CHAPTER IV

THE GRAPH OF THE BINOMIAL COEFFICIENT FUNCTION

4.1 The Graph for a Fixed Non-negative Value of n

From 3.2, we have

$$f(r, n) = \frac{\Gamma(n + 1)}{\Gamma(r + 1)\,\Gamma(n - r + 1)} \,. \qquad \text{.......(1)}$$

Let us find the graph of $f(r, n)$ when $n = 4$, from Fig. 4, we have

....

$f(-3, 4) = 0,$

$f(-2, 4) = 0,$

$f(-1, 4) = 0,$

$f(0, 4) = 1,$

$f(1, 4) = 4,$

$f(2, 4) = 6,$

$f(3, 4) = 4,$

$f(4, 4) = 1,$

$f(5, 4) = 0,$

$f(6, 4) = 0,$

$f(7, 4) = 0,$

....

From (1) in 4.1, (2), (3), (4) in 2.2 , and the table of the gamma function, we have

$$f(-2.5,\ 4) = 0.0054,$$

$$f(-1.5,\ 4) = -0.0235,$$

$$f(-0.5,\ 4) = 0.2587,$$

$$f(0.5,\ 4) = 2.3284,$$

$$f(1.5,\ 4) = 5.4327,$$

$$f(2.5,\ 4) = 5.4327,$$

$$f(3.5,\ 4) = 2.3284,$$

$$f(4.5,\ 4) = 0.2587,$$

$$f(5.5,\ 4) = -0.0235,$$

$$f(6.5,\ 4) = 0.0054,$$

....

and

$$f(1.8,\ 4) = 5.9056,$$

$$f(2.2,\ 4) = 5.9056.$$

From the above data, we can plot the graph shown in Fig. 7.

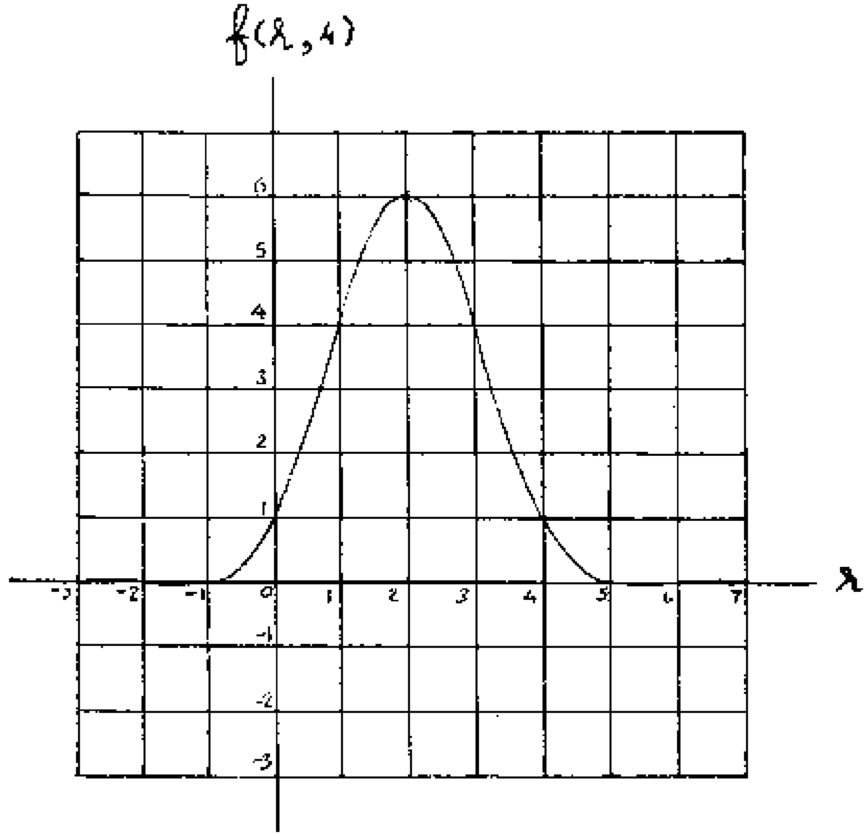

Fig. 7 : The Graph of f(r, 4)

Now, let us consider the values of f(r, n) from Figs. 8 and 9.

