CHAPTER III

THE BINOMIAL COEFFICIENT FUNCTION ON A PLANE

3.1 The Function on the Lattice Points

Let r and n be the two axes of the plane, where the r - axis is the horizontal axis and the n - axis is the vertical axis.

Consider the values of ${}^{n}C_{r}$ when n and r are integers.

We have, $(1 + a)^n = {}^{n}C_{o} \cdot 1 + {}^{n}C_{1}a + {}^{n}C_{2}a^2 + \dots + {}^{n}C_{n}a^n$, ...(1)

where $${}^{n}C_{r} = \frac{n!}{r!\,(n-r)!} .$$

For n zero and positive, we can find ${}^{n}C_{r}$ by using Pascal's triangle in 2.1 .

Therefore, we have the values of the binomial coefficients on the lattice points when n is zero or a positive integer, as in the following figure.

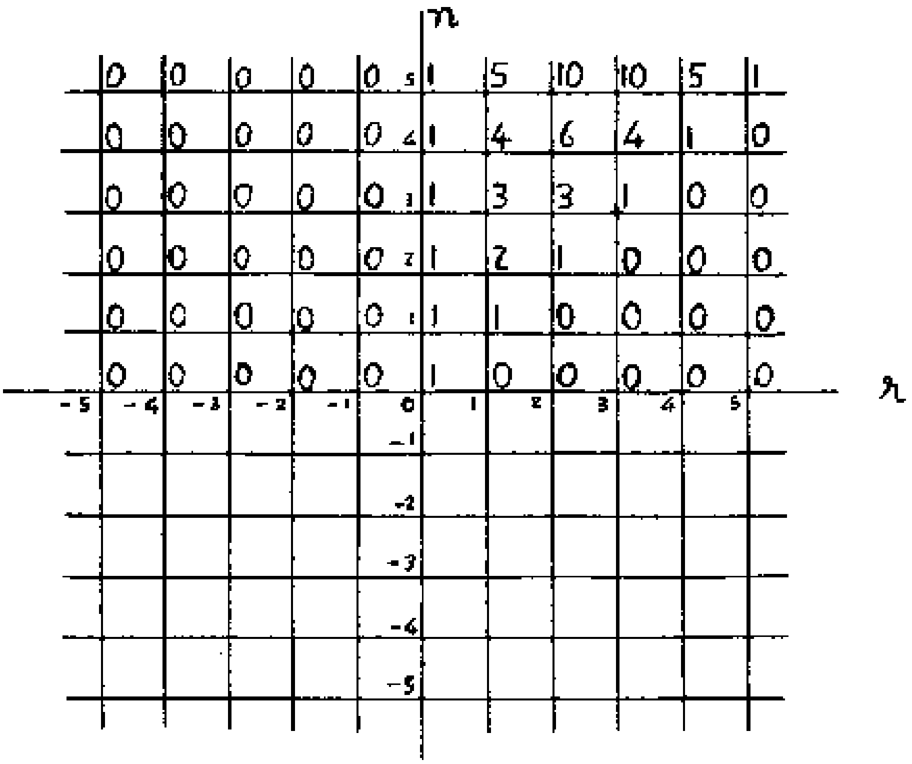

Fig.2 : The Values of the Binomial Coefficient Function on the Lattice Points of the 1st and 2nd Quadrants of the (r, n) Plane

Now, let us consider the values of ${}^{n}C_{r}$ when n is a negative integer.

For $|a| < 1$, we have

$$(1+a)^{-1} = \frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - a^5 + \ldots\ldots ,$$

$$(1+a)^{-2} = \frac{1}{(1+a)^2} = 1 - 2a + 3a^2 - 4a^3 + 5a^4 - 6a^5 + \ldots ,$$

$$(1+a)^{-3} = \frac{1}{(1+a)^3} = 1 - 3a + 6a^2 - 10a^3 + 15a^4 - 21a^5 + \ldots ,$$

$$(1+a)^{-4} = \frac{1}{(1+a)^4} = 1 - 4a + 10a^2 - 20a^3 + 35a^4 - 56a^5 + \ldots ,$$

$$(1+a)^{-5} = \frac{1}{(1+a)^5} = 1 - 5a + 15a^2 - 35a^3 + 70a^4 - 126a^5 \ldots ,$$

....

Therefore, from (1), we have the binomial coefficients in the following figure .

