CHAPTER II

THE RELATION BETWEEN THE BINOMIAL COEFFICIENTS AND THE GAMMA FUNCTION

2.1 The Binomial Coefficients

From the binomial theorem we have :

$$(a + b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots + {}^nC_r a^{n-r} b^r + \dots + b^n ,$$

where $^nC_r = \dfrac{n!}{r!\,(n-r)!}$ and $^nC_1$, $^nC_2$,.... are called the binomial coefficients of the nth orders. By listing the expansions for the first few values of n as follows :

$$(a + b)^0 = \dots\dots + 0.a^1 b^{-1} + 1.a^0 b^0 + 0.a^{-1} b^1 + 0.a^{-2} b^2 + \dots$$

$$(a + b)^1 = \dots\dots + 0.a^2 b^{-1} + 1.a^1 b^0 + 1.a^0 b^1 + 0.a^{-1} b^2 + \dots\dots$$

$$(a + b)^2 = \dots\dots + 0.a^3 b^{-1} + 1.a^2 b^0 + 2.a^1 b^1 + 1.a^0 b^2 + \dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

and abstracting the coefficients, we obtain the familiar pattern known as Pascal's triangle, thus :

....	0	1	0	0	....
....	0	1	1	0	....
....	0	1	2	1	....
....	....	....	....	....	....

Each entry is the sum of two numbers in the row immediately above, one of the numbers being in the same column and the other in the column immediately to the left. All numbers outside the angle are zero.

Let us replace the numbers in the above diagram by ^nC_r, so that , we have

....	${}^{o}C_{-1}$	${}^{o}C_{o}$	${}^{o}C_{1}$	${}^{o}C_{2}$	....
....	${}^{1}C_{-1}$	${}^{1}C_{o}$	${}^{1}C_{1}$	${}^{1}C_{2}$	....
....	${}^{2}C_{-1}$	${}^{2}C_{o}$	${}^{2}C_{1}$	${}^{2}C_{2}$	....
....	....	....	....	....	....

The question of how the diagram can be extended upwards, that is for negative values of n, will be discussed in chapter 3.

2.2 The Gamma Function

The gamma function denoted by $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_{o}^{\infty} x^{n-1} \epsilon^{-x} \, dx \ , \qquad \text{.........(1)}$$

which is convergent for $n > o$.

A recurrence formula for the gamma function is

$$\Gamma(n + 1) = n \, \Gamma(n) \ , \qquad \text{..........(2)}$$

where $\Gamma(1) = 1$. From (2), $\Gamma(n)$ can be determined for all $n > o$ when the values for $1 \leq n < 2$ (or any other interval of unit length) are known.

In particular if n is a positive integer then

$$\Gamma(n + 1) = n! \ , \quad n = 1, 2, 3, \ldots\ldots \ . \qquad \text{.....(3)}$$

The recurrence relation (2) is a difference equation which has (1) as a solution. By taking (1) as a definition of $\Gamma(n)$ for $n > o$, we can generalize the gamma function to $n < o$ by the use of (2) in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} . \qquad \text{...........} (4)$$

The function thus obtained has singularities for n = 0, -1, -2, -3, ... and so on, as shown in Fig. 1.

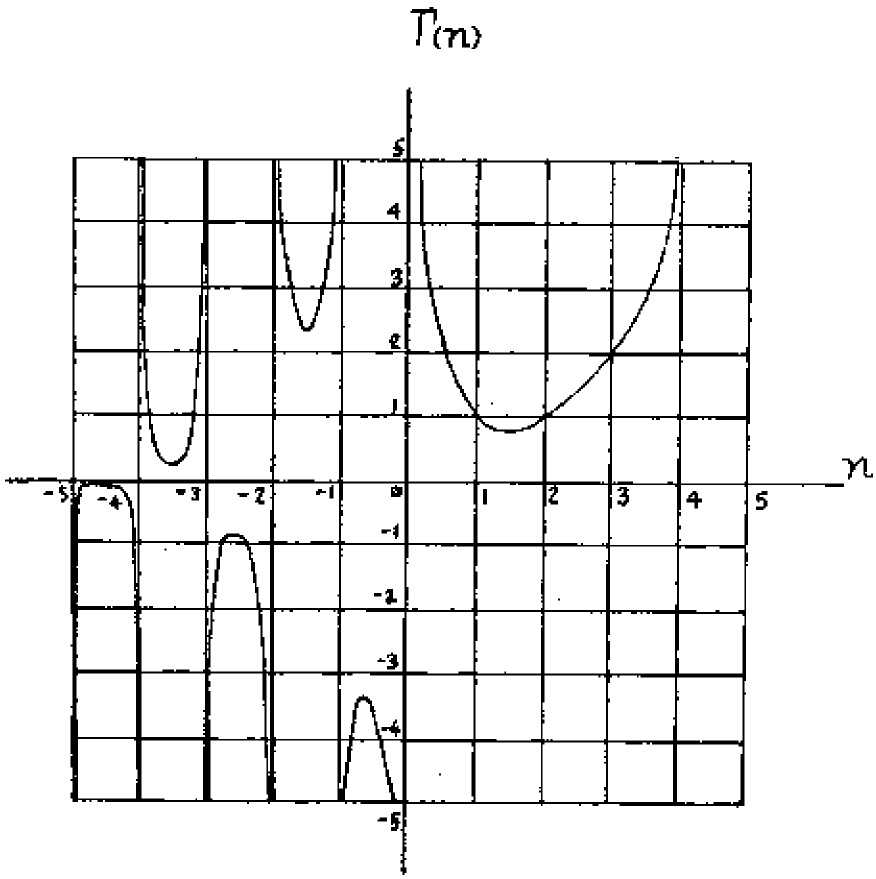

Fig.1 : The Graph of the Gamma Function

2.3 The Binomial Coefficients in the Form of Gamma Functions

From 2.1 , we have

$$ {}^{n}C_{r} = \frac{n!}{r!\,(n-r)!} , \qquad \text{............(1)} $$

and from (3) of 2.2 , we have

$$ \Gamma(n+1) = n! . \qquad \text{..........(2)} $$

Then from (1) and (2) , we have

$$ {}^{n}C_{r} = \frac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)} . \qquad \text{.......... (3)} $$