a \in 2\mathbb{Z}_0^+$ and $b \in \mathbb{Z}_0^+ - \{1\}$. That is $y = (a, b) \in P$. Therefore, P is a subtractive extension of I.

Next, we provide a result showing that the radicals of k-ideals are subtractive extension of those k-ideals.

Proposition 2.2.11. *Let R be a semiring and I a k-ideal of R. Then $\sqrt{I}$ is a subtractive extension of I.*

Proof. Let $a, b \in R$ be such that $a \in I$ and $a + b \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $(a + b)^n \in I$. Since

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

and I is a k-ideal containing a, we obtain $b^n \in I$. Hence $b \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a subtractive extension of I. □

Theorem 2.2.12. [13] *Let R be a semiring, I a Q-ideal of R and P an ideal of R containing I. Then following statements are equivalent.*

(1) P is a subtractive extension of I.

(2) I is a Q'-ideal of P where $Q' = Q \cap P$.

(3) $P/I = \{q + I : q \in Q \cap P\}$ is an ideal of a semiring R/I.

(4) $P/I \subseteq R/I$.

Theorem 2.2.13. [13] *Let I be a Q-ideal of a semiring R. Then L is an ideal of R/I if and only if there exists an ideal P of R such that P is a subtractive extension of I and $P/I = L$.*

Theorem 2.2.14. [13] *Let I be a Q-ideal of a semiring R. Then a subset L of R/I is a k-ideal of R/I if and only if there exists a k-ideal P of R with $I \subseteq P$ and $P/I = L$.*

Therefore, we can conclude that ideals (k-ideals) of a quotient semiring R/I where I is a Q-ideal of a semiring R must be in the form $P/I = \{q+I : q \in Q \cap P\}$ where P is a subtractive extension of I (a k-ideal containing I). Hence, from now on, when we mention about any ideal P/I of a quotient semiring R/I where I is a Q-ideal of a semiring R, we usually assume that P is a subtractive extension of I.

The rest of results in this section are needed in other chapters.

Lemma 2.2.15. *Let R be a semiring, I a Q-ideal of R, P a subtractive extension of I and $a \in R$. If $a + I \in P/I$, then $a \in P$.*

Proof. Let $a+I \in P/I = \{q+I : q \in Q \cap P\}$. Then there exists a $q \in Q \cap P$ such that $a + I = q + I$. Thus there is an $x \in I$ such that $a = q + x$. Since $I \subseteq P$, we obtain $x \in P$. Therefore, $a = q + x \in P$. □

Proposition 2.2.16. *Let R be a semiring, I a Q-ideal of R and P a subtractive extension of I. Then $(q_1+I)(q_2+I)\cdots(q_n+I) \in P/I$ if and only if $q_1q_2\cdots q_n \in P$ for all $q_1, q_2, \ldots, q_n \in Q$.*

Proof. Let $q_1, q_2, \ldots, q_n \in Q$. First, assume that $(q_1+I)(q_2+I)\cdots(q_n+I) \in P/I$. Then $(q_1 + I)(q_2 + I)\cdots(q_n + I) = q + I$ for some unique element $q \in Q \cap P$ such that $q_1q_2\cdots q_n + I \subseteq q + I$. Since $q \in P$ and $I \subseteq P$, we obtain $q + I \subseteq P$. Then $q_1q_2\cdots q_n \in q + I \subseteq P$. Therefore, $q_1q_2\cdots q_n + I \in P$.

Conversely, assume that $q_1q_2\cdots q_n \in P$. Suppose that $(q_1+I)(q_2+I)\cdots(q_n+I)$ $= q + I$ for some unique element $q \in Q$ such that $q_1q_2\cdots q_n + I \subseteq q + I$. Hence $q_1q_2\cdots q_n \in q + I$ and so there exists $y \in I$ such that $q_1q_2\cdots q_n = q + y$. Since $q_1q_2\cdots q_n \in P$, we must have $q + y \in P$. Thus we get $q \in P$ because P is a subtractive extension of I and $y \in I, q + y \in P$. Thus $q \in Q \cap P$. Therefore, $(q_1 + I)(q_2 + I)\cdots(q_n + I) = q + I \in P/I$. □

For an ideal I of a semiring R, the radicals of ideals which are subtractive extensions of I are also subtractive extension of I.

Lemma 2.2.17. *Let R be a semiring and I an ideal of R. If P is a subtractive extension of I, then $\sqrt{P}$ is also a subtractive extension of I*

Proof. Assume that P is a subtractive extension of I. Then $\sqrt{P}$ is an ideal and $I \subseteq P \subseteq \sqrt{P}$. Let $x, y \in R$ be such that $x \in I$ and $x+y \in \sqrt{P}$. Then $(x+y)^n \in P$ for some $n \in \mathbb{N}$. That is

$$x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + y^n \in P.$$

We obtain $x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} \in I$ because $x \in I$. Since P is a subtractive extension of I, it follows that $y^n \in P$. Hence $y \in \sqrt{P}$. Therefore, $\sqrt{P}$ is a subtractive extension of I. □

Consequently, we can conclude that if I is an ideal of a semiring R and P is a subtractive extension of I, then $\sqrt{P}$ is also a subtractive extension of I. Hence, if we assume further that I is a Q-ideal of a semiring R, then not only P/I is an ideal of R/I but also $\sqrt{P}/I$. This raises to a question whether $\sqrt{P}/I$ and $\sqrt{P/I}$ are identical.

Proposition 2.2.18. *Let R be a semiring, I a Q-ideal of R and P a subtractive extension of I. Then $\sqrt{P}/I = \sqrt{P/I}$.*

Proof. First, let $q + I \in \sqrt{P}/I$ where $q \in Q \cap \sqrt{P}$. Then there is $n \in \mathbb{N}$ such that $q^n \in P$. By Proposition 2.2.16, we have $\underbrace{(q+I)(q+I)\cdots(q+I)}_{n \text{ copies}} \in P/I$. That is $(q+I)^n \in P/I$ and hence $q + I \in \sqrt{P/I}$. Thus $\sqrt{P}/I \subseteq \sqrt{P/I}$.

Next, let $q + I \in \sqrt{P/I}$. Then there is an $n \in \mathbb{N}$ such that $(q+I)^n \in P/I$. That is $\underbrace{(q+I)(q+I)\cdots(q+I)}_{n \text{ copies}} \in P/I$. By Proposition 2.2.16, we get $q^n \in P$. Hence $q \in \sqrt{P} \cap Q$ and so $q + I \in \sqrt{P}/I$. Then $\sqrt{P/I} \subseteq \sqrt{P}/I$.

Therefore, $\sqrt{P}/I = \sqrt{P/I}$. □

Throughout this dissertation, the symbol ϕ is assumed to be a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ where $\mathscr{I}(R)$ is the set of ideals of a semiring R. Moreover, if R is a semiring and there is a function $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$, then R is called a ***semiring with $\boldsymbol{\phi}$***.

For a semiring R and any two functions $\varphi_1, \varphi_2 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$, we define $\varphi_1 \leq \varphi_2$ if $\varphi_1(I) \subseteq \varphi_2(I)$ for each $I \in \mathscr{I}(R)$ in the same manner as given in [3].

Let R be a semiring and I a Q-ideal of R. Moreover, let ϕ be a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ such that $\phi(L)$ is a subtractive extension of I for all ideal L of R where L is a subtractive extension of I. We define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\varnothing\}$ by $\phi_I(J/I) = (\phi(J))/I$ for each ideal J of R where J is a subtractive extension of I.

We call R a ***semiring with $\boldsymbol{\phi}$ satisfying the property*** **$(*)$** if R is a semiring with ϕ, I is a Q-ideal of R and ϕ_I is a function from $\mathscr{I}(R/I)$ into $\mathscr{I}(R/I) \cup \{\varnothing\}$ where ϕ and ϕ_I are defined in the previous paragraph.

The following theorem is very important that we use it in proving the main theorems of the last section in every chapter later.

Theorem 2.2.19. *Let R be a semiring with ϕ satisfying the property $(*)$, I a Q-ideal of R and P a subtractive extension of I. Then $(q_1 + I)(q_2 + I) \cdots (q_n + I) \in P/I - \phi_I(P/I)$ if and only if $q_1 q_2 \cdots q_n \in P - \phi(P)$ for all $q_1, q_2, \ldots, q_n \in Q$.*

Proof. The proof is completed by Proposition 2.2.16. □

From the Proposition 2.2.5 and the ideal $2\mathbb{Z}_0^+$ of the semiring $(\mathbb{Z}_0^+, \mathrm{gcd}, \mathrm{lcm})$, we know that every Q-ideal is a k-ideal but not vice versa. Nevertheless, Q-ideals and k-ideals are coincide in some semirings such as strongly Euclidean semirings which were introduced by J. S. Golan in 1999 [17]. In strongly Euclidean semiring, not only Q-ideals and k-ideals are coincide but also principal ideals. Moreover, in every chapter after this, there are results relate to strongly Euclidean semirings.

Definition 2.2.20. [17] A semiring R is called a ***Euclidean semiring*** if there exists a function $d : R - \{0\} \to \mathbb{Z}_0^+$ such that if $a, b \in R$ with $b \neq 0$ then there exist unique elements $q, r \in R$ such that $a = bq + r$ where either $r = 0$ or $d(r) < d(b)$.

Definition 2.2.21. [17] A semiring R is called a ***strongly Euclidean semiring*** if there exists a function $d : R - \{0\} \to \mathbb{Z}_0^+$ such that

(1) $d(ab) \geq d(a)$ for all $a, b \in R - \{0\}$ and

(2) if $a, b \in R$ with $b \neq 0$ then there exist unique elements $q, r \in R$ such that $a = bq + r$ where either $r = 0$ or $d(r) < d(b)$.

By the definition of strongly Euclidean semirings, every strongly Euclidean semiring is a Euclidean semiring.

Theorem 2.2.22. [18] *Let R be a strongly Euclidean semiring. Then the following statements are equivalent.*

(1) I is a Q-ideal of R.

(2) I is a k-ideal of R.

(3) I is a principal ideal of R.

Example 2.2.23. [18] The semiring $\mathbb{Z}_0^+$ is a strongly Euclidean semiring. Hence the ideals $a\mathbb{Z}_0^+$ where $a \in \mathbb{Z}_0^+$ are Q-ideals and k-ideals. Moreover, we can conclude that the ideal $\mathbb{Z}_0^+ - \{1\} = \langle 2, 3 \rangle$ is not a Q-ideal and not a k-ideal because it is not a principal ideal of $\mathbb{Z}_0^+$. All prime k-ideals of the semiring $\mathbb{Z}_0^+$ are $\{0\}$ or $p\mathbb{Z}_0^+$ for some prime number p (see Example 2.1.19 and Example 2.2.4 (1)).

2.3 Fundamental Results in Semirings of Fractions

In 1999, J. S. Golan extended the concept of rings of fractions to the notion of semirings of fractions by using a straightforward adaptation of the method used for rings. In this section, we introduce the idea of semirings of fractions. Besides, our fundamental results of semirings of fractions are given.

Definition 2.3.1. [17] An element a of a semiring R is said to be ***multiplicatively cancellable*** if $ba = ca$ only when $b = c$ for all $b, c \in R$.

Let R be a semiring and S a set of all multiplicatively cancellable elements of R. We would like to show that S is closed under multiplication. Let $a, b \in S$ and $x, y \in R$ be such that $xab = yab$, then $xa = ya$ because $b \in S$, and so $x = y$ since $a \in S$. Hence $ab \in S$. Therefore, we can conclude that S is closed under multiplication. Moreover, it is easy to see that $1 \in S$ and $0 \notin S$.

Example 2.3.2. Consider the semiring $\mathbb{Z}_0^+$. Then $\mathbb{Z}_0^+ - \{0\}$ is the set of all multiplicatively cancellable elements of $\mathbb{Z}_0^+$.

Note that the set of all multiplicatively cancellable elements of semirings must not be empty because all considered semirings containing the identity element and it certainly contained in this set.

In 1999, J. S. Golan [17] gave the construction of semirings of fractions as follows. Let R be a semiring and S the set of all multiplicatively cancellable elements of R. Define a relation $\sim$ on $R \times S$ as follows:

$$(a, s) \sim (b, t) \quad \text{if and only if} \quad at = bs$$

for all $(a, s), (b, t) \in R \times S$. Then $\sim$ is an equivalence relation on $R \times S$.

For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ containing (a, s) by $\frac{a}{s}$, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S forms a semiring under operations

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \qquad \text{and} \qquad \left(\frac{a}{s}\right)\left(\frac{b}{t}\right) = \frac{ab}{st}$$

for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the ***semiring of fractions of R with respect to S***.

Since R is a commutative semiring with nonzero identity 1, it follows that R_S is a commutative semiring with nonzero identity $\frac{1}{1}$; in addition, its zero element is $\frac{0}{1}$, see [5].

In 2008, R. E. Atani and S. E. Atani [5] investigated the ideal theory in semirings. We are interested in many of their results and apply them in this dissertation.

Moreover, R. E. Atani and S. E. Atani [5] also examine ideals of quotient semirings. Let I be an ideal of R. If $a \in I$ and $t \in R_S$, then $t = \dfrac{b}{c}$ for some $b \in R$ and $c \in S$; in addition, $at = a\left(\dfrac{b}{c}\right) = \dfrac{ab}{c}$. The ***ideal of R_S generated by I***, is defined to be the set $\left\{\sum_{i=1}^{n} a_i t_i \,\middle|\, a_i \in I, t_i \in R_S \text{ and } n \in \mathbb{N}\right\}$, and is called the ***extension of I to R_S***, denoted by IR_S.

Definition 2.3.3. [5] Let R be a semiring, S the set of all multiplicatively cancellable elements of R and J an ideal of R_S. Then the ***contraction of J in R***, denoted by $J \cap R$, is defined as

$$J \cap R = \left\{r \in R \,\middle|\, \frac{r}{1} \in J\right\}.$$

Proposition 2.3.4. [5] *Let R be a semiring and S the set of all multiplicatively cancellable elements of R. If J is an ideal of R_S, then $J \cap R$ is an ideal of R.*

Proposition 2.3.5. [5] *Let R be a semiring and S the set of all multiplicatively cancellable elements of R. Assume that I, J and K are ideals of R and let L be an ideal of the semiring R_S. Then the following statements hold.*

(1) $x \in IR_S$ *if and only if it can be written in the form* $x = \dfrac{a}{c}$ *for some* $a \in I$ *and* $c \in S$.

(2) $(L \cap R)R_S = L$.

(3) $(I \cap J)R_S = IR_S \cap JR_S$.

Proposition 2.3.6. [5] *Let R be a semiring, S the set of all multiplicatively cancellable elements of R and I a k-ideal of R. Then IR_S is a k-ideal of the semiring R_S.*

Finally, we provide some results regarding semirings of fractions which are applied to the proof of some results in the last section of Chapter III, Chapter V and Chapter VI.

Proposition 2.3.7. *Let R be a semiring, S the set of all multiplicatively cancellable elements of R and I an ideal of R. Then $\sqrt{I}R_S = \sqrt{IR_S}$.*

Proof. First, let $\frac{x}{t} \in \sqrt{I}R_S$. By Proposition 2.3.5, there exist $a \in \sqrt{I}$ and $u \in S$ such that $\frac{x}{t} = \frac{a}{u}$. Thus $xu = at$. Since $a \in \sqrt{I}$, there is $n \in \mathbb{N}$ such that $a^n \in I$. Hence $x^n u^n = (xu)^n = (at)^n = a^n t^n \in I$. So we get $\left(\frac{x}{t}\right)^n = \frac{x^n u^n}{t^n u^n} \in IR_S$. Therefore, $\frac{x}{t} \in \sqrt{IR_S}$, and then $\sqrt{I}R_S \subseteq \sqrt{IR_S}$.

Next, let $\frac{y}{s} \in \sqrt{IR_S}$. Then there is an $m \in \mathbb{N}$ such that $\left(\frac{y}{s}\right)^m \in IR_S$. Thus there exist $b \in I$ and $v \in S$ such that $\frac{y^m}{s^m} = \frac{b}{v}$. Hence $y^m v = bs^m \in I$, and so $(yv)^m = y^m v^m \in I$. Thus $yv \in \sqrt{I}$, and then $\frac{y}{s} = \frac{yv}{sv} \in \sqrt{I}R_S$. Therefore, $\sqrt{IR_S} \subseteq \sqrt{I}R_S$. □

Let R be a semiring with ϕ and S the set of all multiplicatively cancellable elements of R. We define $\phi_S : \mathscr{I}(R_S) \to \mathscr{I}(R_S) \cup \{\varnothing\}$ in the same manner as seen in [3] by $\phi_S(J) = \phi(J \cap R)R_S$ if $\phi(J \cap R) \in \mathscr{I}(R)$ and $\phi_S(J) = \varnothing$ if $\phi(J \cap R) = \varnothing$ for all $J \in \mathscr{I}(R_S)$.

The following theorem as well as Theorem 2.2.19 are important results because they are main tools for providing one of main results in other chapters.

Theorem 2.3.8. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I an ideal of R with $\phi(I)R_S \subseteq \phi_S(IR_S)$. For $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots, \frac{x_n}{s_n} \in R_S$, if $(\frac{x_1}{s_1})(\frac{x_2}{s_2}) \cdots (\frac{x_n}{s_n}) \in IR_S - \phi_S(IR_S)$, then $x_1 x_2 \cdots x_n v \in I - \phi(I)$ for some $v \in S$.*

Proof. Let $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots, \frac{x_n}{s_n} \in R_S$ be such that $(\frac{x_1}{s_1})(\frac{x_2}{s_2}) \cdots (\frac{x_n}{s_n}) \in IR_S - \phi_S(IR_S)$. Since $\phi(I)R_S \subseteq \phi_S(IR_S)$, we obtain $\frac{x_1 x_2 \cdots x_n}{s_1 s_2 \cdots s_n} \in IR_S - \phi(I)R_S$. Then there exist $a \in I$ and $v \in S$ such that $\frac{x_1 x_2 \cdots x_n}{s_1 s_2 \cdots s_n} = \frac{a}{v}$. Thus $x_1 x_2 \cdots x_n v = a s_1 s_2 \cdots s_n \in I$. If $x_1 x_2 \cdots x_n v \in \phi(I)$, then $\frac{x_1 x_2 \cdots x_n}{s_1 s_2 \cdots s_n} = \frac{x_1 x_2 \cdots x_n v}{s_1 s_2 \cdots s_n v} \in \phi(I)R_S$ which is a contradiction. Therefore, $x_1 x_2 \cdots x_n v \in I - \phi(I)$. □

CHAPTER III
GENERALIZATIONS OF PRIMARY IDEALS OF SEMIRINGS

In ring theory, there are many generalizations of prime ideals and one of those is known as primary ideals. A proper ideal I of a ring R is said to be a ***primary ideal*** if whenever $a, b \in R$ with $ab \in I$, either $a \in I$ or $b^n \in I$ for some positive integer n. Hence prime ideals are primary ideals but not vice versa. For example, $9\mathbb{Z}$ is a primary ideal of the ring $\mathbb{Z}$ but it is not a prime ideal of $\mathbb{Z}$ because $3 \cdot 3 = 9 \in 9\mathbb{Z}$ but $3 \notin 9\mathbb{Z}$. In 2005, S. E. Atani and F. Farzalipour [8] generalized the concept of primary ideals to weakly primary ideals of rings. They defined a ***weakly primary ideal*** I of a ring R to be a proper ideal and if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b^n \in I$ for some positive integer n. Thus every primary ideal is a weakly primary ideal. Nevertheless, weakly primary ideals need not be primary ideals. For example, $\{\bar{0}\}$ is a weakly primary ideal of the ring $\mathbb{Z}_{10}$ and $\bar{2} \cdot \bar{5} \in \{\bar{0}\}$ but $\bar{2} \notin \{\bar{0}\}$ and $\bar{5}^n \notin \{\bar{0}\}$ for all $n \in \mathbb{N}$. Hence $\{\bar{0}\}$ is not a primary ideal of the ring $\mathbb{Z}_{10}$. Therefore, weakly primary ideals are generalizations of primary ideals.

Many types of ideals of rings are generalized to the similar types of ideals of semirings. Primary ideals also play such that role. The notion of primary ideals of a semiring have been introduced and studied by S. E. Atani and M. S. Kohan in 2010 [9]. They defined a ***primary ideal*** I of a semiring R to be a proper ideal and if whenever $a, b \in R$ with $ab \in I$, then $a \in I$ or $b^n \in I$ for some positive integer n. After that, in 2011, J. N. Chaudhari and B. R. Bonde [12] generalized the notion of primary ideals of semirings to weakly primary ideals of semirings. They defined a ***weakly primary ideal*** I of a semiring R to be a proper ideal and if whenever $a, b \in R$ with $0 \neq ab \in I$, then $a \in I$ or $b^n \in I$ for some positive integer n.

A. Y. Darani [15] generalized the notion of primary ideals and weakly primary ideals to ϕ-primary ideals of rings in 2012. He defined a ***ϕ-primary ideal*** I of a ring R to be a proper ideal and if whenever $a, b \in R$ with $ab \in I - \phi(I)$, either $a \in I$ or $b^n \in I$ for some positive integer n.

At this point of view, we extend the concepts of primary ideals, weakly primary ideals and ϕ-prime ideals of semirings and ϕ-primary ideals of rings to ϕ-primary ideals of semirings. We divide this chapter into three sections that are ϕ-primary ideals of semirings, ϕ-primary ideals in decomposable semirings and the last one is ϕ-primary ideals in quotient semirings and semirings of fractions.

3.1 ϕ-Primary Ideals of Semirings

For the sake of completeness, we begin with a definition that is used throughout this chapter. We would like to restate the definitions of primary ideals and weakly primary ideals of semirings; in addition, we define almost primary ideals, n-almost primary ideals and ω-primary ideals of semirings in the same manner as almost primary ideals, n-almost primary ideals and ω-primary ideals of rings given in [15].

A tool that we use most frequently in this chapter is the radicals of ideals. So, first of all, we would like to recall them. The radical of an ideal I of a semiring R is denoted by $\sqrt{I}$ and $\sqrt{I} = \{a \in R \,|\, a^n \in I \text{ for some } n \in \mathbb{N}\}$ is an ideal of R. Hence, for an ideal I of a semiring R containing a, we can write $a \in \sqrt{I}$ in stead of the statement that $a^n \in I$ for some positive integer n; moreover, we use $a \in \sqrt{I}$ from now on.

Definition 3.1.1. Let R be a semiring.

A proper ideal I of R is said to be ***primary*** if whenever $a, b \in R$ and $ab \in I$, then $a \in I$ or $b \in \sqrt{I}$.

A proper ideal I of R is said to be ***weakly primary*** if whenever $a, b \in R$ and $0 \neq ab \in I$, then $a \in I$ or $b \in \sqrt{I}$.

A proper ideal I of R is said to be ***almost primary*** if whenever $a, b \in R$ and $ab \in I - I^2$, then $a \in I$ or $b \in \sqrt{I}$.

A proper ideal I of R is said to be ***n-almost primary*** ($n \in \mathbb{N}$ with $n \geq 2$) if whenever $a, b \in R$ and $ab \in I - I^n$, then $a \in I$ or $b \in \sqrt{I}$.

A proper ideal I of R is said to be ***ω-primary*** if whenever $a, b \in R$ and $ab \in I - \cap_{n=1}^{\infty} I^n$, then $a \in I$ or $b \in \sqrt{I}$.

Proposition 3.1.2. *Let p be a prime and n a positive integer. Then $p^n\mathbb{Z}_0^+$ is a primary ideal of the semiring $\mathbb{Z}_0^+$.*

Proof. Let $a, b \in \mathbb{Z}_0^+$ be such that $ab \in p^n\mathbb{Z}_0^+$. Then $ab = p^n l$ for some $l \in \mathbb{Z}_0^+$.
Case 1: Assume that p^n is a factor of a. Thus $a = p^n h$ for some $h \in \mathbb{Z}_0^+$. Hence $a \in p^n\mathbb{Z}_0^+$.
Case 2: Assume that p^n is not a factor of a. Then p is a factor of b. Thus $b = pm$ for some $m \in \mathbb{Z}_0^+$. Hence $b^n = (pm)^n = p^n m^n \in p^n\mathbb{Z}_0^+$. Therefore, $b \in \sqrt{p^n\mathbb{Z}_0^+}$.

From any cases, we can conclude that $p^n\mathbb{Z}_0^+$ is a primary ideal of $\mathbb{Z}_0^+$. □

As above definitions, it is easy to see that the zero ideal is a weakly primary ideal, an almost primary ideal, an n-almost primary ideal and an ω-primary ideal because $I - \{0\}, I - I^2, I - I^n$ and $I - \cap_{m=1}^{\infty} I^m$ must be the empty set. Nevertheless, the zero ideal may be a primary ideal of some semirings and probably not be a primary ideal of other semirings as shown in the following example.

Example 3.1.3. (1) Consider the semiring $\mathbb{R}_0^+$ and its ideal $\{0\}$. Let $a, b \in \mathbb{R}_0^+$ such that $ab \in \{0\}$. Thus $a = 0$ or $b = 0$, and so $a \in \{0\}$ or $b \in \sqrt{\{0\}}$. Hence the ideal $\{0\}$ is a primary ideal of the semiring $\mathbb{R}_0^+$.

(2) Consider the ideal $\{(0,0)\}$ of the semiring $\mathbb{Q}_0^+ \times \mathbb{Q}_0^+$. Let $a, b \in \mathbb{Q}_0^+ - \{0\}$. Since $(a, 0) \cdot (0, b) = (0, 0) \in \{(0, 0)\}$ but $(a, 0) \notin \{(0, 0)\}$ and $(0, b)^n = (0, b^n) \notin \{(0, 0)\}$ for all $n \in \mathbb{N}$. That is $(a, 0) \notin \{(0, 0)\}$ and $(0, b) \notin \sqrt{\{(0, 0)\}}$. Therefore, the ideal $\{(0, 0)\}$ is not a primary ideal of the semiring $\mathbb{Q}_0^+ \times \mathbb{Q}_0^+$.

From the definition of almost primary ideals and n-almost primary ideals, one can see that 2-almost primary ideals are just almost primary ideals.

In the following, we would like to define the main character of this chapter that is ϕ-primary ideals of semirings which is defined in the same fashion as ϕ-primary ideals of rings given by A. Y. Darani in 2012.

Definition 3.1.4. A proper ideal I of a semiring R with ϕ is said to be ***ϕ-primary*** if whenever $a, b \in R$ and $ab \in I - \phi(I)$, then $a \in I$ or $b \in \sqrt{I}$.

Next, we provide relationships between ϕ-primary ideals and primary ideals (weakly primary ideals, ω-primary ideals, almost primary ideals, n-almost primary ideals) in the same manner as found in [15].

Example 3.1.5. Let R be a semiring.

(1) Define $\phi_\varnothing : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_\varnothing(I) = \varnothing$ for all $I \in \mathscr{I}(R)$. Then I is a $\phi_\varnothing$-primary ideal if and only if I is a primary ideal.

(2) Define $\phi_0 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_0(I) = \{0\}$ for all $I \in \mathscr{I}(R)$. Then I is a ϕ_0-primary ideal if and only if I is a weakly primary ideal.

(3) Define $\phi_1 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_1(I) = I$ for all $I \in \mathscr{I}(R)$, (i.e., ϕ_1 is the identity function). Then I is a ϕ_1-primary ideal if and only if I is a proper ideal.

(4) Define $\phi_2 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_2(I) = I^2$ for all $I \in \mathscr{I}(R)$. Then I is a ϕ_2-primary ideal if and only if I is an almost primary ideal.

(5) Define $\phi_n : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_n(I) = I^n$ for all $I \in \mathscr{I}(R)$ ($n \in \mathbb{N}$ with $n \geq 2$). Then I is a ϕ_n-primary ideal if and only if I is an n-almost primary ideal.

(6) Define $\phi_\omega : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_\omega(I) = \cap_{n=1}^{\infty} I^n$ for all $I \in \mathscr{I}(R)$. Then I is a ϕ_ω-primary ideal if and only if I is an ω-primary ideal.

From the definition of ϕ-primary ideals and Example 3.1.5, we can conclude that ϕ-primary ideals of semirings are a generalization of primary ideals, weakly primary ideals, almost primary ideals, n-almost primary ideals and ω-primary ideals of semirings depending on the defined function ϕ. Moreover, from now on, we use the notation $\phi_\varnothing, \phi_0, \phi_1, \phi_2, \phi_n$ and ϕ_ω instead of functions from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ which are defined as above.

Recall that the notation $\varphi_1 \le \varphi_2$ means $\varphi_1(I) \subseteq \varphi_2(I)$ for all $I \in \mathscr{I}(R)$ where R is a semiring and $\varphi_1, \varphi_2 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ are functions. In the next proposition, we show that if $\varphi_1 \le \varphi_2$, then φ_1-primary ideals imply φ_2-primary ideals.

Proposition 3.1.6. *Let R be a semiring, I a proper ideal of R and $\varphi_1 \le \varphi_2$ where φ_1 and φ_2 are functions from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$. If I is a φ_1-primary ideal, then I is a φ_2-primary ideal.*

Proof. Assume that I is a φ_1-primary ideal. Let $a, b \in R$ be such that $ab \in I - \varphi_2(I)$. Since $\varphi_1(I) \subseteq \varphi_2(I)$, we obtain $ab \in I - \varphi_1(I)$. Then $a \in I$ or $b \in \sqrt{I}$ because I is a φ_1-primary ideal. Therefore, I is a φ_2-primary ideal. □

Relationships between ϕ-primary ideals and primary ideals (weakly primary ideals, almost primary ideals, n-almost primary ideals, ω-primary ideals) are already shown in Example 3.1.5. Furthermore, from Proposition 3.1.6, we obtain relationships among primary ideals, weakly primary ideals, almost primary ideals, n-almost primary ideals and ω-primary ideals.

Corollary 3.1.7. *Let I be a proper ideal of a semiring and $n \in \mathbb{N}$ with $n \ge 2$. Consider the following statements:*

(1) I is a primary ideal.

(2) I is a weakly primary ideal.

(3) I is an ω-primary ideal.

(4) I is an $(n+1)$-almost primary ideal.

(5) I is an n-almost primary ideal.

(6) I is an almost primary.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

Proof. The result follows from the fact that $\varnothing \subseteq \{0\} \subseteq \cap_{n=1}^{\infty} I^n \subseteq I^{n+1} \subseteq I^n \subseteq I^2$ $(\phi_\varnothing \leq \phi_0 \leq \phi_\omega \leq \phi_{n+1} \leq \phi_n \leq \phi_2)$ where $n \in \mathbb{N}$ such that $n \geq 2$. □

As a result of Corollary 3.1.7, we obtain that ω-primary ideals imply n-almost primary ideals for all positive integer $n \geq 2$. Nevertheless, the converse might not be true, although we still cannot find a counter-example. For a positive integer $n \geq 2$, we remark that if an ideal I of a semiring R is a counter-example of this statement, if exists, it, at least, must not be an idempotent ideal and must satisfy the condition that $I^n \neq \cap_{m=1}^{\infty} I^m$ because if I is an idempotent ideal or $I^n = \cap_{m=1}^{\infty} I^m$, then $I - I^n = I - \cap_{m=1}^{\infty} I^m$ so that I is an n-almost primary ideal if and only if I is an ω-primary ideal.

However, if a proper ideal I is assumed to be an n-almost primary ideal for all $n \in \mathbb{N}$ with $n \geq 2$, then I is an ω-primary ideal.

Proposition 3.1.8. *Let R be a semiring and I a proper ideal of R. Then I is an ω-primary ideal if and only if I is an n-almost primary ideal for all $n \geq 2$.*

Proof. Assume that I is an ω-primary ideal. The proof is clear by Corollary 3.1.7.

Conversely, assume that I is an n-almost primary for all $n \geq 2$. Let $a, b \in R$ be such that $ab \in I - \phi_\omega(I) = I - \cap_{n=1}^{\infty} I^n$. Then $ab \in I - I^l$ where $l \in \mathbb{N} - \{1\}$. Since I is an l-almost primary ideal, we obtain $a \in I$ or $b \in \sqrt{I}$. Therefore, I is an ω-primary ideal. □

We would like to point out here that many results in this chapter are concerned with k-ideal I of a semiring R with ϕ such that $\phi(I)$ is also a k-ideal and sometimes including $\phi(I) \subseteq I$. Thus, it is natural to verify whether this situation is reasonable. Note that, for any k-ideal I of a semiring R with ϕ, if the function ϕ is the identity map, then it is clear that $\phi(I)$ is a k-ideal. Moreover, there are many functions ϕ which make $\phi(I)$ k-ideals. In the next example, we provide some functions ϕ that not only make $\phi(I)$ a k-ideal but also make $\phi(I)$ a subset of I.