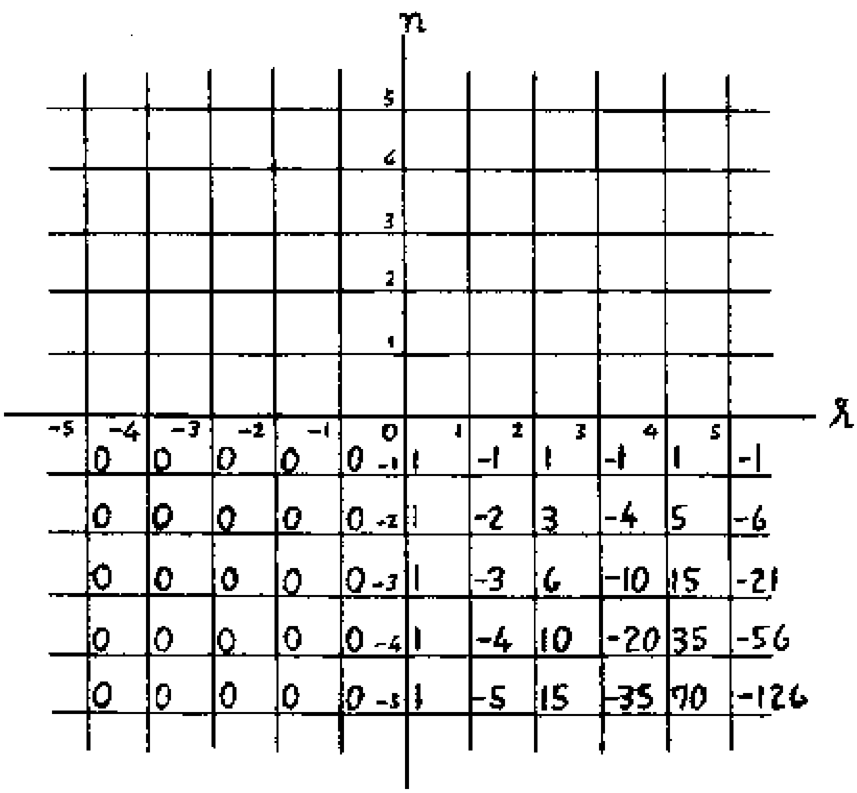

Fig. 3 : The Values of the Binomial Coefficient Function on the Lattice Points of the 3rd and 4th Quadrants of the (r, n) Plane

From Fig. 2 and Fig. 3, we have values of the binomial coefficients for every lattice point of the (r, n) plane , as in the following figue.

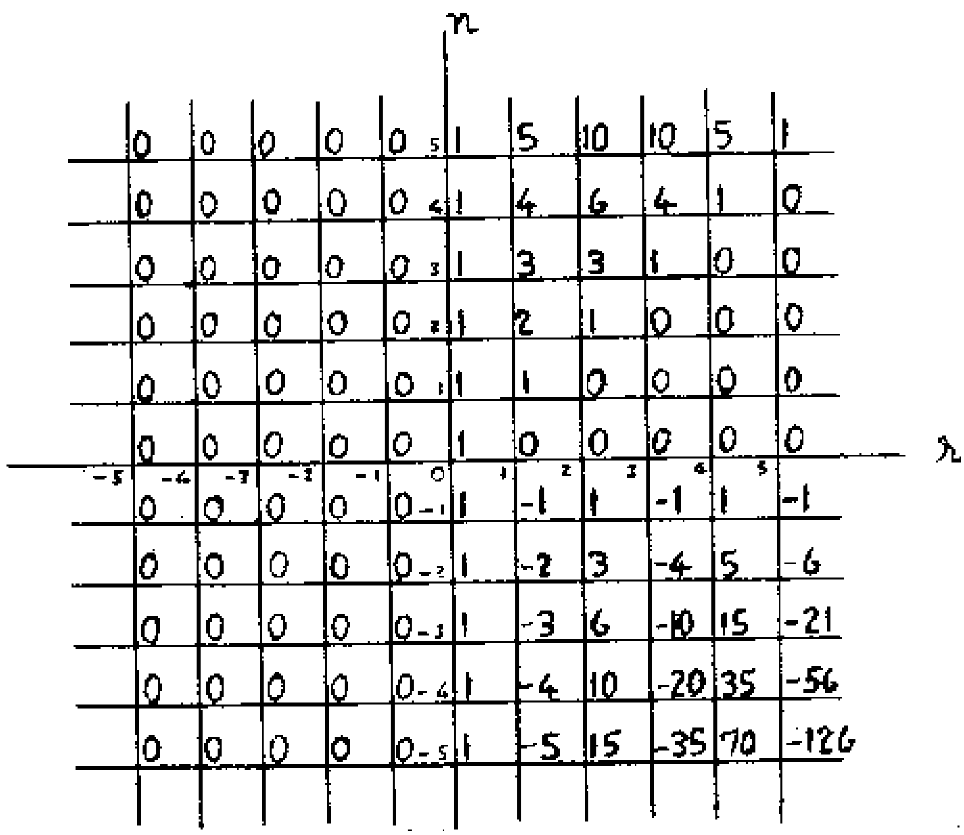

Fig. 4 : The Values of the Binomial Coefficient Function on the Lattice Points of the (r, n) Plane

But there is another way of obtaining values for the binomial coefficients when n is a negative integer.

