CHAPTER III

MEASURABLE FUNCTIONS AND MAPS

This chapter reviews some known results on measurable functions. The new results concern quaternion measurable functions.

3.1 Definition Let $\mathfrak{M}$ be a σ-algebra in a set X. Then X is called a measurable space, and the members of $\mathfrak{M}$ are called the measurable sets in X.

3.2 Definition Let X be a measurable space, Y a topological space, and $f:X \longrightarrow Y$. Then f is said to be measurable if $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

3.3 Theorem Let Y and Z be topological spaces, and let $g:Y \longrightarrow Z$ be continuous. If X is a measurable space, if $f:X \longrightarrow Y$ is measurable, then $g \circ f: X \longrightarrow Z$ is measurable.

Proof Standard. #

3.4 Theorem Let X be a measurable space and Y a topological space. Let $u_i:X \longrightarrow \mathbb{R}$ be measurable for all $i \leq 4$ and $\phi: \mathbb{R}^4 \longrightarrow Y$ continuous. Define $h:X \longrightarrow Y$ by

$$h(x) = \phi(u_1(x), u_2(x), u_3(x), u_4(x))$$

for all $x \in X$. Then h is measurable.

Proof Define $f:X \longrightarrow \mathbb{R}^4$ by

$$f(x) = (u_1(x), u_2(x), u_3(x), u_4(x))$$

$\forall x$ X. Let I_1, I_2, I_3 and I_4 be open intervals in $\mathbb{R}$. Claim that $f^{-1}(I_1 \times I_2 \times I_3 \times I_4) = u_1^{-1}(I_1) \cap u_2^{-1}(I_2) \cap u_3^{-1}(I_3) \cap u_4^{-1}(I_4)$.

Note that

$$a \in f^{-1}(I_1 \times I_2 \times I_3 \times I_4) \Longleftrightarrow f(a) = (u_1(a), u_2(a), u_3(a), u_4(a))$$
$$\in I_1 \times I_2 \times I_3 \times I_4$$
$$\Longleftrightarrow u_i(a) \in I_i, \quad i \le 4$$
$$\Longleftrightarrow a \in u_i^{-1}(I_i), \quad i \le 4$$
$$\Longleftrightarrow a \in \bigcap_{i=1}^{4} u_i^{-1}(I_i) .$$

So we have the claim. Since for each $i \in \{1,2,3,4\}$ u_i is measurable function, $u_i^{-1}(I_i)$ is measurable set in X. Hence $\bigcap_{i=1}^{4} u_i^{-1}(I_i)$ is a measurable set in X. Then $f^{-1}(I_1 \times I_2 \times I_3 \times I_4)$ is a measurable set in X. If V is an open set in $\mathbb{R}^4$, by Remark 1.28 and Theorem 1.29, there exist open intervals $I_{i1}, I_{i2}, I_{i3}, I_{i4} \in \mathbb{R}$ for all $i \in \mathbb{N}$ such that

$$V = \bigcup_{i=1}^{\infty} (I_{i1} \times I_{i2} \times I_{i3} \times I_{i4}) ,$$

so

$$f^{-1}(V) = \bigcup_{i=1}^{\infty} f^{-1}(I_{i1} \times I_{i2} \times I_{i3} \times I_{i4})$$

which is a measurable set in X. Hence f is measurable. By Theorem 3.3, h is measurable. #

3.5 <u>Corollary</u> Let X be a measurable space. Then

(a) If $f = u_1 + iu_2 + ju_3 + ku_4$ where $u_{i'} : X \longrightarrow \mathbb{R}$ is a real measurable function on X for all $i' \le 4$. then f is a quaternion measurable function.

(b) If $f = u_1 + iu_2 + ju_3 + ku_4$ is a quaternion measurable function on X ($u_{i'} : X \longrightarrow \mathbb{R}$, $i' \le 4$), then $u_{i'}$ and $|f|$ are real measurable functions on X.

(c) If f and g are quaternion measurable functions on X, then so are $f+g$ and fg.

(d) If E is a measurable set in X and if

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E, \end{cases}$$

then χ_E is a measurable function.

(e) If f is a quaternion measurable function on X, then there exists a quaternion measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.

Note that the function χ_E which occurs in (d) is called the <u>characteristic</u> <u>function</u> of the set E.

<u>Proof of Corollary 3.5</u> Standard. #

3.6 <u>Definition</u> Let X, Y be topological spaces. Let $\mathcal{B}$ be the set of all Borel sets in X. $f: X \to Y$ is called <u>Borel measurable</u> if $f^{-1}(V) \in \mathcal{B}$ for all open set V in Y.

<u>Remark:</u> Every continuous map is Borel measurable.