Example 3.1.9. Consider the semiring $\mathbb{Z}_0^+$ with ϕ_2. Then $8\mathbb{Z}_0^+$ is a k-ideal of $\mathbb{Z}_0^+$. Recall that ϕ_2 is a function defined by $\phi_2(I) = I^2$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Thus

$\phi_2(8\mathbb{Z}_0^+) = (8\mathbb{Z}_0^+)^2 = (8\mathbb{Z}_0^+)(8\mathbb{Z}_0^+) = 64\mathbb{Z}_0^+$ and hence $\phi_2(8\mathbb{Z}_0^+)$ is a k-ideal. In addition, $\phi_2(8\mathbb{Z}_0^+) = 64\mathbb{Z}_0^+ \subseteq 8\mathbb{Z}_0^+$.

However, there is a semiring R with ϕ in which ϕ is a function such that $\phi(I)$ is not a k-ideal of R and $\phi(I) \nsubseteq I$ while I is a k-ideal of R.

Example 3.1.10. Consider the semiring $\mathbb{Z}_0^+$. Then $3\mathbb{Z}_0^+$ is a k-ideal of $\mathbb{Z}_0^+$. Let $J = 2\mathbb{Z}_0^+$. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(I) = I + J$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Then $\phi(3\mathbb{Z}_0^+) = 3\mathbb{Z}_0^+ + 2\mathbb{Z}_0^+ = \mathbb{Z}_0^+ - \{1\}$, and so $\phi(3\mathbb{Z}_0^+)$ is not a k-ideal of $\mathbb{Z}_0^+$ because $2, 2+1 \in \mathbb{Z}_0^+ - \{1\}$ but $1 \notin \mathbb{Z}_0^+ - \{1\}$. Furthermore, $\phi(3\mathbb{Z}_0^+) = \mathbb{Z}_0^+ - \{1\} \nsubseteq 3\mathbb{Z}_0^+$.

Since the empty set is a subset of any sets and primary ideals are just $\phi_\varnothing$-primary ideals, primary ideals imply ϕ-primary ideals for any ϕ but not vice versa as shown in the following example.

Example 3.1.11. Consider the semiring $\mathbb{Z}_0^+$ and its ideal $30\mathbb{Z}_0^+$. Since $5 \cdot 6 = 30 \in 30\mathbb{Z}_0^+$ but $5 \notin 30\mathbb{Z}_0^+$ and $6^n \notin 30\mathbb{Z}_0^+$ for all $n \in \mathbb{N}$, i.e., $5 \notin 30\mathbb{Z}_0^+$ and $6 \notin \sqrt{30\mathbb{Z}_0^+}$. Thus $30\mathbb{Z}_0^+$ is not a primary ideal of the semiring $\mathbb{Z}_0^+$. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(I) = I + 3\mathbb{Z}_0^+$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Hence $30\mathbb{Z}_0^+ - \phi(30\mathbb{Z}_0^+) = 30\mathbb{Z}_0^+ - (30\mathbb{Z}_0^+ + 3\mathbb{Z}_0^+) = \varnothing$ because $30\mathbb{Z}_0^+ \subseteq 30\mathbb{Z}_0^+ + 3\mathbb{Z}_0^+$. Thus $30\mathbb{Z}_0^+$ is a ϕ-primary ideal of the semiring $\mathbb{Z}_0^+$. Therefore, $30\mathbb{Z}_0^+$ is a ϕ-primary ideal but not a primary ideal of the semiring $\mathbb{Z}_0^+$.

In the next theorem, we provide conditions showing that k-ideals and $I^2 \nsubseteq \phi(I)$ are sufficient for a ϕ-primary ideal I to be a primary ideal.

Theorem 3.1.12. *Let R be a semiring with ϕ and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal. If I is a ϕ-primary ideal with $I^2 \nsubseteq \phi(I)$, then I is a primary ideal.*

Proof. Assume that I is a ϕ-primary ideal with $I^2 \nsubseteq \phi(I)$. Let $a, b \in R$ be such that $ab \in I$. If $ab \in I - \phi(I)$, then $a \in I$ or $b \in \sqrt{I}$ because I is ϕ-primary. So we suppose that $ab \in \phi(I)$.

Case 1: Assume that $aI \nsubseteq \phi(I)$ or $bI \nsubseteq \phi(I)$. Without loss of generality, suppose that $aI \nsubseteq \phi(I)$. Then there exists $x_0 \in I$ such that $ax_0 \notin \phi(I)$. Since $ab, x_0 \in I$, we obtain $ab+ax_0 \in I$. If $ab+ax_0 \in \phi(I)$, then $ax_0 \in \phi(I)$ because $\phi(I)$ is a k-ideal and $ab, ab+ax_0 \in \phi(I)$ which is a contradiction. Thus $a(b+x_0) = ab+ax_0 \in I - \phi(I)$. Since I is ϕ-primary, $a \in I$ or $b + x_0 \in \sqrt{I}$. Since $\sqrt{I}$ is a subtractive extension of I by Proposition 2.2.11, we obtain $a \in I$ or $b \in \sqrt{I}$.

Case 2: Assume that $aI \subseteq \phi(I)$ and $bI \subseteq \phi(I)$. Since $I^2 \nsubseteq \phi(I)$, there exist $x_1, x_2 \in I$ such that $x_1x_2 \notin \phi(I)$. Then $(a + x_2)(b + x_2) = ab + ax_2 + bx_1 + x_1x_2 \in I - \phi(I)$ because $\phi(I)$ is a k-ideal. Since I is ϕ-primary, $a + x_1 \in I$ or $b + x_2 \in \sqrt{I}$. Since $x_1, x_2 \in I$, I is a k-ideal and $\sqrt{I}$ is a subtractive extension of I by Proposition 2.2.11, we obtain $a \in I$ or $b \in \sqrt{I}$.

Therefore, I is a primary ideal. □

In the next example, we show that there is a proper k-ideal I of a semiring R with ϕ such that $\phi(I)$ is a k-ideal and I is a ϕ-primary ideal with $I^2 \nsubseteq \phi(I)$.

Example 3.1.13. Consider the semiring $R = \mathbb{Z}_0^+$ and the ideal $I = p^n\mathbb{Z}_0^+$ where p is a prime number and $n \in \mathbb{N}$. Then I is a proper k-ideal of R. Define $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi(J) = q\mathbb{Z}_0^+$ for all $J \in \mathscr{I}(R)$ where q is a prime number such that $q \neq p$. Thus $\phi(I) = q\mathbb{Z}_0^+$ is a k-ideal. By Proposition 3.1.2, I is a primary ideal of R so is a ϕ-primary ideal of R. Moreover, we obtain $I^2 = p^n\mathbb{Z}_0^+ \cdot p^n\mathbb{Z}_0^+ = p^{2n}\mathbb{Z}_0^+ \nsubseteq q\mathbb{Z}_0^+ = \phi(I)$.

The converse of Theorem 3.1.12 is not true in general and we provide an example to confirm this.

Example 3.1.14. Consider the semiring $R = \mathbb{Z}_0^+$ and the ideal $I = 25\mathbb{Z}_0^+$. Then I is a primary k-ideal of R and $I^2 = (25\mathbb{Z}_0^+)^2 = 625\mathbb{Z}_0^+$. Next, we define $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi(n\mathbb{Z}_0^+) = 5n\mathbb{Z}_0^+$ for all $n \in \mathbb{Z}_0^+$ and $\phi(J) = J$ otherwise. Thus $\phi(I) = \phi(25\mathbb{Z}_0^+) = 125\mathbb{Z}_0^+$, and so $\phi(I)$ is a k-ideal of R. Hence I is a primary ideal of R while $I^2 = 625\mathbb{Z}_0^+ \subseteq 125\mathbb{Z}_0^+ = \phi(I)$. Therefore, the converse of Theorem 3.1.12 is not true as desired.

Next, the consequences of Theorem 3.1.12 are provided.

Corollary 3.1.15. *Let R be a semiring with ϕ and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal and $\phi(I) \subseteq I$. If I is a ϕ-primary ideal but not a primary ideal, then $\sqrt{I} = \sqrt{\phi(I)}$.*

Proof. Assume that I is a ϕ-primary ideal but not a primary ideal. By Theorem 3.1.12, we obtain $I^2 \subseteq \phi(I)$. Since $\sqrt{I} = \sqrt{I^2}$ by Proposition 2.1.24 (2), we have $\sqrt{I} = \sqrt{I^2} \subseteq \sqrt{\phi(I)}$. Since $\phi(I) \subseteq I$, we obtain $\sqrt{\phi(I)} \subseteq \sqrt{I}$. Therefore, $\sqrt{I} = \sqrt{\phi(I)}$. □

The next example shows that the converse of Corollary 3.1.15 is not true.

Example 3.1.16. Consider the semring $\mathbb{Z}_0^+$ and its ideal $I = p^2\mathbb{Z}_0^+$ where p is a prime number. Then I is a primary k-ideal of $\mathbb{Z}_0^+$. We define $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi(J) = pJ$ if J is a principal ideal of $\mathbb{Z}_0^+$ and $\phi(J) = J^2$ otherwise for all $J \in \mathscr{I}(R)$. Thus $\phi(I) = \phi(p^2\mathbb{Z}_0^+) = p(p^2\mathbb{Z}_0^+) = p^3\mathbb{Z}_0^+$ is a k-ideal of $\mathbb{Z}_0^+$; in addition, $\phi(I) = p^3\mathbb{Z}_0^+ \subseteq p^2\mathbb{Z}_0^+ = I$. Moreover, $\sqrt{\phi(I)} = \sqrt{\phi(p^2\mathbb{Z}_0^+)} = \sqrt{p^3\mathbb{Z}_0^+} = p\mathbb{Z}_0^+ = \sqrt{p^2\mathbb{Z}_0^+} = \sqrt{I}$ by Proposition 2.1.25. Hence I is a primary ideal of $\mathbb{Z}_0^+$ and $\sqrt{I} = \sqrt{\phi(I)}$. Therefore, the converse of Corollary 3.1.15 is not true.

Corollary 3.1.17. *Let R be a semiring with $\phi \leq \phi_3$ and I a proper k-ideal of R. If I is a ϕ-primary ideal such that $\phi(I)$ is a k-ideal, then I is an ω-primary ideal.*

Proof. Assume that I is a ϕ-primary ideal. If I is a primary ideal, then I is an ω-primary by Corollary 3.1.7. So assume that I is not a primary ideal. Then $I^2 \subseteq \phi(I)$ by Theorem 3.1.12. Thus $I^2 \subseteq \phi(I) \subseteq \phi_3(I) = I^3 \subseteq I^2$, and hence $I^2 = \phi(I) = I^3$. Thus $\phi(I) = I^n$ for each $n \geq 2$. Therefore, I is an n-almost primary ideal for all $n \geq 2$. By Proposition 3.1.8, we can conclude that I is an ω-primary ideal. □

The next example shows that the converse of Corollary 3.1.17 is not true.

Example 3.1.18. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and its ideal $I = \{(0,0)\}$. Then I is a k-ideal. Since $I^n = \{(0,0)\}^n = \{(0,0)\}$ for all $n \in \mathbb{N}$, it follows that $\cap_{n=1}^{\infty} I^n = \{(0,0)\}$. Then $I - \cap_{n=1}^{\infty} I^n = \{(0,0)\} - \{(0,0)\} = \varnothing$, so that

I is an ω-primary ideal. We define $\phi_\varnothing : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi_\varnothing(J) = \varnothing$ for all $J \in \mathscr{I}(R)$. Then $\phi_\varnothing \leq \phi_3$. Since $(2,0) \cdot (0,3) = (0,0) \in I - \phi_\varnothing(I)$ but $(2,0) \notin \{(0,0)\}$ and $(0,3)^n = (0,3^n) \notin \{(0,0)\}$ for all $n \in \mathbb{N}$. That is $(2,0) \notin \{(0,0)\}$ and $(0,3) \notin \sqrt{\{(0,0)\}}$. Therefore, the ideal $\{(0,0)\}$ is not a $\phi_\varnothing$-primary ideal of the semiring R.

Corollary 3.1.19. *Let R be a semiring. If I is a weakly primary k-ideal but not a primary ideal, then $I^2 = \{0\}$.*

Proof. Assume that I is a weakly primary k-ideal but not a primary ideal. Since I is a weakly primary ideal, I is a ϕ_0-primary ideal. Then $I^2 \subseteq \phi_0(I) = \{0\}$ by Theorem 3.1.12. Hence $I^2 = \{0\}$. □

From Corollary 3.1.19, we realize that the square of a weakly primary k-ideal which is not a primary ideal must be the zero ideal. However, since $\{0\}$ is a primary ideal of the semiring $\mathbb{R}_0^+$ by Example 3.1.3 (1) and $\{0\}^2 = \{0\}$, it follows that the converse of Corollary 3.1.19 is not true.

Before moving to the next theorem, we would like to recall some notation that are used in the following theorem. For given ideals I of a semiring R containing a, $(I : a) = \{x \in R \,|\, xa \in I\}$ is an ideal of R containing I; moreover, if I is a k-ideal, then $(I : a)$ is a k-ideal. Furthermore, if I and J are ideals of a semiring R containing a such that $I \subseteq J$, then $(I : a) \subseteq (J : a)$.

Theorem 3.1.20. *Let R be a semiring with ϕ and I a proper ideal of R and $\phi(I) \subseteq I$. The following statements are equivalent.*

(1) I is a ϕ-primary ideal.

(2) For any $x \in R - \sqrt{I}, (I : x) = I \cup (\phi(I) : x)$.

Proof. To show (1) ⇒ (2), suppose that I is a ϕ-primary ideal. Let $x \in R - \sqrt{I}$. Since $I \subseteq (I : x)$ and $(\phi(I) : x) \subseteq (I : x)$, we obtain $I \cup (\phi(I) : x) \subseteq (I : x)$. Next, let $a \in (I : x)$. Then $ax \in I$. If $ax \notin \phi(I)$, then $a \in I$ because I is a ϕ-primary ideal and $x \notin \sqrt{I}$. If $ax \in \phi(I)$, then $a \in (\phi(I) : x)$. Hence $(I : x) \subseteq I \cup (\phi(I) : x)$.

Therefore, $(I : x) = I \cup (\phi(I) : x)$.

To show (2) $\Rightarrow$ (1), assume that the statement (2) holds. Let $a, b \in R$ be such that $ab \in I - \phi(I)$. If $b \in \sqrt{I}$, then we are done. Suppose that $b \notin \sqrt{I}$. Thus $(I : b) = I \cup (\phi(I) : b)$. Since $ab \in I - \phi(I)$, we acquire $a \in (I : b) - (\phi(I) : b)$. Hence $a \in I$. Therefore, I is a ϕ-primary ideal. □

Recall that a ϕ-primary ideal I of a semiring R is a proper ideal of R in which $ab \in I - \phi(I)$ implies $a \in I$ or $b \in \sqrt{I}$ for $a, b \in R$. We notice that this definition is given via using the elements of those semirings. In the following result, the definition via elements can be replaced by the definition via ideals.

Theorem 3.1.21. *Let R be a semiring with ϕ and I a proper k-ideal of R such that $\phi(I)$ and $\sqrt{I}$ are k-ideals and $\phi(I) \subseteq I$. The following statements are equivalent.*

(1) I is a ϕ-primary ideal.

(2) For any $x \in R - \sqrt{I}, (I : x) = I \cup (\phi(I) : x)$.

(3) For any $x \in R - \sqrt{I}, (I : x) = I$ or $(I : x) = (\phi(I) : x)$.

(4) For ideals A and B of R, $AB \subseteq I$ and $AB \nsubseteq \phi(I)$ imply $A \subseteq I$ or $B \subseteq \sqrt{I}$.

Proof. We obtain (1) $\Leftrightarrow$ (2) by Theorem 3.1.20.

To show (2) $\Rightarrow$ (3), suppose that the statement (2) holds. Let $x \in R - \sqrt{I}$. Since I and $\phi(I)$ are k-ideals, $(I : x)$ and $(\phi(I) : x)$ are k-ideals. Therefore, $(I : x) = I$ or $(I : x) = (\phi(I) : x)$ by Proposition 2.1.13.

To show (3) $\Rightarrow$ (4), assume that the statement (3) holds. Let A and B be ideals of R such that $AB \subseteq I$. Suppose that $A \nsubseteq I$ and $B \nsubseteq \sqrt{I}$. We would like to show that $AB \subseteq \phi(I)$. Let $b \in B$.

Case 1: Assume that $b \notin \sqrt{I}$. Then $(I : b) = I$ or $(I : b) = (\phi(I) : b)$ by the statement (3). Since $Ab \subseteq AB \subseteq I$, we obtain $A \subseteq (I : b)$. Since $A \nsubseteq I$ and $A \subseteq (I : b)$, we obtain $(I : b) \neq I$. Thus $(I : b) = (\phi(I) : b)$. Therefore, $A \subseteq (\phi(I) : b)$, and hence $Ab \subseteq \phi(I)$.

Case 2: Assume that $b \in \sqrt{I}$. Since $B \nsubseteq \sqrt{I}$, there is $b' \in B - \sqrt{I}$. Similarly to Case 1, we obtain $Ab' \subseteq \phi(I)$. It is clear that $b + b' \in B$. If $b + b' \in \sqrt{I}$, then $b' \in \sqrt{I}$ because $b, b + b' \in \sqrt{I}$ and $\sqrt{I}$ is a k-ideal. Thus $b + b' \in B - \sqrt{I}$, and hence $A(b + b') \subseteq \phi(I)$ is obtained similarly to Case 1. Let $a \in A$. Then $ab', ab + ab' \in \phi(I)$. Since $\phi(I)$ is a k-ideal, $ab \in \phi(I)$. Hence $Ab \subseteq \phi(I)$.

Any cases show that $Ab \subseteq \phi(I)$. Therefore, $AB \subseteq \phi(I)$ because b is an arbitrary element of B.

To show (4) $\Rightarrow$ (1), assume that the statement (4) holds. Let $x, y \in R$ be such that $xy \in I - \phi(I)$. Then $\langle x\rangle \langle y\rangle \subseteq I$. If $\langle x\rangle \langle y\rangle \subseteq \phi(I)$, then $xy \in \phi(I)$ which is a contradiction. Then $\langle x\rangle \langle y\rangle \nsubseteq \phi(I)$. By statement (4), we obtain $\langle x\rangle \subseteq I$ or $\langle y\rangle \subseteq \sqrt{I}$. Hence $x \in I$ or $y \in \sqrt{I}$. Therefore, I is ϕ-primary. □

Next, we would like to recall from Chapter II that a strongly Euclidean semiring R is a semiring which is consistent with conditions that for a function $d : R - \{0\} \to \mathbb{Z}_0^+$ such that $d(ab) \geq d(a)$ for all $a, b \in R - \{0\}$ and if $a, b \in R$ with $b \neq 0$ then there exist unique elements $q, r \in R$ such that $a = bq + r$ where either $r = 0$ or $d(r) < d(b)$.

The advantage of strongly Euclidean semirings that we use in this research is that k- ideal and principal ideal are coincide which was studied by V. Gupta in 2006 [18].

Let R be a semiring. Then I is a ϕ_2-primary ideal if and only if I is an almost primary ideal where ϕ_2 is given in Example 3.1.5 (4). In the next theorem, we show that if $\langle a\rangle$ is a ϕ_2-primary ideal, then $\langle a\rangle$ is a primary ideal for any element a in a strongly Euclidean semiring under some conditions.

As a consequence of Proposition 2.1.16 and Example 2.1.17, we gain that $\langle a\rangle \subseteq (\langle a\rangle^2 : a)$ for any element a of a semiring but not vice versa. Thus a semiring in Example 2.1.17 is an example of a semiring such that $(\langle a\rangle^2 : a) \neq \langle a\rangle$; nevertheless, in the following theorem, we suppose the condition that $(\langle a\rangle^2 : a) = \langle a\rangle$ holds.

Theorem 3.1.22. *Let R be a strongly Euclidean semiring and $a \in R$ such that $(\langle a\rangle^2 : a) = \langle a\rangle$. Then $\langle a\rangle$ is a ϕ-primary ideal for some ϕ with $\phi \leq \phi_2$ if and only* *if $\langle a \rangle$ is a primary ideal.*

Proof. If $\langle a \rangle$ is a primary ideal, then $\langle a \rangle$ is ϕ-primary for any ϕ. Next, we assume that $\langle a \rangle$ is a ϕ-primary ideal for some ϕ with $\phi \leq \phi_2$. Then $\langle a \rangle$ is a ϕ_2-primary ideal by Proposition 3.1.6. We would like to show that $\langle a \rangle$ is a primary ideal. Let $x, y \in R$ be such that $xy \in \langle a \rangle$. If $xy \in \langle a \rangle - \langle a \rangle^2$, then $x \in \langle a \rangle$ or $y \in \sqrt{\langle a \rangle}$ because $\langle a \rangle$ is ϕ_2-primary. So we assume that $xy \in \langle a \rangle^2$. Since R is a strongly Euclidean semiring, $\langle a \rangle$ and $\langle a^2 \rangle$ are k-ideals. Note that $(x+a)y = xy + ay \in \langle a \rangle$.
Case 1: Assume that $(x+a)y \in \langle a \rangle - \langle a \rangle^2$. Since $\langle a \rangle$ is ϕ_2-primary, $x + a \in \langle a \rangle$ or $y \in \sqrt{\langle a \rangle}$. Hence $x \in \langle a \rangle$ or $y \in \sqrt{\langle a \rangle}$ because $\langle a \rangle$ is a k-ideal and $a \in \langle a \rangle$.
Case 2: Assume that $(x+a)y \in \langle a \rangle^2 = \langle a^2 \rangle$. Since $\langle a^2 \rangle$ is a k-ideal and $xy, xy + ay \in \langle a^2 \rangle$, we obtain $ay \in \langle a^2 \rangle$. Thus $y \in (\langle a \rangle^2 : a) = \langle a \rangle$.

Therefore, $\langle a \rangle$ is a primary k-ideal. □

Next, we provide an example to confirm that there exists a strongly Euclidean semiring R such that $(\langle a \rangle^2 : a) = \langle a \rangle$ for some $a \in R$.

Example 3.1.23. Let $R = \mathbb{Z}_0^+$ and $a \in \mathbb{Z}_0^+ - \{0\}$. Then R is a strongly Euclidean semiring by Example 2.2.23. Since $\langle a \rangle \subseteq (\langle a \rangle^2 : a)$ by Proposition 2.1.16 (4), it remains to show that $(\langle a \rangle^2 : a) \subseteq \langle a \rangle$. Let $x \in (\langle a \rangle^2 : a)$. Then $xa \in \langle a \rangle^2 = a^2\mathbb{Z}_0^+$. Thus $xa = a^2 r$ for some $r \in \mathbb{Z}_0^+$. Because $a \neq 0$, we obtain $x = ar \in \langle a \rangle$. Hence $(\langle a \rangle^2 : a) \subseteq \langle a \rangle$. Therefore $(\langle a \rangle^2 : a) = \langle a \rangle$.

3.2 ϕ-Primary Ideals in Decomposable Semirings

In this section, we concern with relationships among primary ideals, weakly primary ideals and ϕ-primary ideals of decomposable semirings.

For a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$ ($m \in \mathbb{N}$ with $m \geq 2$) such that R_i is a semiring with φ_i for all $i \in \{1, 2, \ldots, m\}$ and an ideal $I_1 \times I_2 \times \cdots \times I_m$ of R, it follows that $\varphi_1(I_1) \times \varphi_2(I_2) \times \cdots \times \varphi_m(I_m)$ is an ideal of R or the empty set. Hence there is a function $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ such that $\phi(I_1 \times I_2 \times \cdots \times I_m) = \varphi_1(I_1) \times \varphi_2(I_2) \times \cdots \times \varphi_m(I_m)$ for all $I_1 \times I_2 \times \cdots \times I_m \in \mathscr{I}(R)$; in addition, we denote the function ϕ which is defined as the previous by $\phi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_m$.

The property of decomposable semirings frequently used in this section is Proposition 2.1.27: $\sqrt{I_1 \times I_2 \times \cdots \times I_m} = \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$ for any ideal $I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring.

Next, we would like to show that a nonzero weakly primary ideal $I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring $R_1 \times R_2 \times \cdots \times R_m$ which has at least one of I_i must not be proper.

Proposition 3.2.1. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper ideal of R. If I is a weakly primary ideal, then $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$.*

Proof. Suppose that I is a weakly primary ideal of R. Since I is a nonzero ideal, there is $(a_1, a_2, \ldots, a_m) \in I$ such that $(a_1, a_2, \ldots, a_m) \neq (0, 0, \ldots, 0)$. Thus

$$(0, 0, \ldots, 0) \neq (a_1, a_2, \ldots, a_m) = (a_1, a_2, \ldots, a_{m-1}, 1)(1, 1, \ldots, 1, a_m) \in I.$$

Since I is a weakly primary ideal, $(a_1, a_2, \ldots, a_{m-1}, 1) \in I$ or $(1, 1, \ldots, 1, a_m) \in \sqrt{I}$. Since $\sqrt{I} = \sqrt{I_1 \times I_2 \times \cdots \times I_m} = \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$, we obtain $1 \in I_m$ or $1 \in \sqrt{I_i}$ for some $i \in \{1, 2, \ldots, m-1\}$, i.e., $1 \in I_j$ for some $j \in \{1, 2, \ldots, m\}$. Therefore, $I_j = R_j$. □

As a consequence of Corollary 3.1.7 and since $\{\bar{0}\}$ is a weakly primary ideal but not a primary ideal of the semiring $\mathbb{Z}_{12}^{+}$, primary ideals imply weakly primary ideals but not vice versa. Nevertheless, in a decomposable semiring, weakly primary ideals and primary ideals are coincide provided they are nonzero proper k-ideals.

Proposition 3.2.2. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper k-ideal of R. Then I is a weakly primary ideal if and only if I is a primary ideal.*

Proof. Suppose that I is a weakly primary ideal of R. We obtain from Proposition 3.2.1 that $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$. Then $I^2 \neq \{0\}$. Thus I is a primary ideal by Corollary 3.1.19. The converse holds because of Corollary 3.1.7. □

From Proposition 3.2.2, nonzero weakly primary ideals and nonzero primary ideals are coincide because I is a k-ideal and there is at least one of I_i which is equal to R_i. In the following theorem, we assume these conditions hold while the condition that "I is a nonzero ideal"can be omitted. We still obtain the same result; in addition, we also show that any proper components of I are primary ideals.

Theorem 3.2.3. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R which at least one $I_i = R_i$ where $i \in \{1, 2, \ldots, m\}$. Consider the following statements:*

(1) I is a weakly primary ideal of R.

(2) I is a primary ideal of R.

(3) If $I_j \neq R_j$ where $j \in \{1, 2, \ldots, m\}$, then I_j is a primary ideal of R_j.

Then (1) and (2) are equivalent and (2) implies (3).

Proof. Obviously, (2) $\Rightarrow$ (1).

To show (1) $\Rightarrow$ (2), assume that I is a weakly primary ideal of R. Then $I^2 \neq \{0\}$ because $I_i = R_i$. Thus I is a primary ideal of R by Corollary 3.1.19.

To show (2) $\Rightarrow$ (3), assume that I is a primary ideal of R. Furthermore, suppose that $I_j \neq R_j$ for some $j \in \{1, 2, \ldots, m\}$. To show that I_j is a primary ideal of R_j, let $a, b \in R_j$ be such that $ab \in I_j$. Then

$$(0, \ldots, 0, a, 0, \ldots, 0)(0, \ldots, 0, b, 0, \ldots, 0) = (0, \ldots, 0, ab, 0, \ldots, 0) \in I.$$

Since I is a primary ideal, $(0, \ldots, 0, a, 0, \ldots, 0) \in I$ or $(0, \ldots, 0, b, 0, \ldots, 0) \in \sqrt{I} = \sqrt{I_1} \times \cdots \times \sqrt{I_m}$. Hence $a \in I_j$ or $b \in \sqrt{I_j}$. Therefore, I_j is a primary ideal of R_j. □

From the previous theorem, one should suspect whether the statement (3) implies the statement (1) and the statement (2) or not. The next example clarifies this.

Example 3.2.4. Let p, q be prime numbers (not necessary distinct) and n a positive integer. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $I = p^n\mathbb{Z}_0^+ \times q^m\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. We know that $p^n\mathbb{Z}_0^+$ and $q^m\mathbb{Z}_0^+$ are primary ideals of the semiring $\mathbb{Z}_0^+$. Since $(0,0,0) \neq (p^n,1,2)(1,q^m,3) = (p^n,q^m,6) \in I$ but $(p^n,1,2) \notin I$ and $(1,q^m,3) \notin \sqrt{I}$ because $1 \notin q^m\mathbb{Z}_0^+$ and $1 \notin \sqrt{p^n\mathbb{Z}_0^+}$. Therefore, I is not a weakly primary ideal of R, and so I is not a primary ideal of R.

Example 3.2.4 confirms that the conditions in Theorem 3.2.3 are not enough to make (3) imply (1) and (2). In the next result, we assume that there is exactly one I_i such that $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$ instead of the condition that $I_i \neq R_i$ for some $i \in \{1, 2, \ldots, m\}$ in Theorem 3.2.3 for making (3) imply (1) and (2).

Theorem 3.2.5. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. The following statements are equivalent.*

(1) I is a weakly primary ideal of R

(2) I is a primary ideal of R.

(3) I_i is a primary ideal of R_i.

Proof. We obtain (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) by Theorem 3.2.3.

To show (3) $\Rightarrow$ (2), assume that I_i is a primary ideal of R_i. Let $(a_1, a_2, \ldots, a_m)$, $(b_1, b_2, \ldots, b_m) \in R$ be such that $(a_1b_1, \ldots, a_{i-1}b_{i-1}, a_ib_i, a_{i+1}b_{i+1}, \ldots, a_mb_m) \in I$. Note that $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$. Since $a_ib_i \in I_i$ and I_i is a primary ideal of R_i, we have $a_i \in I_i$ or $b_i \in \sqrt{I_i}$. Hence $(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m) \in R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m = I$ or $(b_1 \ldots, b_{i-1}, b_i, b_{i+1}, \ldots, b_m) \in R_1 \times \cdots \times R_{i-1} \times \sqrt{I_i} \times R_{i+1} \times \cdots \times R_m = \sqrt{I}$. Therefore, I is a primary ideal of R. □

Corollary 3.2.6. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring with ϕ and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. If I_i is a primary ideal of R_i, then I is a ϕ-primary ideal of R.*

Proof. The proof is completed by the fact that every primary ideal is a ϕ-primary ideal. □

The converse of Corollary 3.2.6 is not true and we provide an example to support this.

Example 3.2.7. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$ and its ideal $I = 20\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$. Then I is a k-ideal of R. Define $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi(J) = \sqrt{J}$ for all $J \in \mathscr{I}(R)$. Thus $\phi(I) = \sqrt{I} = \sqrt{20\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+} = \sqrt{20\mathbb{Z}_0^+} \times \sqrt{\mathbb{Z}_0^+} \times \cdots \times \sqrt{\mathbb{Z}_0^+} = 10\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$. Hence $I - \phi(I) = (20\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+) - (10\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+) = \varnothing$ because $20\mathbb{Z}_0^+ \subseteq 10\mathbb{Z}_0^+$. Therefore, the ideal I is a ϕ-primary ideal of R. However, $20\mathbb{Z}_0^+$ is not a primary ideal of $\mathbb{Z}_0^+$ because $4 \cdot 5 = 20 \in 20\mathbb{Z}_0^+$ but $4 \notin 20\mathbb{Z}_0^+$ and $5^n \notin 20\mathbb{Z}_0^+$ for all $n \in \mathbb{N}$.

Corollary 3.2.6 shows that if I_i is a primary k-ideal of a semiring R_i, then the ideal $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$ is a ϕ-primary ideal of the decomposable semiring $R_1 \times R_2 \times \cdots \times R_m$ with ϕ. Next, we take care of the case that I_i is a weakly primary ideal of R_i under the same conditions as in Corollary 3.2.6.

Theorem 3.2.8. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. If I_i is a weakly primary ideal of R_i, then I is a ϕ-primary ideal of R for all $\phi_\omega \leq \phi$.*

Proof. Without loss of generality, we assume that $i = 1$. Then $I = I_1 \times R_2 \times \cdots \times R_m$. Since I is a k-ideal, I_1 is a k-ideal by Proposition 2.1.28. Assume further that I_1 is a weakly primary ideal of R_1. We show that $I = I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-primary ideal of all $\phi_\omega \leq \phi$. If I_1 is a primary ideal of R_1, then I is a ϕ_ω-primary ideal of R by Corollary 3.2.6. So assume that I_1 is not a primary ideal. Thus $I_1^2 = \{0\}$ by Corollary 3.1.19. Consider the element $(x_1, x_2, \ldots, x_m) \in \phi_\omega(I) = \bigcap_{n=1}^{\infty} I^n \subseteq I^2 = (I_1 \times R_2 \times \cdots \times R_m)^2 \subseteq I_1^2 \times R_2 \times \cdots \times R_m = \{0\} \times R_2 \times \cdots \times R_m$. Let $(a_1, a_2, \ldots, a_m), (b_1, b_2, \ldots, b_m) \in R$ be such that $(a_1, a_2, \ldots, a_m)(b_1, b_2, \ldots, b_m) =$

$(a_1b_1, a_2b_2, \ldots, a_mb_m) \in I - \phi_\omega(I)$. Then $a_1b_1 \in I_1 - \{0\}$. Thus $a_1 \in I_1$ or $b_1 \in \sqrt{I_1}$ because I_1 is a weakly primary ideal of R_1. Hence $(a_1, a_2, \ldots, a_m) \in I_1 \times R_2 \times \cdots \times R_m$ or $(b_1, b_2, \ldots, b_m) \in \sqrt{I_1} \times R_2 \times \cdots \times R_m = \sqrt{I_1 \times R_2 \times \cdots \times R_m} = \sqrt{I}$. Thus I is a ϕ_ω-primary ideal of R. Therefore, in any cases, I is a ϕ_ω-primary ideal, and hence I is a ϕ-primary ideal for all $\phi_\omega \leq \phi$. □

Finally, we obtain a generalization of ϕ-primary ideals of decomposable semirings with two components.