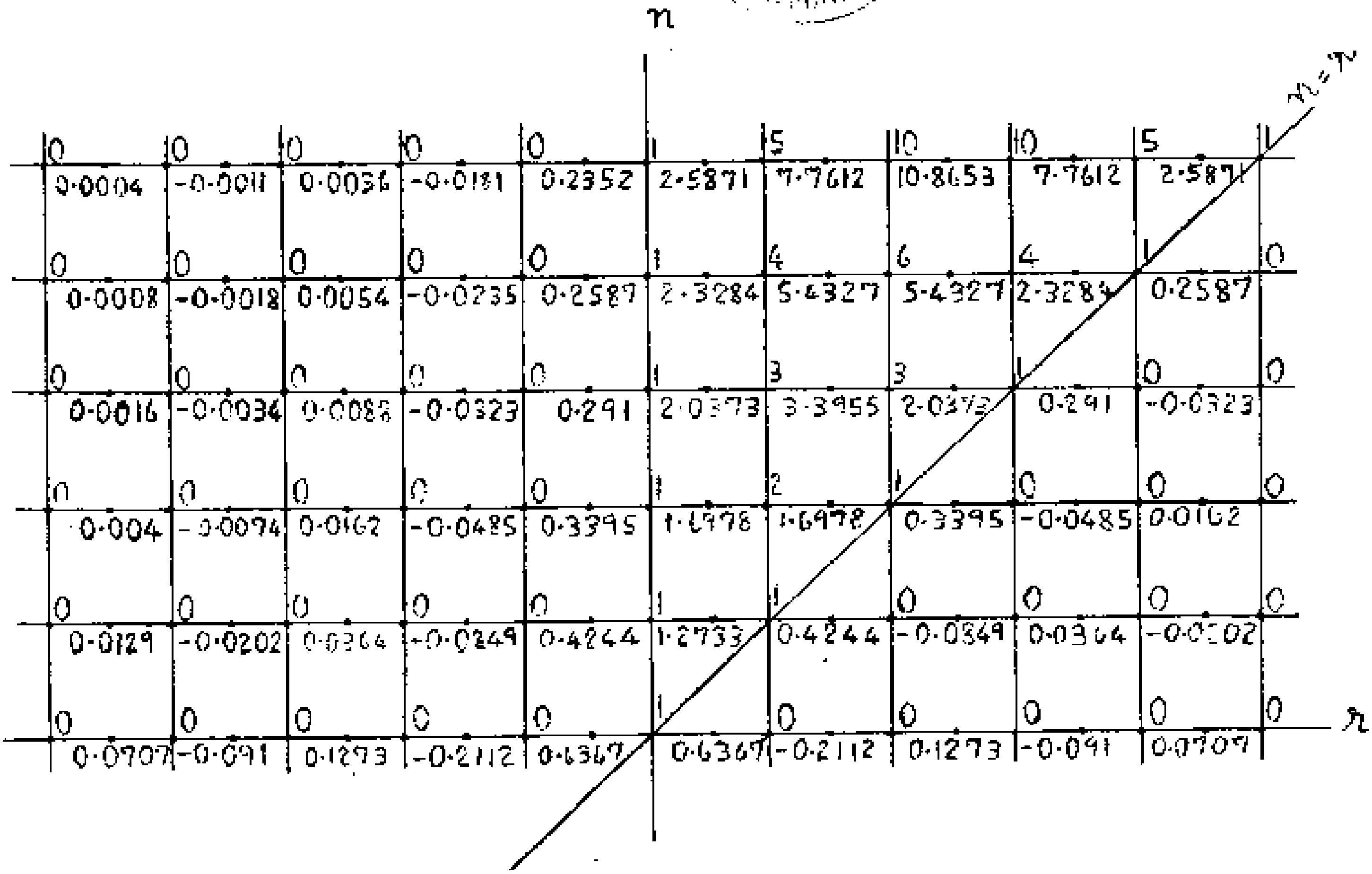

Fig. 8 : Some Values of the Function f(r, n) between Lattice Points in the 1st Two Quadrants of the (r, n) Plane