3.7 <u>Theorem</u> Suppose $\mathfrak{M}$ is a σ-algebra in X and Y a set. Assume $f: X \to Y$.

(a) If $\Omega = \{E \subseteq Y / f^{-1}(E) \in \mathfrak{M}\}$, then Ω is a σ-algebra in Y.

(b) If Y is a topological space, f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.

(c) If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ ($f^{-1}([-\infty, \alpha)) \in \mathfrak{M}$) for all $\alpha \in \mathbb{R}$, then f is measurable.

<u>Proof of (a)</u> Standard.

Proof of (b)[9] Let $\Omega = \{E \subseteq Y / f^{-1}(E) \in \mathfrak{M}\}$. Since f is measurable, $f^{-1}(V) \in \mathfrak{M}$ for all open set V in Y, thus Ω contains all open sets in Y. Since, by (a), Ω is a σ-algebra in Y, Ω contains all Borel sets in Y. Hence $E \in \Omega$, so $f^{-1}(E) \in \mathfrak{M}$.

Proof of (c)[9] Let $\Omega = \{E \subseteq [-\infty, \infty] / f^{-1}(E) \in \mathfrak{M}\}$. By (a), Ω is a σ-algebra in $[-\infty, \infty]$. By assumption, $(\alpha, \infty] \in \Omega$ for all $\alpha \in \mathbb{R}$. For $\alpha \in \mathbb{R}$,

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha - \tfrac{1}{n}] = \bigcup_{n=1}^{\infty} (\alpha - \tfrac{1}{n}, \infty]^c \in \Omega .$$

Then for $\alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty] \in \Omega$. Because every open set in $[-\infty, \infty]$ is a countable union of segments of the types $(\alpha, \infty]$, $[-\infty, \alpha)$ and (α, β), Ω contains every open set in $[-\infty, \infty]$, so f is measurable. #

3.8 Corollary Let $\mathfrak{M}$ be a σ-algebra in X and Y,Z topological spaces. Let $f: X \longrightarrow Y$ be measurable and $g: Y \longrightarrow Z$ Borel measurable Then $g \circ f: X \longrightarrow Z$ is measurable.

Proof Standard. #

3.9 Definition Let X be a non empty set and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of extended real-valued functions on X, i.e.,

$$f_n : X \longrightarrow [-\infty, \infty] \quad \text{for all } n \in \mathbb{N} .$$

We define $\sup_n f_n$ and $\inf_n f_n : X \longrightarrow [-\infty, \infty]$ by

$$\sup_n f_n(x) = \sup_n (f_n(x)) = \sup\{f_n(x) / n \in \mathbb{N}\}$$

$$\inf_n f_n(x) = \inf_n (f_n(x)) = \inf\{f_n(x) / n \in \mathbb{N}\}$$

and define $\limsup_{n \to \infty} f_n$ and $\liminf_{n \to \infty} f_n : X \longrightarrow [-\infty, \infty]$ by

$$\limsup_{n\to\infty} f_n(x) = \limsup_{n\to\infty}(f_n(x)), \quad \liminf_{n\to\infty} f_n(x) = \liminf_{n\to\infty}(f_n(x)).$$

3.10 <u>Theorem</u> If $f_n : X \to [-\infty, \infty]$ is measurable for all $n \in \mathbb{N}$, then $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable.

<u>Proof</u>[9] Let $g = \sup_n f_n$. Let $\alpha \in \mathbb{R}$. Claim that $g^{-1}((\alpha, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$. To prove this, let $x \in g^{-1}(\alpha, \infty]$. Then $g(x) \in (\alpha, \infty]$, so $\sup\{f_1(x), f_2(x), \ldots\} \in (\alpha, \infty]$. Thus $\alpha < \sup\{f_1(x), f_2(x), \ldots\}$. Then there exists $m \in \mathbb{N}$ such that $f_m(x) > \alpha$. Hence $x \in f_m^{-1}(\alpha, \infty]$. Hence $g^{-1}(\alpha, \infty] \subseteq \bigcup_{n=1}^{\infty} f_n^{-1}(\alpha, \infty]$. Conversely, let $y \in f_k^{-1}(\alpha, \infty]$ for some $k \in \mathbb{N}$. Then $f_k(y) \in (\alpha, \infty]$, so

$$g(y) = \sup\{f_1(y), f_2(y), \ldots\} \geq f_k(y) > \alpha,$$

that is $g(y) \in (\alpha, \infty]$ and thus $y \in g^{-1}(\alpha, \infty]$. Hence $\bigcup_{n=1}^{\infty} f_n^{-1}(\alpha, \infty] \subseteq g^{-1}(\alpha, \infty]$. So we have claim. Since $f_n^{-1}(\alpha, \infty]$ is measurable for all $n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} f_n^{-1}(\alpha, \infty]$ is measurable, hence $g^{-1}(\alpha, \infty]$ is measurable. By Theorem 3.7 (c), $\sup_n f_n$ is measurable.