Theorem 3.2.9. *Let $R = R_1 \times R_2$ be a decomposable semiring and $\phi = \varphi_1 \times \varphi_2$ where each $\varphi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\varnothing\}$ is a function. Then the ϕ-primary ideals of R have exactly one of the following three types:*

(1) $I_1 \times I_2$ where $I_j \subseteq \varphi_j(I_j)$ for all $j \in \{1, 2\}$ and at least one I_i is a proper ideal of R_i for some $i \in \{1, 2\}$.

(2) $I_1 \times R_2$ where I_1 is a φ_1-primary ideal of R_1 which must be primary if $\varphi_2(R_2) \neq R_2$.

(3) $R_1 \times I_2$ where I_2 is a φ_2-primary ideal of R_2 which must be primary if $\varphi_1(R_1) \neq R_1$.

Proof. First, we would like to show that an ideal of R having one of these three types is a ϕ-primary ideal.

(1) Assume that (1) holds. Then $I_1 \times I_2 - \phi(I_1 \times I_2) = \varnothing$, and so $I_1 \times I_2$ is a ϕ-primary ideal.

(2) Assume that (2) holds. If I_1 is primary, then $I_1 \times R_2$ is primary and hence is ϕ-primary. So suppose that I_1 is a φ_1-primary ideal of R_1 and $\varphi_2(R_2) = R_2$. Let $(a, b), (c, d) \in R_1 \times R_2$ be such that $(ac, bd) \in I_1 \times R_2 - \phi(I_1 \times R_2) = (I_1 - \varphi_1(I_1)) \times R_2$. Since I_1 is φ_1-primary, $a \in I_1$ or $c \in \sqrt{I_1}$. Hence $(a, b) \in I_1 \times R_2$ or $(c, d) \in \sqrt{I_1} \times R_2 = \sqrt{I_1 \times R_2}$. Therefore, $I_1 \times R_2$ is a ϕ-primary ideal of R.

The other case is similar to the previous one.

Next, we suppose that $I_1 \times I_2$ is a ϕ-primary ideal of R. Thus I_1 or I_2 is a proper ideal of R. Without loss of generality, assume that I_1 is a proper ideal. We would like to show that $I_1 \times I_2$ is exactly one of these three types. Assume $a, b \in R_1$ with $ab \in I_1 - \varphi_1(I_1)$. Then $(a,0)(b,0) = (ab,0) \in I_1 \times I_2 - \phi(I_1 \times I_2)$. Since $I_1 \times I_2$ is a ϕ-primary ideal of R, we obtain $(a,0) \in I_1 \times I_2$ or $(b,0) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$. Hence $a \in I_1$ or $b \in \sqrt{I_1}$. Therefore, I_1 is a φ_1-primary ideal of R_1. If $I_j \subseteq \varphi_j(I_j)$ for all $j \in \{1,2\}$, then (1) is obtained. Suppose that $I_1 \nsubseteq \varphi_1(I_1)$ or $I_2 \nsubseteq \varphi_2(I_2)$. Without loss of generality, assume that $I_1 \nsubseteq \varphi_1(I_1)$. Then there is $x \in I_1 - \varphi_1(I_1)$. Let $y \in I_2$. Then $(x,1)(1,y) = (x,y) \in I_1 \times I_2 - \phi(I_1 \times I_2)$. Since $I_1 \times I_2$ is a ϕ-primary ideal of R, we gain $(x,1) \in I_1 \times I_2$ or $(1,y) \in \sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$. Hence $I_2 = R_2$ or $I_1 = R_1$. Since I_1 is a proper ideal, $I_2 = R_2$. Now, we can conclude that $I_1 \times R_2$ is a ϕ-primary ideal of R where I_1 is a φ_1-primary ideal of R_1. It remains to show that I_1 is actually primary if $\varphi_2(R_2) \neq R_2$. Assume that $\varphi_2(R_2) \neq R_2$. Then $1 \notin \varphi_2(R_2)$. Let $a, b \in R_1$ be such that $ab \in I_1$. Thus $(a,1)(b,1) = (ab,1) \in I_1 \times R_2 - \phi(I_1 \times R_2)$. Since $I_1 \times R_2$ is a ϕ-primary ideal of R, we have $(a,1) \in I_1 \times R_2$ or $(b,1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times \sqrt{I_2}$. Hence $a \in I_1$ or $b \in \sqrt{I_1}$. Therefore, I_1 is a primary ideal of R_1. □

3.3 ϕ-Primary Ideals in Quotient Semirings and in Semirings of Fractions

In this final section, we are interested in ϕ-primary ideals of quotient semirings and ϕ-primary ideals of semirings of fractions.

Recall that if R is a semiring, I is a Q-ideal of R and ϕ is a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ such that $\phi(L)$ is a subtractive extension of I for all ideal L of R where L is a subtractive extension of I, then we define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\varnothing\}$ by $\phi_I(J/I) = (\phi(J))/I$ for each ideal J of R where J is a subtractive extension of I.

Recall further that R is a semiring with ϕ satisfying the property $(*)$ if R is a semiring with ϕ, I is a Q-ideal of R and ϕ_I is a function from $\mathscr{I}(R/I)$ into $\mathscr{I}(R/I) \cup \{\varnothing\}$ where ϕ and ϕ_I are defined as in the above paragraph.

First of all, we would like to present relationships between ϕ-primary ideals of semirings and ϕ-primary ideals of quotient semirings.

Theorem 3.3.1. *Let R be a semiring with ϕ satisfying the property $(*)$, I a Q-ideal of R and P a subtractive extension of I. Then P is a ϕ-primary ideal of R if and only if P/I is a ϕ_I-primary ideal of R/I.*

Proof. Suppose that P is a ϕ-primary ideal of R. Then P/I is an ideal of R/I because P is a subtractive extension of I. Next, we would like to show that P/I is a ϕ_I-primary ideal of R/I. Let $q_1+I, q_2+I \in R/I$ be such that $(q_1+I)(q_2+I) \in P/I - \phi_I(P/I)$. By Theorem 2.2.19, we have $q_1q_2 \in P - \phi(P)$. Since P is ϕ-primary, $q_1 \in P$ or $q_2 \in \sqrt{P}$. Hence $q_1 + I \in P/I$ or $q_2 + I \in \sqrt{P}/I = \sqrt{P/I}$ by Proposition 2.2.18. Therefore, P/I is a ϕ_I-primary k-ideal of R/I.

Conversely, assume that P/I is a ϕ_I-primary ideal of R/I. We show that P is a ϕ-primary ideal of R. Let $a, b \in R$ be such that $ab \in P - \phi(P)$. Then there exist $q_1, q_2 \in Q$ such that $a \in q_1 + I$ and $b \in q_2 + I$. Thus there are $x, y \in I$ such that $a = q_1+x$ and $b = q_2+y$. Since $q_1q_2+q_1y+q_2x+xy = (q_1+x)(q_2+y) = ab \in P-\phi(P)$ and P and $\phi(P)$ are subtractive extensions of I, we acquire $q_1q_2 \in P - \phi(P)$. By Theorem 2.2.19, we obtain $(q_1 + I)(q_2 + I) \in P/I - \phi_I(P/I)$. Since P/I is ϕ_I-primary, $(q_1 + I) \in P/I$ or $(q_2 + I) \in \sqrt{P/I} = \sqrt{P}/I$ by Proposition 2.2.18. Thus $q_1 \in P$ or $q_2 \in \sqrt{P}$ by Lemma 2.2.15. Hence $a = q_1 + x \in P$ or $b = q_2 + y \in \sqrt{P}$. Therefore, P is a ϕ-primary ideal of R. ☐

Example 3.3.2. Consider the semiring $\mathbb{Z}_0^+$. Let $P = 4\mathbb{Z}_0^+$ and $I = 12\mathbb{Z}_0^+$. Then P is a k-ideal of $\mathbb{Z}_0^+$ containing I and I is a Q-ideal of $\mathbb{Z}_0^+$ where $Q = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Thus P is a subtractive extension of I. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(J) = 3\mathbb{Z}_0^+$ if J is a subtractive extension of I and $\phi(J) = J$ otherwise for all $J \in \mathscr{I}(\mathbb{Z}_0^+)$. Moreover, we define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\varnothing\}$ by $\phi_I(J/I) = (3\mathbb{Z}_0^+)/I$ for each ideal J of R where J is a subtractive extension of I. Hence $\mathbb{Z}_0^+$ is a semiring with ϕ satisfying the property $(*)$. Since P is a primary ideal by Proposition 3.1.2, P is a ϕ-primary ideal of R. Therefore, $P/I = 4\mathbb{Z}_0^+/12\mathbb{Z}_0^+$ is a ϕ_I-primary ideal of the quotient semiring $\mathbb{Z}_0^+/12\mathbb{Z}_0^+$.

Corollary 3.3.3. *Let R be a semiring with ϕ satisfying the property $(*)$, I a Q-ideal of R. Then I is a ϕ-primary ideal of R if and only if the zero ideal of R/I is a ϕ_I-primary ideal.*

Proof. The result follows from the fact that I is a Q-ideal, then I is a k-ideal by Proposition 2.2.5 and so I is a subtractive extension of itself. □

Toward the end of this section, we deal with semirings of fractions. First, we would like to recall that for an ideal I of a semiring R, the ideal generated by I of R_S where S is the set of all multiplicatively cancellable elements of R is the set of all finite sums $a_1s_1 + a_2s_2 + \cdots + a_ns_n$ where $a_i \in I$ and $s_i \in R_S$ and is denoted by IR_S; in addition, we know that $x \in IR_S$ if and only if it can be written in the form $x = \dfrac{a}{c}$ for some $a \in I$ and $c \in S$. Recall further that for an ideal J of R_S, the contraction of J in R is $J \cap R = \left\{ r \in R \,\middle|\, \dfrac{r}{1} \in J \right\}$ which is an ideal of R.

Let R be a semiring with ϕ and I an ideal of R. Then either $\phi(I)$ is an ideal of R or $\phi(I) = \varnothing$. If $\phi(I)$ is an ideal, then $\phi(I)R_S$ is the set of finite sums given as above. Otherwise, $\phi(I)R_S = \varnothing$.

Proposition 3.3.4. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I a ϕ-primary ideal of R with $\phi(I) \subsetneq I$ and $\sqrt{I} \cap S = \varnothing$. If $IR_S \neq \phi(I)R_S$, then $IR_S \cap R = I$.*

Proof. Assume that $IR_S \neq \phi(I)R_S$. Since $I \subseteq IR_S \cap R$, it remains to show that $IR_S \cap R \subseteq I$. Let $x \in IR_S \cap R$. Then $\dfrac{x}{1} \in IR_S$. Thus there exist $a \in I$ and $s \in S$ such that $\dfrac{x}{1} = \dfrac{a}{s}$. Hence $xs = a \in I$. If $xs \notin \phi(I)$, then $x \in I$ because I is ϕ-primary and $\sqrt{I} \cap S = \varnothing$. So assume that $xs \in \phi(I)$. Then $\dfrac{x}{1} = \dfrac{xs}{1s} \in \phi(I)R_S$, and hence $x \in \phi(I)R_S \cap R$. Then $IR_S \cap R \subseteq I$ or $IR_S \cap R \subseteq \phi(I)R_S \cap R$. Since $I \subseteq IR_S \cap R$ and $\phi(I)R_S \cap R \subseteq IR_S \cap R$, we obtain $I = IR_S \cap R$ or $\phi(I)R_S \cap R = IR_S \cap R$. If $\phi(I)R_S \cap R = IR_S \cap R$, then $\phi(I)R_S = IR_S$ and leads to a contradiction. Therefore, $IR_S \cap R = I$. □

We end this chapter with relationships between ϕ-primary ideals of semirings and ϕ-primary ideals of semirings of fractions. Recall that for a semiring R with ϕ, we define $\phi_S : \mathscr{I}(R_S) \to \mathscr{I}(R_S) \cup \{\varnothing\}$ by $\phi_S(J) = \phi(J \cap R)R_S$ if $\phi(J \cap R) \in \mathscr{I}(R)$ and $\phi_S(J) = \varnothing$ if $\phi(J \cap R) = \varnothing$ for all $J \in \mathscr{I}(R_S)$.

Theorem 3.3.5. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I an ideal of R with $I \cap S = \varnothing$ and $\phi(I)R_S \subseteq \phi_S(IR_S)$. If I is a ϕ-primary ideal of R, then IR_S is a ϕ_S-primary ideal of R_S.*

Proof. Assume that I is a ϕ-primary ideal of R. Then IR_S is a proper ideal of R_S since $I \cap S = \varnothing$. Let $\frac{x}{s}, \frac{y}{t} \in R_S$ be such that $\frac{xy}{st} \in IR_S - \phi_S(IR_S)$. By Theorem 2.3.8, there is $v \in S$ such that $xyv \in I - \phi(I)$. Since I is ϕ-primary, it follows that $x \in I$ or $yv \in \sqrt{I}$ and so $\frac{x}{s} \in IR_S$ or $\frac{y}{t} = \frac{yv}{tv} \in \sqrt{I}R_S = \sqrt{IR_S}$ by Proposition 2.3.7. Therefore, IR_S is a ϕ_S-primary ideal of R_S. $\square$

CHAPTER IV
GENERALIZATIONS OF n-ABSORBING IDEALS OF SEMIRINGS

In rings, there are other ways to generalize prime ideals besides primary ideals, for instance, 2-absorbing ideals. In 2007, A. Badawi [10] introduced the concept of 2-absorbing ideals of a ring. He defined a ***2-absorbing ideal*** I of a ring R to be a proper ideal and if whenever $a, b, c \in R$ with $abc \in I$, either $ab \in I$ or $ac \in I$ or $bc \in I$. Thus every prime ideal is a 2-absorbing ideal. Nevertheless, 2-absorbing ideals need not be prime ideals. For example, $21\mathbb{Z}$ is a 2-absorbing ideal of the ring $\mathbb{Z}$ and $3 \cdot 7 = 21 \in 21\mathbb{Z}$ but $3 \notin 21\mathbb{Z}$ and $7 \notin 21\mathbb{Z}$. Then $21\mathbb{Z}$ is not a prime ideal of the ring $\mathbb{Z}$. Hence 2-absorbing ideals are generalizations of prime ideals. In 2011, D. F. Anderson and A. Badawi [2] generalized the concept of 2-absorbing ideals to n-absorbing ideals (with integer $n \geq 2$) of a ring. They defined an ***n-absorbing ideal*** I of a ring R to be a proper ideal and if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ with $x_1 x_2 \cdots x_{n+1} \in I$, then $x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. From the definition of n-absorbing ideals, it is easy to see that if n, n' are positive integers such that $n \leq n'$ and I is an n-absorbing ideal, then I is an n'-absorbing ideal. Moreover, if $n = 1$, then a 1-absorbing ideal is just a prime ideal. However, n'-absorbing ideals need not be n-absorbing ideals for any $n, n' \in \mathbb{N}$ with $n \leq n'$. For example, $42\mathbb{Z}$ is a 3-absorbing ideal but is not a 2-absorbing ideal because $2 \cdot 3 \cdot 7 = 42 \in 42\mathbb{Z}$ but $2 \cdot 3 = 6 \notin 42\mathbb{Z}$, $2 \cdot 7 = 14 \notin 42\mathbb{Z}$ and $3 \cdot 7 = 21 \notin 42\mathbb{Z}$. Therefore, n'-absorbing ideals are generalizations of n-absorbing ideals for any $n, n' \in \mathbb{N}$ with $n \leq n'$.

After that, in 2012, M. Ebrahimpour and R. Nekooei [16] introduced the concept of $(n-1, n)$-ϕ-prime ideals (with integer $n \geq 2$) of a ring. They defined an ***$(n-1, n)$-ϕ-prime ideal*** I of a ring R to be a proper ideal and if whenever

$x_1, x_2, \ldots, x_n \in R$ with $x_1x_2\cdots x_n \in I - \phi(I)$, then $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_n \in I$ for some $i \in \{1, 2, \ldots, n\}$. Then an $(n-1, n)$-ϕ-prime ideal is just an $(n-1)$-absorbing ideal if $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ is a function with $\phi[\mathscr{I}(R)] = \{\varnothing\}$.

In this chapter, we also extend n-absorbing ideals and $(n-1, n)$-ϕ-prime ideals of a ring to n-absorbing ideals and ϕ-n-absorbing ideals of a semiring. Like Chapter III, we divide this chapter into three sections. They are ϕ-n-absorbing ideals of semirings, ϕ-n-absorbing ideals in decomposable semirings and the last section is ϕ-n-absorbing ideals in quotient semirings and semirings of fractions. Some results of this chapter are parallel to the results of Chapter III. Besides, we obtain relationships between ϕ-n-absorbing ideals and ϕ-n'-absorbing ideals for any $n, n' \in \mathbb{N}$ with $n' \neq n$.

4.1 ϕ-n-Absorbing Ideals of Semirings

We start this chapter with definitions that we use throughout this chapter like Chapter III. In this chapter, we define n-absorbing ideals of semirings in the same fashion as n-absorbing ideals of rings given in [2]; moreover, we define weakly n-absorbing ideals, almost n-absorbing ideals, m-almost n-absorbing ideals and ω-n-absorbing ideals of semirings in the same manner as weakly primary ideals, almost primary ideals, m-almost primary ideals and ω-primary ideals of semirings given in Chapter III.

Let n and m be positive integers. We denote $\hat{x}_{i,n+1}$ the element of R obtained by eliminating x_i from the product $x_1x_2\cdots x_{n+1}$ where $x_1, x_2, \ldots, x_{n+1} \in R$; in addition, we denote $\hat{x}_{\{i_1,\ldots,i_m\},n+1}$ the element of R obtained by eliminating $x_{i_1}, \ldots, x_{i_m}$ from the product $x_1x_2\cdots x_{n+1}$ where $x_1, x_2, \ldots, x_{n+1} \in R$ and $\{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n+1\}$. For an ideal I of a semiring R containing $x_1, x_2, \ldots, x_{n+1}$, from now on we use the statement $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$ in stead of the statement that $x_1x_2\cdots x_{i-1}x_{i+1}\cdots x_{n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$.

Definition 4.1.1. Let R be a semiring and n a positive integer.

A proper ideal I of R is said to be ***n-absorbing*** if whenever $x_1, x_2, \ldots, x_{n+1} \in R$

and $x_1x_2\cdots x_{n+1} \in I$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Moreover, we denote 0-absorbing ideal the ideal R.

A proper ideal I of R is said to be ***weakly n-absorbing*** if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I-\{0\}$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$.

A proper ideal I of R is said to be ***almost n-absorbing*** if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - I^2$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$.

A proper ideal I of R is said to be ***m-almost n-absorbing*** ($m \in \mathbb{N}$ with $m \geq 2$) if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - I^m$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$.

A proper ideal I of R is said to be ***ω-n-absorbing*** if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2\cdots x_{n+1} \in I - \cap_{m=1}^{\infty} I^m$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$.

Because $I - \{0\}, I - I^2, I - I^n$ and $I - \cap_{i=1}^{\infty} I^i$ are equal to the empty set if I is the zero ideal, it follows that the zero ideal is a weakly n-absorbing ideal, an almost n-absorbing ideal, an m-almost n-absorbing ideal and an ω-n-absorbing ideal in the same manner as given in Chapter III. Moreover, in Chapter III, we show that the zero ideal is not a primary ideal of some semirings while it may be a primary ideal of other some semirings, so is an n-absorbing ideal.

Example 4.1.2. (1) Let n be a positive integer. Consider the semiring $\mathbb{Q}_0^+$ and its ideal $\{0\}$. Let $x_1, x_2, \ldots, x_{n+1} \in \mathbb{Q}_0^+$ be such that $x_1x_2\cdots x_{n+1} \in \{0\}$. Then there exists $x_i = 0$ for some $i \in \{1, 2, \ldots, n+1\}$. Hence $\hat{x}_{j,n+1} = 0$ where $j \in \{1, 2, \ldots, n+1\} - \{i\}$. Therefore, $\hat{x}_{j,n+1} \in \{0\}$ and so $\{0\}$ is an n-absorbing ideal of the semiring $\mathbb{Q}_0^+$.

(2) Consider the semiring $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and its ideal $\{(0,0,0)\}$. Let $a, b, c \in \mathbb{R}_0^+ - \{0\}$. Since $(a,b,0)(a,0,c)(0,b,c) = (0,0,0) \in \{(0,0,0)\}$ but $(a,b,0)(a,0,c) = (a^2,0,0) \notin \{(0,0,0)\}$, $(a,b,0)(0,b,c) = (0,b^2,0) \notin \{(0,0,0)\}$ and $(a,0,c)(0,b,c) = (0,0,c^2) \notin \{(0,0,0)\}$, it follows that $\{(0,0,0)\}$ is not a 2-absorbing ideal of the semiring $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+$.

From the definition of n-absorbing ideals, one can see that 1-absorbing ideals are just prime ideals; moreover, we call prime ideals instead of 1-absorbing ideals from now on.

From Definition 4.1.1, it is easy to see that if n and n' are positive integers such that $n \leq n'$ and I is an n-absorbing ideal (a weakly n-absorbing ideal, an almost n-absorbing ideal, an m-almost n-absorbing ideal and an ω-n-absorbing ideal), then I is an n'-absorbing ideal (a weakly n'-absorbing ideal, an almost n'-absorbing ideal, an m-almost n'-absorbing ideal and an ω-n'-absorbin g ideal). However, the converse of this statement is not true in general and we provide an example to confirm.

Example 4.1.3. Consider the semiring $\mathbb{Z}_0^+$. The ideal $36\mathbb{Z}_0^+$ is a 4-absorbing ideal of the semiring $\mathbb{Z}_0^+$ but is not a 3-absorbing ideal of the semiring $\mathbb{Z}_0^+$ because $2^2 \cdot 3^2 \in 36\mathbb{Z}_0^+$ but $2^2 \cdot 3 \notin 36\mathbb{Z}_0^+$ and $2 \cdot 3^2 \notin 36\mathbb{Z}_0^+$.

In the following proposition, we provide a result that helps us find an example of n-absorbing ideals but not $(n-1)$-absorbing ideals of the semiring $\mathbb{Z}_0^+$ more easily.

Proposition 4.1.4. *Let n be a positive integer with $n \geq 2$ and $p_1, p_2, \ldots, p_n$ prime numbers (not necessary distinct). Then $p_1 p_2 \cdots p_n \mathbb{Z}_0^+$ is an n-absorbing ideal but not an $(n-1)$-absorbing ideal of the semiring $\mathbb{Z}_0^+$.*

Proof. First, we would like to show that the ideal $p_1 p_2 \cdots p_n \mathbb{Z}_0^+$ is an n-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Let $x_1, x_2, \ldots, x_{n+1} \in \mathbb{Z}_0^+$ be such that $x_1 x_2 \cdots x_{n+1} \in p_1 p_2 \cdots p_n \mathbb{Z}_0^+$. Then $x_1 x_2 \cdots x_{n+1} = p_1 p_2 \cdots p_n a$ for some $a \in \mathbb{Z}_0^+$. Since p_i is a prime number for all $i \in \{1, 2, \ldots, n\}$, it follows that p_i is a factor of x_j for some $j \in \{1, 2, \ldots, n+1\}$. Hence there is $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{n-m}}\} \subseteq \{x_1, x_2, \ldots, x_{n+1}\}$ for some $m \in \mathbb{Z}_0^+$ and for some distinct $i_1, i_2, \ldots, i_{n-m} \in \{1, 2, \ldots, n+1\}$ such that $x_{i_1} x_{i_2} \cdots x_{i_{n-m}} = p_1 p_2 \cdots p_n h$ for some $h \in \mathbb{Z}_0^+$. By choosing all distinct $x_{i_{n-m+1}}, x_{i_{n-m+2}}, \ldots, x_{i_n} \in \{x_1, x_2, \ldots, x_{n+1}\} - \{x_{i_1}, x_{i_2}, \ldots, x_{i_{n-m}}\}$ and by multiplying, $x_{i_1} x_{i_2} \cdots x_{i_n} = (x_{i_1} x_{i_2} \cdots x_{i_{n-m}})(x_{i_{n-m+1}} x_{i_{n-m+2}} \cdots x_{i_n}) = p_1 p_2 \cdots p_n hl$ for some $l \in \mathbb{Z}_0^+$. Hence $x_{i_1} x_{i_2} \cdots x_{i_n} \in p_1 p_2 \cdots p_n \mathbb{Z}_0^+$. Therefore, $p_1 p_2 \cdots p_n \mathbb{Z}_0^+$ is an n-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

Next, it remains to show that $p_1 p_2 \cdots p_n \mathbb{Z}_0^+$ is not an $(n-1)$-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Since $p_1, p_2, \ldots, p_n \in \mathbb{Z}_0^+$ but $\hat{p}_{i,n} \notin p_1p_2\cdots p_n\mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, n\}$, the ideal $p_1p_2\cdots p_n\mathbb{Z}_0^+$ is not an $(n-1)$-absorbing ideal of the semiring $\mathbb{Z}_0^+$. $\square$

Example 4.1.5. Consider the semiring $\mathbb{Z}_0^+$.

(1) Since $32 = 2^5, 48 = 2^4 \cdot 3^1$ and $72 = 2^3 \cdot 3^2$, it follows that $32\mathbb{Z}_0^+$, $48\mathbb{Z}_0^+$ and $72\mathbb{Z}_0^+$ are 5-absorbing ideals but are not 4-absorbing ideals of $\mathbb{Z}_0^+$.

(2) Since $128 = 2^7, 288 = 2^5 \cdot 3^2$ and $1080 = 2^3 \cdot 3^3 \cdot 5^1$, it follows that $128\mathbb{Z}_0^+, 288\mathbb{Z}_0^+$ and $1080\mathbb{Z}_0^+$ are 7-absorbing ideals but are not 6-absorbing ideals of $\mathbb{Z}_0^+$.

In Chapter III, ϕ-primary ideal is the main character. Similarly, in this chapter we have a main character as well, it is a ϕ-n-absorbing ideal.

Definition 4.1.6. Let R be a semiring with ϕ and n a positive integer. A proper ideal I of R is said to be ***ϕ-n-absorbing*** if whenever $x_1, \ldots, x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I - \phi(I)$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, \ldots, n+1\}$.

From the definition of ϕ-n-absorbing ideals, we can define a function ϕ in several ways and we also can substitute n by any positive integers. This is the difference from the main characters of Chapter III which has exactly one thing that can be changed that is function ϕ. So we are interested in relationships between ϕ-n-absorbing ideals and ϕ-n'-absorbing ideals where $n, n' \in \mathbb{N}$ with $n \neq n'$. In addition, we call ϕ-prime ideals in stead of ϕ-1-absorbing ideals in the same fashion as we call prime ideals in stead of 1-absorbing ideals.

In the following result, we give the equivalent definition of ϕ-n-absorbing ideals.

Theorem 4.1.7. *Let R be a semiring with ϕ, I a proper ideal of R and n, n' positive integers with $n' > n$. Then I is a ϕ-n-absorbing ideal if and only if whenever $x_1x_2\cdots x_{n'} \in I - \phi(I)$ for any $x_1, x_2, \ldots, x_{n'} \in R$, then $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.*

Proof. First, let I be a ϕ-n-absorbing ideal of R and $x_1, x_2, \ldots, x_{n'} \in R$ be such that $x_1x_2\cdots x_{n'} = x_1x_2\cdots x_n(x_{n+1}x_{n+2}\cdots x_{n'}) \in I - \phi(I)$. Since I is a ϕ-n-absorbing ideal, $x_1x_2\cdots x_n \in I$ or $\hat{x}_{i,n}(x_{n+1}x_{n+2}\cdots x_{n'}) \in I$ for some $i \in \{1,2,\ldots,n\}$. If $x_1x_2\cdots x_n \in I$, then we are done. So we suppose that $\hat{x}_{i,n}x_{n+1}x_{n+2}\cdots x_{n'} \in I$. Since $x_1x_2\cdots x_{n'} \notin \phi(I)$, we obtain $\hat{x}_{i,n}x_{n+1}x_{n+2}\cdots x_{n'} = \hat{x}_{i,n}x_{n+1}(x_{n+2}\cdots x_{n'}) \in I-\phi(I)$. Thus $\hat{x}_{i,n}x_{n+1} \in I$ or $\hat{x}_{\{i,j\},n+1}(x_{n+2}\cdots x_{n'}) \in I$ for some $j \in \{1,2,\ldots,n+1\}-\{i\}$ because I is a ϕ-n-absorbing ideal. If $\hat{x}_{i,n}x_{n+1} \in I$, then we are done. If not, we continue this process, and hence we obtain $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1,i_2,\ldots,i_n \in \{1,2,\ldots,n'\}$.

Conversely, the proof is clear by choosing $n' = n+1$. □

Corollary 4.1.8. *Let R be a semiring, I a proper ideal of R and n, n' positive integers with $n' > n$. Then I is an n-absorbing ideal if and only if whenever $x_1x_2\cdots x_{n'} \in I$ for $x_1,x_2,\ldots,x_{n'} \in R$, then $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1,i_2,\ldots,i_n \in \{1,2,\ldots,n'\}$.*

Proof. The proof is completed by the fact that an n-absorbing ideal is just a $\phi_\varnothing$-n-absorbing ideal. □

We know that n-absorbing ideals imply n'-absorbing ideals for any $n, n' \in \mathbb{N}$ with $n \leq n'$; moreover, this statement is also true for ϕ-n-absorbing ideals as shown in the next result.

Proposition 4.1.9. *Let R be a semiring with ϕ, I a proper ideal of R and n a positive integer. If I is a ϕ-n-absorbing ideal, then I is a ϕ-n'-absorbing ideal for all $n' \in \mathbb{N}$ with $n' \geq n$.*

Proof. Assume that I is a ϕ-n-absorbing ideal of R. Let $n' \in \mathbb{N}$ be such that $n' \geq n$. Note that, if $n' = n$, then there is nothing to do. So we assume that $n' > n$. Let $x_1,x_2,\ldots,x_{n'+1} \in R$ be such that $x_1x_2\cdots x_{n'+1} \in I-\phi(I)$. We obtain from Theorem 4.1.7 that $x_{i_1}x_{i_2}\cdots x_{i_n} \in I$ for some distinct $i_1,i_2,\ldots,i_n \in \{1,2,\ldots,n'+1\}$. By choosing all distinct

$$i_{n+1}, i_{n+2}, \ldots, i_{n'} \in \{1,2,\ldots,n'+1\} - \{i_1,i_2,\ldots,i_n\}$$

and by multiplying, $x_{i_1}x_{i_2}\cdots x_{i_{n'}} = (x_{i_1}x_{i_2}\cdots x_{i_n})\left(x_{i_{n+1}}x_{i_{n+2}}\cdots x_{n'}\right) \in I$. Hence I is a ϕ-n'-absorbing ideal of R. Therefore, I is a ϕ-n'-absorbing ideal for all $n' \geq n$. □

The converse of Proposition 4.1.9 is not true in the same fashion as n'-absorbing ideals do not imply n-absorbing ideals where $n', n \in \mathbb{N}$ with $n' \geq n$.

Example 4.1.10. Consider the semiring $\mathbb{Z}_0^+$ with ϕ_2 and its ideal $24\mathbb{Z}_0^+$. Recall that ϕ_2 is the function defined by $\phi_2(I) = I^2$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Since $24 = 2^3 \cdot 3^1$, it follows that $24\mathbb{Z}_0^+$ is a 4-absorbing ideal of $\mathbb{Z}_0^+$ by Proposition 4.1.4. Hence $24\mathbb{Z}_0^+$ is a ϕ_2-4-absorbing ideal of $\mathbb{Z}_0^+$. Since $2 \cdot 3 \cdot 4 = 24 \in 24\mathbb{Z}_0^+ - \phi(24\mathbb{Z}_0^+) = 24\mathbb{Z}_0^+ - (24\mathbb{Z}_0^+)^2 = 24\mathbb{Z}_0^+ - 576\mathbb{Z}_0^+$ but $2 \cdot 3 = 6 \notin 24\mathbb{Z}_0^+$, $2 \cdot 4 = 8 \notin 24\mathbb{Z}_0^+$ and $3 \cdot 4 = 12 \notin 24\mathbb{Z}_0^+$. Therefore, $24\mathbb{Z}_0^+$ is not a ϕ_2-2-absorbing ideal of $\mathbb{Z}_0^+$.