For $|a| > 1$, we have

$$(a + 1)^{-1} = \frac{1}{a+1} = \frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} - \frac{1}{a^4} + \frac{1}{a^5} - \cdots ,$$

$$(a + 1)^{-2} = \frac{1}{(a+1)^2} = \frac{1}{a^2} - \frac{2}{a^3} + \frac{3}{a^4} - \frac{4}{a^5} + \cdots\cdots ,$$

$$(a+1)^{-3} = \frac{1}{(a+1)^3} = \frac{1}{a^3} - \frac{3}{a^4} + \frac{6}{a^5} - \cdots\cdots ,$$

$$(a+1)^{-4} = \frac{1}{(a+1)^4} = \frac{1}{a^4} - \frac{4}{a^5} + \cdots\cdots ,$$

$$(a+1)^{-5} = \frac{1}{(a+1)^5} = \frac{1}{a^5} - \cdots\cdots ,$$

....

Therefore, from (1), we have the binomial coefficients in Fig.5 .

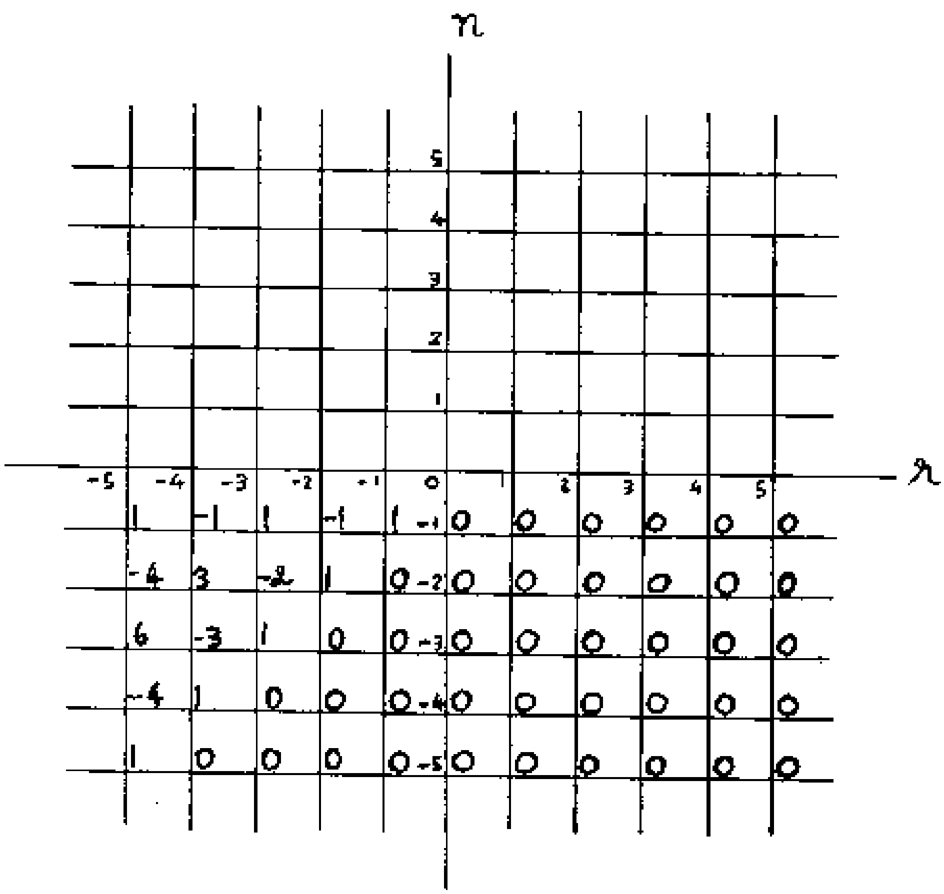

Fig. 5 : Another Set of Values for the Binomial Coefficient Function on the Lattice Points of the 3rd and 4th Quadrants of the (r, n) Plane

From Fig.2 and Fig.5, we now have values for the binomial coefficients for every lattice point of the (r, n) plane as in Fig. 6.