Similarly, $\inf_n f_n$ is measurable, since $(\inf_n f_n)^{-1}[-\infty, \alpha) = \bigcup_{n=1}^{\infty} f_n^{-1}[-\infty, \alpha)$ for all $\alpha \in \mathbb{R}$.

Since $$\limsup_{n\to\infty} f_n = \inf_{k\geq 1}(\sup_{i\geq k} f_i)$$

and

$$\liminf_{n\to\infty} f_n = \sup_{k\geq 1}(\inf_{i\geq k} f_i),$$

it follows that they are both measurable. #

3.11 Theorem Let X be a measurable space. If $f_n : X \longrightarrow \mathbb{H}$ is measurable for all $n \in \mathbb{N}$ and there exists $f : X \longrightarrow \mathbb{H}$ such that $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for all $x \in X$, then f is measurable.

Proof For each $n \in \mathbb{N}$, let $f_n = u_n + iv_n + jw_n + kt_n$ for some $u_n, v_n, w_n, t_n : X \longrightarrow \mathbb{R}$ and $f = u + iv + jw + kt$ for some u, $v, w, t : X \longrightarrow \mathbb{R}$. Since for all $x \in X$,

$$\lim_{n \to \infty}(u_n(x) + iv_n(x) + jw_n(x) + kt_n(x)) = \lim_{n \to \infty} f_n(x) = f(x) =$$

$u(x) + iv(x) + jw(x) + kt(x)$, so $\lim\limits_{n \to \infty} u_n(x) = u(x)$, $\lim\limits_{n \to \infty} v_n(x) = v(x)$, $\lim\limits_{n \to \infty} w_n(x) = w(x)$ and $\lim\limits_{n \to \infty} t_n(x) = t(x)$. By Corollary 3.5 (b), u_n, v_n, w_n and t_n are real measurable for all $n \in \mathbb{N}$. Since $u(x) = \lim\limits_{n \to \infty} u_n(x) = \limsup\limits_{n \to \infty} u_n(x)$ for all $x \in X$, by Theorem 3.10, hence u is real measurable, Similarly, v, w and t are real measurable. By Corollarly 3.5 (a), $f = u + iv + jw + kt$ is measurable. #

3.12 Definition Let $f : X \longrightarrow [-\infty, \infty]$. Define

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0, \end{cases}$$

and

$$f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) \geq 0, \end{cases}$$

for all $x \in X$. The functions f^+ and f^- are called the positive and negative parts of f. We have $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

3.13 Theorem Let X be a measurable space and $f:X \longrightarrow [-\infty, \infty]$ real measurable. Then f^+ and f^- are non negative measurable functions.

Proof Clearly, f^+ and f^- are non negative. Let $A = \{x / f(x) \geq 0\}$ and $B = \{x / f(x) < 0\}$. Then $A = f^{-1}[0, \infty]$ and $B = f^{-1}[-\infty, 0)$, hence A and B are measurable. Therefore $f^+ = f\chi_A$ and $f^- = -f\chi_B$. Hence f^+ and f^- are measurable. #

3.14 Definition Let X be a measurable space. A function $s:X \longrightarrow (-\infty, \infty)$ is called a simple function if $s(X)$ is finite.

Let $\alpha_1, \ldots, \alpha_n$ be distinct values of a simple function $s:X \longrightarrow (-\infty, \infty)$. For each $i \in \{1, 2, \ldots, n\}$, let

$$A_i = \{x \in X / s(x) = \alpha_i\}.$$

Then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where χ_{A_i} as defined in Corollary 3.5 (d). Observe that s is measurable function if and only if each A_i is a measurable set .

3.15 Theorem Let $f:X \longrightarrow [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that

(a) $0 \leq s_1 \leq s_2 \leq \ldots \leq f$.

(b) $\lim_{n \to \infty} s_n(x) = f(x)$ for all $x \in X$.