Corollary 4.1.11. *Let R be a semiring with ϕ. Then every ϕ-prime ideal of R is a ϕ-n-absorbing ideal of R for all $n \in \mathbb{N}$.*

Recall that the radical of an ideal I of a semiring R is denoted by $\sqrt{I}$ and $\sqrt{I} = \{a \in R \,|\, a^n \in I \text{ for some } n \in \mathbb{N}\}$ is an ideal of R.

Lemma 4.1.12. *Let R be a semiring with ϕ, n a positive integer and I a proper ideal of R with $\phi(\sqrt{I}) = \sqrt{(\phi(I))}$. If I is a ϕ-n-absorbing ideal, then $x^n \in I - \phi(I)$ for all $x \in \sqrt{I} - \phi(\sqrt{I})$.*

Proof. Assume that I is a ϕ-n-absorbing ideal of R. Let $x \in \sqrt{I} - \phi(\sqrt{I})$. Then $x \in \sqrt{I} - \sqrt{\phi(I)}$ because $\phi(\sqrt{I}) = \sqrt{(\phi(I))}$. Since $x \notin \sqrt{\phi(I)}$, we obtain $x^m \notin \phi(I)$ for all positive integer m. Since $x \in \sqrt{I}$, we acquire $x^l \in I$ for some $l \in \mathbb{N}$. Then $x^l \in I - \phi(I)$. If $l \leq n$, then $x^n \in I - \phi(I)$. Suppose that $l > n$. Thus $x^n \in I$ by Theorem 4.1.7. Therefore, $x^n \in I - \phi(I)$. □

In the following example, we show that there is a ϕ-n-absorbing ideal I of a semiring with ϕ such that $\phi(\sqrt{I}) = \sqrt{\phi(I)}$; in addition, we also provide an example of ϕ-n-absorbing ideal of R such that $\sqrt{\phi(I)} \neq \phi(\sqrt{I})$.

Example 4.1.13. Consider the semiring $\mathbb{Z}_0^+$ and its ideals $125\mathbb{Z}_0^+$ and $45\mathbb{Z}_0^+$. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(I) = 3I$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Note that $125\mathbb{Z}_0^+$ and $45\mathbb{Z}_0^+$ are ϕ-3-absorbing ideals of the semiring $\mathbb{Z}_0^+$ because $125 = 5^3$ and $45 = 3^2 \cdot 5$. Since $\sqrt{\phi(125\mathbb{Z}_0^+)} = \sqrt{3(125\mathbb{Z}_0^+)} = \sqrt{375\mathbb{Z}_0^+} = \sqrt{3 \cdot 5^3\mathbb{Z}_0^+} = 15\mathbb{Z}_0^+$ and $\phi(\sqrt{125\mathbb{Z}_0^+}) = \phi(5\mathbb{Z}_0^+) = 3(5\mathbb{Z}_0^+) = 15\mathbb{Z}_0^+$, we obtain $\sqrt{\phi(125\mathbb{Z}_0^+)} = \phi(\sqrt{125\mathbb{Z}_0^+})$. Next, we consider the ideal $45\mathbb{Z}_0^+$. Because $\sqrt{\phi(45\mathbb{Z}_0^+)} = \sqrt{3(45\mathbb{Z}_0^+)} = \sqrt{135\mathbb{Z}_0^+} = \sqrt{3^3 \cdot 5\mathbb{Z}_0^+} = 15\mathbb{Z}_0^+$ but $\phi(\sqrt{45\mathbb{Z}_0^+}) = \phi(\sqrt{3^2 \cdot 5\mathbb{Z}_0^+}) = \phi(15\mathbb{Z}_0^+) = 3(15\mathbb{Z}_0^+) = 45\mathbb{Z}_0^+$, it follows that $\sqrt{\phi(45\mathbb{Z}_0^+)} \neq \phi(\sqrt{45\mathbb{Z}_0^+})$.

Proposition 4.1.14. *Let R be a semiring with ϕ, n a positive integer and I a proper ideal of R such that $\phi(\sqrt{I}) = \sqrt{(\phi(I))}$. If I is a ϕ-n-absorbing ideal, then $\sqrt{I}$ is a ϕ-n-absorbing ideal.*

Proof. Assume that I is a ϕ-n-absorbing ideal. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in \sqrt{I} - \phi(\sqrt{I})$. Then $(x_1x_2\cdots x_{n+1})^n \in I - \phi(I)$ by Lemma 4.1.12. That is $x_1^n x_2^n \cdots x_{n+1}^n \in I - \phi(I)$. Since I is ϕ-n-absorbing, $\hat{x}_{i,n+1}^n \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Hence $\hat{x}_{i,n+1} \in \sqrt{I}$. Therefore, $\sqrt{I}$ is a ϕ-n absorbing ideal of R. $\square$

Next, we would like to show that ϕ-primary ideals and ϕ-n-absorbing ideals do not imply each other as shown in the next example.

Example 4.1.15. Consider the semiring $\mathbb{Z}_0^+$ with ϕ_0. Recall that ϕ_0 is the function defined by $\phi_0(I) = \{0\}$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$.

(1) Consider the ideal $64\mathbb{Z}_0^+$ of $\mathbb{Z}_0^+$. We know that $64\mathbb{Z}_0^+$ is a primary ideal of $\mathbb{Z}_0^+$ because $64 = 2^6$, so it is a ϕ_0-primary ideal of $\mathbb{Z}_0^+$. Since $2^6 \in 64\mathbb{Z}_0^+ - \{0\} = 64\mathbb{Z}_0^+ - \phi_0(64\mathbb{Z}_0^+)$ but $2^5 \notin 64\mathbb{Z}_0^+$, it follows that $64\mathbb{Z}_0^+$ is not a ϕ_0-5-absorbing ideal of $\mathbb{Z}_0^+$.

(2) Consider the ideal $70\mathbb{Z}_0^+$ of $\mathbb{Z}_0^+$. We know that $70\mathbb{Z}_0^+$ is a 3-absorbing ideal because $70 = 2 \cdot 5 \cdot 7$, so it is a ϕ_0-3-absorbing ideal. Since $14 \cdot 5 = 70 \in 70\mathbb{Z}_0^+ - \{0\} = 70\mathbb{Z}_0^+ - \phi_0(70\mathbb{Z}_0^+)$ but $14 \notin 70\mathbb{Z}_0^+$ and $5^m \notin 70\mathbb{Z}_0^+$ for all $m \in \mathbb{N}$, i.e., $14 \notin 70\mathbb{Z}_0^+$ and $5 \notin \sqrt{70\mathbb{Z}_0^+}$, it follows that $70\mathbb{Z}_0^+$ is not a ϕ_0-primary ideal of $\mathbb{Z}_0^+$.

Nevertheless, in case of $n = 1$, every ϕ-prime ideal (ϕ-1-absorbing ideal) is, in fact, a ϕ-primary ideal.

Proposition 4.1.16. *Let R be a semiring with ϕ. Then every ϕ-prime ideal of R is a ϕ-primary ideal of R.*

Proof. Assume that I is a ϕ-prime ideal of R. Let $a, b \in R$ be such that $ab \in I - \phi(I)$. Then $a \in I$ or $b \in I \subseteq \sqrt{I}$. Therefore, I is a ϕ-primary ideal. $\square$

Now, we know that every ϕ-prime ideal (prime ideal) is a ϕ-primary ideal (primary ideal). However, the converse of this statement is not true as shown in the next example.

Example 4.1.17. Consider the semiring $\mathbb{Z}_0^+$.

(1) From Proposition 3.1.2, the ideal $9\mathbb{Z}_0^+$ is a primary ideal of the semiring $\mathbb{Z}_0^+$. Since $3 \cdot 3 = 9 \in 9\mathbb{Z}_0^+$ but $3 \notin 9\mathbb{Z}_0^+$, it follows that $9\mathbb{Z}_0^+$ is not a prime ideal of the semiring $\mathbb{Z}_0^+$.

(2) The ideal $49\mathbb{Z}_0^+$ is a primary ideal of the semiring $\mathbb{Z}_0^+$ by Proposition 3.1.2. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(I) = I \cap 5\mathbb{Z}_0^+$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Since $49\mathbb{Z}_0^+$ is a primary ideal, it is a ϕ-primary ideal. We would like to show that $49\mathbb{Z}_0^+$ is not a ϕ-prime ideal of the semiring $\mathbb{Z}_0^+$. Since $7 \cdot 7 = 49 \in 49\mathbb{Z}_0^+ - \phi(49\mathbb{Z}_0^+) = 49\mathbb{Z}_0^+ - (49\mathbb{Z}_0^+ \cap 5\mathbb{Z}_0^+) = 49\mathbb{Z}_0^+ - 245\mathbb{Z}_0^+$ but $7 \notin 49\mathbb{Z}_0^+$, it follows that $49\mathbb{Z}_0^+$ is not a ϕ-prime ideal of the semiring $\mathbb{Z}_0^+$ as desired.

In rings and semirings, one can show that if I is a primary ideal, then $\sqrt{I}$ is a prime ideal. This leads us to consider in sense of ϕ-primary ideals and ϕ-prime ideals of semirings. In the next proposition, we show that if I is a ϕ-primary ideal of a semiring R with ϕ under some conditions, then $\sqrt{I}$ is a ϕ-prime ideal of R.

Proposition 4.1.18. *Let R be a semiring with ϕ. If I is a ϕ-primary ideal of R with $\sqrt{\phi(I)} = \phi(\sqrt{I})$, then $\sqrt{I}$ is a ϕ-prime ideal of R, so that $\sqrt{I}$ is a ϕ-primary ideal of R.*

Proof. Suppose that I is a ϕ-primary ideal of R with $\sqrt{\phi(I)} = \phi(\sqrt{I})$. Let $a, b \in R$ be such that $ab \in \sqrt{I} - \phi(\sqrt{I})$. Then there is $n \in \mathbb{N}$ such that $(ab)^n \in I$. If

$(ab)^n \in \phi(I)$, then $ab \in \sqrt{\phi(I)} = \phi(\sqrt{I})$ which is a contradiction. Thus $a^n b^n = (ab)^n \in I - \phi(I)$. Since I is ϕ-primary, $a^n \in I$ or $b^n \in \sqrt{I}$. Hence $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Therefore, $\sqrt{I}$ is ϕ-prime so is a ϕ-primary ideal of R. □

We provide an example to confirm that there is a ϕ-primary ideal of a semiring R with ϕ such that $\sqrt{\phi(I)} = \phi(\sqrt{I})$; moreover, in this example we also provide an example of ϕ-primary ideal of R such that $\sqrt{\phi(I)} \neq \phi(\sqrt{I})$.

Example 4.1.19. Consider the semiring $\mathbb{Z}_0^+$ and its ideals $7\mathbb{Z}_0^+$ and $8\mathbb{Z}_0^+$. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(n\mathbb{Z}_0^+) = 2n\mathbb{Z}_0^+$ for all $n \in \mathbb{Z}_0^+$ and $\phi(J) = \{0\}$ otherwise. Since $7\mathbb{Z}_0^+$ and $8\mathbb{Z}_0^+$ are primary ideals of the semirimg $\mathbb{Z}_0^+$, the ideals $7\mathbb{Z}_0^+$ and $8\mathbb{Z}_0^+$ are ϕ-primary ideals of the semirimg $\mathbb{Z}_0^+$. Since $\sqrt{\phi(7\mathbb{Z}_0^+)} = \sqrt{14\mathbb{Z}_0^+} = 14\mathbb{Z}_0^+ = \phi(7\mathbb{Z}_0^+) = \phi(\sqrt{7\mathbb{Z}_0^+})$, it follows that $\sqrt{\phi(7\mathbb{Z}_0^+)} = \phi(\sqrt{7\mathbb{Z}_0^+})$. Because $\sqrt{\phi(8\mathbb{Z}_0^+)} = \sqrt{16\mathbb{Z}_0^+} = 2\mathbb{Z}_0^+$ and $\phi(\sqrt{8\mathbb{Z}_0^+}) = \phi(2\mathbb{Z}_0^+) = 4\mathbb{Z}_0^+$, we obtain that $\sqrt{\phi(8\mathbb{Z}_0^+)} \neq \phi(\sqrt{8\mathbb{Z}_0^+})$.

In the same fashion as in Chapter III, we give relationships between ϕ-n-absorbing ideals and n-absorbing ideals (weakly n-absorbing ideals, almost n-absorbing ideals, m-almost n-absorbing ideals, ω-n-absorbing ideals) by using the notation $\phi_0, \phi_1, \phi_2, \phi_n$ and ϕ_ω given in Example 3.1.5.

Example 4.1.20. Let R be a semiring and n a positive integer. Then

(1) I is a $\phi_\varnothing$-n-absorbing ideal if and only if I is a n-absorbing ideal,

(2) I is a ϕ_0-n-absorbing ideal if and only if I is a weakly n-absorbing ideal,

(3) I is a ϕ_1-n-absorbing ideal if and only if I is a proper ideal,

(4) I is a ϕ_2-n-absorbing ideal if and only if I is an almost n-absorbing ideal,

(5) I is a ϕ_m-n-absorbing ideal if and only if I is an m-almost n-absorbing ideal, and

(6) I is a ϕ_ω-n-absorbing ideal if and only if I is an ω-n-absorbing ideal.

Almost all of the results that we show from now on are parallel to the results of Chapter III.

Proposition 4.1.21. *Let R be a semiring, n a positive integer, I a proper ideal of R and $\varphi_1 \leq \varphi_2$ where φ_1 and φ_2 are functions from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$. If I is a φ_1-n-absorbing ideal, then I is a φ_2-n-absorbing ideal.*

Proof. The proof is similar to that of Proposition 3.1.6. □

Corollary 4.1.22. *Let I be a proper ideal of a semiring and $n, m \in \mathbb{N}$ with $m \geq 2$. Consider the following statements:*

(1) I is an n-absorbing ideal.

(2) I is a weakly n-absorbing ideal.

(3) I is an ω-n-absorbing ideal.

(4) I is an $(m+1)$-almost n-absorbing ideal.

(5) I is an m-almost n-absorbing ideal.

(6) I is an almost n-absorbing ideal.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

From the above corollary, we know that ω-n-absorbing ideals imply m-almost n-absorbing ideals. In the next proposition, we would like to show that if I is an m-almost n-absorbing ideal for all $m \geq 2$, then I is an ω-n-absorbing ideal.

Proposition 4.1.23. *Let R be a semiring, n a positive integer and I a proper ideal of R. Then I is an ω-n-absorbing ideal if and only if I is an m-almost n-absorbing ideal for all $m \geq 2$.*

Proof. The proof for the first direction is clear by Corollary 4.1.22.

Conversely, the proof is similar to one of Proposition 3.1.8. □

From Chapter III, we show that being the k-ideals of I and $\phi(I)$ and $I^2 \nsubseteq \phi(I)$ are necessary conditions for making ϕ-primary ideals imply primary ideals. In the following theorem, we only change the condition $I^2 \nsubseteq \phi(I)$ to $I^{n+1} \nsubseteq \phi(I)$ in order to get the similar result.

Theorem 4.1.24. *Let R be a semiring with ϕ, n a positive integer and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal. If I is a ϕ-n-absorbing ideal with $I^{n+1} \nsubseteq \phi(I)$, then I is an n-absorbing ideal.*

Proof. Assume that I is a ϕ-n-absorbing ideal with $I^{n+1} \nsubseteq \phi(I)$. We show that I is an n-absorbing ideal of R. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in I$. If $x_1x_2\cdots x_{n+1} \in I-\phi(I)$, then $\hat{x}_{i,n+1} \in I$ for some $i \in \{1,2,\ldots,n+1\}$ because I is a ϕ-n-absorbing ideal, and hence we are done. Suppose that $x_1x_2\cdots x_{n+1} \in \phi(I)$.
Case 1: Assume that $\hat{x}_{i,n+1}I \nsubseteq \phi(I)$ for some $i \in \{1,2,\ldots,n+1\}$. Then there exists $p_1 \in I$ such that $\hat{x}_{i,n+1}p_1 \in I - \phi(I)$. Since $\phi(I)$ is a k-ideal, we obtain $\hat{x}_{i,n+1}(x_i + p_1) \in I - \phi(I)$. Since I is a ϕ-n-absorbing ideal, $\hat{x}_{i,n+1} \in I$ or $\hat{x}_{\{i,j\},n+1}(x_i + p_1) \in I$ for some $j \in \{1,2,\ldots,n+1\} - \{i\}$. Thus $\hat{x}_{i,n+1} \in I$ or $\hat{x}_{j,n+1} \in I$ because I is a k-ideal. Hence $\hat{x}_{l,n+1} \in I$ for some $l \in \{1,2,\ldots,n+1\}$.
Case 2: Assume that $\hat{x}_{i,n+1}I \subseteq \phi(I)$ for all $i \in \{1,2,\ldots,n+1\}$.

Subcase 2.1: Suppose that $\hat{x}_{\{i,j\},n+1}I^2 \nsubseteq \phi(I)$ for some $j \in \{1,2,\ldots,n+1\} - \{i\}$. Then there are $p_1, p_2 \in I$ such that $\hat{x}_{\{i,j\},n+1}p_1p_2 \notin \phi(I)$. Since $\phi(I)$ is a k-ideal, we gain $\hat{x}_{\{i,j\},n+1}(x_i + p_1)(x_j + p_2) \in I - \phi(I)$. Because I is a ϕ-n-absorbing ideal, $\hat{x}_{\{i,j\},n+1}(x_i + p_1) \in I$ or $\hat{x}_{\{i,j\},n+1}(x_j + p_2) \in I$ or $\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_j + p_2) \in I$ for some $l \in \{1,2,\ldots,n+1\} - \{i,j\}$. Hence $\hat{x}_{i,n+1} \in I$ or $\hat{x}_{j,n+1} \in I$ or $\hat{x}_{l,n+1} \in I$ because I is a k-ideal. Therefore, $\hat{x}_{h,n+1} \in I$ for some $h \in \{1,2,\ldots,n+1\}$.

Subcase 2.2: Suppose that $\hat{x}_{\{i,j\},n+1}I^2 \subseteq \phi(I)$ for all $j \in \{1,2,\ldots,n+1\} - \{i\}$.

Subcase 2.2.1: Assume that $\hat{x}_{\{i,j,l\},n+1}I^3 \nsubseteq \phi(I)$ for some $l \in \{1,2,\ldots,n+1\} - \{i,j\}$. Then $\hat{x}_{\{i,j,l\},n+1}p_1p_2p_3 \notin \phi(I)$ for some $p_1, p_2, p_3 \in I$. Since $\phi(I)$ is a k-ideal, we obtain $\hat{x}_{\{i,j,l\},n+1}(x_i+p_1)(x_j+p_2)(x_l+p_3) \in I-\phi(I)$. Then $\hat{x}_{i,n+1} \in I$ or $\hat{x}_{j,n+1} \in I$ or $\hat{x}_{l,n+1} \in I$ because I is a ϕ-n-absorbing k-ideal. Therefore, $\hat{x}_{h,n+1} \in I$ for some $h \in \{1,2,\ldots,n+1\}$.

Subcase 2.2.2: Assume that $\hat{x}_{\{i,j,l\},n+1}I^3 \subseteq \phi(I)$ for all $l \in \{1, 2, \ldots, n+1\} - \{i, j\}$.

Continue this process, it remains to show the following case.

Assume that $x_{i_1}x_{i_2}\cdots x_{i_{n+1-m}}I^m \subseteq \phi(I)$ for all $\{i_1, i_2, \ldots, i_{n+1-m}\} \in \{1, 2, \ldots, n+1\}$ for $1 \leq m \leq n$. Since $I^{n+1} \nsubseteq \phi(I)$, there exist $p_1, p_2, \ldots, p_{n+1} \in I$ such that $p_1p_2\cdots p_{n+1} \notin \phi(I)$. Then $(x_1+p_1)(x_2+p_2)\cdots(x_{n+1}+p_{n+1}) \in I - \phi(I)$. Since I is ϕ-n-absorbing, $(x_1+p_1)(x_2+p_2)\cdots(x_{i-1}+p_{i-1})(x_{i+1}+p_{i+1})\cdots(x_{n+1}+p_{n+1})$ for some $i \in \{1, 2, \ldots, n+1\}$. Hence $\hat{x}_{i,n+1} \in I$.

Therefore, from any cases, we can conclude that I is n-absorbing. □

In fact, the proof of Theorem 4.1.24 use the same idea as the proof of Theorem 3.1.12 but it is more complicated because n can be arbitrary positive integer.

Corollary 4.1.25. *Let R be a semiring, n a positive integer and I a proper k-ideal of R. If I is a ϕ-n-absorbing ideal for some ϕ with $\phi \leq \phi_{n+2}$ such that $\phi(I)$ is a k-ideal, then I is an m-almost n-absorbing ideal for all $m \geq n+1$.*

Proof. Assume that I is a ϕ-n-absorbing ideal for some ϕ with $\phi \leq \phi_{n+2}$ such that $\phi(I)$ is a k-ideal. If I is an n-absorbing ideal, then I is an m-almost n-absorbing ideal for all $m \geq n+1$. So suppose that I is not an n-absorbing ideal. Then $I^{n+1} \subseteq \phi(I)$ by Theorem 4.1.24. Thus $I^{n+1} \subseteq \phi(I) \subseteq \phi_{n+2}(I) = I^{n+2} \subseteq I^{n+1}$, and so $I^{n+1} = \phi(I) = I^{n+2}$. Hence $\phi(I) = I^m$ for all $m \geq n+1$. Therefore, I is m-almost n-absorbing for all $m \geq n+1$. □

From Corollary 4.1.25, if we consider in case of $n = 1$, then we obtain the same result as in Corollary 3.1.17.

Corollary 4.1.26. *Let R be a semiring and n a positive integer. If I is a weakly n-absorbing k-ideal but is not an n-absorbing ideal, then $I^{n+1} = \{0\}$.*

Proof. Assume that I is a weakly n-absorbing k-ideal but is not an n-absorbing ideal. Since I is a weakly n-absorbing ideal, I is ϕ_0-n-absorbing. Then we obtain $I^{n+1} \subseteq \phi_0(I) = \{0\}$ by Theorem 4.1.24. Therefore, $I^{n+1} = \{0\}$. □

The converse of Corollary 4.1.26 is not true because $\{0\}$ is an n-absorbing ideal of the semiring $\mathbb{Q}_0^+$ by Example 4.1.2 (1) and $\{0\}^{n+1} = \{0\}$ for all $n \in \mathbb{N}$.

The following result is parallel to Theorem 3.1.20.

Theorem 4.1.27. *Let R be a semiring with ϕ, n a positive integer and I a proper ideal such that $\phi(I) \subseteq I$. Then the following statements are equivalent.*

(1) I is a ϕ-n-absorbing ideal.

(2) $(I : x_1x_2\cdots x_n) = \cup_{i=1}^n (I : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n)$ for any $x_1x_2\cdots x_n \in R - I$.

Proof. To show (1) $\Rightarrow$ (2), assume that the ideal I is a ϕ-n-absorbing ideal. Let $x_1, x_2, \ldots, x_n \in R$ be such that $x_1x_2\cdots x_n \in R - I$. Let $y \in (I : x_1x_2\cdots x_n)$. Then $x_1x_2\cdots x_ny \in I$. If $x_1x_2\cdots x_ny \in I - \phi(I)$, then $\hat{x}_{i,n}y \in I$ for some $i \in \{1, 2, \ldots, n\}$ because $x_1x_2\cdots x_n \notin I$. Hence $y \in (I : \hat{x}_{i,n})$. Otherwise, we assume that $x_1x_2\cdots x_ny \in \phi(I)$. Thus $y \in (\phi(I) : x_1x_2\cdots x_n)$. This shows that

$$(I : x_1x_2\cdots x_n) \subseteq \cup_{i=1}^n (I : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n).$$

On the other hand, we gain $(\phi(I) : x_1x_2\cdots x_n) \subseteq (I : x_1x_2\cdots x_n)$ because $\phi(I) \subseteq I$. Let $y \in (I : \hat{x}_{i,n})$ for some $i \in \{1, 2, \ldots, n\}$. Thus $\hat{x}_{i,n}y \in I$, and so $x_1x_2\cdots x_ny \in I$. Hence $y \in (I : x_1x_2\cdots x_n)$. Therefore,

$$\cup_{i=1}^n (I : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n) \subseteq (I : x_1x_2\cdots x_n).$$

So, we can conclude that $(I : x_1x_2\cdots x_n) = \cup_{i=1}^n (I : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n)$.

To show (2) $\Rightarrow$ (1), suppose that (2) holds. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in I - \phi(I)$. If $x_1x_2\cdots x_n \in I$, then we are done. Suppose that $x_1x_2\cdots x_n \notin I$. By (2), it follows that

$$(I : x_1x_2\cdots x_n) = \cup_{i=1}^n (I : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n).$$

Then $x_{n+1} \in (I : x_1x_2\cdots x_n) - (\phi(I) : x_1x_2\cdots x_n)$ since $x_1x_2\cdots x_{n+1} \in I - \phi(I)$. Hence $x_{n+1} \in (I : \hat{x}_{i,n})$ for some $i \in \{1, 2, \ldots, n\}$, and so $\hat{x}_{i,n}x_{n+1} \in I$. Therefore, I is a ϕ-n-absorbing ideal. □

Next, we are interested in the case $n = 1$. We obtain the result which is parallel to Theorem 3.1.21 because ϕ-prime ideals and ϕ-primary ideals have a similar structure.

Proposition 4.1.28. *Let R be a semiring with ϕ and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal and $\phi(I) \subseteq I$. The following statements are equivalent.*

(1) I is a ϕ-prime ideal.

(2) For any $x \in R - I, (I : x) = I \cup (\phi(I) : x)$.

(3) For any $x \in R - I, (I : x) = I$ or $(I : x) = (\phi(I) : x)$.

(4) For ideals A and B of R, $AB \subseteq I$ and $AB \nsubseteq \phi(I)$ imply $A \subseteq I$ or $B \subseteq I$.

Proof. To show (1) $\Rightarrow$ (2), suppose that I is a ϕ-prime ideal. Let $x \in R - I$. Since $I \subseteq (I : x)$ and $(\phi(I) : x) \subseteq (I : x)$, we obtain $I \cup (\phi(I) : x) \subseteq (I : x)$. Let $a \in (I : x)$. Then $ax \in I$. If $ax \notin \phi(I)$, then $a \in I$ because I is a ϕ-prime and $x \in R - I$. So, we assume that $ax \in \phi(I)$, then $a \in (\phi(I) : x)$. Hence $(I : x) \subseteq I \cup (\phi(I) : x)$. Therefore, $(I : x) = I \cup (\phi(I) : x)$.

To show (2) $\Rightarrow$ (3), assume that the statement (2) holds. Let $x \in R - I$. Since I and $\phi(I)$ are k-ideals, $(I : x)$ and $(\phi(I) : x)$ are k-ideals. It follows that $(I : x) = I$ or $(I : x) = (\phi(I) : x)$ by Proposition 2.1.13.

To show (3) $\Rightarrow$ (4), suppose that the statement (3) holds. Assume that A and B are ideals of R such that $AB \subseteq I$. Assume further that $A \nsubseteq I$ and $B \nsubseteq I$. We would like to show that $AB \subseteq \phi(I)$. Let $a \in A$.

Case 1: Assume that $a \notin I$. Then $(I : a) = I$ or $(I : a) = (\phi(I) : a)$ by (3). Since $AB \subseteq I$, we obtain $aB \subseteq I$. Thus $B \subseteq (I : a)$. Since $B \nsubseteq I$ but $B \subseteq (I : a)$, we have that $(I : a) \neq I$. Hence $(I : a) = (\phi(I) : a)$. Then $B \subseteq (I : a) = (\phi(I) : a)$, and so $aB \subseteq \phi(I)$.

Case 2: Assume that $a \in I$. Since $A \nsubseteq I$, there is $a' \in A - I$. Then $a'B \subseteq \phi(I)$ is obtained similarly to the previous case. Note that $a + a' \in A$ because $a, a' \in A$. If $a + a' \in I$, then $a' \in I$ because $a \in I$ and I is a k-ideal which is a contradiction. Hence $a + a' \in A - I$, and so $(a + a')B \subseteq \phi(I)$ is obtained. Let $b \in B$. Then

$a'b, ab+a'b \in \phi(I)$ because $a'B, (a+a')B \subseteq \phi(I)$. Since $\phi(I)$ is a k-ideal, $ab \in \phi(I)$. Hence $aB \subseteq \phi(I)$.

All cases show that $aB \subseteq \phi(I)$. Therefore, $AB \subseteq \phi(I)$ because a is an arbitrary element of A.

To show (4) $\Rightarrow$ (1), assume that the statement (4) holds. Let $x, y \in R$ be such that $xy \in I - \phi(I)$. Then $\langle x \rangle \langle y \rangle \subseteq I$. If $\langle x \rangle \langle y \rangle \subseteq \phi(I)$, then $xy \in \langle x \rangle \langle y \rangle \subseteq \phi(I)$ which is a contradiction. Thus $\langle x \rangle \langle y \rangle \nsubseteq \phi(I)$. Hence $x \in \langle x \rangle \subseteq I$ or $y \in \langle y \rangle \subseteq I$ by (4). Therefore, I is a ϕ-prime ideal. □

From the assumption of Proposition 4.1.28 that I is a proper k-ideal of a semiring R with ϕ such that $\phi(I)$ is a k-ideal with $\phi(I) \subseteq I$ and by the fact that $\phi(I) \subseteq \sqrt{\phi(I)}$; however, in case that I is a ϕ-prime ideal but is not a prime ideal, we obtain $I\sqrt{\phi(I)} \subseteq \phi(I)$.

Corollary 4.1.29. *Let R be a semiring with ϕ and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal and $\phi(I) \subseteq I$. If I is a ϕ-prime ideal but is not a prime ideal, then $I\sqrt{\phi(I)} \subseteq \phi(I)$.*

Proof. Assume that I is a ϕ-prime ideal but is not a prime ideal. Let $x \in \sqrt{\phi(I)}$. We would like to show that $Ix \subseteq \phi(I)$. If $x \in I$, then $Ix \subseteq I^2 \subseteq \phi(I)$ by Theorem 4.1.24. So suppose that $x \notin I$. By Proposition 4.1.28, we have $(I : x) = I$ or $(I : x) = (\phi(I) : x)$. We claim that $(I : x) \neq I$. Suppose that $(I : x) = I$. Since $x \in \sqrt{\phi(I)}$, there exists $m \in \mathbb{N}$ such that $x^m \in \phi(I)$. By well-ordering principle, there is the smallest integer n such that $x^n \in \phi(I) \subseteq I$. Since $x \notin I$, we obtain $n > 1$. Then $x^{n-1} \in (I : x) = I$. Hence $x^{n-1} \in I - \phi(I)$. Since I is a ϕ-prime ideal, $x \in I$ which is a contradiction. Thus $(I : x) \neq I$ as desired. Then we obtain $(I : x) = (\phi(I) : x)$. Hence $I \subseteq (I : x) = (\phi(I) : x)$, and so $Ix \subseteq \phi(I)$. Since x is an arbitrary element in $\sqrt{\phi(I)}$, we obtain $\{ab \,|\, a \in I \text{ and } b \in \sqrt{\phi(I)}\} \subseteq \phi(I)$. Therefore, $I\sqrt{\phi(I)} = \{\sum_{i=1}^{n} a_i b_i \,|\, a_i \in I \text{ and } b_i \in \sqrt{\phi(I)}\} \subseteq \phi(I)$ because $\phi(I)$ is an ideal. □

The converse of Corollary 4.1.29 is not true in general and we provide an example to confirm this statement as follows.

Example 4.1.30. Consider the semiring $R = \mathbb{Z}_0^+$ and the ideal $I = 5\mathbb{Z}_0^+$ of R. Then I is a k-ideal of R; in addition, $\sqrt{I} = \sqrt{5\mathbb{Z}_0^+} = 5\mathbb{Z}_0^+$ by Proposition 2.1.25. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(J) = J \cap 2\mathbb{Z}_0^+$ for all $J \in \mathscr{I}(\mathbb{Z}_0^+)$. Then $\phi(I) = 5\mathbb{Z}_0^+ \cap 2\mathbb{Z}_0^+ = 10\mathbb{Z}_0^+ \subseteq 5\mathbb{Z}_0^+ = I$. By Proposition 2.1.25, we obtain $\sqrt{10\mathbb{Z}_0^+} = 10\mathbb{Z}_0^+$. Hence $I\sqrt{\phi(I)} = 5\mathbb{Z}_0^+\sqrt{10\mathbb{Z}_0^+} = (5\mathbb{Z}_0^+)(10\mathbb{Z}_0^+) = 50\mathbb{Z}_0^+ \subseteq 10\mathbb{Z}_0^+ = \phi(5\mathbb{Z}_0^+) = \phi(I)$. Since $I\sqrt{\phi(I)} \subseteq \phi(I)$ and $I = 5\mathbb{Z}_0^+$ is a prime ideal of R and then I is a ϕ-prime ideal of R, the converse of Corollary 4.1.29 is not true.