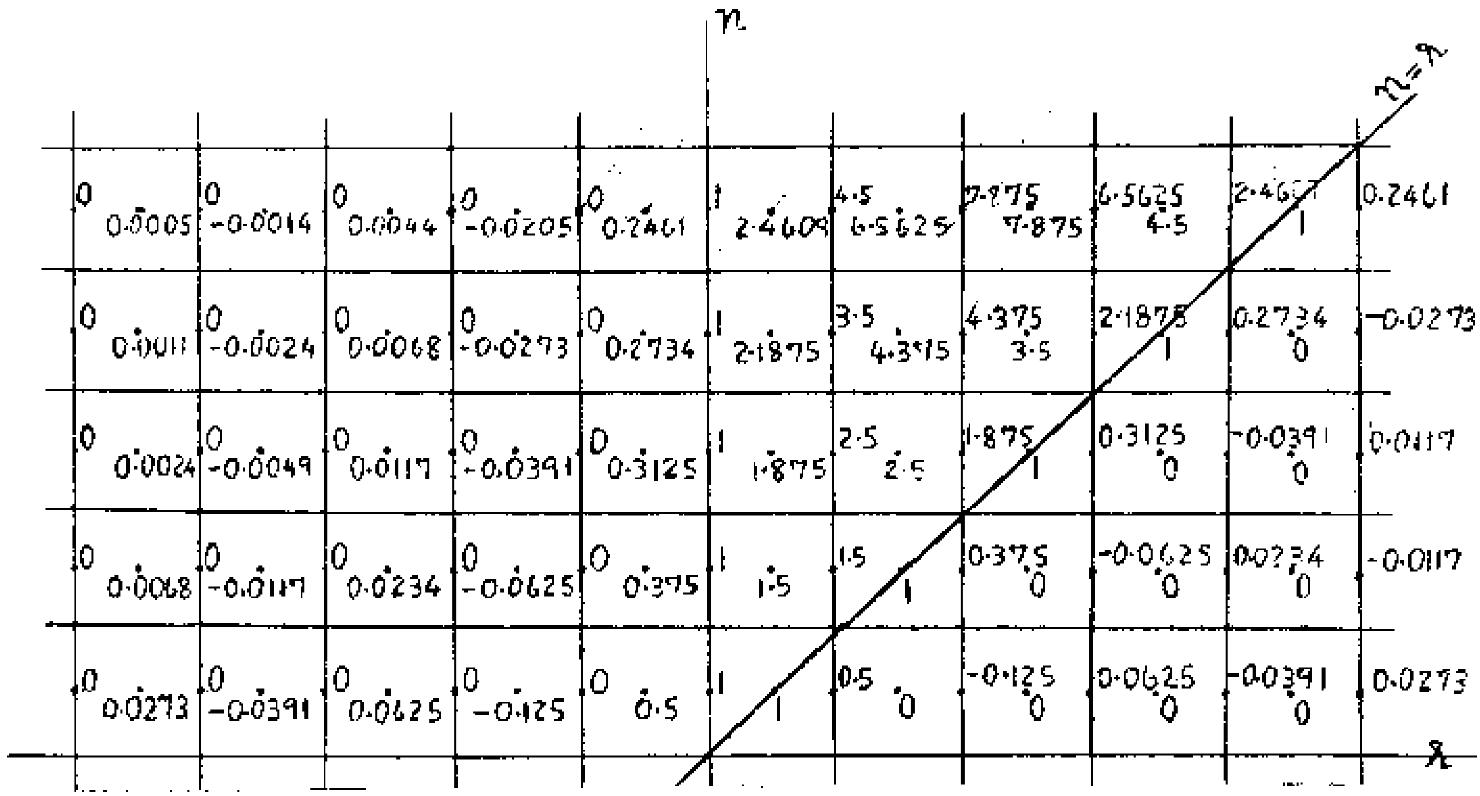

Fig. 9 : Another Set of Values of the Function f(r, n) between Lattice Points in the 1st Two Quadrants of the (r, n) Plane

From Figs. 8 and 9, we see that, when n is constant, the graph of f(r, n) is similar to the graph of f(r, 4) in Fig.7.

Let us consider the values of f(r, n) from Figs. 8 and 9 again, we see that, the maximum value for f(r, n), when n is constant, is on the line n = 2r, or $\frac{\partial f}{\partial r} = 0$ on the line n = 2r, which we shall prove below.

we have
$$f(r, n) = \frac{\Gamma(n+1)}{\Gamma(r+1)\ \Gamma(n-r+1)}, \qquad \text{.......(1)}$$

and from (1) in 2.2, we have

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad \ldots\ldots\ldots\ldots(2)$$

so that
$$\frac{d\,\Gamma(x)}{dx} = \frac{d}{dx}\int_0^{\infty} t^{x-1} e^{-t} dt$$

$$= \int_0^{\infty} \frac{d}{dx}\, t^{x-1} e^{-t} dt . \qquad \ldots\ldots\ldots\ldots(3)$$

Since
$$\frac{d\, t^{x-1}}{dx} = t^{x-1} \log t, \qquad \ldots\ldots\ldots\ldots(4)$$

substitute (4) in (3) ;

$$\frac{d\,\Gamma(x)}{dx} = \int_0^{\infty} t^{x-1} \log t\; e^{-t} dt . \qquad \ldots\ldots\ldots\ldots(5)$$

From (1) and (5) , we have

$$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left[\frac{\Gamma(n+1)}{\Gamma(r+1)\;\Gamma(n-r+1)}\right]$$

$$= -\frac{\Gamma(n+1)}{\Gamma(r+1)\,\Gamma(n-r+1)}\left[-\frac{1}{\Gamma(n-r+1)}\int_0^{\infty} t^{n-r}\log t\; e^{-t}dt + \frac{1}{\Gamma(r+1)}\int_0^{\infty} t^{r}\log t\; e^{-t}dt\right].$$

Therefore, if $n = 2r$, we have

$$\frac{\partial f}{\partial r} = -\frac{\Gamma(2r+1)}{\Gamma(r+1)\,\Gamma(r+1)}\left[-\frac{1}{\Gamma(r+1)}\int_0^{\infty} t^{r}\log t\; e^{-t}dt + \frac{1}{\Gamma(r+1)}\int_0^{\infty} t^{r}\log t\; e^{-t}dt\right]$$

$$= 0 .$$

4.2 The Graph for a Fixed Integral Value of r

From 3.2, when r is constant integer, we have

(1) $r < 0$, $f(r, n) = 0$,

(2) $r = 0$, $f(r, n) = 1$,

(3) $r > 0$, $f(r, n) = \dfrac{n(n-1)(n-2) \ldots (n-r+1)}{r!}$.