Proof[9] For each $n \in \mathbb{N}$ and each i such that $1 \leq i \leq n2^n$, define

$$E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right)$$

and

$$F_n = f^{-1}([n, \infty]),$$

and let

$$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}.$$

Then $X = (\bigcup_{i=1}^{n2^n} E_{n,i}) \cup F_n$ which is a disjoint union for all $n \in \mathbb{N}$. By Theorem 3.7 (b), each $E_{n,i}$ and each F_n are measurable sets. Since $s_1 = \frac{1}{2}\chi_{E_{1,2}} + \chi_{F_1}$, $s_1 \geq 0$. Let $n \in \mathbb{N}$. Let $x \in X$. Since $\bigcup_{i=1}^{n2^n} [\frac{i-1}{2^n}, \frac{i}{2^n}) \cup [n, \infty] = [0, \infty]$, $f(x) \in [n, \infty]$ or $f(x) \in [\frac{i_o-1}{2^n}, \frac{i_o}{2^n})$ for some $i_o \in \{1, 2, \ldots, n2^n\}$. Then $x \in f^{-1}[n, \infty]$ or $x \in f^{-1}[\frac{i_o-1}{2^n}, \frac{i_o}{2^n})$. Thus if $x \in f^{-1}[n, \infty]$, then $s_n(x) = n \leq f(x)$, and if $x \in f^{-1}[\frac{i_o-1}{2^n}, \frac{i_o}{2^n})$, then $s_n(x) = \frac{i_o-1}{2^n} \leq f(x)$. This proves that $s_n \leq f$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, $\frac{i'-1}{2^{n+1}} = n \Longleftrightarrow i' = n2^{n+1} + 1 \in \{1, 2, \ldots, (n+1)2^{n+1}\}$ and for each $i \in \{1, 2, \ldots, n2^n\}$,

$$\frac{i'-1}{2^{n+1}} = \frac{i-1}{2^n} \Longleftrightarrow i' = 2i-1 \in \{1, 2, \ldots, (n+1)2^{n+1}\}.$$

Let $n \in \mathbb{N}$. Let $x \in X$. If $f(x) \in [n, \infty] = [\frac{n2^{n+1}+1-1}{2^{n+1}}, \infty]$, then $s_n(x) = n$ and $s_{n+1}(x) \geq \frac{n2^{n+1}+1-1}{2^{n+1}} = n$. If $f(x) \in [\frac{i_o-1}{2^n}, \frac{i_o}{2^n})$ for some $i_o \in \{1, 2, \ldots, n2^n\}$, then $f(x) \in [\frac{(2i_o-1)-1}{2^{n+1}}, \infty]$, so $s_n(x) = \frac{i_o-1}{2^n}$ and $s_{n+1}(x) \geq \frac{2i_o-1-1}{2^{n+1}} = \frac{i_o-1}{2^n}$. Hence $s_n \leq s_{n+1}$. This proves (a).

Next, we shall prove (b). Let $x \in X$. If $f(x) = \infty$, then $f(x) \in [n, \infty]$ for all $n \in \mathbb{N}$. Thus $s_n(x) = n$ for all $n \in \mathbb{N}$. Hence $\lim_{n \to \infty} s_n(x) = \infty = f(x)$. Assume $f(x) < \infty$. Then there exists $N_1 \in \mathbb{N}$ such that $N_1 > f(x)$. Thus $x \in f^{-1}[0, N_1)$, so for all $n \geq N_1$ there exists $i \in \{1, 2, \ldots, n2^n\}$ such that $f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})$, since $\bigcup_{i=1}^{n2^n} [\frac{i-1}{2^n}, \frac{i}{2^n}) = [0, n) \supseteq [0, N_1)$.

Hence

$$|s_n(x) - f(x)| = f(x) - s_n(x) < \frac{1}{2^n}$$

for all $n \geq N_1$. Therefore $\lim_{n \to \infty} s_n(x) = f(x)$. #

<u>Remark</u>: Observe that if f is bounded, then $(s_n)_{n \in \mathbb{N}}$ converges to f uniformly.

3.16 <u>Definition</u> Let $\mathfrak{M}$ be a σ-algebra in X. Let μ be an arbitrary measure on $\mathfrak{M}$, i.e., μ may be a positive or a real or a complex or a quaternion measure. Let $E \in \mathfrak{M}$ and P be a property which a point x may or may not have. The statement " P holds <u>almost</u> <u>everywhere</u> (a.e.) on E " means that there exists $N \in \mathfrak{M}$ such that $N \subseteq E$, $\mu(N) = 0$ and P holds at every point of $E \sim N$.

<u>Example</u> If f and g are measurable functions on X such that

$$\mu(\{x \in X / \ f(x) \neq g(x)\}) = 0 \ ,$$

we say that $f = g$ a.e. $[\mu]$ on X, and say that f and g are equivalent .