We end this section with the result that is parallel to the Theorem 3.1.22.

Theorem 4.1.31. *Let R be a strongly Euclidean semiring, n a positive integer and $a \in R$ such that $(\langle a\rangle^2 : a) = \langle a\rangle$. Then $\langle a\rangle$ is a ϕ-n-absorbing ideal for some ϕ with $\phi \leq \phi_2$ if and only if $\langle a\rangle$ is an n-absorbing ideal.*

Proof. If $\langle a\rangle$ is an n-absorbing ideal, then $\langle a\rangle$ is a ϕ-n-absorbing ideal for any ϕ. So we assume that $\langle a\rangle$ is a ϕ-n-absorbing ideal for some ϕ with $\phi \leq \phi_2$. Then $\langle a\rangle$ is a ϕ_2-n-absorbing ideal. We would like to show that $\langle a\rangle$ is an n-absorbing ideal. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in \langle a\rangle$. If $x_1x_2\cdots x_{n+1} \in \langle a\rangle - \langle a\rangle^2$, then $\hat{x}_{i,n+1} \in \langle a\rangle$ for some $i \in \{1, 2, \ldots, n+1\}$ because $\langle a\rangle$ is a ϕ_2-n-absorbing ideal. So we can assume that $x_1x_2\cdots x_{n+1} \in \langle a\rangle^2$. Since R is a strongly Euclidean semiring, $\langle a\rangle$ and $\langle a\rangle^2$ are k-ideals. Now we have $(x_1 + a)x_2\cdots x_{n+1} = x_1x_2\cdots x_{n+1} + ax_2x_3\cdots x_{n+1} \in \langle a\rangle$.

Case 1: Assume that $(x_1 + a)x_2\cdots x_{n+1} \in \langle a\rangle - \langle a\rangle^2$. Since $\langle a\rangle$ is ϕ_2-n-absorbing, $x_2x_3\cdots x_{n+1} \in \langle a\rangle$ or $(x_1 + a)\hat{x}_{i,n+1} \in \langle a\rangle$ for some $i \in \{2, 3, \ldots n+1\}$. Hence $x_2x_3\cdots x_{n+1} \in \langle a\rangle$ or $x_1\hat{x}_{i,n+1} \in \langle a\rangle$ because $\langle a\rangle$ is a k-ideal.

Case 2: Assume that $(x_1 + a)x_2\cdots x_{n+1} \in \langle a\rangle^2 = \langle a^2\rangle$. Since $\langle a^2\rangle$ is a k-ideal and $x_1x_2\cdots x_{n+1}, x_1x_2\cdots x_{n+1} + ax_2x_3\cdots x_{n+1} \in \langle a^2\rangle$, we obtain $ax_2x_3\cdots x_{n+1} \in \langle a^2\rangle$. Thus $x_2x_3\cdots x_{n+1} \in (\langle a\rangle^2 : a) = \langle a\rangle$.

Therefore, $\langle a\rangle$ is an n-absorbing k-ideal. □

4.2 ϕ-n-Absorbing Ideals in Decomposable Semirings

In this section, we not only investigate n-absorbing ideals, weakly n-absorbing ideals and ϕ-n-absorbing ideals of decomposable semirings in the same sense as Section 3.2 but also obtain the different results from Chapter III such as Theorem 4.2.9 and Theorem 4.2.10.

We begin with some results which are parallel to the results in Section 3.2. The following proposition is parallel to Proposition 3.2.1.

Proposition 4.2.1. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ where $m, n \in \mathbb{N}$ with $m \geq n+1$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper ideal of R. If I is a weakly n-absorbing ideal, then $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$.*

Proof. Assume that I is a weakly n-absorbing ideal. Since I is a nonzero ideal, there is $(x_1, x_2, \ldots, x_m) \in I$ such that $(x_1, x_2, \ldots, x_m) \neq (0, 0, \ldots, 0)$. Then

$$\begin{aligned}(0, 0, \ldots, 0) &\neq (x_1, x_2, \ldots, x_m) \\ &= (x_1, 1, \ldots, 1)(1, x_2, 1, \ldots, 1) \cdots (1, \ldots, 1, x_{n+1}, \ldots, x_m) \in I.\end{aligned}$$

Thus $(x_1, x_2, \ldots, x_n, 1, 1, \ldots, 1) \in I$ or $(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_{n+1}, \ldots, x_m) \in I$ for some $i \in \{1, 2, \ldots, n\}$ because I is a weakly n-absorbing ideal. Hence $1 \in I_i$ for some $i \in \{1, 2, \ldots, m\}$. Therefore, $I_i = R_i$. □

We know that n-absorbing ideals imply weakly n-absorbing ideals but not vice versa in general. However, the converse of this statement is true if we assume those ideals are nonzero proper k-ideals.

Proposition 4.2.2. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ where $m, n \in \mathbb{N}$ with $m \geq n+1$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper k-ideal of R. Then I is a weakly n-absorbing ideal if and only if I is an n-absorbing ideal.*

Proof. Assume that I is a weakly n-absorbing ideal of R. Then $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$ by Proposition 4.2.1. Thus $I^{n+1} \neq \{0\}$. Therefore, I is an n-absorbing ideal by Corollary 4.1.26. The converse is clear by Corollary 4.1.22. □

From Proposition 4.2.2, we can conclude that weakly n-absorbing ideals and n-absorbing ideals are coincide if we provided they are nonzero proper k-ideals of decomposable semirings with m components where $m \geq n+1$. In the following theorem, we omit those conditions and add the condition that at least one $I_i = R_i$. These lead us to get that not only weakly n-absorbing ideals and n-absorbing ideals are coincide but also get that if $I = I_1 \times I_2 \times \cdots \times I_m$ is an n-absorbing ideal of $R = R_1 \times R_2 \times \cdots \times R_m$ and each I_i is a proper ideal, then I_i is an n-absorbing ideal of R_i.

Theorem 4.2.3. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with at least one $I_i = R_i$ where $i \in \{1, 2, \ldots, m\}$. Consider the following statements:*

(1) I is a weakly n-absorbing ideal of R.

(2) I is an n-absorbing ideal of R.

(3) If $I_j \neq R_j$ where $j \in \{1, 2, \ldots, m\}$, then I_j is an n-absorbing ideal of R_j.

Then (1) and (2) are equivalent and (2) implies (3).

Proof. To show (1) $\Leftrightarrow$ (2), if I is an n-absorbing ideal, then I is a weakly n-absorbing ideal. Conversely, assume that I is a weakly n-absorbing ideal of R. Since $I_i = R_i$, we obtain $I^{n+1} \neq \{0\}$. Then I is an n-absorbing ideal of R by Corollary 4.1.26.

To show (2) $\Rightarrow$ (3), assume that I is an n-absorbing ideal of R and $I_j \neq R_j$ for some $j \in \{1, 2, \ldots, m\}$. Let $x_1, x_2, \ldots, x_{n+1} \in R_j$ be such that $x_1 x_2 \cdots x_{n+1} \in I_j$. Then

$$(0, \ldots, 0, x_1, 0, \ldots, 0)(0, \ldots, 0, x_2, 0, \ldots, 0) \cdots (0, \ldots, 0, x_{n+1}, 0, \ldots, 0)$$
$$= (0, \ldots, 0, x_1 x_2 \cdots x_{n+1}, 0, \ldots, 0) \in I.$$

Since I is an n-absorbing ideal, $(0, \ldots, 0, x_1, 0, \ldots, 0) \cdots (0, \ldots, 0, x_{l-1}, 0, \ldots, 0)$ $(0, \ldots, 0, x_{l+1}, 0, \ldots, 0) \cdots (0, \ldots, 0, x_{n+1}, 0, \ldots, 0) \in I$ for some $l \in \{1, 2, \ldots, m\}$. Thus $(0, \ldots, 0, \hat{x}_{l,n+1}, 0, \ldots, 0) \in I$. Hence $\hat{x}_{l,n+1} \in I_j$. Therefore, I_j is an n-absorbing ideal of R_j. $\square$

From Theorem 4.2.3, we can conclude that if $I_1 \times I_2 \times \cdots \times I_m$ is an n-absorbing ideal (weakly n-absorbing ideal) of $R_1 \times R_2 \times \cdots \times R_m$, then I_j with $I_j \neq R_j$ is an n-absorbing ideal of R_j where $j \in \{1, 2, \ldots, m\}$. Nevertheless, the converse of this statement is not true in general as we show in the next example.

Example 4.2.4. Let $R = R_1 \times R_2 \times \cdots \times R_m = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$ and n a positive integer. Let $I_1 = p_1 p_2 \cdots p_n \mathbb{Z}_0^+$ and $I_2 = q_1 q_2 \cdots q_n \mathbb{Z}_0^+$ where $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$ are positive primes. Thus I_1 and I_2 are n-absorbing ideals of $\mathbb{Z}_0^+$. Consider

$$\begin{aligned}(p_1, 1, 1, \ldots, 1)&(p_2, q_1, 1, \ldots, 1) \cdots (p_n, q_{n-1}, 1, \ldots, 1)(1, q_n, 1, \ldots, 1) \\ &= (p_1 p_2 \cdots p_n, q_1 q_2 \cdots q_n, 1, 1, \ldots, 1) \in I_1 \times I_2 \times R_3 \times \cdots \times R_m.\end{aligned}$$

Since $\hat{p}_{i,n} \notin I_1$ and $\hat{q}_{j,n} \notin I_2$ for all $i, j \in \{1, 2, \ldots, n\}$, the ideal $I_1 \times I_2 \times R_3 \times \cdots \times R_m$ is not an n-absorbing ideal.

In the following result, we assume a stronger condition than conditions given in Theorem 4.2.3 in order to make (1), (2) and (3) be equivalent.

Theorem 4.2.5. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. The following statements are equivalent.*

(1) I is a weakly n-absorbing ideal of R.

(2) I is an n-absorbing ideal of R.

(3) I_i is an n-absorbing ideal of R_i.

Proof. We obtain (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) by Theorem 4.2.3. Thus it remains to show (3) $\Rightarrow$ (2).

Assume I_i is an n-absorbing ideal of R_i. Let $(x_{11}, \ldots, x_{1m})$, $(x_{21}, \ldots, x_{2m}), \ldots,$ $(x_{(n+1)1}, \ldots, x_{(n+1)m}) \in R$ be such that

$$(x_{11}, \ldots, x_{1m})(x_{21}, \ldots, x_{2m}) \cdots (x_{(n+1)1}, \ldots, x_{(n+1)m}) \in I.$$

Note that $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$. Thus

$$(x_{11}x_{21}\cdots x_{(n+1)1},\ldots,x_{1i}x_{2i}\cdots x_{(n+1)i},\ldots,x_{1m}x_{2m}\cdots x_{(n+1)m})\in I.$$

Since I_i is an n-absorbing ideal of R_i, we obtain $\hat{x}_{ji,(n+1)i}\in I_i$ for some $j\in\{1,2,\ldots,n+1\}$. Hence $(x_{11},\ldots,x_{1m})\cdots(x_{(j-1)1},\ldots,x_{(j-1)m})(x_{(j+1)1},\ldots,x_{(j+1)m})\cdots(x_{(n+1)1},\ldots,x_{(n+1)m})\in I$. Therefore, I is an n-absorbing ideal of R. □

Corollary 4.2.6. *Let $R=R_1\times R_2\times\cdots\times R_m$ be a decomposable semiring with ϕ, n a positive integer and $I=I_1\times I_2\times\cdots\times I_m$ a proper k-ideal of R with exactly one $I_i\neq R_i$ where $i\in\{1,2,\ldots,m\}$. If I_i is an n-absorbing ideal of R_i, then I is a ϕ-n-absorbing ideal of R.*

The next example shows that the converse of Corollary 4.2.6 is not true.

Example 4.2.7. Consider the semiring $R=\mathbb{Z}_0^+\times\mathbb{Z}_0^+\times\mathbb{Z}_0^+$ and its ideal $I=729\mathbb{Z}_0^+\times\mathbb{Z}_0^+\times\mathbb{Z}_0^+$. Since $3^6=729\in 729\mathbb{Z}_0^+$ but $3^5=243\notin 729\mathbb{Z}_0^+$, it follows that $729\mathbb{Z}_0^+$ is not a 5-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Define $\phi:\mathscr{I}(R)\to\mathscr{I}(R)\cup\{\varnothing\}$ by $\phi(J)=J+(2\mathbb{Z}_0^+\times 3\mathbb{Z}_0^+\times 5\mathbb{Z}_0^+)$ for all $J\in\mathscr{I}(R)$. Then $\phi(I)=(729\mathbb{Z}_0^+\times\mathbb{Z}_0^+\times\mathbb{Z}_0^+)+(2\mathbb{Z}_0^+\times 3\mathbb{Z}_0^+\times 5\mathbb{Z}_0^+)$. Thus $I\subseteq\phi(I)$. Hence $I-\phi(I)=\varnothing$. Therefore, I is a ϕ-5-absorbing ideal of R.

From Corollary 4.2.6, we can conclude that if I_i is an n-absorbing ideal of a semiring R_i, then the ideal $I=R_1\times R_2\times\cdots\times R_{i-1}\times I_i\times R_{i+1}\times\cdots\times R_m$ of the decomposable semiring $R=R_1\times R_2\times\cdots\times R_m$ with ϕ is a ϕ-n-absorbing ideal when we provide that I is a proper k-ideal of R. In the next result, we also assume $I=R_1\times R_2\times\cdots\times R_{i-1}\times I_i\times R_{i+1}\times\cdots\times R_m$ is a proper k-ideal but we change to study in case of I_i is a weakly n-absorbing ideal of R_i.

Theorem 4.2.8. *Let $R=R_1\times R_2\times\cdots\times R_m$ be a decomposable semiring, n a positive integer and $I=I_1\times I_2\times\cdots\times I_m$ a proper k-ideal of R with exactly one $I_i\neq R_i$ where $i\in\{1,2,\ldots,m\}$. If I_i is a weakly n-absorbing ideal of R_i, then I is a ϕ-n-absorbing ideal of R for all $\phi_\omega\leq\phi$.*

Proof. In fact, $I=R_1\times\cdots\times R_{i-1}\times I_i\times R_{i+1}\times\cdots\times R_m$ for some $i\in\{1,2,\ldots,m\}$. Without loss of generality, we assume that $i=1$. We would like to show that $I=$

$I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-n-absorbing ideal of R for all $\phi_\omega \leq \phi$. By Proposition 2.1.28, we can conclude that I_1 is a k-ideal because I is a k-ideal. If I_1 is an n-absorbing ideal of R_1, then I is an n-absorbing ideal of R by Theorem 4.2.5, and so I is a ϕ_ω-n-absorbing ideal of R. Assume that I_1 is not an n-absorbing ideal of R_1. Thus $I_1^{n+1} = \{0\}$ from Corollary 4.1.26. Consider the element $(x_1, \ldots, x_m) \in \phi_\omega(I) = \cap_{l=1}^{\infty} I^l \subseteq I^{n+1} = (I_1 \times R_2 \times \cdots \times R_m)^{n+1} \subseteq I_1^{n+1} \times R_2 \times \cdots \times R_m = \{0\} \times R_2 \times \cdots \times R_m$. Let $(x_{11}, \ldots, x_{1m}), (x_{21}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, \ldots, x_{(n+1)m}) \in R$ be such that $(x_{11}x_{21} \cdots x_{(n+1)1}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I - \phi_\omega(I)$. Then $x_{11}x_{21} \cdots x_{(n+1)1} \in I_1 - \{0\}$. Since I_1 is a weakly n-absorbing ideal, we obtain $\hat{x}_{j1,(n+1)1} \in I_1$ for some $j \in \{1, 2, \ldots, n+1\}$. Hence $(\hat{x}_{j1,(n+1)1}, \hat{x}_{j2,(n+1)2}, \ldots, \hat{x}_{jm,(n+1)m}) \in I$. Thus I is a ϕ_ω-n-absorbing ideal. Therefore, in any cases, I is a ϕ_ω-n-absorbing ideal, and so I is a ϕ-n-absorbing ideal for all $\phi_\omega \leq \phi$. □

The rest of all results in this section except Theorem 4.2.14 is not analogous to the results in Section 3.2.

Theorem 4.2.9. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer with $n \geq 2$ and $I = I_1 \times I_2 \times \cdots \times I_m$ where $I_i \neq \{0\}$ for all $i \in \{1, 2, \ldots, m\}$ is a weakly n-absorbing k-ideal. Then I is an n-absorbing ideal of R or I_i is an $(n-1)$-absorbing ideal of R_i for all $i \in \{1, 2, \ldots, m\}$.*

Proof. If I is an n-absorbing ideal of R, then we are done. Suppose that I is not an n-absorbing ideal of R. Then $I^{n+1} = \{0\}$ by Corollary 4.1.26. Hence $I_j \neq R_j$ for all $j \in \{1, 2, \ldots, m\}$. Let $i, j \in \{1, 2, \ldots, m\}$. Without loss of generality, we assume that $j < i$. We show that I_j is an $(n-1)$-absorbing ideal of R_j. Let $x_1, x_2, \ldots, x_n \in R_j$ be such that $x_1x_2 \cdots x_n \in I_j$. Since $I_i \neq \{0\}$, there exists $0 \neq y_i \in I_i$. So $(0, 0, \ldots, 0) \neq (0, \ldots, 0, x_1x_2 \cdots x_n, 0, \ldots, 0, y_i, 0, \ldots, 0) \in I$. Thus

$$(0, 0, \ldots, 0) \neq (0, \ldots, 0, x_1, 0, \ldots, 0, 1, 0, \ldots, 0)(0, \ldots, 0, x_2, 0, \ldots, 0, 1, 0, \ldots, 0) \cdots (0, \ldots, 0, x_n, 0, \ldots, 0, 1, 0, \ldots, 0)(0, \ldots, 0, 1, 0, \ldots, 0, y_i, 0, \ldots, 0) \in I.$$

Since I is weakly n-absorbing, $1 \in I_i$ or $\hat{x}_{l,n} \in I_j$ for some $l \in \{1, 2, \ldots, n\}$. Since $I_i \neq R_i$, we obtain $\hat{x}_{l,n} \in I_j$. Therefore, I_j is an $(n-1)$-absorbing ideal of R_j. □

From Theorem 4.2.9, if we consider in case of $n = 2$, then we can conclude that if $I = I_1 \times I_2 \times \cdots \times I_m$ where $I_i \neq \{0\}$ for all $i \in \{1, 2, \ldots, m\}$ is a weakly 2-absorbing k-ideal of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$, then I is a 2-absorbing ideal of R or I_i is a prime ideal of R_i for all $i \in \{1, 2, \ldots, m\}$.

Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. We obtain from Theorem 4.2.5 that if I_i is an n-absorbing ideal of R_i, then I is an n-absorbing ideal of R. In the next result, we consider in case of every component I_i of I is an n_i-absorbing ideal of R_i, then we obtain an interesting result which is I is an n-absorbing ideal where $n = n_1 + n_2 + \cdots + n_m$; in addition, in this theorem n_i can be zero. Recall that a 0-absorbing ideal of a semiring R is R.

Theorem 4.2.10. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ an ideal of R. If I_i is an n_i-absorbing ideal of R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$, then I is an n-absorbing ideal of R where $n = n_1 + n_2 + \cdots + n_m$, so that I is a ϕ-n-absorbing ideal of R.*

Proof. Assume that I_i is an n_i-absorbing ideal of R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$. Let $n = n_1 + n_2 + \cdots + n_m$. We show that I is an n-absorbing ideal of R. Let $(x_{11}, x_{12}, \ldots, x_{1m}), (x_{21}, x_{22}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in R$ be such that

$$(x_{11}, x_{12}, \ldots, x_{1m})(x_{21}, x_{22}, \ldots, x_{2m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in I.$$

Then $(x_{11}x_{21} \cdots x_{(n+1)1}, x_{12}x_{22} \cdots x_{(n+1)2}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I$. Since I_i is an n_i-absorbing ideal, $x_{1i}x_{2i} \cdots x_{(n+1)i} \in I_i$ and $n_i < n+1$, we obtain $x_{j_1 i}x_{j_2 i} \cdots x_{j_{n_i} i} \in I_i$ for some distinct $j_1, j_2, \ldots, j_{n_i} \in \{1, 2, \ldots, n+1\}$ by Corollary 4.1.8. Suppose that $\cup_{i=1}^{m}\{j_1, j_2, \ldots, j_{n_i}\} = \{j'_1, j'_2, \ldots, j'_h\}$. Thus $\{j'_1, j'_2, \ldots, j'_h\} \subseteq \{1, 2, \ldots, n+1\}$ and $h \leq n$ since $n_1 + n_2 + \cdots + n_m = n$. Since $\{j_1, j_2, \ldots, j_{n_i}\} \subseteq \{j'_1, j'_2, \ldots, j'_h\}$ and $x_{j_1 i}x_{j_2 i} \cdots x_{j_{n_i} i} \in I_i$ for all $i \in \{1, 2, \ldots, m\}$, we obtain

$$x_{j'_1 i}x_{j'_2 i} \cdots x_{j'_h i} \in I_i.$$

By choosing all distinct $j'_{h+1}, j'_{h+2}, \ldots, j'_n \in \{1, 2, \ldots, n+1\} - \{j'_1, j'_2, \ldots, j'_h\}$, hence

$$x_{j'_1 i} x_{j'_2 i} \cdots x_{j'_n i} = (x_{j'_1 i} x_{j'_2 i} \cdots x_{j'_h i})(x_{j'_{h+1} i} x_{j'_{h+2} i} \cdots x_{j'_n i}) \in I_i.$$

Then we obtain

$$\begin{aligned}(x_{j'_1 1}, x_{j'_1 2}, \ldots, x_{j'_1 m})(x_{j'_2 1}, x_{j'_2 2}, \ldots, x_{j'_2 m}) \cdots (x_{j'_n 1}, x_{j'_n 2}, \ldots, x_{j'_n m}) \\ = (x_{j'_1 1} x_{j'_2 1} \cdots x_{j'_n 1}\,,\ x_{j'_1 2} x_{j'_2 2} \cdots x_{j'_n 2}\,,\ \ldots,\ x_{j'_1 m} x_{j'_2 m} \cdots x_{j'_n m}) \in I.\end{aligned}$$

Therefore, I is an n-absorbing ideal of R, and hence I is a ϕ-n-absorbing ideal of R. □

Example 4.2.11. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

(1) Then $2\mathbb{Z}_0^+ \times 6\mathbb{Z}_0^+ \times 30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ is a 6-absorbing ideal of R because $2\mathbb{Z}_0^+$ is a 1-absorbing ideal, $6\mathbb{Z}_0^+$ is a 2-absorbing ideal, $30\mathbb{Z}_0^+$ is a 3-absorbing ideal and $\mathbb{Z}_0^+$ is a 0-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

(2) Then $2^2\mathbb{Z}_0^+ \times 2^3\mathbb{Z}_0^+ \times 2^4\mathbb{Z}_0^+ \times 2^5\mathbb{Z}_0^+$ is a 14-absorbing ideal of R because $2^l\mathbb{Z}_0^+$ is an l-absorbing ideal of the semiring $\mathbb{Z}_0^+$ for all $l \in \mathbb{N}$.

From the above theorem, we can conclude that, for an ideal $I = I_1 \times I_2 \times \cdots \times I_m$ of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$, if every component of I is a prime ideal of its semiring, then I is an m-absorbing ideal of R.

Next, we provide the last theorems concerning ϕ-n-absorbing ideals of decomposable semirings.

Theorem 4.2.12. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $\phi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_m$ where each $\varphi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\varnothing\}$ is a function. Then the following statements hold.*

(1) $I_1 \times I_2 \times \cdots \times I_m$ is a ϕ-n-absorbing ideal of R where $I_j \subseteq \varphi_j(I_j)$ for all $j \in \{1, 2, \ldots, m\}$ and at least one I_i is a proper ideal of R_i for some $i \in \{1, 2, \ldots, m\}$.

(2) $R_1 \times R_2 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$ is a ϕ-n-absorbing ideal of R where I_i is a φ_i-n-absorbing ideal of R_i which must be an n-absorbing ideal if $\varphi_j(R_j) \neq R_j$ for some $j \in \{1, 2, \ldots, m\} - \{i\}$.

Proof. (1) The result follows from the fact that $I_1 \times I_2 \times \cdots \times I_m - \phi(I_1 \times I_2 \times \cdots \times I_m) = \varnothing$.

(2) Without loss of generality, we assume that I_1 is a proper ideal of R_1. If I_1 is an n-absorbing ideal of R_1, then $I_1 \times R_2 \times \cdots \times R_m$ is an n-absorbing ideal of R by Theorem 4.2.5. Thus $I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-n-absorbing ideal of R. Moreover, assume that I_1 is a φ_1-n-absorbing ideal of R_1 and $\varphi_j(R_j) = R_j$ for all $j \in \{1, 2, \ldots, m\}$. We show that $I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-n-absorbing ideal of R. Let $(x_{11}, x_{12}, \ldots, x_{1m}), (x_{21}, x_{22}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in R$ be such that $(x_{11}, x_{12}, \ldots, x_{1m})(x_{21}, x_{22}, \ldots, x_{2m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in I_1 \times R_2 \times \cdots \times R_m - \phi(I_1 \times R_2 \times \cdots \times R_m)$. Then

$$\begin{aligned}
&(x_{11}x_{21} \cdots x_{(n+1)1}, x_{21}x_{22} \cdots x_{(n+1)2}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \\
&\qquad \in I_1 \times R_2 \times \cdots \times R_m - \phi(I_1 \times R_2 \times \cdots \times R_m) \\
&\qquad = I_1 \times R_2 \times \cdots \times R_m - (\varphi_1(I_1) \times \varphi_2(R_2) \times \cdots \times \varphi_m(R_m)) \\
&\qquad = I_1 \times R_2 \times \cdots \times R_m - (\varphi_1(I_1) \times R_2 \times \cdots \times R_m) \\
&\qquad = (I_1 - \varphi_1(I_1)) \times R_2 \times \cdots \times R_m.
\end{aligned}$$

Thus $x_{11}x_{21} \cdots x_{(n+1)1} \in I_1 - \varphi_1(I_1)$. Since I_1 is a φ_1-n-absorbing ideal of R_1, we gain $\hat{x}_{i1,(n+1)1} \in I_1$ for some $i \in \{1, 2, \ldots, n+1\}$. It follows that

$$(x_{11}, x_{12}, \ldots, x_{1m}) \cdots (x_{(i-1)1}, x_{(i-1)2}, \ldots, x_{(i-1)m})(x_{(i+1)1}, x_{(i+1)2}, \ldots, x_{(i+1)m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in I_1 \times R_2 \times \cdots \times R_m.$$

Therefore, $I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-n-absorbing ideal of R. $\square$

If I_i is a φ_i-n-absorbing ideal but is not n-absorbing and $\varphi_j(R_j) \neq R_j$ for some $j \in \{1, 2, \ldots, m\}$, then $R_1 \times R_2 \times \cdots R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$ does not have to be a ϕ-n-absorbing ideal of $R_1 \times R_2 \times \cdots \times R_m$ as we show in the following example.

Example 4.2.13. Consider the semiring $\underbrace{\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+}_{m \text{ copies}}$. Let $p_1, p_2, \ldots, p_{n+1}$ are positive primes and $I_1 = p_1 p_2 \cdots p_{n+1}\mathbb{Z}_0^+$. Let $\varphi_1 : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ be a function such that I_1 is a φ_1-n-absorbing ideal. Since $I_1 = p_1p_2 \cdots p_{n+1}\mathbb{Z}_0^+$, it is easy to see that I_1 is not an n-absorbing ideal. Define $\varphi_j : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\varphi_j(I) = \{0\}$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$ and for all $j \in \{2, 3, \ldots, m\}$. Then $\varphi_j(\mathbb{Z}_0^+) = \{0\} \neq \mathbb{Z}_0^+$ for all $j \in \{2, 3, \ldots, m\}$. Let $\phi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_m$. Consider

$$\begin{aligned}
(p_1, 1, \ldots, 1)&(p_2, 1, \ldots, 1) \cdots (p_{n+1}, 1, \ldots, 1) \\
&= (p_1p_2 \cdots p_{n+1}, 1, \ldots, 1) \\
&\in (I_1 \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+) - (\varphi_1(I_1) \times \{0\} \times \cdots \times \{0\}) \\
&= (I_1 \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+) - \phi(I_1 \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+)
\end{aligned}$$

but

$$\begin{aligned}
(p_1, 1, \ldots, 1) \cdots (p_{i-1}, 1, \ldots, 1)&(p_{i+1}, 1, \ldots, 1) \cdots (p_{n+1}, 1, \ldots, 1) \\
&= (\hat{p}_{i,n+1}, 1, \ldots, 1) \notin I_1 \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+
\end{aligned}$$

for all $i \in \{1, 2, \ldots, m\}$ because $\hat{p}_{i,n+1} \notin I_1$ for all $i \in \{1, 2, \ldots, m\}$. Therefore, $I_1 \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$ is not a ϕ-n-absorbing ideal of $\mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \cdots \times \mathbb{Z}_0^+$.

We obtain from Theorem 4.2.12 that the ideals in (1) and (2) are ϕ-n-absorbing ideals of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$ but we cannot guarantee that there are two categories of ϕ-n-absorbing ideals of R. However, in case $n = 1$, we can conclude that ϕ-prime ideals of decomposable semirings with two components need to be in three formats only as shown in the following theorem.

Theorem 4.2.14. *Let $R = R_1 \times R_2$ be a decomposable semiring and $\phi = \varphi_1 \times \varphi_2$ where each $\varphi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\varnothing\}$ is a function. Then the ϕ-prime ideals of R have exactly one of the following three types:*

(1) $I_1 \times I_2$ where $I_j \subseteq \varphi_j(I_j)$ for all $j \in \{1, 2\}$ and at least one I_i is a proper ideal of R_i for some $i \in \{1, 2\}$.

(2) $I_1 \times R_2$ where I_1 is a φ_1-prime ideal of R_1 which must be prime if $\varphi_2(R_2) \neq R_2$.

(3) $R_1 \times I_2$ where I_2 is a φ_2-prime ideal of R_2 which must be prime if $\varphi_1(R_1) \neq R_1$.

Proof. By Theorem 4.2.12, we can conclude that the ideals in the statements (1), (2) and (3) are ϕ-prime ideals.

For the other direction, assume that $I_1 \times I_2$ is a ϕ-prime ideal of R. Thus I_1 or I_2 is a proper ideal of R. Without loss of generality, assume that I_1 is a proper ideal. Let $a, b \in R_1$ be such that $ab \in I_1 - \varphi_1(I_1)$. Then $(a, 0)(b, 0) = (ab, 0) \in I_1 \times I_2 - \phi(I_1 \times I_2)$. Since $I_1 \times I_2$ is a ϕ-prime ideal of R, we obtain $(a, 0) \in I_1 \times I_2$ or $(b, 0) \in I_1 \times I_2$. Hence $a \in I_1$ or $b \in I_1$. Therefore, I_1 is a φ_1-prime ideal of R_1. If $I_j \subseteq \varphi_j(I_j)$ for all $j \in \{1, 2\}$, then (1) is obtained. Suppose that $I_1 \nsubseteq \varphi_1(I_1)$ or $I_2 \nsubseteq \varphi_2(I_2)$. Without loss of generality, assume that $I_1 \nsubseteq \varphi_1(I_1)$. Then there is $x \in I_1 - \varphi_1(I_1)$. Let $y \in I_2$. Thus $(x, 1)(1, y) = (x, y) \in I_1 \times I_2 - \phi(I_1 \times I_2)$. Since $I_1 \times I_2$ is a ϕ-prime ideal of R, we have $(x, 1) \in I_1 \times I_2$ or $(1, y) \in I_1 \times I_2$. Hence $I_2 = R_2$ or $I_1 = R_1$. Since I_1 is a proper ideal, $I_2 = R_2$. Then $I_1 \times R_2$ is a ϕ-prime ideal of R where I_1 is a φ_1-prime ideal of R_1. It remains to show that I_1 is actually prime if $\varphi_2(R_2) \neq R_2$. Assume further that $\varphi_2(R_2) \neq R_2$. Then $1 \notin \varphi_2(R_2)$. Let $a, b \in R_1$ be such that $ab \in I_1$. Thus $(a, 1)(b, 1) = (ab, 1) \in I_1 \times R_2 - \phi(I_1 \times R_2)$. Since $I_1 \times R_2$ is a ϕ-prime ideal of R, we have $(a, 1) \in I_1 \times R_2$ or $(b, 1) \in I_1 \times R_2$. Hence $a \in I_1$ or $b \in I_1$. Therefore, I_1 is a prime ideal of R_1. So, the statement (2) is obtained. In the same way, if we assume I_2 is a proper ideal, then the statement (3) holds. □

4.3 ϕ-n-Absorbing Ideals in Quotient Semirings and in Semirings of Fractions

In this last section, ϕ-n-absorbing ideals of quotient semirings and ϕ-n-absorbing ideals of semirings of fractions are discussed. All results of this section are parallel to the results in Section 3.3.