Let us consider the graphs of $f(r, n)$ in (3) , which are the graphs of polynomials in n.

(a) When $r = 1$, $f(1, n) = n$

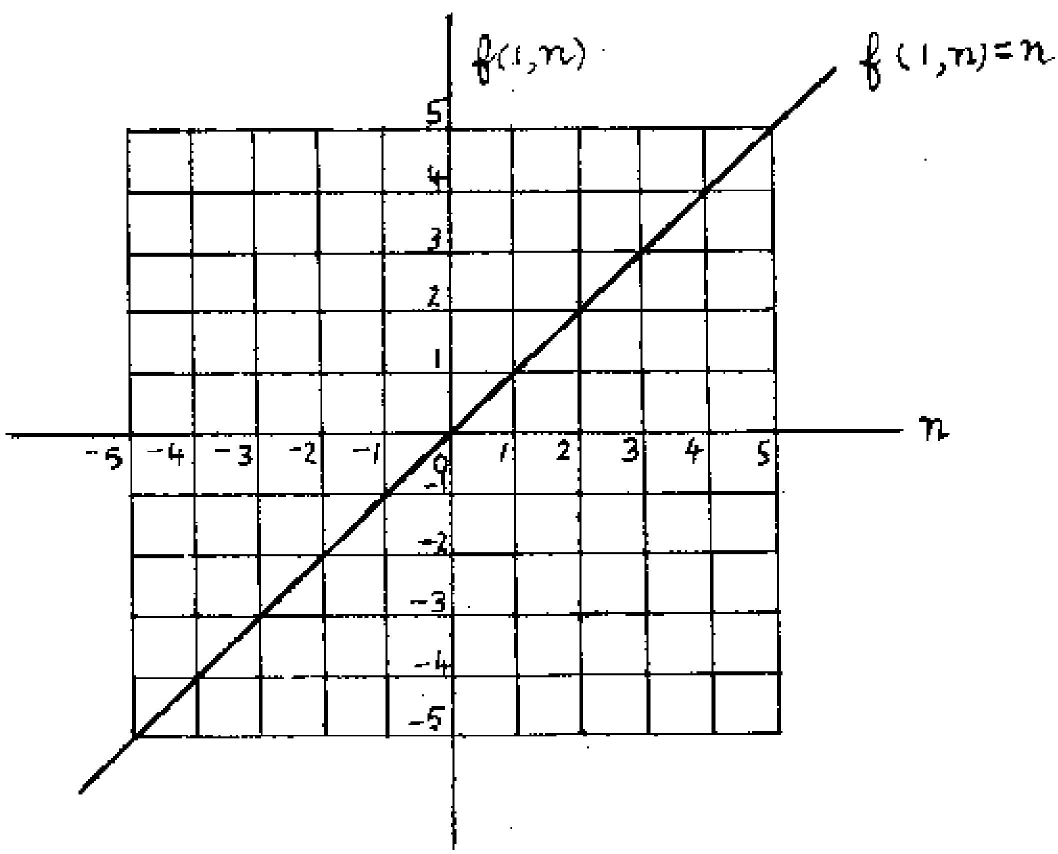

Fig.10 : The Graph of f(1, n)