3.17 <u>Theorem</u>(Lebesgue) Let $\mathfrak{M}$ be a σ-algebra in X and $E \in \mathfrak{M}$. Let μ be a positive measure such that $\mu(E) < \infty$. Let there be given a sequence $(f_n)_{n\in N}$ of measurable functions on E, all of which are finite almost everywhere. Suppose that $f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$ almost everywhere on E and that f(x) is finite almost everywhere. Then

$$\lim_{n\to\infty} [\mu(\{x \in E / \ |f_n(x)-f(x)| \geq \sigma\})] = 0$$

for all $\sigma > 0$.

Proof [5] Let

$$A = \{x \in E / \ |f|(x) = \infty\}, \quad A_n = \{x \in E / \ |f_n|(x) = \infty\},$$

$$B = \{x \in E / \ f_n(x) \not\longrightarrow f(x)\}, \quad Q = A \cup (\bigcup_{n=1}^{\infty} A_n) \cup B.$$

It is clear that $\mu(Q) = 0$. Furthermore, let

$$E_k(\sigma) = \{x \in E / \ |f_k(x)-f(x)| \geq \sigma\},$$

$$R_n(\sigma) = \bigcup_{k=n}^{\infty} E_k(\sigma), \quad M = \bigcap_{n=1}^{\infty} R_n(\sigma).$$

All of these sets are measurable. Since $R_1(\sigma) \subseteq E$ and $\mu(E) < \infty$, $\mu(R_1(\sigma)) < \infty$. Since $R_1(\sigma) \supseteq R_2(\sigma) \supseteq \ldots$, we have

$$\lim_{n\to\infty} \mu(R_n(\sigma)) = \mu(M).$$

If $x_o \notin Q$, then $\lim_{k\to\infty} f_k(x_o) = f(x_o)$ where all the numbers $f_1(x_o), f_2(x_o), \ldots$ and their limit $f(x_o)$, are finite. Thus we can find an n such that

$$|f_k(x_o)-f(x_o)| < \sigma$$

for all $k \geq n$. In other words,

$$x_o \notin E_k(\sigma)$$

for all $k \geq n$. Therefore $x_o \notin R_n(\sigma)$ and so $x_o \notin M$. This shows that $M \subseteq Q$. Since $\mu(Q) = 0$, $\mu(M) = 0$. Thus

$$\lim_{n\to\infty} \mu(R_n(\sigma)) = 0.$$

Since $E_n(\sigma) \subseteq R_n(\sigma)$, we have $\lim_{n\to\infty} \mu(E_n(\sigma)) \leq$ $\lim_{n\to\infty} \mu(R_n(\sigma)) = 0$, hence $\lim_{n\to\infty} \mu(E_n(\sigma)) = 0$. #

3.18 <u>Definition</u> Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let the sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ be defined and finite almost everywhere on a measurable set E. Let f be a quaternion measurable function which is finite almost everywhere. If

$$\lim_{n\to\infty} \mu(\{x\in E / \ |f_n(x)-f(x)| \geq \sigma\}) = 0$$

for all positive number σ, the sequence $(f_n)_{n\in\mathbb{N}}$ is said to <u>converge in measure</u> to the function f on E.

From Theorem 3.17, we can conclude that if a sequence of measurable functions converges almost everywhere, it converges in measure to the same limit function. But the converse is false. For example:

Let m be the Lebesgue measure on $\mathbb{R}$. On the half-open interval $[0,1)$ define the k functions

$$f_1^{(k)}(x), f_2^{(k)}(x), \ldots, f_k^{(k)}(x),$$

for every natural number k, according to the following definition:

$$f_i^{(k)}(x) = \begin{cases} 1 & \text{for } x \in [\frac{i-1}{k}, \frac{i}{k}), \\ 0 & \text{for } x \notin [\frac{i-1}{k}, \frac{i}{k}). \end{cases}$$

In particular, $f_1^{(1)}(x) \equiv 1$ on $[0,1)$. Numbering all these functions with the indices 1,2,3,..., we obtain the sequence

$$\varphi_1(x) = f_1^{(1)}(x),\ \varphi_2(x) = f_1^{(2)}(x),\ \varphi_3(x) = f_2^{(2)}(x),$$

$\varphi_4(x) = f_1^{(3)}(x)$, $\varphi_5(x) = f_2^{(3)}(x)$, ...

It is easy to see that the sequence of functions φ_n converges in measure to zero. In fact, if $\varphi_n(x) = f_i^{(k)}(x)$, we have

$$\{x \in [0,1) / \ |\varphi_n(x)| \geqslant \sigma\} = [\tfrac{i-1}{k}, \tfrac{i}{k})$$

for all σ such that $0 < \sigma \leq 1$ and the measure of this set, which is equal to $\frac{1}{k}$, tend to zero as $n \longrightarrow \infty$. But for each $x \in [0,1)$, $\varphi_n(x) \not\longrightarrow 0$. In fact, if $x_0 \in [0,1)$, for every k we can find an i such that

$$x_0 \in [\tfrac{i-1}{k}, \tfrac{i}{k})$$

so that $f_i^{(k)}(x_0) = 1$. This proves our assertion.