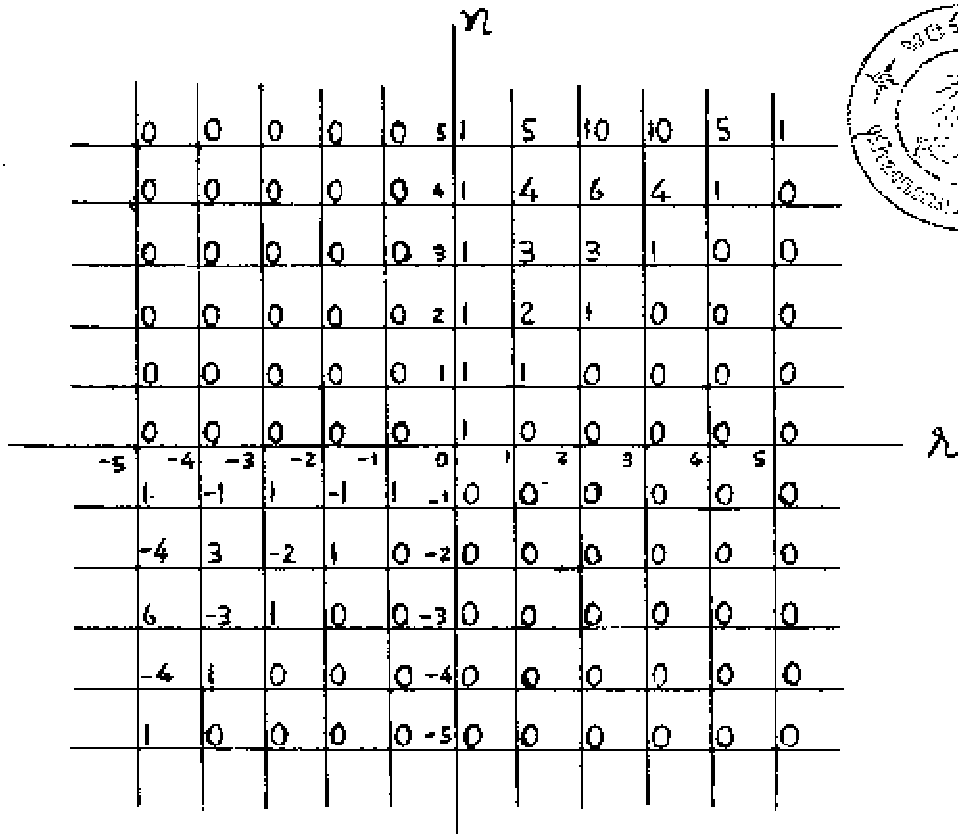

Fig. 6 : Another Set of Values for the Binomial Coefficient Function on the Lattice Points of the (r, n) Plane

From Fig. 4 and Fig. 6, we see that we have two sets of values for the binomial coefficients, each of which may be regarded as an extension of Pascal's triangle to cover all the lattice points (r, n) on the plane.

3.2 The Function on the Plane

Let us replace ${}^nC_r$ by f(r, n), so that from (3) in 2.3, we have

$$f(r, n) = \frac{\Gamma(n + 1)}{\Gamma(r + 1)\,\Gamma(n - r + 1)} \quad \text{.........} \quad (1)$$

From 2.2, we know that $\Gamma(n)$ is singular when n = 0, - 1, -2, -3, ... Now, let us consider $\Gamma(z)$, when z is a complex number. The singularities of $\Gamma(z)$ at z = 0, -1, -2,............ are simple poles. The following two theorems are from the theory of complex variables. (See for example Nehari "Introduction to Complex Analysis")

Theorem A If f(z) has a simple pole at z = a, then $\frac{1}{f(z)}$ has a simple zero at z = a, and is analytic there.

Theorem B If $f_1(z)$ and $f_2(z)$ are analytic at z = a, then $f_1(z) \cdot f_2(z)$ is analytic at z = a, and $\frac{f_1(z)}{f_2(z)}$ is analytic at z = a. provided $f_2(a) \neq 0$.

In (1) regard r and n as complex variables.

(a) If Re (n + 1) > 0, or Re (n) > -1, then $\Gamma(n + 1)$ is analytic.

(b) $\Gamma(r + 1)$ is never zero, and it is analytic except when r + 1 = 0, -1, -2, ... or r = -1, -2, -3,....... .