(b) When $r = 2$, $f(2, n) = \frac{n(n-1)}{2!}$

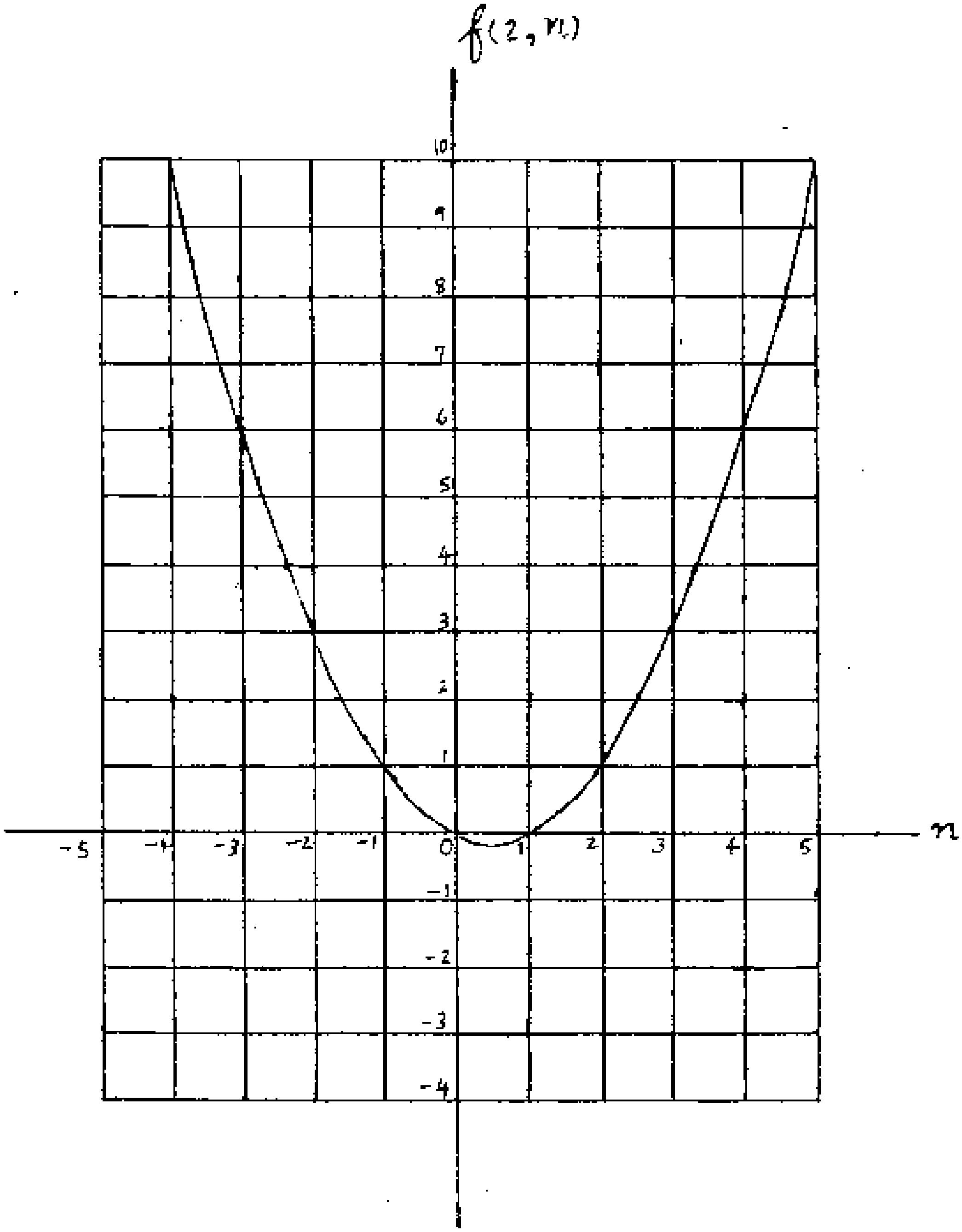

Fig.11 : The Graph of f(2, n)