3.19 __Theorem__ Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. If the sequence of functions f_n converges in measure to the function f, then the same sequence converges in measure to every function g which is equivalent to the function f.

__Proof__[5] For all $\sigma > 0$,

$$\{x \in X / \ |f_n(x)-g(x)| \geqslant \sigma\} \subseteq \{x \in X / \ f(x) \neq g(x)\} \cup \{x \in X / \ |f_n(x)-f(x)| \geqslant \sigma\}$$

and since $\mu(\{x \in X / \ f(x) \neq g(x)\}) = 0$, we have

$$\mu(\{x \in X / \ |f_n(x)-g(x)| \geqslant \sigma\}) \leq \mu(\{x \in X / \ |f_n(x)-f(x)| \geqslant \sigma\})$$

This proves the theorem. #

3.20 __Theorem__ Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. If the sequence of the functions f_n converges in measure to two functions f and g, then these limit functions are equivalent.

Proof[5] It is easy to see that

$$(*) \quad \{x\in X/\ |f(x)-g(x)|\geqslant \sigma\} \subseteq \{x\in X/\ |f_n(x)-f(x)|\geqslant \tfrac{\sigma}{2}\}\cup \{x\in X/\ |f_n(x)-g(x)|\geqslant \tfrac{\sigma}{2}\}$$

for all $\sigma > 0$, because a point not in the set on the right hand side of this inclusion cannot possibly be in the set on the left hand side. Since $(f_n)_{n\in\mathbb{N}}$ converges in measure to f and g, we have the measure of the right hand member of (*) tends to zero as $n\longrightarrow\infty$, from which it is clear that

$$\mu(\{x\in X/\ |f(x)-g(x)|\geqslant\sigma\}) = 0.$$

Since $\{x\in X/\ f(x)\neq g(x)\} \subseteq \bigcup_{n=1}^{\infty}\{x\in X/\ |f(x)-g(x)|\geqslant \frac{1}{n}\}$, we see that f = g a.e. . #

3.21 Theorem Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. If $\lim\limits_{n\longrightarrow\infty} f_n(x) = f_o(x)$ in measure and if $\lim\limits_{n\longrightarrow\infty} g_n(x) = g_o(x)$ in measure, then $\lim\limits_{n\longrightarrow\infty}(f_n+g_n)(x) = (f_o+g_o)(x)$ in measure.

Proof[4] Given $\sigma>0$, it follows from the inequality

$$|f_n(x)+g_n(x)-f_o(x)-g_o(x)| \leq |f_n(x)-f_o(x)| + |g_n(x)-g_o(x)|$$

that

$$\{x\in X/\ |f_n(x)+g_n(x)-f_o(x)-g_o(x)|\geqslant\sigma\}$$

$$\subseteq \{x\in X/\ |f_n(x)-f_o(x)|\geqslant\tfrac{\sigma}{2}\}\cup\{x\in X/\ |g_n(x)-g_o(x)|\geqslant\tfrac{\sigma}{2}\}.$$

But $\lim\limits_{n\to\infty}\mu(\{x\in X/\ |f_n(x)-f_o(x)|\geqslant\frac{\sigma}{2}\}) = 0 =$ $\lim\limits_{n\to\infty}\mu(\{x\in X/\ |g_n(x)-g_o(x)|\geqslant\frac{\sigma}{2}\})$. Hence

$$\lim_{n\to\infty}\mu(\{x\in X/\ |f_n(x)+g_n(x)-f_o(x)-g_o(x)|\geqslant\sigma\}) = 0.\ \#$$

3.22 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let θ be the function which is identically zero. If $\lim_{n\to\infty} f_n(x) = \theta(x)$ in measure and if $\lim_{n\to\infty} g_n(x) = \theta(x)$ in measure, then $\lim_{n\to\infty} f_n g_n(x) = \theta(x)$ in measure.

<u>Proof</u>[4] It follows immediately from the fact that $\{x \in X/\ |f_n(x)g_n(x)| \geq \sigma\} \subseteq \{x \in X/\ |f_n(x)| \geq \sqrt{\sigma}\} \cup \{x \in X/\ |g_n(x)| \geq \sqrt{\sigma}\}$ for all $\sigma > 0$. #

3.23 <u>Theorem</u> Let μ be a positive measure on a σ-algebra $\mathcal{M}$ in X. Let $E \in \mathcal{M}$ be such that $\mu(E) < \infty$. Let θ be the function which is identically zero. If $\lim_{n\to\infty} f_n(x) = \theta(x)$ in measure on E and if g_o is a measurable function which is finite almost everywhere, then $\lim_{n\to\infty} f_n g_o(x) = \theta(x)$ in measure on E.