Therefore by theorem B : $\frac{1}{\Gamma(r + 1)}$ is analytic everywhere except possibly at r = -1, -2, -3,..... . But by theorem A, $\frac{1}{\Gamma(r + 1)}$ has simple zeroes at r = -1, -2, -3, ... and is analytic there.

Therefore $\frac{1}{\Gamma(r+1)}$ is analytic everywhere.

Similarly $\frac{1}{\Gamma(n-r+1)}$ is analytic everywhere.

Therefore $f(r, n) = \Gamma(n+1) \cdot \frac{1}{\Gamma(r+1)} \cdot \frac{1}{\Gamma(n-r+1)}$ is analytic everywhere except at the simple poles of $\Gamma(n+1)$, where $f(r, n)$ has simple poles : that is where $n = -1, -2, -3, \ldots\ldots$.

Returning to the real r, n - plane, we have the result that $f(r, n)$ is single valued, continuous, and has continuous derivatives of all orders at every point on the plane, except on the lines $n = -1, -2, -3, \ldots$.

But in Figs. 4 and 6 in 3.1 we have assigned values to ${}^nC_r$ of which $f(r, n)$ is intended to be a generalization . We shall show below that the singularities of $f(r, n)$ can be removed along certain lines passing through the lattice points by assigning values to $f(r, n)$ either from Fig. 4 or from **Fig. 6.**

As an example consider the point (3, -4) , and let ϵ be such that $0 < |\epsilon| < 1$.

From $$f(r,n) = \frac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)},$$

we have $$f(3,-4+\epsilon) = \frac{\Gamma(-4+\epsilon+1)}{\Gamma(3+1)\,\Gamma(-4+\epsilon-3+1)}$$

$$= \frac{\Gamma(-3+\epsilon)}{\Gamma(4)\quad\Gamma(-6+\epsilon)}$$

$$= \frac{\Gamma(-3+\epsilon)}{3! \times \dfrac{\Gamma(-3+\epsilon)}{(-6+\epsilon)(-5+\epsilon)(-4+\epsilon)}}$$, by (3) and (4), section 2.2,

$$= \frac{(-6+\epsilon)(-5+\epsilon)(-4+\epsilon)}{6}$$

$$= \frac{-120 + 74\epsilon - 15\epsilon^2 + \epsilon^3}{6}$$

$$= -20 + \frac{37}{3}\epsilon - \frac{5}{2}\epsilon^2 + \frac{\epsilon^3}{6}.$$

Therefore $\lim_{\epsilon \to 0} f(3, -4 + \epsilon) = -20$.

This result holds when $(r, n) \longrightarrow (3, -4)$ along the line $r = 3$ from both directions. Therefore we can remove the singularity of $f(r, n)$ along this line at $(3, -4)$ by putting $f(3, -4) = -20$, which is the same value as in Fig. 4 in 3.1 .

We may now generalize this result. We shall prove that, at any lattice point (r_1, n_1) , the limit of $f(r_1, n_1)$ exists, where the limit is taken along the line $r = r_1$ to the point (r_1, n_1) from either direction, and the value of this limit is the same value as in Fig. 4 in 3.1 .

Consider the lattice point (r_1, n_1) , and suppose ϵ satisfies the condition $0 < |\epsilon| < 1$.

From $\quad f(r, n) = \dfrac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)}$,

we have $\quad f(r_1, n_1 + \epsilon) = \dfrac{\Gamma(n_1+\epsilon+1)}{\Gamma(r_1+1)\,\Gamma(n_1+\epsilon-r_1+1)}$.

<u>Case 1</u> $\qquad n_1 + \epsilon - r_1 + 1 < n_1 + \epsilon + 1$

This occurs when $r_1 > 0$.

$$\Gamma(n_1+\epsilon-r_1+1) = \frac{\Gamma(n_1+\epsilon+1)}{(n_1+\epsilon-r_1+1)(n_1+\epsilon-r_1+2)\ldots(n_1+\epsilon)} \quad \text{etc,}$$

as in the example above.