(c) When $r = 3$, $f(3, n) = \frac{n(n-1)(n-2)}{3!}$

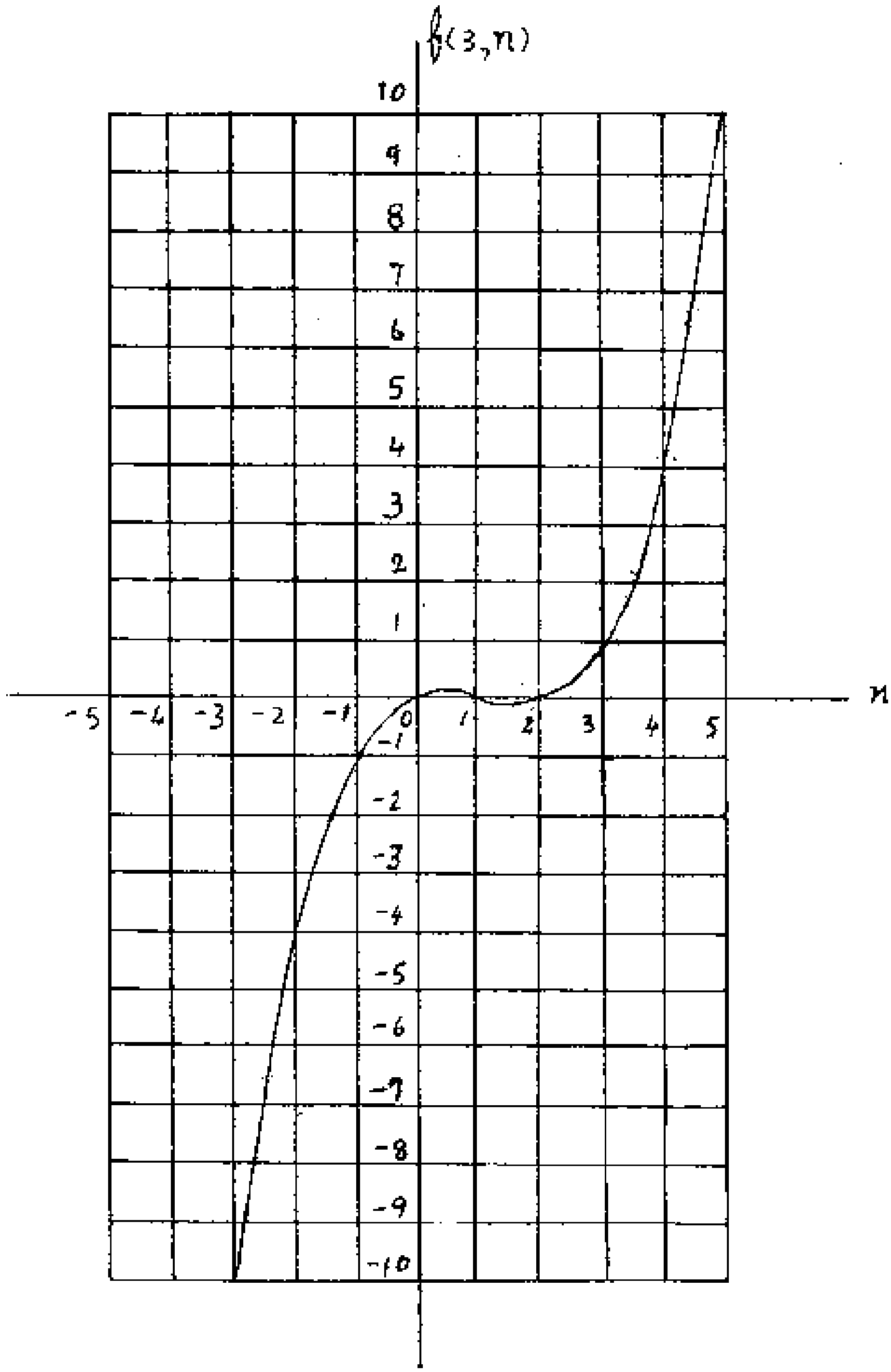

Fig. 12 : The Graph of f(3, n)

(d) When $r = 4$, $f(4, n) = \dfrac{n(n-1)(n-2)(n-3)}{4!}$

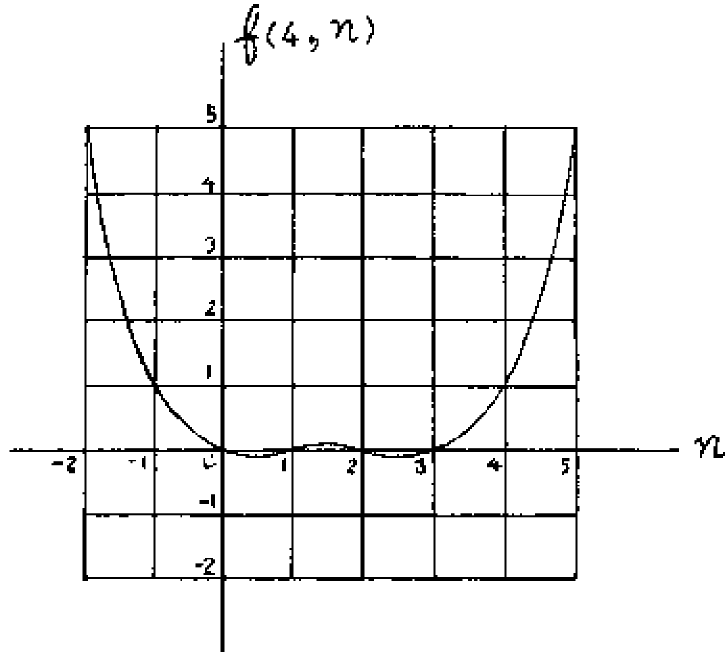

Fig.13 : The Graph of f(4, n)