<u>Proof</u>[4] For each $n \in \mathbb{N}$, let

$$E_n = \{x \in E/\ |g_o(x)| \geq n\} \text{ and } A = \bigcap_{n=1}^{\infty} E_n .$$

Then $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ and $A = \{x \in E/\ |g_o(x)| = \infty\}$. Hence $\lim_{n\to\infty} \mu(E_n) = \mu(A) = 0$. Let $\varepsilon > 0$ be given. Then there exists $k_o \in \mathbb{N}$ such that $\mu(E_{k_o}) < \frac{\varepsilon}{2}$. Let $\sigma > 0$ be given. We have

$$\{x \in E/\ |f_n(x)g_o(x)| \geq \sigma\} \subseteq \{x \in E/\ |f_n(x)| \geq \frac{\sigma}{k_o}\} \cup \{x \in E/\ |g_o(x)| \geq k_o\}.$$

Since $\lim_{n\to\infty} f_n(x) = \theta(x)$ in measure, there exists $N \in \mathbb{N}$ such that $\mu(\{x \in E/\ |f_n(x)| \geq \frac{\sigma}{k_o}\}) < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence for all $n \in \mathbb{N}$, $n \geq \max\{N, k_o\}$ implies that

$$\mu(\{x\in E/\ |f_n(x)g_o(x)|\geqslant \sigma\})$$
$$\leq \mu(\{x\in E/\ |f_n(x)|\geqslant \frac{\sigma}{k_o}\}\cup\{x\in E/\ |g_o(x)|\geqslant k_o\})$$
$$\leq \mu(\{x\in E/\ |f_n(x)|\geqslant \frac{\sigma}{k_o}\}) + \mu(\{x\in E/\ |g_o(x)|\geqslant k_o\})$$
$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \ . \ \#$$

3.24 Corollary Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E\in\mathfrak{M}$ be such that $\mu(E)<\infty$. If $\lim_{n\to\infty} f_n(x) = f_o(x)$ in measure on E and if $\lim_{n\to\infty} g_n(x) = g_o(x)$ in measure on E, then $\lim_{n\to\infty} f_ng_n(x) = f_og_o(x)$ in measure on E.

Proof[4] Since $f_ng_n-f_og_o = (f_n-f_o)(g_n-g_o)+f_o(g_n-g_o)+(f_n-f_o)g_o$, we are done by appling Theorem 3.21, 3.22 and 3.23 . #

3.25 Theorem(F. Riesz) Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E\in\mathfrak{M}$ be such that $\mu(E)<\infty$. If the sequence of functions f_n converges in measure to the function f on E. Then there exists a subsequence

$$f_{n_1}, f_{n_2}, f_{n_3}, \ldots \quad (n_1 < n_2 < n_3 < \ldots),$$

which converges to the function f almost everywhere on E.

Proof[5] Consider a sequence of positive numbers

$$\sigma_1 > \sigma_2 > \sigma_3 > \ldots,$$

for which $\lim_{k\to\infty} \sigma_k = 0$. Furthur, let

$$\eta_1 + \eta_2 + \eta_3 + \ldots \qquad (\eta_k > 0)$$

be a convergent series of positive terms. We select the required sequence of indices

(*) $$n_1 < n_2 < n_3 < \dots$$

in the following way. Let n_1 be a natural number for which

$$\mu(\{x \in E / |f_{n_1}(x) - f(x)| \geq \sigma_1\}) < \eta_1.$$

Such a number must necessarily exist, because

$$\mu(\{x \in E / |f_n(x) - f(x)| \geq \sigma_1\}) \longrightarrow 0$$

as $n \longrightarrow \infty$. Let n_2 be a natural number for which

$$\mu(\{x \in E / |f_{n_2}(x) - f(x)| \geq \sigma_2\}) < \eta_2, \quad n_1 < n_2 .$$

In general, we choose the number n_k so that

$$\mu(\{x \in E / |f_{n_k}(x) - f(x)| \geq \sigma_k\}) < \eta_k, \quad n_{k-1} < n_k .$$

Thus the sequence (*) is defined.