But the values of $f(r, n)$ when n is a negative integer from Fig. 4 , are calculated from $(1 + a)^n$ when n is a negative integer and $|a| < 1$, since

$$(1 + a)^n = 1 + na + \frac{n(n-1)a^2}{2!} + \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)a^r}{r!} + \ldots, \quad \ldots\ldots(2)$$

holds when n is a negative integer or a fraction as well as when n is a positive integer. (See for example of Frederic H. Miller "Analytic Geometry And Calculus'') Therefore, from Fig. 4 , when n is a negative integer,

we have $${}^nC_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!} .$$

Now, in this case, we have

$$\Gamma(n_1+\epsilon-r_1+1) = \frac{\Gamma(n_1+\epsilon+1)}{(n_1+\epsilon-r_1+1)(n_1+\epsilon-r_1+2)\ldots(n_1+\epsilon)} ,$$

so that $$f(r_1, n_1+\epsilon) = \frac{\Gamma(n_1+\epsilon+1)}{\Gamma(r_1+1) \cdot \dfrac{\Gamma(n_1+\epsilon+1)}{(n_1+\epsilon-r_1+1)(n_1+\epsilon-r_1+2)\ldots(n_1+\epsilon)}}$$

$$= \frac{(n_1+\epsilon-r_1+1)(n_1+\epsilon-r_1+2)\ldots(n_1+\epsilon)}{r_1!} \quad \text{by (3), section 2.2,}$$

and $$\lim_{\epsilon \to 0} f(r_1, n_1+\epsilon) = \frac{(n_1-r_1+1)(n_1-r_1+2)\ldots\ldots\ldots(n_1)}{r_1!} .$$

Therefore $\lim_{\epsilon \to 0} f(r_1, n_1+\epsilon) = \dfrac{n_1(n_1-1)(n_1-2)\ldots(n_1-r_1+1)}{r_1!} = {}^{n_1}C_{r_1}$,

which is the same value as in Fig. 4.

Case 2 $\quad n_1 + \epsilon - r_1 + 1 = n_1 + \epsilon + 1$

This occurs when $r_1 = 0$.

The formula $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ is not used, and $f(0, n) = 1$ for all n as in Fig. 4.

Case 3 $\quad n_1 + \epsilon - r_1 + 1 > n_1 + \epsilon + 1$

This occurs when $r_1 < 0$, but (r_1, n_1) is a lattice point, so that, in this case r_1 is a negative integer. Therefore $f(r_1, n_1 + \epsilon) = 0$, and $\lim_{\epsilon \to 0} f(r_1, n_1 + \epsilon) = 0$ as in Fig. 4. For, as r_1 approaches a negative integer, $\Gamma(r_1 + 1)$ approaches $\pm\infty$, while $\Gamma(n_1 + \epsilon + 1)$ approaches a finite value and $\Gamma(n_1 + \epsilon - r_1 + 1)$ approaches a non-zero value.

Let us now consider the limit of $f(r, n)$ as (r, n) approaches the point $(3, -4)$ along the line $n = r - 7$. Let ϵ be such that $0 < |\epsilon| < 1$.

From $\quad f(r,n) = \dfrac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)}$,

we have

$$f(3+\epsilon, -4+\epsilon) = \frac{\Gamma(-4+\epsilon+1)}{\Gamma(3+\epsilon+1)\,\Gamma(-4+\epsilon-3-\epsilon+1)}$$

$$= \frac{\Gamma(-3+\epsilon)}{\Gamma(4+\epsilon)\,\Gamma(-6)} = 0 .$$

Therefore $\lim_{\epsilon \to 0} f(3+\epsilon, -4+\epsilon) = 0$.

This result holds when $(r, n) \longrightarrow (3, -4)$ along the diagonal line $n = r - 7$ from either direction, the limit of $f(3, -4)$ is zero, which is the same value as in Fig. 6 in 3.1 .

We may now generalize this result. We shall prove that at any lattice point (r_2, n_2) , the limit of $f(r_2, n_2)$ exists, where the limit is taken along the line $n_2 = r_2 + k$, when k is an integer, to the point (r_2, n_2) from either direction, and the value of this limit is the same value as in Fig. 6 in 3.1 .

Consider the lattice point (r_2, n_2) and suppose ϵ satisfies the condition $0 < |\epsilon| < 1$.

From $f(r, n) = \dfrac{\Gamma(n + 1)}{\Gamma(r + 1)\ \Gamma(n - r + 1)}$,

we have
$$f(r_2+\epsilon, n_2+\epsilon) = \frac{\Gamma(n_2+\epsilon+1)}{\Gamma(r_2+\epsilon+1)\ \Gamma(n_2+\epsilon-r_2-\epsilon+1)}$$

$$= \frac{\Gamma(n_2+\epsilon+1)}{\Gamma(r_2+\epsilon+1)\ \Gamma(n_2-r_2+1)} .$$

<u>Case 1</u> $r_2 + \epsilon + 1 < n_2 + \epsilon + 1$

This occurs when $r_2 < n_2$.