(e) When r = 5, $f(5, n) = \dfrac{n(n-1)(n-2)(n-3)(n-4)}{5!}$

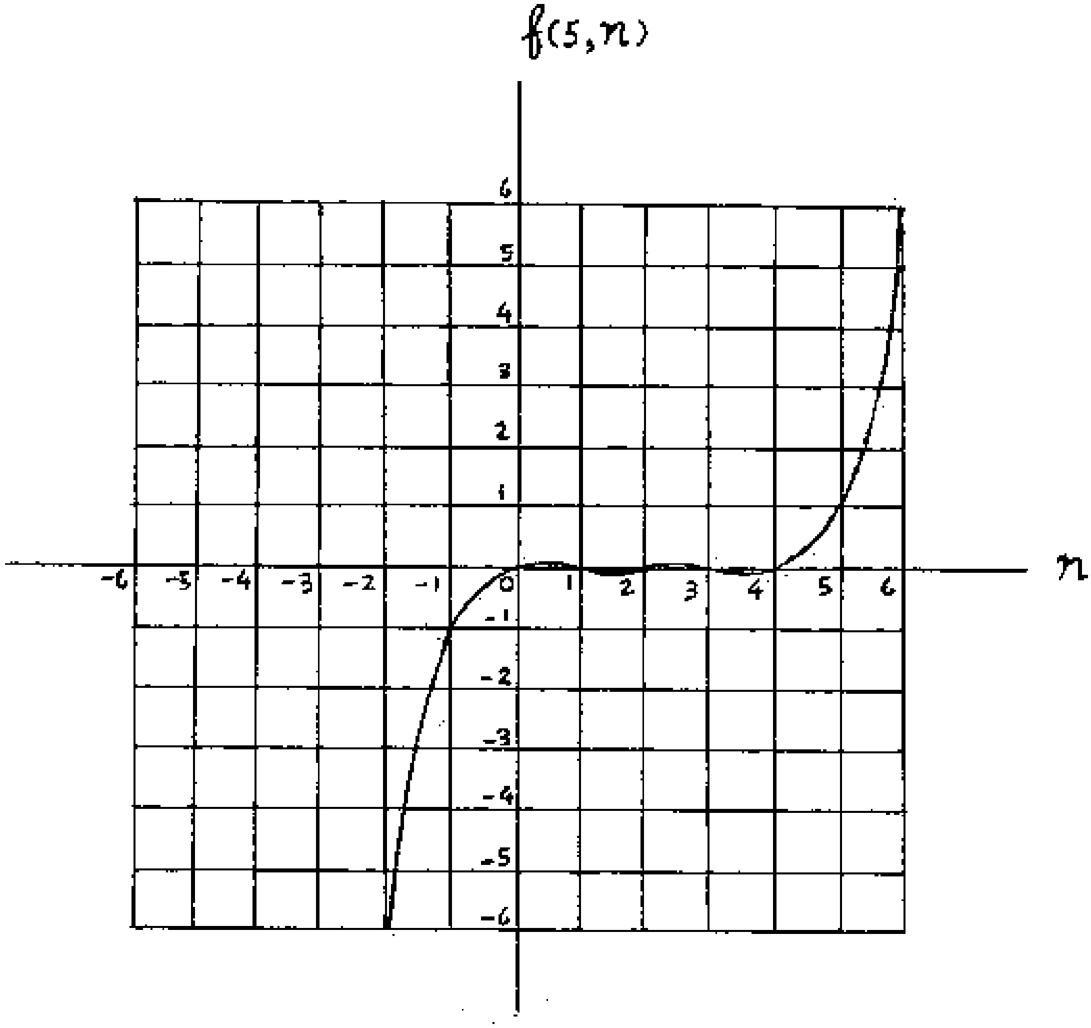

Fig.14 : The Graph of f(5, n)

4.3 The Graph for a Fixed Non-integral Value of r

Two examples are given below.