Recall that if R is a semiring, I is a Q-ideal of R and ϕ is a function from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$ such that $\phi(L)$ is a subtractive extension of I for all ideal L of R where L is a subtractive extension of I, then we define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\varnothing\}$ by $\phi_I(J/I) = (\phi(J))/I$ for each ideal J of R where J is a subtractive extension of I.

Recall further that R is a semiring with ϕ satisfying the property $(*)$ if R is a semiring with ϕ, I is a Q-ideal of R and ϕ_I is a function from $\mathscr{I}(R/I)$ into $\mathscr{I}(R/I) \cup \{\varnothing\}$ where ϕ and ϕ_I are defined as above paragraph.

Theorem 4.3.1. *Let R be a semiring with ϕ satisfying the property $(*)$, n a positive integer, I a Q-ideal of R and P a subtractive extension of I. Then P is a ϕ-n-absorbing ideal of R if and only if P/I is a ϕ_I-n-absorbing ideal of R/I.*

Proof. First, assume that P is a ϕ-n-absorbing ideal of R. Then P/I is an ideal of R/I because P is a subtractive extension of I. Let $q_1+I, q_2+I, \ldots, q_{n+1}+I \in R/I$ be such that $(q_1+I)(q_2+I)\cdots(q_{n+1}+I) \in P/I - \phi_I(P/I)$. By Theorem 2.2.19, we obtain $q_1q_2\cdots q_{n+1} \in P-\phi(P)$. Since P is a ϕ-n-absorbing ideal, $\hat{q}_{i,n+1} \in P$ for some $i \in \{1,2,\ldots,n+1\}$. Hence $(q_1+I)\cdots(q_{i-1}+I)(q_{i+1}+I)\cdots(q_{n+1}+I) \in P/I$. Therefore, P/I is a ϕ_I-n-absorbing k-ideal of R/I.

Conversely, assume that P/I is a ϕ_I-n-absorbing ideal of R/I. We show that P is a ϕ-n-absorbing ideal of R. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in P-\phi(P)$. Then there exist $q_1, q_2 \ldots, q_{n+1} \in Q$ such that $x_i \in q_i + I$ for all $i \in \{1,2,\ldots,n+1\}$. So there is $y_i \in I$ such that $x_i = q_i+y_i$ for all $i \in \{1,2,\ldots,n+1\}$. Hence we obtain $(q_1+y_1)(q_2+y_2)\cdots(q_{n+1}+y_{n+1}) \in P-\phi(P)$. Then $q_1q_2\cdots q_{n+1} \in P-\phi(P)$ because P and $\phi(P)$ are subtractive extensions of I. Thus $(q_1+I)(q_2+I)\cdots(q_{n+1}+I) \in P/I - \phi_I(P/I)$ by Theorem 2.2.19. Hence $(q_1+I)\cdots(q_{i-1}+I)(q_{i+1}+I)\cdots(q_{n+1}+I) \in P/I$ for some $i \in \{1,2,\ldots,n+1\}$ since P/I is a ϕ_I-n-absorbing ideal. Then $\hat{q}_{i,n+1} \in P$. Thus $\hat{x}_{i,n+1} = (q_1+y_1)\cdots(q_{i-1}+y_{i-1})(q_{i+1}+y_{i+1})\cdots(q_{n+1}+y_{n+1}) \in P$. Therefore, P is a ϕ-n-absorbing ideal of R. □

Example 4.3.2. Consider the semiring $\mathbb{Z}_0^+$. Let $P = 28\mathbb{Z}_0^+$ and $I = 56\mathbb{Z}_0^+$. Then P is a 3-absorbing k-ideal of $\mathbb{Z}_0^+$ containing I and I is a Q-ideal of $\mathbb{Z}_0^+$ where $Q = \{0,1,2,3,\ldots,55\}$. Thus P is a subtractive extension of I. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(J) = 7\mathbb{Z}_0^+$ for all $J \in \mathscr{I}(\mathbb{Z}_0^+)$. Certainly, $\phi(L) = 7\mathbb{Z}_0^+$ is a subtractive extension of $I = 56\mathbb{Z}_0^+$ for all $L \in \mathscr{I}(R)$ where L is a subtractive extension of I. Define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\varnothing\}$ by $\phi_I(J/I) = (7\mathbb{Z}_0^+)/I$

for each ideal J of R where J is a subtractive extension of I. Thus $\mathbb{Z}_0^+$ is the semiring with ϕ satisfying the property $(*)$. Since P is a 3-absorbing ideal, P is a ϕ-3-absorbing ideal. Therefore, $P/I = 28\mathbb{Z}_0^+/56\mathbb{Z}_0^+$ is a ϕ_I-3-absorbing ideal of the quotient semiring $\mathbb{Z}_0^+/56\mathbb{Z}_0^+$.

Corollary 4.3.3. *Let R be a semiring with ϕ satisfying the property $(*)$, n a positive integer and I a Q-ideal of R. Then I is a ϕ-n-absorbing ideal of R if and only if the zero ideal of R/I is a ϕ_I-n-absorbing ideal.*

Like Chapter III, this chapter is ended with results regarding ϕ-n-absorbing ideals of semirings of fractions.

Recall that for a semiring R with ϕ, we define $\phi_S : \mathscr{I}(R_S) \to \mathscr{I}(R_S) \cup \{\varnothing\}$ by $\phi_S(J) = \phi(J \cap R)R_S$ if $\phi(J \cap R) \in \mathscr{I}(R)$ and $\phi_S(J) = \varnothing$ if $\phi(J \cap R) = \varnothing$ for all $J \in \mathscr{I}(R_S)$.

Proposition 4.3.4. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I a ϕ-prime ideal of R with $\phi(I) \subsetneq I$ and $I \cap S = \varnothing$. If $IR_S \neq \phi(I)R_S$, then $IR_S \cap R = I$.*

Proof. Assume that $IR_S \neq \phi(I)R_S$. Since $I \subseteq IR_S \cap R$, it remains to show that $IR_S \cap R \subseteq I$. Let $x \in IR_S \cap R$. Then $\frac{x}{1} \in IR_S$. Thus there exist $a \in I$ and $s \in S$ such that $\frac{x}{1} = \frac{a}{s}$. Hence $xs = a \in I$. If $xs \in I - \phi(I)$, then $x \in I$ because I is ϕ-prime and $I \cap S = \varnothing$. So assume that $xs \in \phi(I)$. Then $\frac{x}{1} = \frac{xs}{1s} \in \phi(I)R_S$, and hence $x \in \phi(I)R_S \cap R$. Then $IR_S \cap R \subseteq I$ or $IR_S \cap R \subseteq \phi(I)R_S \cap R$. Since $I \subseteq IR_S \cap R$ and $\phi(I)R_S \cap R \subseteq IR_S \cap R$, we obtain $I = IR_S \cap R$ or $\phi(I)R_S \cap R = IR_S \cap R$. If $\phi(I)R_S \cap R = IR_S \cap R$, then $\phi(I)R_S = IR_S$ which is a contradiction. Therefore, $IR_S \cap R = I$. □

In the last theorem, we would like to show that if I is a ϕ-n-absorbing ideal of R under some conditions, then IR_S is a ϕ_S-n-absorbing ideal of R_S.

Theorem 4.3.5. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I an ideal of R with $I \cap S = \varnothing$ and $\phi(I)R_S \subseteq$*

$\phi_S(IR_S)$. If I is a ϕ-n-absorbing ideal of R, then IR_S is a ϕ_S-n-absorbing ideal of R_S.

Proof. Assume that I is a ϕ-n-absorbing ideal of R. Since $I \cap S = \varnothing$, it follows that IR_S is a proper ideal of R_S. Let $\frac{x_1}{s_1}, \frac{x_2}{s_2}, \ldots, \frac{x_{n+1}}{s_{n+1}} \in R_S$ be such that $\frac{x_1x_2\cdots x_{n+1}}{s_1s_2\cdots s_{n+1}} \in IR_S - \phi_S(IR_S)$. Theorem 2.3.8 yields that there is $v \in S$ such that $x_1x_2\cdots x_{n+1}v \in I - \phi(I)$. Since I is ϕ-n-absorbing, $x_1x_2\cdots x_n \in I$ or $\hat{x}_{i,n}x_{n+1}v \in I$ for some $i \in \{1, 2, \ldots, n\}$. Thus $\frac{x_1x_2\cdots x_n}{s_1s_2\cdots s_n} \in IR_S$ or $\frac{\hat{x}_{i,n}x_{n+1}v}{\hat{s}_{i,n}s_{n+1}v} \in IR_S$. Hence $\frac{\hat{x}_{j,n+1}}{\hat{s}_{j,n+1}} \in IR_S$ for some $j \in \{1, 2, \ldots, n+1\}$. Therefore, IR_S is a ϕ_S-n-absorbing ideal of R_S. □

CHAPTER V

GENERALIZATIONS OF Gn-ABSORBING IDEALS OF SEMIRINGS

This chapter devotes to the last main results of our research. In 2015 [14], S. Chinwarakorn and S. Pianskool defined a new type of ideals which is still a generalization of primary ideals and n-absorbing ideals of a ring. They defined a ***generalized n-absorbing ideal*** (simply ***Gn-absorbing ideal***) I of a ring R to be a proper ideal and if whenever $x_1, x_2, \dots, x_{n+1} \in R$ with $x_1 x_2 \cdots x_{n+1} \in I$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \dots, n+1\}$. Thus every primary ideal is a Gn-absorbing ideal but not vice versa. For example, $30\mathbb{Z}$ is a $G3$-absorbing ideal of the ring $\mathbb{Z}$ and $6 \cdot 5 \in 30\mathbb{Z}$ but $6 \notin 30\mathbb{Z}$ and $5 \notin \sqrt{30\mathbb{Z}}$ so that $30\mathbb{Z}$ is not a primary ideal of the semiring $\mathbb{Z}$. Hence Gn-absorbing ideals are a generalization of primary ideals. Moreover, every n-absorbing ideal is a Gn-absorbing ideal. However, Gn-absorbing ideals need not be n-absorbing ideals. For example, the ideal $\{(\bar{0}, \bar{0})\}$ is a $G2$-absorbing ideal but is not a 2-absorbing ideal of the ring $\mathbb{Z}_6 \times \mathbb{Z}_9$ since $(\bar{2}, \bar{1})(\bar{1}, \bar{3})(\bar{3}, \bar{3}) \in \{(\bar{0}, \bar{0})\}$ but $(\bar{2}, \bar{1})(\bar{1}, \bar{3}) = (\bar{2}, \bar{3}) \notin \{(\bar{0}, \bar{0})\}$, $(\bar{2}, \bar{1})(\bar{3}, \bar{3}) = (\bar{0}, \bar{3}) \notin \{(\bar{0}, \bar{0})\}$ and $(\bar{1}, \bar{3})(\bar{3}, \bar{3}) = (\bar{3}, \bar{0}) \notin \{(\bar{0}, \bar{0})\}$. Therefore, Gn-absorbing ideals are a generalization of n-absorbing ideals.

In this chapter, we extend the notion of generalized n-absorbing ideals of a ring to ϕ-generalized-n-absorbing ideals of a semiring . We define a ***ϕ-generalized n-absorbing ideal*** (simply ***ϕ-Gn-absorbing***) I of a semiring R with ϕ to be a proper ideal and if whenever $x_1, x_2, \dots, x_{n+1} \in R$ with $x_1 x_2 \cdots x_{n+1} \in I - \phi(I)$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \dots, n+1\}$.

Like Chapter III and Chapter IV, we divide this chapter into three sections. They are ϕ-Gn-absorbing ideals of semirings, ϕ-Gn-absorbing ideals in decomposable semirings and ϕ-Gn-absorbing ideals in quotient semirings and semirings of fractions. Note further that almost all of the results of this chapter are parallel to the results of Chapter IV.

5.1 ϕ-Gn-Absorbing Ideals of Semirings

In the same fashion as in the previous chapters, we begin this chapter with the definitions that we use throughout this chapter. First, we define Gn-absorbing ideals of semirings similarly to Gn-absorbing ideals of rings given by S. Chinwarakorn and S. Pianskool in [14]. Moreover, we define weakly Gn-absorbing ideals, almost Gn-absorbing ideals, m-almost Gn-absorbing ideals and ω-Gn-absorbing ideals of semirings in the same way as weakly n-absorbing ideals, almost n-absorbing ideals, m-almost n-absorbing ideals and ω-n-absorbings given in Chapter IV.

Definition 5.1.1. Let R be a semiring and n a positive integer.

A proper ideal I of R is said to be ***generalized n-absorbing***, or simply ***Gn-absorbing***, if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2 \cdots x_{n+1} \in I$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$. Moreover, we denote 0-$Gn$-absorbing the ideal R.

A proper ideal I of R is said to be ***weakly generalized n-absorbing***, or simply ***weakly Gn-absorbing***, if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $0 \neq x_1x_2 \cdots x_{n+1} \in I$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$.

A proper ideal I of R is said to be ***almost generalized n-absorbing***, or simply ***almost Gn-absorbing***, if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2 \cdots x_{n+1} \in I - I^2$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$.

A proper ideal I of R is said to be ***m-almost generalized n-absorbing*** ($m \in \mathbb{N}$ with $m \geq 2$), or simply ***m-almost Gn-absorbing***, if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2 \cdots x_{n+1} \in I - I^m$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$.

A proper ideal I of R is said to be ***ω-generalized n-absorbing***, or simply ***ω-Gn-absorbing***, if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1x_2 \cdots x_{n+1} \in I - \cap_{n=1}^{\infty} I^n$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$.

Hence, the zero ideal is a weakly Gn-absorbing ideal, an almost Gn-absorbing ideal, an m-almost Gn absorbing ideal and an ω-Gn-absorbing ideal like the previous chapters.

In the following result, we provide a characterization of being Gn-absorbing ideals of semirings.

Theorem 5.1.2. *Let R be a semiring, I a proper ideal of R and n, n' positive integers with $n' > n$. Then I is a Gn-absorbing ideal if and only if whenever $x_1x_2\cdots x_{n'} \in I$ for $x_1, x_2, \ldots, x_{n'} \in R$, then $x_{i_1}x_{i_2}\cdots x_{i_n} \in \sqrt{I}$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.*

Proof. First, assume that I is a Gn-absorbing ideal of R and $x_1, x_2, \ldots, x_{n'} \in R$ with $n' > n$ be such that $x_1x_2\cdots x_{n'} \in I$. Then $x_1x_2\cdots x_n(x_{n+1}x_{n+2}\cdots x_{n'}) \in I$. Since I is a Gn-absorbing ideal, $x_1x_2\cdots x_n \in \sqrt{I}$ or $\hat{x}_{i,n}(x_{n+1}x_{n+2}\cdots x_{n'}) \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n\}$. If $x_1x_2\cdots x_n \in \sqrt{I}$, then we are done. Assume that the other case is yielded. Without loss of generality, we suppose that $x_2x_3\cdots x_{n'} \in \sqrt{I}$, i.e., $(x_2x_3\cdots x_{n'})^\alpha \in I$ for some $\alpha \in \mathbb{N}$. Thus we write

$$x_2^\alpha x_3^\alpha \cdots x_{n+1}^\alpha (x_{n+2}^\alpha x_{n+3}^\alpha \cdots x_{n'}^\alpha) = (x_2x_3\cdots x_{n'})^\alpha \in I.$$

Then $x_2^\alpha x_3^\alpha \cdots x_{n+1}^\alpha \in \sqrt{I}$ or $x_2^\alpha \cdots x_{i-1}^\alpha x_{i+1}^\alpha \cdots x_{n+1}^\alpha (x_{n+2}^\alpha x_{n+3}^\alpha \cdots x_{n'}^\alpha) \in \sqrt{I}$ for some $i \in \{2, \ldots, n+1\}$ because I is a Gn-absorbing ideal. Then we obtain $x_2x_3\cdots x_{n+1} \in \sqrt{\sqrt{I}} = \sqrt{I}$ or $x_2\cdots x_{i-1}x_{i+1}\cdots x_{n+1}(x_{n+2}x_{n+3}\cdots x_{n'}) \in \sqrt{\sqrt{I}} = \sqrt{I}$. If $x_2x_3\cdots x_{n+1} \in \sqrt{I}$, then we are done. If not, by repeating the same process as above, we obtain $x_{i_1}x_{i_2}\cdots x_{i_n} \in \sqrt{I}$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.

Conversely, the proof is clear by choosing $n' = n + 1$. □

Next, we show that if I is a Gn-absorbing ideal, then I is a Gn'-absorbing ideal for all integer $n' \geq n$.

Proposition 5.1.3. *Let R be a semiring, I a proper ideal of R and n a positive integer. If I is a Gn-absorbing ideal, then I is a Gn'-absorbing ideal for all $n' \in \mathbb{N}$ with $n' \geq n$.*

Proof. Assume that I is a Gn-absorbing ideal of R. Let $n' \in \mathbb{N}$ be such that $n' \geq n$. Note that, if $n' = n$, then there is nothing to do. So we assume that $n' > n$. Let $x_1, x_2, \ldots, x_{n'+1} \in R$ be such that $x_1 x_2 \cdots x_{n'+1} \in I$. Applying Theorem 5.1.2 yields $x_{i_1} x_{i_2} \cdots x_{i_n} \in \sqrt{I}$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'+1\}$. By choosing all distinct

$$i_{n+1}, i_{n+2}, \ldots, i_{n'} \in \{1, 2, \ldots, n'+1\} - \{i_1, i_2, \ldots, i_n\}$$

and by multiplying, we get $x_{i_1} x_{i_2} \cdots x_{i_{n'}} = (x_{i_1} x_{i_2} \cdots x_{i_n}) \left(x_{i_{n+1}} x_{i_{n+2}} \cdots x_{n'}\right) \in \sqrt{I}$. Hence, I is a Gn'-absorbing ideal of R. Therefore, I is a Gn'-absorbing ideal for all $n' \geq n$ as desired. □

Nevertheless, the converse of the Proposition 5.1.3 is not true as we shown in the following example.

Example 5.1.4. Consider the semiring $\mathbb{Z}_0^+$. Then $20\mathbb{Z}_0^+$ is a $G2$-absorbing ideal of the semiring $\mathbb{Z}_0^+$ but is not a $G1$-absorbing ideal of the semiring $\mathbb{Z}_0^+$ because $4 \cdot 5 = 20 \in 20\mathbb{Z}_0^+$ but $4^\alpha \notin 20\mathbb{Z}_0^+$ for all $\alpha \in \mathbb{N}$ and $5^\beta \notin 20\mathbb{Z}_0^+$ for all $\beta \in \mathbb{N}$, i.e., $4 \notin \sqrt{20\mathbb{Z}_0^+}$ and $5 \notin \sqrt{20\mathbb{Z}_0^+}$.

From the Definition 5.1.1, we can conclude that every primary ideal (weakly primary ideal, almost primary ideal, m-almost primary ideal and ω primary ideal) is a Gn-absorbing ideal (weakly Gn-absorbing ideal, almost Gn-absorbing ideal, m-almost Gn-absorbing ideal and ω-Gn-absorbing ideal). Moreover, every n-absorbing ideal (weakly n-absorbing ideal, almost n-absorbing ideal, m-almost n-absorbing ideal and ω-n-absorbing ideal) is a Gn-absorbing ideal (weakly Gn-absorbing ideal, almost Gn-absorbing ideal, m-almost Gn-absorbing ideal and ω-Gn-absorbing ideal). Certainly, the converse of both statements are not true in general and we provide an example to confirm.

Example 5.1.5. Consider the semiring $\mathbb{Z}_0^+$.

(1) Consider the ideal $100\mathbb{Z}_0^+$. Form Chapter IV, we know that $100\mathbb{Z}_0^+$ is a 4-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Then $100\mathbb{Z}_0^+$ is a $G4$-absorbing ideal of the semring $\mathbb{Z}_0^+$. Since $4 \cdot 25 = 100 \in 100\mathbb{Z}_0^+$ but $4 \notin 100\mathbb{Z}_0^+$ and $25 \notin \sqrt{100\mathbb{Z}_0^+}$, the

ideal $100\mathbb{Z}_0^+$ is not a primary ideal of the semiring $\mathbb{Z}_0^+$.

(2) Consider the ideal $900\mathbb{Z}_0^+$. The ideal $900\mathbb{Z}_0^+$ is a $G3$-absorbing ideal of the semiring $\mathbb{Z}_0^+$ but is not a 3-absorbing ideal of the semiring $\mathbb{Z}_0^+$ because $2 \cdot 5 \cdot 9 \cdot 10 = 900 \in 900\mathbb{Z}_0^+$ but $2 \cdot 5 \cdot 9 = 90 \notin 900\mathbb{Z}_0^+$, $2 \cdot 5 \cdot 10 = 100 \notin 900\mathbb{Z}_0^+$, $2 \cdot 9 \cdot 10 = 180 \notin 900\mathbb{Z}_0^+$ and $5 \cdot 9 \cdot 10 = 450 \notin 900\mathbb{Z}_0^+$.

Consequently, Gn-absorbing ideals are a generalization of primary ideals and of n-absorbing ideals.

From the fact that Gn-absorbing ideals are a generalization of n-absorbing ideals, in the following proposition, we provide a result that helps us find examples of Gn-absorbing ideals which are not n-absorbing ideals.

Proposition 5.1.6. *Let $n, \alpha_1, \alpha_2, \ldots, \alpha_n$ be positive integers and $p_1, p_2, \ldots, p_n$ prime numbers (not necessary distinct). Then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ is a Gn-absorbing ideal but not an n-absorbing ideal of the semiring $\mathbb{Z}_0^+$ if there is $\alpha_i > 1$ for some $i \in \{1, 2, \ldots, n\}$.*

Proof. First, we show that the ideal $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ is a Gn-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Let $x_1, x_2, \ldots, x_{n+1} \in \mathbb{Z}_0^+$ be such that $x_1 x_2 \cdots x_{n+1} \in p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$. Then $x_1 x_2 \cdots x_{n+1} = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} a$ for some $a \in \mathbb{Z}_0^+$. Thus p_i is a factor of x_j for some $j \in \{1, 2, \ldots, n+1\}$. Hence there is $\{x_{i_1}, x_{i_2}, \ldots, x_{i_{n-m}}\} \subseteq \{x_1, x_2, \ldots, x_{n+1}\}$ for some $m \in \mathbb{Z}_0^+$ and for some distinct $i_1, i_2, \ldots, i_{n-m} \in \{1, 2, \ldots, n+1\}$ such that $x_{i_1} x_{i_2} \cdots x_{i_{n-m}} = p_1 p_2 \cdots p_n h$ for some $h \in \mathbb{Z}_0^+$. By choosing all distinct

$$i_{n-m+1}, i_{n-m+2}, \ldots, i_n \in \{1, 2, \ldots, n+1\} - \{i_1, i_2, \ldots, i_{n-m}\}$$

and by multiplying,

$$x_{i_1} x_{i_2} \cdots x_{i_n} = (x_{i_1} x_{i_2} \cdots x_{i_{n-m}})(x_{i_{n-m+1}} x_{i_{n-m+2}} \cdots x_{i_n}) = p_1 p_2 \cdots p_n hl$$

for some $l \in \mathbb{Z}_0^+$. Let $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then $\alpha \in \mathbb{N}$. Hence

$$\begin{aligned}(x_{i_1} x_{i_2} \cdots x_{i_n})^{\alpha} &= (p_1 p_2 \cdots p_n hl)^{\alpha} \\ &= (p_1 p_2 \cdots p_n hl)^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}\end{aligned}$$

$$= p_1^{\alpha_1+\alpha_2+\cdots+\alpha_n} p_2^{\alpha_1+\alpha_2+\cdots+\alpha_n} \cdots p_n^{\alpha_1+\alpha_2+\cdots+\alpha_n} (hl)^{\alpha_1+\alpha_2+\cdots+\alpha_n}$$
$$= p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} (p_1^{\alpha_2+\alpha_3+\cdots+\alpha_n} p_2^{\alpha_1+\alpha_3+\cdots+\alpha_n} \cdots p_n^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}} (hl)^{\alpha_1+\alpha_2+\cdots+\alpha_n})$$
$$\in p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+.$$

Thus $x_{i_1} x_{i_2} \cdots x_{i_n} \in \sqrt{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+}$. Therefore, $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ is a Gn-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

Next, it remains to show that $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ is not an n-absorbing ideal of the semiring $\mathbb{Z}_0^+$ if there is $\alpha_i > 1$ for some $i \in \{1, 2, \ldots, n\}$. Without loss of generality, we suppose that $\alpha_n > 1$. Since

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n - 1} p_n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \in p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$$

but $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n - 1} \notin p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$, $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}} p_n \notin p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ and $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}} p_{i+1}^{\alpha_{i+1}} \cdots p_n^{\alpha_n - 1} p_n \notin p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, n-1\}$, we can conclude that $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ is not an n-absorbing ideal of the semiring $\mathbb{Z}_0^+$. □

Example 5.1.7. Consider the semiring $\mathbb{Z}_0^+$.

(1) $1296\mathbb{Z}_0^+ = 2^4 \cdot 3^4 \mathbb{Z}_0^+$ is a $G2$-absorbing ideal but is not a 2-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

(2) $490\mathbb{Z}_0^+ = 2 \cdot 5 \cdot 7^2 \mathbb{Z}_0^+$ is a $G3$-absorbing ideal but is not a 3-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

(3) $3500\mathbb{Z}_0^+ = 2 \cdot 2 \cdot 5^3 \cdot 7 \mathbb{Z}_0^+$ is a $G4$-absorbing ideal but is not a 4-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

From Proposition 5.1.6, it is clear that if $p_1 = p_2 = \cdots = p_n$, then we obtain that $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+ = p_1^{\alpha_1+\alpha_2+\cdots+\alpha_n} \mathbb{Z}_0^+$ is a primary ideal of the semiring $\mathbb{Z}_0^+$. It makes us wonder that if there are $p_i, p_j \in \{p_1, p_2, \ldots, p_n\}$ such that $p_i \neq p_j$, then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+$ is still a primary ideal of the semiring $\mathbb{Z}_0^+$ or not. The next proposition is an answer to this. For the convenience, we write $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ in the form $q_1^{\beta_1} q_2^{\beta_2} \cdots q_l^{\beta_l}$ where $q_1, q_2, \ldots, q_l$ are all distinct prime numbers and $\beta_1, \beta_2, \ldots, \beta_l \in \mathbb{N}$; in addition, it is clear that $l \leq n$. Then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \mathbb{Z}_0^+ =$

$q_1^{\beta_1} q_2^{\beta_2} \cdots q_l^{\beta_l}\mathbb{Z}_0^+$ is a Gl-absorbing ideal of the semiring $\mathbb{Z}_0^+$; nonetheless, by the fact that $l \leq n$, so it is a Gn-absorbing ideal.

Proposition 5.1.8. *Let $n, \alpha_1, \alpha_2, \ldots, \alpha_n$ be positive integers and $p_1, p_2, \ldots, p_n$ all distinct prime numbers . Then $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ is a Gn-absorbing ideal but not a primary ideal of the semiring $\mathbb{Z}_0^+$ if $n \geq 2$.*

Proof. By Proposition 5.1.6, $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ is a Gn-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Thus it remains to show that $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ is not a primary ideal of the semiring $\mathbb{Z}_0^+$ if $n \geq 2$. Assume that $n \geq 2$. Since

$$(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}})(p_n^{\alpha_n}) = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \in p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$$

but $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}} \notin p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ and $(p_n^{\alpha_n})^{\alpha} \notin p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ for all $\alpha \in \mathbb{N}$ because $n \geq 2$ and $p_1, p_2, \ldots, p_n$ are all distinct prime numbers. Then we obtain $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n-1}^{\alpha_{n-1}} \notin p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ and $p_n^{\alpha_n} \notin \sqrt{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+}$. Therefore, we can conclude that $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}\mathbb{Z}_0^+$ is not a primary ideal of the semiring $\mathbb{Z}_0^+$ where $n \geq 2$. □

Example 5.1.9. Consider the semiring $\mathbb{Z}_0^+$.

(1) $484\mathbb{Z}_0^+ = 2^2 \cdot 11^2\mathbb{Z}_0^+$ is a $G2$-absorbing ideal but is not a primary ideal of the semiring $\mathbb{Z}_0^+$.

(2) $150\mathbb{Z}_0^+ = 2 \cdot 3 \cdot 5^2$ is a $G3$-absorbing ideal but is not a primary ideal of the semiring $\mathbb{Z}_0^+$.

Next, we define ϕ-generalized-n-absorbing ideals of semirings which is a main character of this chapter.

Definition 5.1.10. A proper ideal I of a semiring R is said to be ***ϕ-generalized-n-absorbing***, or simply ***ϕ-Gn-absorbing***, if whenever $x_1, x_2, \ldots, x_{n+1} \in R$ and $x_1 x_2 \cdots x_{n+1} \in I - \phi(I)$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$.

Let R be a semiring with ϕ. From Chapter IV, we obtain that ϕ-primary ideals and ϕ-n-absorbing ideals do not imply each other. In the following result, we would like to show that every ϕ-primary ideal and ϕ-n-absorbing ideal are ϕ-Gn-absorbing ideals for all $n \in \mathbb{N}$.

Proposition 5.1.11. *Let R be a semiring with ϕ and n a positive integer. Then*

(1) every ϕ-primary ideal of R is a ϕ-Gn-absorbing ideal of R, and

(2) every ϕ-n-absorbing ideal of R is a ϕ-Gn-absorbing ideal of R.

Proof. (1) Assume that I is a ϕ-primary ideal of R. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1 x_2 \cdots x_{n+1} \in I - \phi(I)$. Since I is a ϕ-primary ideal of R and $x_1(x_2 x_3 \cdots x_{n+1}) \in I - \phi(I)$, we obtain $x_1 \in I$ or $x_2 x_3 \cdots x_{n+1} \in \sqrt{I}$. If $x_1 \in I$, then $x_1 \hat{x}_{i,n+1} \in I \subseteq \sqrt{I}$ for all $i \in \{2, 3, \ldots, n+1\}$. If $x_2 x_3 \cdots x_{n+1} \in \sqrt{I}$, then we are done. Hence we can conclude that $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$. Therefore, I is a ϕ-Gn-absorbing ideal of R.

(2) Assume that I is a ϕ-n-absorbing ideal of R. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1 x_2 \cdots x_{n+1} \in I - \phi(I)$. Since I is a ϕ-n-absorbing ideal of R, we obtain $\hat{x}_{i,n+1} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$. Then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$ because $I \subseteq \sqrt{I}$. Therefore, I is a ϕ-Gn-absorbing ideal of R. □

However, the converse of each statement of the above proposition is not true in general as shown in the next example.

Example 5.1.12. Consider the semiring $\mathbb{Z}_0^+$ with ϕ_2. Recall that ϕ_2 is the function defined by $\phi_2(I) = I^2$ for all $I \in \mathscr{I}(\mathbb{Z}_0^+)$. Consider the ideal $363\mathbb{Z}_0^+$ of $\mathbb{Z}_0^+$. Since $363 = 3 \cdot 11^2$, it follows that $363\mathbb{Z}_0^+$ is a $G2$-absorbing ideal of $\mathbb{Z}_0^+$, so it is a ϕ_2-$G2$-absorbing ideal of $\mathbb{Z}_0^+$.

Since $33 \cdot 11 = 363 \in 363\mathbb{Z}_0^+ - (363\mathbb{Z}_0^+)^2 = 363\mathbb{Z}_0^+ - 131769\mathbb{Z}_0^+$ but $33 \notin 363\mathbb{Z}_0^+$ and $11^\alpha \notin 363\mathbb{Z}_0^+$ for all $\alpha \in \mathbb{N}$, i.e., $33 \notin 363\mathbb{Z}_0^+$ and $11 \notin \sqrt{363\mathbb{Z}_0^+}$, it follows that $363\mathbb{Z}_0^+$ is not a ϕ_2-primary ideal of $\mathbb{Z}_0^+$.

Because $3 \cdot 11 \cdot 11 = 363 \in 363\mathbb{Z}_0^+ - (363\mathbb{Z}_0^+)^2 = 363\mathbb{Z}_0^+ - 131769\mathbb{Z}_0^+$ but $3 \cdot 11 = 33 \notin 363\mathbb{Z}_0^+$, $11 \cdot 11 = 121 \notin 363\mathbb{Z}_0^+$, it follows that $363\mathbb{Z}_0^+$ is not a ϕ_2-2-absorbing ideal of $\mathbb{Z}_0^+$.