$$\Gamma(r_2+\epsilon+1) = \frac{\Gamma(n_2+\epsilon+1)}{(r_2+\epsilon+1)(r_2+\epsilon+2)\ldots.(n_2+\epsilon)} \text{ etc.}$$

Let us consider the formula of $f(r, n)$ in Fig. 6 in 3.1 , from (2), we have

$$(1 + a)^n = 1 + na + \frac{n(n - 1)}{2!}a^2 + \ldots\ldots + \frac{n(n - 1)\ldots(n-r+1)a^r}{r!} + \ldots ,$$

which holds when $|a| < 1$ and n is a negative integer or a fraction as well as when n is a positive integer. The values of f (r, n) when n is a negative integer from Fig. 6, are calculated from $(a+1)^n$ when n is a negative integer and $|a| > 1$.

Write $(a+1)^n$ in the form of (2), we have

$$(a+1)^n = a^n(1+\frac{1}{a})^n$$

$$= a^n\left\{1 + n(\frac{1}{a}) + \frac{n(n-1)}{2!}(\frac{1}{a})^2 + \dots + \frac{n(n-1)\dots(n-s+1)}{s!}(\frac{1}{a})^s + \dots\right\}$$

$$= a^n + na^{n-1} + \frac{n(n-1)a^{n-2}}{2!} + \dots + \frac{n(n-1)\dots(n-s+1)}{s!}a^{n-s} + \dots .$$

Therefore $f(r, n) = \frac{n(n-1)\ \dots\dots (r+1)}{(n-r)!}$, by putting $s = n - r$ in the above result.

In this case, we have

$$f(r_2+\epsilon, n_2+\epsilon) = \frac{\Gamma(n_2+\epsilon+1)}{\frac{\Gamma(n_2+\epsilon+1)}{(r_2+\epsilon+1)(r_2+\epsilon+2)\dots(n_2+\epsilon)} \times \Gamma(n_2-r_2+1)}$$

$$= \frac{(r_2+\epsilon+1)(r_2+\epsilon+2)\dots\dots(n_2+\epsilon)}{(n_2-r_2)!}$$, by (3) and (4) in 2.2.

Therefore $\lim_{\epsilon \to 0} f(r_2+\epsilon, n_2+\epsilon) = \frac{(r_2+1)(r_2+2)\dots\dots(n_2)}{(n_2-r_2)!}$, which is the same value as in Fig. 6 .

<u>Case 2</u> $r_2 + \epsilon + 1 = n_2 + \epsilon + 1$

This occurs when $r_2 = n_2$.

The formula $\Gamma(n) = \dfrac{\Gamma(n + 1)}{n}$ is not used, and $f(n_2, n_2) = 1$ for all n as in Fig. 6.

<u>Case 3</u> $r_2 + \epsilon + 1 > n_2 + \epsilon + 1$

This occurs when $r_2 > n_2$, but (r_2, n_2) is a lattice point, so that, in this case $(n_2 - r_2)$ is a negative integer. Therefore $f(r_2 + \epsilon , n_2 + \epsilon) = 0$, and $\lim_{\epsilon \to 0} f(r_2 + \epsilon , n_2 + \epsilon) = 0$ as in Fig.6. For , as $(n_2 - r_2)$ approaches a negative integer, $\Gamma(n_2 - r_2 + 1)$ approaches $\pm \infty$, while $\Gamma(n_2 + \epsilon + 1)$ approaches a finite value and $\Gamma(r_2 + \epsilon + 1)$ approaches a non-zero value.

We see that, from the preceding proof, at any lattice point (r_1, n_1) , the limit of $f(r_1, n_1)$ exists, where the limit is taken along the line $r = r_1$ to the point (r_1, n_1) from either direction. Now, let us consider the function f(r, n) when r is any integer, given $r = r_1$ where $r_1 = \ldots\ldots\ldots, -3, -2, -1, 0\ 1, 2, 3, \ldots$.

From $$f(r, n) = \frac{\Gamma(n + 1)}{\Gamma(r + 1)\,\Gamma(n - r + 1)} ,$$

we have $$f(r_1, n) = \frac{\Gamma(n + 1)}{\Gamma(r_1 + 1)\,\Gamma(n - r_1 + 1)} .$$

(a) If $r_1 < 0$, or r_1 is a negative integer, then when n is not a negative integer, we have $f(r_1, n) = 0$. For, as r_1 approaches a negative integer, $\Gamma(r_1 + 1)$ approaches $\pm \infty$, while $\Gamma(n + 1)$ approaches a finite value and $\Gamma(n - r_1 + 1)$ approaches a non-zero value. And, when n is a negative integer, we already have $\lim_{\epsilon \to 0} f(r_1, n + \epsilon) = 0$, so we shall put $f(r_1, n) = 0$ as in Fig. 4. Then for all n, $f(r_1, n) = 0$, and f is analytic in n.