(1) The graph for r = - 2.5

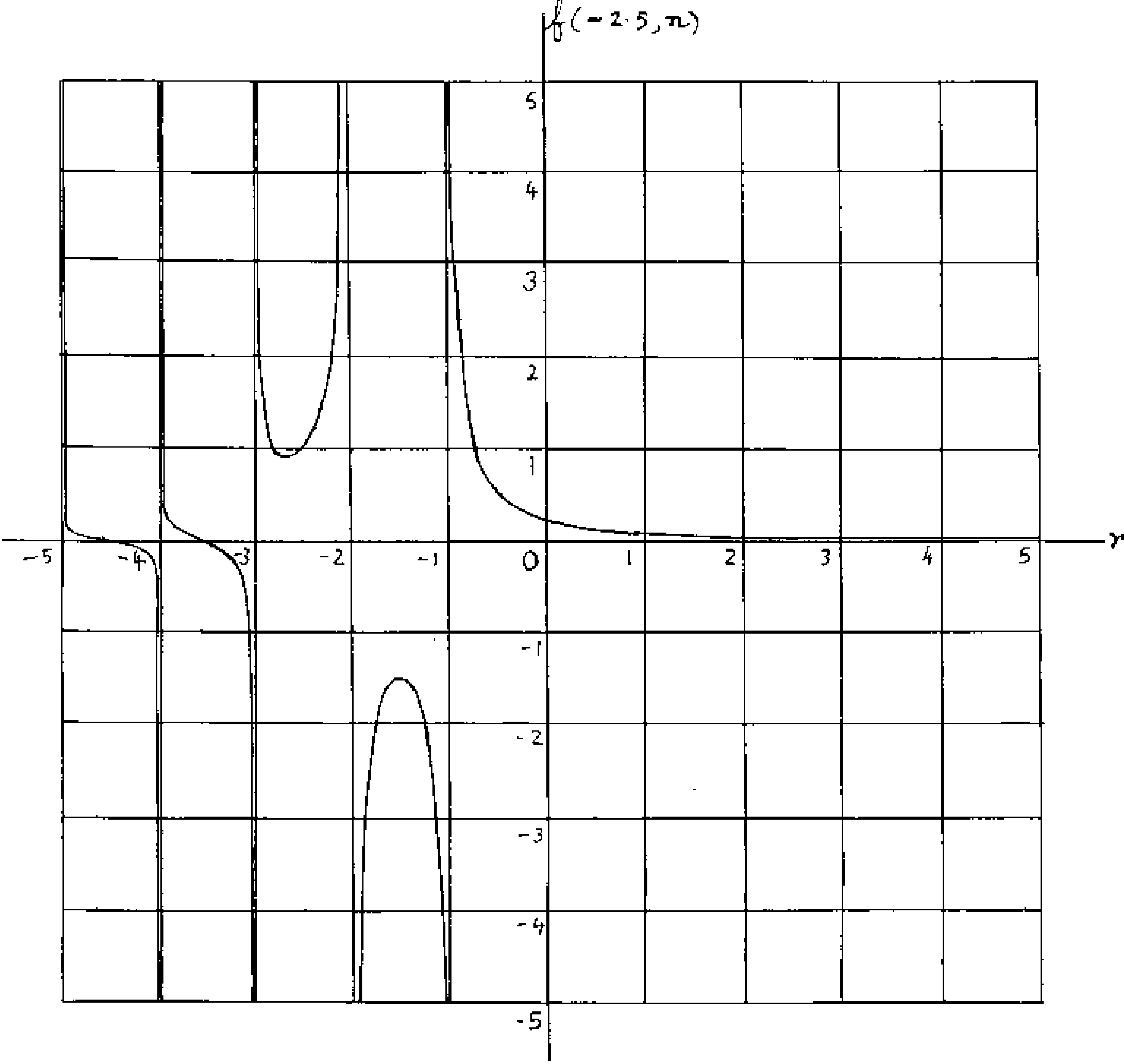

Fig.15 : The Graph of f(-2.5, n)

(2) the graph for r = 2.5

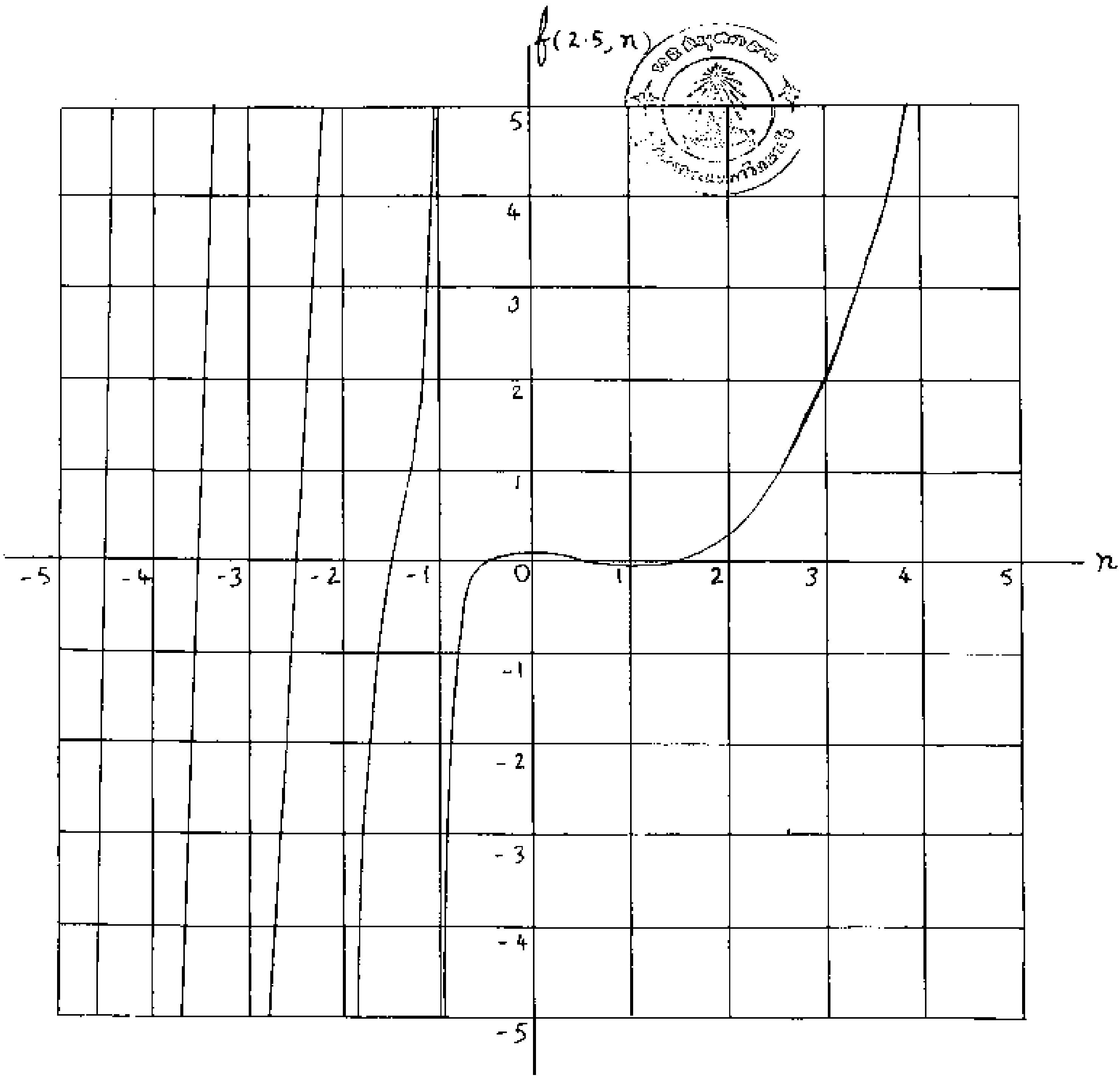

Fig. 16 : The Graph of f(2.5, n)