Therefore, we can conclude that $363\mathbb{Z}_0^+$ is a ϕ_2-$G2$-absorbing ideal but is not a ϕ_2-primary ideal and is not a ϕ_2-2-absorbing ideal of $\mathbb{Z}_0^+$.

From Proposition 5.1.11 and Example 5.1.12, we acquire that ϕ-Gn-absorbing ideals are a generalization of both ϕ-primary ideals and ϕ-n-absorbing ideals.

Theorem 5.1.13. *Let R be a semiring with ϕ, I a proper ideal of R such that $\phi(\sqrt{I}) = \sqrt{\phi(I)}$ and n, n' positive integers with $n' > n$. Then $\sqrt{I}$ is a ϕ-Gn-absorbing ideal if and only if whenever $x_1x_2\cdots x_{n'} \in \sqrt{I} - \phi(\sqrt{I})$ for any $x_1, x_2, \ldots, x_{n'} \in R$, then $x_{i_1}x_{i_2}\cdots x_{i_n} \in \sqrt{I}$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.*

Proof. First, let $\sqrt{I}$ be a ϕ-Gn-absorbing ideal of R and $x_1, x_2, \ldots, x_{n'} \in R$ with $n' > n$ be such that $x_1x_2\cdots x_{n'} \in \sqrt{I} - \phi(\sqrt{I})$. Since $\phi(\sqrt{I}) = \sqrt{\phi(I)}$, we gain $x_1x_2\cdots x_{n'} \in \sqrt{I} - \sqrt{\phi(I)}$. Because $\sqrt{I}$ is a ϕ-Gn-absorbing ideal and $x_1x_2\cdots x_n(x_{n+1}x_{n+2}\cdots x_{n'}) \in \sqrt{I} - \phi(\sqrt{I})$, we obtain $x_1x_2\cdots x_n \in \sqrt{\sqrt{I}}$ or $\hat{x}_{i,n}(x_{n+1}x_{n+2}\cdots x_{n'}) \in \sqrt{\sqrt{I}}$ for some $i \in \{1, 2, \ldots, n\}$. Thus we divide this proof into two cases.

Case 1: If $x_1x_2\cdots x_n \in \sqrt{\sqrt{I}} = \sqrt{I}$, then we are done.

Case 2: Assume that $\hat{x}_{i,n}x_{n+1}x_{n+2}\cdots x_{n'} \in \sqrt{\sqrt{I}} = \sqrt{I}$. Because $x_1x_2\cdots x_{n'} \notin \sqrt{\phi(I)}$, we obtain $\hat{x}_{i,n}x_{n+1}(x_{n+2}\cdots x_{n'}) \in \sqrt{I} - \sqrt{\phi(I)} = \sqrt{I} - \phi(\sqrt{I})$. Since $\sqrt{I}$ is a ϕ-Gn-absorbing ideal, $\hat{x}_{i,n}x_{n+1} \in \sqrt{\sqrt{I}}$ or $\hat{x}_{\{i,j\},n+1}(x_{n+2}\cdots x_{n'}) \in \sqrt{\sqrt{I}}$ for some $j \in \{1, 2, \ldots, n+1\} - \{i\}$. We divide this case into two subcases.

Subcase 2.1: If $\hat{x}_{i,n}x_{n+1} \in \sqrt{\sqrt{I}} = \sqrt{I}$, then we are done.

Subcase 2.2: Assume that $\hat{x}_{\{i,j\},n+1}(x_{n+2}\cdots x_{n'}) \in \sqrt{\sqrt{I}}$. By repeating the same process as above, we must have $x_{i_1}x_{i_2}\cdots x_{i_n} \in \sqrt{\sqrt{I}} = \sqrt{I}$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$.

Conversely, the proof is clear by choosing $n' = n + 1$. □

Proposition 5.1.14. *Let R be a semiring with ϕ, I a proper ideal of R such that $\phi(\sqrt{I}) = \sqrt{\phi(I)}$ and n a positive integer. If $\sqrt{I}$ is a ϕ-Gn-absorbing ideal, then $\sqrt{I}$ is a ϕ-Gn'-absorbing ideal for all $n' \geq n$.*

Proof. Assume that $\sqrt{I}$ is a ϕ-Gn-absorbing ideal. Let $n' \in \mathbb{N}$ be such that $n' \geq n$. Let $x_1, x_2, \ldots, x_{n'+1} \in R$ be such that $x_1x_2\cdots x_{n'+1} \in \sqrt{I} - \phi(\sqrt{I})$. By Theorem 5.1.13, we obtain that $x_{i_1}x_{i_2}\cdots x_{i_n} \in \sqrt{\sqrt{I}}$ for some distinct $i_1, i_2, \ldots, i_n \in \{1, 2, \ldots, n'\}$. By choosing all distinct

$$i_{n+1}, i_{n+2}, \ldots, i_{n'} \in \{1, 2, \ldots, n'\} - \{i_1, i_2, \ldots, i_n\}$$

and by multiplying, we have

$$x_{i_1}x_{i_2}\cdots x_{i_{n'}} = (x_{i_1x_{i_2}}\cdots x_{i_n})(x_{i_{n+1}}x_{i_{n+2}}\cdots x_{i_{n'}}) \in \sqrt{\sqrt{I}}.$$

Therefore, $\sqrt{I}$ is a ϕ-Gn'-absorbing ideal for all $n' \geq n$. □

Proposition 5.1.15. *Let R be a semiring with ϕ and I a proper ideal of R such that $\phi(\sqrt{I}) = \sqrt{\phi(I)}$. If I is a ϕ-Gn-absorbing ideal, then $\sqrt{I}$ is a ϕ-Gn-absorbing ideal.*

Proof. Assume that I is a ϕ-Gn-absorbing ideal of R such that $\phi(\sqrt{I}) = \sqrt{\phi(I)}$. If $1 \in \sqrt{I}$, then $1 \in I$, which is a contradiction. Thus $1 \notin \sqrt{I}$, and hence $\sqrt{I}$ is a proper ideal. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in \sqrt{I} - \phi(\sqrt{I})$. Since $\phi(\sqrt{I}) = \sqrt{\phi(I)}$, we have $x_1x_2\cdots x_{n+1} \in \sqrt{I} - \sqrt{\phi(I)}$. Then $x_1^{\alpha}x_2^{\alpha}\cdots x_{n+1}^{\alpha} = (x_1x_2\cdots x_{n+1})^{\alpha} \in I - \phi(I)$ for some $\alpha \in \mathbb{N}$. Since I is a ϕ-Gn-absorbing ideal, $(\hat{x}_{i,n+1})^{\alpha} \in I$ for some $i \in \{1, 2, \ldots, n+1\}$, i.e., $\hat{x}_{i,n+1} \in \sqrt{I} = \sqrt{\sqrt{I}}$ for some $i \in \{1, 2, \ldots, n+1\}$. Therefore, $\sqrt{I}$ is a ϕ-Gn-absorbing ideal. □

In the following example, we provide relationships between ϕ-Gn-absorbing ideals and Gn-absorbing ideals (weakly Gn-absorbing ideals, almost Gn-absorbing ideals, m-almost Gn-absorbing ideals, ω-Gn-absorbing ideals) in the same manner as Chapter III and Chapter IV.

Example 5.1.16. Let R be a semiring. Then

(1) I is a $\phi_{\varnothing}$-Gn-absorbing ideal if and only if I is a Gn-absorbing ideal,

(2) I is a ϕ_0-Gn-absorbing ideal if and only if I is a weakly Gn-absorbing ideal,

(3) I is a ϕ_1-Gn-absorbing ideal if and only if I is a proper ideal,

(4) I is a ϕ_2-Gn-absorbing ideal if and only if I is an almost Gn-absorbing ideal,

(5) I is a ϕ_m-Gn-absorbing ideal if and only if I is an m-almost Gn-absorbing ideal, and

(6) I is a ϕ_ω-Gn-absorbing ideal if and only if I is an ω-Gn-absorbing ideal.

Proposition 5.1.17. *Let R be a semiring, n a positive integer, I a proper ideal of R and $\varphi_1 \leq \varphi_2$ where φ_1 and φ_2 are functions from $\mathscr{I}(R)$ into $\mathscr{I}(R) \cup \{\varnothing\}$. If I is a φ_1-Gn-absorbing ideal, then I is a φ_2-Gn-absorbing ideal.*

Proof. The proof is similar to one of Proposition 3.1.6. □

Corollary 5.1.18. *Let I be a proper ideal of a semiring and $n, m \in \mathbb{N}$ with $m \geq 2$. Consider the following statements:*

(1) I is a Gn-absorbing ideal.

(2) I is a weakly Gn-absorbing ideal.

(3) I is an ω-Gn-absorbing ideal.

(4) I is an $(m+1)$-almost Gn-absorbing ideal.

(5) I is an m-almost Gn-absorbing ideal.

(6) I is an almost Gn-absorbing ideal.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

The next result is parallel to the result derived from Proposition 3.1.8.

Proposition 5.1.19. *Let R be a semiring, n a positive integer and I a proper ideal of R. Then I is an ω-Gn-absorbing ideal if and only if I is an m-almost Gn-absorbing ideal for all $m \geq 2$.*

Proof. The proof for the first direction is clear by Corollary 5.1.18.

Conversely, the proof is similar to the proof of Proposition 3.1.8. □

The following theorem is analogous to the Theorem 4.1.24.

Theorem 5.1.20. *Let R be a semiring with ϕ, n a positive integer and I a proper k-ideal of R such that $\phi(I)$ is a k-ideal. If I is a ϕ-Gn-absorbing ideal with $I^{n+1} \nsubseteq \phi(I)$, then I is a Gn-absorbing ideal.*

Proof. Suppose that I is a ϕ-Gn-absorbing ideal with $I^{n+1} \nsubseteq \phi(I)$. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in I$. If $x_1x_2\cdots x_{n+1} \in I - \phi(I)$, then $\hat{x}_{i,n+1} \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$. Assume that $x_1x_2\cdots x_{n+1} \in \phi(I)$.

Case 1: Assume that $\hat{x}_{i,n+1}I \nsubseteq \phi(I)$ for some $i \in \{1, 2, \ldots, n+1\}$. Then there exists $p_1 \in I$ such that $\hat{x}_{i,n+1}p_1 \in I - \phi(I)$. Thus $\hat{x}_{i,n+1}(x_i + p_1) \in I - \phi(I)$ because $\phi(I)$ is a k-ideal. Since I is ϕ-Gn-absorbing, $\hat{x}_{i,n+1} \in \sqrt{I}$ or $\hat{x}_{\{i,j\},n+1}(x_i + p_1) \in \sqrt{I}$ for some $j \in \{1, 2, \ldots, n+1\} - \{i\}$. If $\hat{x}_{i,n+1} \in \sqrt{I}$, then we are done. Suppose that $\hat{x}_{\{i,j\},n+1}(x_i + p_1) \in \sqrt{I}$ for some $j \in \{1, 2, \ldots, n+1\} - \{i\}$. That is $(\hat{x}_{\{i,j\},n+1}(x_i + p_1))^\alpha \in I$ for some $\alpha \in \mathbb{N}$. Since I is a k-ideal and $p_1 \in I$, we obtain $(\hat{x}_{j,n+1})^\alpha \in I$, i.e., $\hat{x}_{j,n+1} \in \sqrt{I}$.

Case 2: Assume that $\hat{x}_{i,n+1}I \subseteq \phi(I)$ for all $i \in \{1, 2, \ldots, n+1\}$.

Subcase 2.1: Suppose that $\hat{x}_{\{i,j\},n+1}I^2 \nsubseteq \phi(I)$ for some $j \in \{1, 2, \ldots, n+1\} - \{i\}$. Then there are $p_1, p_2 \in I$ such that $\hat{x}_{\{i,j\},n+1}p_1p_2 \notin \phi(I)$. Because $\phi(I)$ is a k-ideal, we gain $\hat{x}_{\{i,j\},n+1}(x_i + p_1)(x_j + p_2) \in I - \phi(I)$. Since I is a ϕ-Gn-absorbing ideal, $\hat{x}_{\{i,j\},n+1}(x_i + p_1) \in \sqrt{I}$ or $\hat{x}_{\{i,j\},n+1}(x_j + p_2) \in \sqrt{I}$ or $\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_2 + p_2) \in \sqrt{I}$ for some $l \in \{1, 2, \ldots, n+1\} - \{i, j\}$, i.e., $(\hat{x}_{\{i,j\},n+1}(x_i + p_1))^\alpha \in I$ or $(\hat{x}_{\{i,j\},n+1}(x_j + p_2))^\beta \in I$ or $(\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_2 + p_2))^\gamma \in I$ for some $\alpha, \beta, \gamma \in \mathbb{N}$. Hence $(\hat{x}_{i,n+1})^\alpha \in I$ or $(\hat{x}_{j,n+1})^\beta \in I$ or $(\hat{x}_{l,n+1})^\gamma \in I$ because I is a k-ideal. Thus we obtain $\hat{x}_{i,n+1} \in \sqrt{I}$ or $\hat{x}_{j,n+1} \in \sqrt{I}$ or $\hat{x}_{l,n+1} \in \sqrt{I}$.

Subcase 2.2: Suppose that $\hat{x}_{\{i,j\},n+1}I^2 \subseteq \phi(I)$ for all $j \in \{1, 2, \ldots, n+1\} - \{i\}$.

Subcase 2.2.1: Assume $\hat{x}_{\{i,j,l\},n+1}I^3 \nsubseteq \phi(I)$ for some $l \in \{1, 2, \ldots, n+1\} - \{i, j\}$. Then there exist $p_1, p_2, p_3 \in I$ such that $\hat{x}_{\{i,j,l\},n+1}p_1p_2p_3 \notin \phi(I)$. Thus $\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_j + p_2)(x_l + p_3) \in I - \phi(I)$ because $\phi(I)$ is a k-ideal. Since I is a ϕ-Gn-absorbing ideal, we obtain $\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_j + p_2) \in \sqrt{I}$ or $\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_l + p_3) \in \sqrt{I}$ or $\hat{x}_{\{i,j,l\},n+1}(x_j + p_2)(x_l + p_3) \in \sqrt{I}$ or $\hat{x}_{\{i,j,l,h\},n+1}(x_i + p_1)(x_j +$

$p_2)(x_l + p_3) \in \sqrt{I}$ for some $h \in \{1, 2, \ldots, n+1\} - \{i, j, l\}$. Then $(\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_j + p_2))^{\alpha} \in I$ or $(\hat{x}_{\{i,j,l\},n+1}(x_i + p_1)(x_l + p_3))^{\beta} \in I$ or $(\hat{x}_{\{i,j,l\},n+1}(x_j + p_2)(x_l + p_3))^{\gamma} \in I$ or $(\hat{x}_{\{i,j,l,h\},n+1}(x_i + p_1)(x_j + p_2)(x_l + p_3))^{\delta} \in I$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{N}$. Since I is a k-ideal, we obtain $(\hat{x}_{i,n+1})^{\alpha} \in I$ or $(\hat{x}_{j,n+1})^{\beta} \in I$ or $(\hat{x}_{l,n+1})^{\gamma} \in I$ or $(\hat{x}_{h,n+1})^{\delta} \in I$. That is $\hat{x}_{i,n+1} \in \sqrt{I}$ or $\hat{x}_{j,n+1} \in \sqrt{I}$ or $\hat{x}_{l,n+1} \in \sqrt{I}$ or $\hat{x}_{h,n+1} \in \sqrt{I}$.

Subcase 2.2.2: Assume $\hat{x}_{\{i,j,l\},n+1} I^3 \subseteq \phi(I)$ for all $l \in \{1, 2, \ldots, n+1\} - \{i, j\}$.

Continue this process, it remains to show the following case.

Assume $x_{i_1} x_{i_2} \cdots x_{i_{n+1-m}} I^m \subseteq \phi(I)$ for all $\{i_1, i_2, \ldots, i_{n+1-m}\} \subseteq \{1, 2, \ldots, n+1\}$ where $1 \leq m \leq n$. Since $I_{n+1} \nsubseteq \phi(I)$, there exist $p_1, p_2, \ldots, p_{n+1} \in I$ such that $p_1 p_2 \cdots p_{n+1} \notin \phi(I)$. Then $(x_1 + p_1)(x_2 + p_2) \cdots (x_{n+1} + p_{n+1}) \in I - \phi(I)$. Since I is ϕ-Gn-absorbing, $(x_1 + p_1)(x_2 + p_2) \cdots (x_{i-1} + p_{i-1})(x_{i+1} + p_{i+1}) \cdots (x_{n+1} + p_{n+1}) \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n+1\}$. Then $((x_1 + p_1)(x_2 + p_2) \cdots (x_{i-1} + p_{i-1})(x_{i+1} + p_{i+1}) \cdots (x_{n+1} + p_{n+1}))^{\alpha} \in I$. Hence $(\hat{x}_{i,n+1})^{\alpha} \in I$ because I is a k-ideal. Thus $\hat{x}_{i,n+1} \in \sqrt{I}$.

Therefore, from any cases, we can conclude that I is a Gn-absorbing ideal. □

Corollary 5.1.21. *Let R be a semiring, n a positive integer and I a proper k-ideal of R. If I is a ϕ-Gn-absorbing ideal for some ϕ with $\phi \leq \phi_{n+2}$ such that $\phi(I)$ is a k-ideal, then I is an m-almost Gn-absorbing ideal for all $m \geq n+1$.*

Proof. The proof is similar to the proof of Corollary 4.1.25. □

Corollary 5.1.22. *Let R be a semiring. If I is a weakly Gn-absorbing k-ideal but is not Gn-absorbing, then $I^{n+1} = \{0\}$.*

Proof. Assume that I is a weakly Gn-absorbing k-ideal but is not a Gn-absorbing ideal. Since I is a weakly Gn-absorbing ideal, I is ϕ_0-Gn-absorbing. By Theorem 5.1.20, we have $I^{n+1} \subseteq \phi_0(I) = \{0\}$. Hence $I^{n+1} = \{0\}$. □

For the ideal $\{0\}$ of the semiring $\mathbb{Q}_0^+$ and a positive integer n, we know that $\{0\}^{n+1} = \{0\}$ and $\{0\}$ is an n-absorbing ideal of $\mathbb{Q}_0^+$ by Example 4.1.2 (1). Since every n-absorbing ideal is a Gn-absorbing ideal, it follows that the converse of

Corollary 5.1.22 is not true.

The next result is similar to the result derived from Theorem 4.1.27.

Theorem 5.1.23. *Let R be a semiring with ϕ, n a positive integer and I a proper ideal such that $\phi(I) \subseteq I$. Then the following statements are equivalent.*

(1) I is a ϕ-Gn-absorbing ideal.

(2) $(I : x_1x_2\cdots x_n) \subseteq \cup_{i=1}^{n}(\sqrt{I} : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n)$ for any $x_1x_2\cdots x_n \in R - \sqrt{I}$.

Proof. To show (1) $\Rightarrow$ (2), assume I is ϕ-Gn-absorbing. Let $x_1, x_2, \ldots, x_n \in R$ be such that $x_1x_2\cdots x_n \in R - \sqrt{I}$. Let $y \in (I : x_1x_2\cdots x_n)$. Then $x_1x_2\cdots x_ny \in I$. If $x_1x_2\cdots x_ny \in I - \phi(I)$, then $\hat{x}_{i,n}y \in \sqrt{I}$ for some $\{1, 2, \ldots, n\}$ because I is a ϕ-Gn-absorbing ideal and $x_1x_2\cdots x_n \notin \sqrt{I}$. Hence $y \in (\sqrt{I} : \hat{x}_{i,n})$. Otherwise, we assume that $x_1x_2\cdots x_ny \in \phi(I)$. Thus $y \in (\phi(I) : x_1x_2\cdots x_n)$. Therefore,

$$(I : x_1x_2\cdots x_n) \subseteq \cup_{i=1}^{n}(\sqrt{I} : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n).$$

To show (2) $\Rightarrow$ (1), suppose that (2) holds. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in I - \phi(I)$. If $x_1x_2\cdots x_n \in \sqrt{I}$, then we are done. Suppose that $x_1x_2\cdots x_n \notin \sqrt{I}$. By (2),

$$(I : x_1x_2\cdots x_n) \subseteq \cup_{i=1}^{n}(\sqrt{I} : \hat{x}_{i,n}) \cup (\phi(I) : x_1x_2\cdots x_n).$$

Then $x_{n+1} \in (I : x_1x_2\cdots x_n) - (\phi(I) : x_1x_2\cdots x_n)$ because $x_1x_2\cdots x_{n+1} \in I - \phi(I)$. Hence $x_{n+1} \in (\sqrt{I} : \hat{x}_{i,n})$ for some $i \in \{1, 2, \ldots, n\}$. Then we acquire $\hat{x}_{i,n}x_{n+1} \in \sqrt{I}$. Therefore, I is a ϕ-Gn-absorbing ideal. □

This section is completed by providing the result of ϕ-Gn-absorbing ideals of strongly Euclidean semirings.

Theorem 5.1.24. *Let R be a strongly Euclidean semiring, n a positive integer and $a \in R$ such that $(\langle a \rangle^2 : a) = \langle a \rangle$. Then $\langle a \rangle$ is a ϕ-Gn-absorbing ideal for some ϕ with $\phi \leq \phi_2$ if and only if $\langle a \rangle$ is a Gn-absorbing ideal.*

Proof. If $\langle a \rangle$ is a Gn-absorbing ideal, then $\langle a \rangle$ is ϕ-Gn-absorbing for any ϕ. So we assume that $\langle a \rangle$ is a ϕ-Gn-absorbing ideal for some ϕ with $\phi \leq \phi_2$. Then $\langle a \rangle$ is ϕ_2-Gn-absorbing. We show that $\langle a \rangle$ is a Gn-absorbing ideal. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1 x_2 \cdots x_{n+1} \in \langle a \rangle$. If $x_1 x_2 \cdots x_{n+1} \in \langle a \rangle - \langle a \rangle^2$, then $\hat{x}_{i,n+1} \in \sqrt{\langle a \rangle}$ for some $i \in \{1, 2, \ldots, n+1\}$ because $\langle a \rangle$ is ϕ_2-Gn-absorbing. So we can assume that $x_1 x_2 \cdots x_{n+1} \in \langle a \rangle^2$. Since R is strongly Euclidean, $\langle a \rangle$ and $\langle a^2 \rangle$ are k-ideals. Now, $(x_1 + a) x_2 \cdots x_{n+1} = x_1 x_2 \cdots x_{n+1} + a x_2 x_3 \cdots x_{n+1} \in \langle a \rangle$.

Case 1: Assume that $(x_1 + a) x_2 \cdots x_{n+1} \in \langle a \rangle - \langle a \rangle^2$. Since $\langle a \rangle$ is ϕ_2-Gn-absorbing, $x_2 x_3 \cdots x_{n+1} \in \sqrt{\langle a \rangle}$ or $(x_1 + a)\hat{x}_{i,n+1} \in \sqrt{\langle a \rangle}$ for some $i \in \{2, 3, \ldots n+1\}$. If $x_2 x_3 \cdots x_{n+1} \in \sqrt{\langle a \rangle}$, then we are done. Assume that $(x_1 + a)\hat{x}_{i,n+1} = x_1 \hat{x}_{i,n+1} + a \hat{x}_{i,n+1} \in \sqrt{\langle a \rangle}$. Since $\sqrt{\langle a \rangle}$ is a subtractive extension of $\langle a \rangle$, we obtain $x_1 \hat{x}_{i,n+1} \in \sqrt{\langle a \rangle}$.

Case 2: Assume that $(x_1 + a) x_2 \cdots x_{n+1} \in \langle a \rangle^2 = \langle a^2 \rangle$. Since $\langle a^2 \rangle$ is a k-ideal and $x_1 x_2 \cdots x_{n+1}, x_1 x_2 \cdots x_{n+1} + a x_2 x_3 \cdots x_{n+1} \in \langle a^2 \rangle$, we obtain $a x_2 x_3 \cdots x_{n+1} \in \langle a^2 \rangle$. Thus $x_2 x_3 \cdots x_{n+1} \in (\langle a \rangle^2 : a) = \langle a \rangle$.

Therefore $\langle a \rangle$ is a Gn-absorbing k-ideal. □

5.2 ϕ-Gn-Absorbing Ideals in Decomposable Semirings

In this section, Gn-absorbing ideals, weakly Gn-absorbing ideals and ϕ-Gn-absorbing ideals of decomposable semirings are taken care of. Almost all of the results in this section are parallel to the results in Section 4.2.

The following proposition is parallel to Proposition 3.2.1 and Proposition 4.2.1.

Proposition 5.2.1. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ where $m, n \in \mathbb{N}$ with $m \geq n+1$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper ideal of R. If I is a weakly Gn-absorbing ideal, then $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$.*

Proof. The proof is similar to Proposition 3.2.1 and Proposition 4.2.1. □

In general, Gn-absorbing ideals implies weakly Gn-absorbing ideals but not vice versa. Nevertheless, in a decomposable semiring with at least $n+1$ components, weakly Gn-absorbing ideals and Gn-absorbing ideals are coincide if they are nonzero proper k-ideals.

Proposition 5.2.2. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ where $m, n \in \mathbb{N}$ with $m \geq n+1$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a nonzero proper k-ideal of R. Then I is a weakly Gn-absorbing ideal if and only if I is a Gn-absorbing ideal of R.*

Proof. Assume that I is a weakly Gn-absorbing ideal of R. By Proposition 5.2.1, we obtain $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$. Thus $I^{n+1} \neq \{0\}$. Therefore, I is a Gn-absorbing ideal by Corollary 5.1.22. The converse follows from Corollary 5.1.18. □

We obtain from Proposition 5.2.1 that being a nonzero ideal of I and the condition that $m \geq n+1$ give that there is at least one of I_i must not be proper and it leads us to conclude that weakly Gn-absorbing ideals and Gn-absorbing ideals are coincide in Proposition 5.2.2. In the next theorem, we assume that $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$. Hence the condition that I is a nonzero ideal and $m \geq n+1$ need not be assumed.

Theorem 5.2.3. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R which at least one $I_i = R_i$ where $i \in \{1, 2, \ldots, m\}$. Consider the following statements:*

(1) I is a weakly Gn-absorbing ideal of R.

(2) I is a Gn-absorbing ideal of R.

(3) If $I_j \neq R_j$ where $j \in \{1, 2, \ldots, m\}$, then I_j is a Gn-absorbing ideal of R_j.

Then (1) *and* (2) *are equivalent and* (2) *implies* (3).

Proof. Obviously, (2) $\Rightarrow$ (1).

To show (1) $\Rightarrow$ (2), assume that I is a weakly Gn-absorbing ideal of R. Note that $I^{n+1} \neq \{0\}$ since $I_i = R_i$ for some $i \in \{1, 2, \ldots, m\}$. Thus I is a Gn-absorbing ideal of R by Corollary 5.1.22.

To show (2) $\Rightarrow$ (3), assume that I is a Gn-absorbing ideal of R and $I_j \neq R_j$ for some $j \in \{1, 2, \ldots, m\}$. Let $x_1, x_2, \ldots, x_{n+1} \in R_j$ be such that $x_1 x_2 \cdots x_{n+1} \in I_j$. Then

$$(0, \ldots, 0, x_1, 0, \ldots, 0)(0, \ldots, 0, x_2, 0, \ldots, 0) \cdots (0, \ldots, 0, x_{n+1}, 0, \ldots, 0)$$

$$= (0, \ldots, 0, x_1 x_2 \cdots x_{n+1}, 0, \ldots, 0) \in I.$$

Thus $(0, \ldots, 0, \hat{x}_{i,n+1}, 0, \ldots, 0) \in \sqrt{I} = \sqrt{I_1 \times I_2 \times \cdots \times I_m} = \sqrt{I_1} \times \sqrt{I_2} \times \cdots \times \sqrt{I_m}$ for some $i \in \{1, 2, \ldots n+1\}$ because I is a Gn-absorbing ideal. Hence $\hat{x}_{i,n+1} \in \sqrt{I_j}$. Therefore, I_j is a Gn-absorbing ideal of R_j. $\square$

In Theorem 5.2.3, we show that if I is a Gn-absorbing ideal (weakly Gn-absorbing ideal) of R, then any proper ideal I_j of R_j is a Gn-absorbing ideal of R_j where $j \in \{1, 2, \ldots, m\}$. Next example confirms that the converse is not true.

Example 5.2.4. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and its ideal $I = 30\mathbb{Z}_0^+ \times 70\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. We know that $30\mathbb{Z}_0^+$ and $70\mathbb{Z}_0^+$ are $G3$-absorbing ideal of the semiring $\mathbb{Z}_0^+$. We would like to show that I is not a weakly $G3$-absorbing ideal of the semiring $\mathbb{Z}_0^+$. Since

$$0 \neq (2, 1, 1)(3, 2, 1)(5, 5, 1)(1, 7, 1) = (30, 70, 1) \in I$$

but

$$((2, 1, 1)(3, 2, 1)(5, 5, 1))^\alpha = (30, 10, 1)^\alpha \notin I \text{ because } 10^\alpha \notin 70\mathbb{Z}_0^+,$$
$$((2, 1, 1)(3, 2, 1)(1, 7, 1))^\beta = (6, 14, 1)^\beta \notin I \text{ because } 6^\beta \notin 30\mathbb{Z}_0^+,$$
$$((2, 1, 1)(5, 5, 1)(1, 7, 1))^\gamma = (10, 35, 1)^\gamma \notin I \text{ because } 10^\gamma \notin 30\mathbb{Z}_0^+ \text{ and}$$
$$((3, 2, 1)(5, 5, 1)(1, 7, , 1))^\delta = (6, 70, 1)^\delta \notin I \text{ because } 6^\delta \notin 30\mathbb{Z}_0^+$$

for all $\alpha, \beta, \gamma \in \mathbb{N}$. Then $(2, 1, 1)(3, 2, 1)(5, 5, 1) \notin \sqrt{I}$, $(2, 1, 1)(3, 2, 1)(1, 7, 1) \notin \sqrt{I}$, $(2, 1, 1)(5, 5, 1)(1, 7, 1) \notin \sqrt{I}$ and $(3, 2, 1)(5, 5, 1)(1, 7, , 1) \notin \sqrt{I}$. Therefore, we can conclude that I is not a weakly $G3$-absorbing ideal of R, and so I is not a $G3$-absorbing ideal of R.

In Theorem 5.2.3, at least one $I_i = R_i$ for $i \in \{1, 2, \ldots, m\}$ is assumed which is not the sufficient condition to make (3) imply (1) or (2). In the next theorem, we assume a stronger condition in order to make (1), (2) and (3) be equivalent.

Theorem 5.2.5. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. Then the following statements are equivalent.*

(1) I is a weakly Gn-absorbing ideal of R.

(2) I is a Gn-absorbing ideal of R.

(3) I_i is a Gn-absorbing ideal of R_i.

Proof. Theorem 5.2.3 yields (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3).

To show (3) $\Rightarrow$ (2), assume I_i is a Gn-absorbing ideal of R_i. Let $(x_{11}, \ldots, x_{1m})$, $(x_{21}, \ldots, x_{2m})$, $\ldots$, $(x_{(n+1)1}, \ldots, x_{(n+1)m}) \in R$ be such that

$$(x_{11}x_{21} \cdots x_{(n+1)1}, \ldots, x_{1i}x_{2i} \cdots x_{(n+1)i}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I.$$

In fact, $I = R_1 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$. Since I_i is a Gn-absorbing ideal of R_i, we obtain $\hat{x}_{ji,(n+1)i} \in \sqrt{I_i}$ for some $j \in \{1, 2, \ldots, n+1\}$. Hence $(x_{11}, \ldots, x_{1m}) \cdots (x_{(j-1)1}, \ldots, x_{(j-1)m})(x_{(j+1)1}, \ldots, x_{(j+1)m}) \cdots (x_{(n+1)1}, \ldots, x_{(n+1)m}) \in R_1 \times \cdots \times R_{i-1} \times \sqrt{I_i} \times R_{i+1} \times \cdots \times R_m = \sqrt{I}$. Therefore, I is a Gn-absorbing ideal of R. $\square$

Corollary 5.2.6. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring with ϕ, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R which exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. If I_i is a Gn-absorbing ideal of R_i, then I is a ϕ-Gn-absorbing ideal of R.*

Corollary 5.2.6 shows that if I_i is a Gn-absorbing ideal of R_i, then $I_1 \times I_2 \times \cdots \times I_m$ is a ϕ-Gn-absorbing ideal of $R_1 \times \cdots \times R_m$ but not vice versa.