(b) If $r_1 = 0$, then when n is not a negative integer, we have $f(r_1, n) = 1$, and when n is a negative integer, we already have $\lim_{\epsilon \to 0} f(r_1, n + \epsilon) = 1$, so we shall put $f(r_1, n) = 1$. Then for all n $f(r_1, n) = 1$, and f is analytic in n.

(c) If $r_1 > 0$, or r_1 is a positive integer, then when n is not a negative integer,

we have
$$\Gamma(n-r_1+1) = \frac{\Gamma(n+1)}{(n-r_1+1)(n-r_1+2)\ldots\ldots n},$$

so that
$$f(r_1, n) = \frac{\Gamma(n+1)}{\Gamma(r_1+1) \times \dfrac{\Gamma(n+1)}{(n-r_1+1)(n-r_1+2)\ldots\ldots n}}$$
$$= \frac{(n-r_1+1)(n-r_1+2)\ldots\ldots n}{r_1!}.$$

And, when n is a negative integer, we have

$$\lim_{\epsilon \to 0} f(r_1, n+\epsilon) = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r_1!},$$

so we shall put
$$f(r_1, n) = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r_1!}.$$

But $r_1!$ is constant, therefore $f(r_1, n)$ is a polynomial in n for all n, which is analytic in n.

Summarizing, we see that, when r is a constant integer, and f(r, n) is given values at the lattice points as in Fig. 4, f(r, n) is analytic in n.

Similarly, we have proved that at any lattice point (r_2, n_2), the limit of $f(r_2, n_2)$ exists, where the limit is taken along the line $n_2 = r_2 + k$, wher k is an integer, to the point (r_2, n_2) from either direction. Now we shall prove that the function $f(r, n)$ is analytic in r (or in n) when $n = r + k$, where k is any constant integer.

From
$$f(r, n) = \frac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)},$$

we have
$$f(r, r+k) = \frac{\Gamma(r+k+1)}{\Gamma(r+1)\,\Gamma(r+k-r+1)}$$
$$= \frac{\Gamma(r+k+1)}{\Gamma(r+1)\,\Gamma(k+1)}.$$

(a) If $k < 0$, or k is a negative integer, then when r is not a negative integer, we have $f(r, r+k) = 0$. For, as k approaches a negative integer, $\Gamma(k+1)$ approaches $\pm\infty$, while $\Gamma(r+k+1)$ approaches a finite value and $\Gamma(r+1)$ approaches a non-zero value. And, when r is a negative integer, we already have $\lim_{\epsilon \to 0} f(r+\epsilon, r+k+\epsilon) = 0$, so we shall put $f(r, r+k) = 0$ as in Fig. 6. Then for all r, $f(r, r+k) = 0$, and f is analytic in r.

(b) If $k = 0$, then when r is not a negative integer, we have $f(r, r+k) = 1$, and when r is a negative integer, we already have $\lim_{\epsilon \to 0} f(r+\epsilon, r+k+\epsilon) = 1$, so we shall put $f(r, r+k) = 1$. Then for all r $f(r, r+k) = 1$, and f is analytic in r.

(c) If $k > 0$, or k is a positive integer, then when r is not a negative integer, we have

$$\Gamma(r+1) = \frac{\Gamma(r+k+1)}{(r+1)(r+2)\,\ldots\ldots(r+k)},$$

so that

$$f(r, r+k) = \frac{\Gamma(r+k+1)}{\dfrac{\Gamma(r+k+1)}{(r+1)(r+2)\ldots\ldots(r+k)} \times \Gamma(k+1)}$$

$$= \frac{(r+1)(r+2)\ldots\ldots(r+k)}{k!}.$$

And, when r is a negative integer, we have

$$\lim_{\epsilon \to 0} f(r+\epsilon, r+k+\epsilon) = \frac{(r+1)(r+2)\ldots\ldots(r+k)}{k!},$$

so we shall put

$$f(r, r+k) = \frac{(r+1)(r+2)\,\ldots\ldots\,(r+k)}{k!}.$$

But $k!$ is constant, therefore $f(r, r+k)$ is a polynomial in r for all r, which is analytic in r.

Summarizing, we see that, when $n = r + k$, where k is a constant integer, and $f(r, n)$ is given values at the lattice points as in Fig. 6, $f(r, n)$ is analytic in r.