Example 5.2.7. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and the ideal $I = 30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$. Since $2 \cdot 3 \cdot 5 = 30 \in 30\mathbb{Z}_0^+$ but $(2 \cdot 3)^\alpha = 6^\alpha \notin 30\mathbb{Z}_0^+$ and $(2 \cdot 5)^\beta = 10^\beta \notin 30\mathbb{Z}_0^+$ and $(3 \cdot 5)^\gamma = 15^\gamma \notin 30\mathbb{Z}_0^+$ for all $\alpha, \beta, \gamma \in \mathbb{N}$, i.e., $2 \cdot 3 \notin \sqrt{30\mathbb{Z}_0^+}$, $2 \cdot 5 \notin \sqrt{30\mathbb{Z}_0^+}$ and $3 \cdot 5 \notin \sqrt{30\mathbb{Z}_0^+}$, it follows that the ideal $30\mathbb{Z}_0^+$ is not a $G2$-absorbing ideal of the semiring $\mathbb{Z}_0^+$. We define $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ by $\phi(30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+) = 15\mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ and $\phi(J) = J$ otherwise. Then $I - \phi(I) = (30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+) - \phi(30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+) = (30\mathbb{Z}_0^+ \times \mathbb{Z}_0^+) - (15\mathbb{Z}_0^+ \times \mathbb{Z}_0^+) = \varnothing$. Thus the ideal I is a ϕ-$G2$-absorbing ideal of R.

Let $R = R_1 \times \cdots \times R_m$ be a decomposable semiring with ϕ and $I = I_1 \times \cdots \times I_m$ a proper ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. We know that, if I_i is a Gn-absorbing ideal of R_i, then I is a ϕ-Gn-absorbing ideal of R. However, if I_i is a weakly Gn-absorbing ideal of R_i, then I need not be a weakly Gn-absorbing ideal of R but it must be a ϕ-Gn-absorbing ideal of R of for all $\phi_\omega \leq \phi$ if I_i is a k-ideal.

Theorem 5.2.8. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper k-ideal of R with exactly one $I_i \neq R_i$ where $i \in \{1, 2, \ldots, m\}$. If I_i is a weakly Gn-absorbing ideal of R_i, then I is a ϕ-Gn-absorbing ideal of R for all $\phi_\omega \leq \phi$.*

Proof. Without loss of generality, we assume that $i = 1$. We would like to show that $I = I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-Gn-absorbing ideal of R for all $\phi_\omega \leq \phi$. Since I is a k-ideal, it follows that I_1 is a k-ideal by Proposition 2.1.28. First, suppose that I_1 is a Gn-absorbing ideal. By Theorem 5.2.5, I is a Gn-absorbing ideal. Hence I is a ϕ_ω-Gn-absorbing ideal. So assume that I_1 is not a Gn-absorbing ideal. Then $I_1^{n+1} = \{0\}$ from Corollary 5.1.22. Consider the element $(x_1, \ldots, x_m) \in \phi_\omega(I) = \cap_{n=1}^{\infty} I^n \subseteq I^{n+1} = (I_1 \times R_2 \times \cdots \times R_m)^{n+1} \subseteq I_1^{n+1} \times R_2 \times \cdots \times R_m = \{0\} \times R_2 \times \cdots \times R_m$. We show that I is a ϕ_ω-Gn-absorbing ideal. Let $(x_{11}, \ldots, x_{1m}), (x_{21}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, \ldots, x_{(n+1)m}) \in R$ be such that $(x_{11}x_{21} \cdots x_{(n+1)1}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I - \phi_\omega(I)$. Then $x_{11}x_{21} \cdots x_{(n+1)1} \in I_1 - \{0\}$. Since I_1 is a weakly Gn-absorbing ideal, we obtain $\hat{x}_{j1,(n+1)1} \in \sqrt{I_1}$ for some $j \in \{1, 2, \ldots, n+1\}$. Hence $(\hat{x}_{j1,(n+1)1}, \hat{x}_{j2,(n+1)2}, \ldots, \hat{x}_{jm,(n+1)m}) \in \sqrt{I_1} \times$

$R_2 \times \cdots \times R_m = \sqrt{I}$. Thus I is a ϕ_ω-Gn-absorbing ideal. Therefore, in any cases, I is a ϕ_ω-Gn-absorbing ideal, and so I is a ϕ-Gn-absorbing ideal for all $\phi_\omega \leq \phi$. □

The following result is parallel to Theorem 4.2.9.

Theorem 5.2.9. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer with $n \geq 2$ and $I = I_1 \times I_2 \times \cdots \times I_m$ where $I_i \neq \{0\}$ for all $i \in \{1, 2, \ldots, m\}$ is a weakly Gn-absorbing k-ideal. Then I is a Gn-absorbing ideal of R or I_i is a $G(n-1)$-absorbing ideal of R_i for all $i \in \{1, 2, \ldots, m\}$.*

Proof. If I is an Gn-absorbing ideal of R, then we are done. Suppose that I is not a Gn-absorbing ideal of R. Then $I^{n+1} = \{0\}$ by Corollary 5.1.22. Hence $I_j \neq R_j$ for all $j \in \{1, 2, \ldots, m\}$. Let $j \in \{1, 2, \ldots, m\}$. We would like to show that I_j is an $(n-1)$-absorbing ideal of R_j. Let $x_1, x_2, \ldots, x_n \in R_j$ be such that $x_1 x_2 \cdots x_n \in I_j$. Let $i \in \{1, 2, \ldots, m\}$ be such that $i \neq j$, without lost of generality, we assume that $j < i$. Since $I_i \neq \{0\}$, there exists $0 \neq y_i \in I_i$. Then

$$(0, \ldots, 0) \neq (0, \ldots, 0, x_1 x_2 \cdots x_n, 0, \ldots, 0, y_i, 0, \ldots, 0) \in I.$$

Thus

$$(0, \ldots, 0) \neq (0, \ldots, 0, x_1, 0, \ldots, 0, 1, 0, \ldots, 0)(0, \ldots, 0, x_2, 0, \ldots, 0, 1, 0, \ldots, 0) \cdots$$
$$(0, \ldots, 0, x_n, 0, \ldots, 0, 1, 0, \ldots, 0)(0, \ldots, 0, 1, 0, \ldots, 0, y_i, 0, \ldots, 0) \in I.$$

Since I is a weakly Gn-absorbing ideal, $1 \in \sqrt{I_i}$ or $\hat{x}_{l,n} \in \sqrt{I_j}$ for some $l \in \{1, 2, \ldots, n\}$. Since $I_i \neq R_i$, we obtain $1 \notin \sqrt{I_i}$, and hence $\hat{x}_{l,n} \in \sqrt{I_j}$. Therefore, I_j is a $G(n-1)$-absorbing ideal of R_j. □

In Chapter IV, we obtain that if $I = I_1 \times I_2 \times \cdots \times I_m$ is a proper ideal of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$ with exactly two prime proper ideals I_i and I_j of R_i and R_j, respectively, then I is a 2-absorbing ideal of R. In the following result, we change the condition that I_i and I_j are prime ideals to I_i and I_j are primary ideals of R_i and R_j, respectively, then we obtain the similar result.

Proposition 5.2.10. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ a proper ideal of R which exactly two $I_i \neq R_i$ and*

$I_j \neq R_j$ where $i, j \in \{1, 2, \ldots, m\}$. If I_i and I_j are primary ideals of R_i and R_j, respectively, then I is a $G2$-absorbing ideal of R so that I is a ϕ-$G2$-absorbing ideal. As a result, this I is a Gn-absorbing ideal of R so that I is a ϕ-Gn-absorbing ideal where $n \in \mathbb{N}$ with $n \geq 2$.

Proof. Assume that I_i and I_j are primary ideals of R_i and R_j, respectively. To show that I is a $G2$-absorbing ideal of R, let $(x_{11}, \ldots, x_{1m}), (x_{21}, \ldots, x_{2m}), (x_{31}, \ldots, x_{3m}) \in R$ be such that $(x_{11}x_{21}x_{31}, \ldots, x_{1m}x_{2m}x_{3m}) \in I$. Then at least one of x_{ni}^{α} belongs to I_i for some $n \in \{1, 2, 3\}$ and for some $\alpha \in \mathbb{N}$ and at least one of x_{lj}^{β} belongs to I_j for some $l \in \{1, 2, 3\}$ and for some $\beta \in \mathbb{N}$. Thus $(x_{ni}x_{li})^{\alpha\beta} \in I_i$ and $(x_{nj}x_{lj})^{\alpha\beta} \in I_j$. Hence $((x_{n1}, \ldots, x_{nm})(x_{l1}, \ldots, x_{lm}))^{\alpha\beta} \in I$, i.e, $(x_{n1}, \ldots, x_{nm})(x_{l1}, \ldots, x_{lm}) \in \sqrt{I}$. Therefore, I is a $G2$-absorbing ideal of R so that I is a ϕ-$G2$-absorbing ideal of R. Moreover, we can conclude that I is a Gn-absorbing ideal of R for $n \geq 2$ and then I is a ϕ-Gn-absorbing ideal of R for all $n \geq 2$. □

The next theorem is parallel to Theorem 4.2.10.

Theorem 5.2.11. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring and $I = I_1 \times I_2 \times \cdots \times I_m$ an ideal of R. If I_i is a Gn_i-absorbing ideal of R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$, then I is a Gn-absorbing ideal of R where $n = n_1 + n_2 + \cdots + n_m$, so that I is a ϕ-Gn-absorbing ideal of R.*

Proof. Assume that I_i is a Gn_i-absorbing ideal of R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$. Let $n = n_1 + n_2 + \cdots + n_m$.We show that I is a Gn-absorbing ideal. Let $(x_{11}, x_{12}, \ldots, x_{1m}), (x_{21}, x_{22}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in R$ be such that

$$(x_{11}, x_{12}, \ldots, x_{1m})(x_{21}, x_{22}, \ldots, x_{2m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m})$$
$$= (x_{11}x_{21} \cdots x_{(n+1)1}, x_{12}x_{22} \cdots x_{(n+1)2}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \in I.$$

For each i, since I_i is a Gn_i-absorbing ideal, $x_{1i}x_{2i} \cdots x_{(n+1)i} \in I_i$ and $n_i < n+1$, we obtain $x_{j_1 i}x_{j_2 i} \cdots x_{j_{n_i} i} \in \sqrt{I_i}$ for some distinct $j_1, j_2, \ldots, j_{n_i} \in \{1, 2, \ldots, n+1\}$ by Theorem 5.1.2. Suppose that $\cup_{i=1}^{m}\{j_1, j_2, \ldots, j_{n_i}\} = \{j'_1, j'_2, \ldots, j'_h\}$. Thus

$\{j'_1, j'_2, \ldots, j'_h\} \subseteq \{1, 2, \ldots, n+1\}$ and $h \leq n$ since $n_1 + n_2 + \cdots + n_m = n$. Since $\{j_1, j_2, \ldots, j_{n_i}\} \subseteq \{j'_1, j'_2, \ldots, j'_h\}$ and $x_{j_1 i} x_{j_2 i} \cdots x_{j_{n_i} i} \in \sqrt{I_i}$ for all $i \in \{1, 2, \ldots, m\}$, we obtain

$$x_{j'_1 i} x_{j'_2 i} \cdots x_{j'_h i} \in \sqrt{I_i}.$$

By choosing all distinct $j'_{h+1}, j'_{h+2}, \ldots, j'_n \in \{1, 2, \ldots, n+1\} - \{j'_1, j'_2, \ldots, j'_h\}$,

$$x_{j'_1 i} x_{j'_2 i} \cdots x_{j'_n i} = (x_{j'_1 i} x_{j'_2 i} \cdots x_{j'_h i})(x_{j'_{h+1} i} x_{j'_{h+2} i} \cdots x_{j'_n i}) \in \sqrt{I_i}.$$

Then we obtain

$$\begin{aligned}
(x_{j'_1 1}, x_{j'_1 2}, \ldots, x_{j'_1 m})(x_{j'_2 1}, x_{j'_2 2}, \ldots, x_{j'_2 m}) \cdots (x_{j'_n 1}, x_{j'_n 2}, \ldots, x_{j'_n m}) \\
&= (x_{j'_1 1} x_{j'_2 1}, \cdots x_{j'_n 1},\ x_{j'_1 2} x_{j'_2 2} \cdots x_{j'_n 2},\ \ldots,\ x_{j'_1 m} x_{j'_2 m} \cdots x_{j'_n m}) \\
&\in \sqrt{I_1} \times \cdots \times \sqrt{I_m} = \sqrt{I}.
\end{aligned}$$

Therefore, I is a Gn-absorbing ideal of R, and hence I is a ϕ-Gn-absorbing ideal of R. □

Example 5.2.12. Consider the semiring $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$.

(1) Then $2^3\mathbb{Z}_0^+ \times 2^2 3^4 \mathbb{Z}_0^+ \times 2^2 3^4 5^3 \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ is a $G6$-absorbing ideal of R because $2^2\mathbb{Z}_0^+$ is a $G1$-absorbing ideal, $2^2 3^4 \mathbb{Z}_0^+$ is a $G2$-absorbing ideal, $2^2 3^4 5^3 \mathbb{Z}_0^+$ is a $G3$-absorbing ideal and $\mathbb{Z}_0^+$ is a $G0$-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

(2) Then $2^2\mathbb{Z}_0^+ \times 2^3\mathbb{Z}_0^+ \times 2^4\mathbb{Z}_0^+ \times 2^5\mathbb{Z}_0^+$ is a $G4$-absorbing ideal of R because $2^2\mathbb{Z}_0^+$, $2^3\mathbb{Z}_0^+$, $2^4\mathbb{Z}_0^+$ and $2^5\mathbb{Z}_0^+$ are $G1$-absorbing ideal of the semiring $\mathbb{Z}_0^+$.

In the last result of this section, we consider ϕ-Gn-absorbing ideals of decomposable semirings.

Theorem 5.2.13. *Let $R = R_1 \times R_2 \times \cdots \times R_m$ be a decomposable semiring, n a positive integer and $\phi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_m$ where each $\varphi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\varnothing\}$ is a function. Then the following statements hold.*

(1) $I_1 \times I_2 \times \cdots \times I_m$ is a ϕ-Gn-absorbing ideal of R where $I_j \subseteq \varphi_j(I_j)$ for all $j \in \{1, 2, \ldots, m\}$ and at least one I_i is a proper ideal of R_i for some $i \in \{1, 2, \ldots, m\}$.

(2) $R_1 \times R_2 \times \cdots \times R_{i-1} \times I_i \times R_{i+1} \times \cdots \times R_m$ *is a* ϕ*-*Gn*-absorbing ideal of* R *where* I_i *is a* φ_i*-*Gn*-absorbing ideal of* R_i *which must be a* Gn*-absorbing ideal if* $\varphi_j(R_j) \neq R_j$ *for some* $j \in \{1, 2, \ldots, m\} - \{i\}$*.*

Proof. (1) The result follows from the fact that $I_1 \times I_2 \times \cdots \times I_m - \phi(I_1 \times I_2 \times \cdots \times I_m) = \varnothing$.

(2) Without loss of generality, we assume that I_1 is a proper ideal of R_1. If I_1 is a Gn-absorbing ideal of R_1, then $I_1 \times R_2 \times \cdots \times R_m$ is a Gn-absorbing ideal of R by Theorem 5.2.5. Hence $I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-Gn-absorbing ideal of R. Suppose that I_1 is a φ_1-Gn-absorbing ideal of R_1 and $\varphi_j(R_j) = R_j$ for all $j \in \{1, 2, \ldots, m\}$. Let $(x_{11}, x_{12}, \ldots, x_{1m}), (x_{21}, x_{22}, \ldots, x_{2m}), \ldots, (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in R_1 \times R_2 \times \cdots \times R_m$ be such that

$$\begin{aligned}
(x_{11},&x_{12}, \ldots, x_{1m})(x_{21}, x_{22}, \ldots, x_{2m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \\
&= (x_{11}x_{21} \cdots x_{(n+1)1}, x_{21}x_{22} \cdots x_{(n+1)2}, \ldots, x_{1m}x_{2m} \cdots x_{(n+1)m}) \\
&\in I_1 \times R_2 \times \cdots \times R_m - \phi(I_1 \times R_2 \times \cdots \times R_m) \\
&= I_1 \times R_2 \times \cdots \times R_m - (\varphi_1(I_1) \times \varphi_2(R_2) \times \cdots \times \varphi_m(R_m)) \\
&= I_1 \times R_2 \times \cdots \times R_m - (\varphi_1(I_1) \times R_2 \times \cdots \times R_m) \\
&= (I_1 - \varphi_1(I_1)) \times R_2 \times \cdots \times R_m.
\end{aligned}$$

Since I_1 is a φ_1-Gn-absorbing ideal of R_1, we obtain $\hat{x}_{i1,(n+1)1} \in \sqrt{I_1}$ for some $i \in \{1, 2, \ldots, n+1\}$. Thus $(x_{11}, x_{12}, \ldots, x_{1m}) \cdots (x_{(i-1)1}, x_{(i-1)2}, \ldots, x_{(i-1)m})(x_{(i+1)1}, x_{(i+1)2}, \ldots, x_{(i+1)m}) \cdots (x_{(n+1)1}, x_{(n+1)2}, \ldots, x_{(n+1)m}) \in \sqrt{I_1} \times R_2 \times \cdots \times R_m = \sqrt{I_1 \times R_2 \times \cdots \times R_m}$. Therefore, $I_1 \times R_2 \times \cdots \times R_m$ is a ϕ-Gn-absorbing ideal of R. □

5.3 ϕ-Gn-Absorbing Ideals in Quotient Semirings and in Semirings of Fractions

In this final section, we investigate ϕ-Gn-absorbing ideals of quotient semirings and ϕ-Gn-absorbing ideals of semirings of fractions.

Theorem 5.3.1. *Let R be a semiring with ϕ satisfying the property $(*)$, n a positive integer, I a Q-ideal of R and P a subtractive extension of I. Then P is a ϕ-Gn-absorbing ideal of R if and only if P/I is a ϕ_I-Gn-absorbing ideal of R/I.*

Proof. First, assume that P is a ϕ-Gn-absorbing ideal of R. Since P is a subtractive extension of I, we obtain P/I is a k-ideal. Let $q_1 + I, q_2 + I, \ldots, q_{n+1} + I \in R/I$ be such that $(q_1 + I)(q_2 + I)\cdots(q_{n+1} + I) \in P/I - \phi_I(P/I)$. Then $q_1q_2\cdots q_{n+1} \in P - \phi(P)$ by Theorem 2.2.19. Since P is a ϕ-Gn-absorbing ideal, $\hat{q}_{i,n+1} \in \sqrt{P}$ for some $i \in \{1, 2, \ldots, n+1\}$. Then $(q_1+I)\cdots(q_{i-1,n+1}+I)(q_{i+1,n+1}+I)\cdots(q_{n+1}+I) \in \sqrt{P}/I = \sqrt{P/I}$ by Proposition 2.2.18. Therefore, P/I is a ϕ_I-Gn-absorbing k-ideal of R/I.

Conversely, suppose that P/I is a ϕ_I-Gn-absorbing ideal of R/I. We show that P is a ϕ-Gn-absorbing ideal of R. Let $x_1, x_2, \ldots, x_{n+1} \in R$ be such that $x_1x_2\cdots x_{n+1} \in P - \phi(P)$. Then there exist $q_1, q_2, \ldots, q_{n+1} \in Q$ such that $x_i \in q_i + I$ for all $i \in \{1, 2, \ldots, n+1\}$. Thus there are $y_i \in I$ such that $x_i = q_i + y_i$ for all $i \in \{1, 2, \ldots, n+1\}$. Then we obtain $(q_1+y_1)(q_2+y_2)\cdots(q_{n+1}+y_{n+1}) \in P - \phi(P)$. Since P and $\phi(P)$ are subtractive extensions of I, we acquire $q_1q_2\cdots q_{n+1} \in P - \phi(P)$. By Theorem 2.2.19, we obtain $(q_1 + I)(q_2 + I)\cdots(q_{n+1} + I) \in P/I - \phi_I(P/I)$. Since P/I is a ϕ_I-Gn-absorbing ideal, $(q_1 + I)\cdots(q_{i-1} + I)(q_{i+1} + I)\cdots(q_{n+1} + I) \in \sqrt{P/I} = \sqrt{P}/I$ for some $i \in \{1, 2, \ldots, n+1\}$ by Proposition 2.2.18. Then $\hat{q}_{i,n+1} \in \sqrt{P}$. Hence $\hat{x}_{i,n+1} = (q_1 + y_1)\cdots(q_{i-1} + y_{i-1})(q_{i+1} + y_{i+1})\cdots(q_{n+1} + y_{n+1}) \in \sqrt{P}$. Therefore, P is a ϕ-Gn-absorbing ideal of R. □

Example 5.3.2. Consider the semiring $\mathbb{Z}_0^+$. Let $P = 20\mathbb{Z}_0^+$ and $I = 60\mathbb{Z}_0^+$. Then P is a $G2$-absorbing k-ideal of $\mathbb{Z}_0^+$ containing I and I is a Q-ideal of $\mathbb{Z}_0^+$ where $Q = \{0, 1, 2, 3, \ldots, 59\}$. Then P is a subtractive extension of I. Define $\phi : \mathscr{I}(\mathbb{Z}_0^+) \to \mathscr{I}(\mathbb{Z}_0^+) \cup \{\varnothing\}$ by $\phi(J) = 5\mathbb{Z}_0^+$ for all $J \in \mathscr{I}(\mathbb{Z}_0^+)$ where J is a subtractive extension of I and $\phi(J) = \{0\}$ otherwise. Since $5\mathbb{Z}_0^+$ is a subtractive extension of $I = 60\mathbb{Z}_0^+$, $\phi(L)$ is a subtractive extension of I for al $L \in \mathscr{I}(R)$ where L is a subtractive extension of I. Moreover, we define $\phi_I : \mathscr{I}(R/I) \to \mathscr{I}(R/I) \cup \{\varnothing\}$ by $\phi_I(J/I) = (5\mathbb{Z}_0^+)/I$ for each ideal J of R where J is a subtractive extension of I. Hence $\mathbb{Z}_0^+$

is the semiring with ϕ satisfying the property $(*)$. Since P s a $G2$-absorbing ideal, P is a ϕ-$G2$-absorbing ideal. Therefore, $P/I = 20\mathbb{Z}_0^+/60\mathbb{Z}_0^+$ is a ϕ_I-$G2$-absorbing ideal of the quotient semiring $\mathbb{Z}_0^+/60\mathbb{Z}_0^+$.

Corollary 5.3.3. *Let R be a semiring with ϕ satisfying the property $(*)$, n a positive integer and I a Q-ideal of R. Then I is a ϕ-Gn-absorbing ideal of R if and only if the zero ideal of R/I is a ϕ_I-Gn-absorbing ideal.*

Finally, we show that if I is a ϕ-Gn-absorbing ideal of R under some conditions, then IR_S is a ϕ_S-Gn-absorbing ideal of R_S.

Theorem 5.3.4. *Let R be a semiring with ϕ, S the set of all multiplicatively cancellable elements of R and I an ideal of R with $I \cap S = \varnothing$ and $\phi(I)R_S \subseteq \phi_S(IR_S)$. If I is a ϕ-Gn-absorbing ideal of R, then IR_S is a ϕ_S-Gn-absorbing ideal of R_S.*

Proof. Assume that I is a ϕ-Gn-absorbing ideal of R. Then IR_S is a proper ideal of R_S because $I \cap S = \varnothing$. Let $\frac{x_1}{s_1}, \frac{x_2}{s_2} \ldots, \frac{x_{n+1}}{s_{n+1}} \in R_S$ be such that $\frac{x_1 x_2 \cdots x_{n+1}}{s_1 s_2 \cdots s_{n+1}} \in IR_S - \phi_S(IR_S)$. By Theorem 2.3.8, we have $x_1 x_2 \cdots x_{n+1} v = x_1 x_2 \cdots x_n (x_{n+1} v) \in I - \phi(I)$ for some $v \in S$. Since I is ϕ-Gn-absorbing, $x_1 \cdots x_n \in \sqrt{I}$ or $\hat{x}_{i,n} x_{n+1} v \in \sqrt{I}$ for some $i \in \{1, 2, \ldots, n\}$. Thus $\frac{x_1 \cdots x_n}{s_1 \cdots s_n} \in \sqrt{I} R_S = \sqrt{IR_S}$ or $\frac{\hat{x}_{i,n} x_{n+1} v}{\hat{s}_{i,n} s_{n+1} v} \in \sqrt{I} R_S = \sqrt{IR_S}$. Hence $\frac{\hat{x}_{j,n+1}}{\hat{s}_{j,n+1}} \in \sqrt{IR_S}$ for some $j \in \{1, 2, \ldots, n+1\}$. Therefore, IR_S is a ϕ_S-Gn-absorbing ideal of R_S. □

CHAPTER VI
CONCLUSIONS

In this dissertation, we introduce many new algebraic objects in semirings and ones of those important are ϕ-primary ideals, ϕ-n-absorbing ideals and ϕ-generalized-n-absorbing ideals of semirings. The given concept of ϕ-primary ideals of semirings that sustains the concepts of primary ideals and weakly primary ideals of semirings that are defined by others. Moreover, the notion of ϕ-n-absorbing ideals sustains the notion of prime ideals, weakly prime ideals, almost prime ideals, 2-absorbing ideals and n-absorbing ideals of rings which are defined before. In this research, we find that ϕ-primary ideals and ϕ-n-absorbing ideals do not imply each other; nevertheless, all of them imply ϕ-generalized-n-absorbing ideals. In our work, ϕ-primary ideals, ϕ-n-absorbing ideals and ϕ-generalized-n-absorbing ideals are investigated in four categories that are semirings, decomposable semirings, quotient semirings and semirings of fractions.

In semirings, we can conclude that being k-ideals of I and $\phi(I)$ and the condition that $I^2 \subseteq \phi(I)$ $(I^{n+1} \subseteq \phi(I))$ are sufficient conditions for ϕ-primary ideals (ϕ-n-absorbing ideals, ϕ-generalized-n-absorbing ideals) to be primary ideals (n-absorbing ideals, generalized n-absorbing ideals). Moreover, we observe that n-absorbing ideals are n'-absorbing ideals for all positive integers $n' \geq n$. This leads us to consider in the case of ϕ-n-absorbing ideals and it follows that ϕ-n-absorbing ideals are ϕ-n'-absorbing ideals for all positive integer $n' \geq n$. In addition, we provide some forms of n-absorbing ideals and generalized n-absorbing ideals of the particular semiring $\mathbb{Z}_0^+$. The attractiveness is that if whatever principal ideal I of the semiring $\mathbb{Z}_0^+$ is considered, then we can find $n, m \in \mathbb{N}$ such that the ideal I is both an n-absorbing ideal and a generalized m-absorbing ideal of $\mathbb{Z}_0^+$.

In decomposable semirings, relationships between ϕ-primary ideals (ϕ-n-absorbing ideals, ϕ-generalized-n-absorbing ideals) of a direct product of semirings and weakly primary ideals (weakly n-absorbing ideals, weakly generalized n-absorbing ideals) of some components of such direct product are inspected. In addition, we obtain more beautiful results when we find that if I_i is an n_i-absorbing ideal (a generalized n_i-absorbing ideal) of a semiring R_i where $n_i \in \mathbb{Z}_0^+$ for all $i \in \{1, 2, \ldots, m\}$, then $I = I_1 \times I_2 \times \cdots \times I_m$ is an n-absorbing ideal (a generalized n-absorbing ideal) of a decomposable semiring $R = R_1 \times R_2 \times \cdots \times R_m$ where $n = n_1 + n_2 + \cdots + n_m$ so that I is a ϕ-n-absorbing ideal (ϕ-generalized n-absorbing ideal).

In quotient semirings and semirings of fractions, we associate relations between ϕ-primary ideals (ϕ-n-absorbing ideals, ϕ-generalized-n-absorbing ideals) of semirings and ϕ-primary ideals (ϕ-n-absorbing ideals, ϕ-generalized-n-absorbing ideals) of quotient semirings and semirings of fractions.

Finally, we present some ideas for extending our results. Since modules are a generalization of rings, many concepts of rings are naturally extended to modules, e.g., prime ideals (weakly prime ideals) and 2-absorbing ideals (weakly 2-absorbing ideals) of rings are extended to prime submodules (weakly prime submodules) and 2-absorbing submodules (weakly 2-absorbing submodules). In addition, now, ϕ-prime ideals of rings are also extended to the ϕ-prime submodules. Similarly, a semiring R is also an R-semimodule. Moreover, semimodules are another generalization of modules and several concepts of semimodules are extended from the concepts of modules. Therefore, we expect that all concepts in this dissertation can be extended to semimodules.

Open Problems

As a consequence of Theorem 3.2.9 and Theorem 4.2.14, we gain a characterization of ϕ-primary ideals and ϕ-prime ideals of decomposable semirings with two components. However, we do not have a characterization of them in decomposable semirings with more than two conponents. In addition, from Theorem 4.2.12 and Theorem 5.2.13, we do not obtain a characterization of ϕ-n-absorbing ideals and ϕ-generalized-n-absorbing ideals of decomposable semirings. Hence there are

4 open problems for this research.

(1) What is a characterization of ϕ-primary ideals of decomposable semirings with more than two components ?.

(2) What is a characterization of ϕ-prime ideals of decomposable semirings with more than two components ?.

(3) What is a characterization of ϕ-n-absorbing ideals of decomposable semirings ?.

(4) What is a characterization of ϕ-generalized-n-absorbing ideals of decomposable semirings ?.

REFERENCES

[1] Allen, P. J.:A fundamental theorem of homomorphisms for semirings, *Proc. Amer. Math. Soc.* **21**(2), 412–416 (1969).

[2] Anderson, D. F., Badawi, A.: On n-absorbing ideals of commutative rings, *Comm. Algebra* **39**(5), 1646–1672 (2011).

[3] Anderson, D. D., Batanieh, M.: Generalizations of prime ideals, *Comm. Algebra* **36**(2), 686–696 (2008).

[4] Anderson, D. D., Smith, E.: Weakly prime ideals, *Houston J. Math.* **29**, 831–840 (2003).

[5] Atani, R. E., Atani, S. E.: Ideal theory in commutative semirings, *Bul. Acad. Stiinte Repub. Mod. Mat.* **2**(57), 14–23 (2008).

[6] Atani, S. E.: The ideal theory in quotients of commutative semirings, *Glas. Mat.* **42**(62), 301–308 (2007).

[7] Atani, S. E., Atani, R. E.: Some remarks on partitioning semirings, *An. Stiint. Univ. Ovidius Constanta Ser. Mat.* **18**(1), 49–62 (2010).

[8] Atani, S. E., Farzalipour, F.: On weakly primary ideals, *Georgian Math. J.* **12**(3), 423–429 (2005).

[9] Atani, S. E., Kohan, M. S.: A note on finitely generated multiplication semimodules over commutative semirings, *Int. J. Algebra* **4**(8), 389–396 (2010).

[10] Badawi, A.: On 2-absorbing ideals of commutative rings, *Bull. Aust. Math. Soc.* **75**, 417–429 (2007).

[11] Bhatwadekar, S. M., Sharma, P. K.: Unique factorization and birth of almost prime, *Comm. Algebra* **33**, 43–49 (2005).

[12] Chaudhari, J. N., Bonde, D. R.: Weakly prime subsemimodules of semimodules over semirings, *Int. J. Algebra* **5**, 167–174 (2011).

[13] Chaudhari, J. N., Bonde, D. R.: Ideal theory in quotient semirings, *Thai J. Math.* **12**(1), 95–101 (2014).

[14] Chinwarakorn, S., Pianskool, S.: On almost generalized 2-absorbing and weakly almost generalized 2-absorbing structures, *ScienceAsia* **41**(1), 64–72 (2015).

[15] Darani, A. Y.: Generalizations of primary ideals in commutative rings, *Novi Sad J. Math.* **42**(1), 27–35 (2012).

[16] Ebrahimpour, M., Nekooei, R.: On generalizations of prime ideals, *Comm. Algebra* **40**(4), 1268–1279 (2012).

[17] Golan, J. S.: Semirings and Their Applications, *Springer Science & Business Media* (1999).

[18] Gupta, V.: On partitioning ideals of semirings, *Kyungpook Math. J.* **46**, 181–184 (2006).

[19] Gupta, V., Chaudhari, J. N.: Some remarks on semirings, *Rad. Mat.* **12**, 13–18 (2003).

[20] Gupta, V., Chaudhari, J. N.: Characterization of weakly prime subtractive ideals in semirings, *Bull. Inst. Math.* **3**(2), 347–352 (2008).

[21] Gupta, V., Chaudhari, J. N.: Prime ideals in semirings, *Bull. Malays. Math. Sci. Soc.(2)* **34**(2), 417–421 (2011).

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Publication

- Petchkhaew, P., Chinram, R.: Fuzzy, rough and rough fuzzy ideals in ternary semigroups, *International Journal of Pure and Applied Mathematics* **56**(1), 21–36 (2009).
- Wasanawichit, A., Petchkhaew, P.: Ideal theory in some ternary semirings, *Proceedings of the AMM 2012*, 165–174 (2012).
- Petchkaew, P., Wasanawichit, A., Pianskool, S.: Generalizations of $G2$-absorbing ideals of commutative semirings, *Novi sad J. Math.*, submitted.
- Petchkaew, P., Wasanawichit, A., Pianskool, S.: Generalizations of n-absorbing ideals of commutative semirings, *Thai J. Math.*, submitted.