To show that $\lim_{k \longrightarrow \infty} f_{n_k}(x) = f(x)$ almost everywhere on E, let

$$R_i = \bigcup_{k=i}^{\infty} \{x \in E / |f_{n_k}(x) - f(x)| \geq \sigma_k\}, \quad Q = \bigcap_{i=1}^{\infty} R_i .$$

Since $R_1 \supseteq R_2 \supseteq R_3 \supseteq \dots$, we have

$$\lim_{i \longrightarrow \infty} \mu(R_i) = \mu(Q).$$

On the other hand, it is clear that

$$\mu(R_i) \leq \sum_{k=i}^{\infty} \eta_k ,$$

so that $\lim_{i \longrightarrow \infty} \mu(R_i) = 0$, and hence $\mu(Q) = 0$. It remains to verify that $\lim_{k \longrightarrow \infty} f_{n_k}(x) = f(x)$ for all $x \in E \sim Q$. Let $x_o \in E \sim Q$. Then $x_o \notin R_{i_o}$ for some natural number i_o. In other words,

$$x_o \notin \{x \in E / |f_{n_k}(x) - f(x)| \geq \sigma_k\},$$

for all $k \geq i_o$, and hence

$$|f_{n_k}(x_o) - f(x_o)| < \sigma_k ,$$

for all $k \geq i_o$. Let $\varepsilon > 0$ be given. Since $\lim_{k \longrightarrow \infty} \sigma_k = 0$, there exists $N_0 \in \mathbb{N}$ such that $\sigma_k < \varepsilon$ for all $k \geq N_0$. Choose $N = \max\{N_0, i_0\}$. Let $k \in \mathbb{N}$ be such that $k \geq N$. Then

$$|f_{n_k}(x_0) - f(x_0)| < \sigma_k < \varepsilon \,. \#$$

3.26 Theorem(D.F. Egorov) Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in X. Let $E \in \mathfrak{M}$ be such that $\mu(E) < \infty$. Let a sequence of measurable and almost everywhere finite functions

$$f_1, f_2, f_3, \ldots$$

be defined on E. Suppose that $(f_n)_{n \in \mathbb{N}}$ converges almost everywhere to the measurable function f which is finite almost everywhere on E:

(*) $$\lim_{n \to \infty} f_n(x) = f(x) \,.$$

Then, for every $\delta > 0$, there exists a measurable set $E_\delta \subseteq E \ni$

(1) $\mu(E_\delta) > \mu(E) - \delta$,

(2) The limit (*) is uniform on the set E_δ.

Proof[5] In the proof of Theorem 3.17, we showed that

(1) $$\lim_{n \to \infty} \mu(R_n(\sigma)) = 0,$$

for all $\sigma > 0$, where

$$R_n(\sigma) = \bigcup_{k=n}^{\infty} \{x \in E / \ |f_k(x) - f(x)| \geq \sigma\} \,.$$

We consider a convergent series of positive terms

$$\eta_1 + \eta_2 + \eta_3 + \ldots \qquad (\eta_i > 0)$$

and a sequence of positive numbers tending to zero:

$$\sigma_1 > \sigma_2 > \sigma_3 > \ldots, \qquad \lim_{i \to \infty} \sigma_i = 0.$$

From (1), we can find for every natural number i another natural number n_i such that

$$\mu(R_{n_i}(\sigma)) < \eta_i .$$

Let $\delta > 0$ be given. Since $\sum_{i=1}^{\infty} \eta_i$ converges, we can choose an i_o such that

$$\sum_{i=i_o}^{\infty} \eta_i < \delta ,$$

and we set

$$A = \bigcup_{i=i_o}^{\infty} R_{n_i}(\sigma_i).$$

Then

$$\mu(A) \leq \sum_{i=i_o}^{\infty} \mu(R_{n_i}(\sigma_i) \leq \sum_{i=i_o}^{\infty} \eta_i < \delta,$$

so

$$\mu(A) < \delta.$$

Let $E_\delta = E \setminus A$. Then $E = E_\delta \cup A$ and $E_\delta \cap A = \phi$, hence $\mu(E) = \mu(E_\delta) + \mu(A) < \mu(E_\delta) + \delta$ which implies that

$$\mu(E_\delta) > \mu(E) - \delta .$$

To show that $\lim_{n \to \infty} f_n(x) = f(x)$ is uniform on the set E_δ, let $\varepsilon > 0$ be given. Since $\lim_{i \to \infty} \sigma_i = 0$, there exists $i^* \in \mathbb{N}$ such that $\sigma_{i^*} < \varepsilon$. Let $i = \max\{i^*, i_o\}$, so $i \geq i_o$ and $\sigma_i < \varepsilon$. Let $k \geq n_i$, and let $x \in E_\delta$. Then $x \notin A$ which implies that $x \notin R_{n_i}(\sigma_i)$. In other words,

$x \notin \{x \in E / \ |f_k(x) - f(x)| \geq \sigma_i\}$ so that $|f_k(x) - f(x)| < \sigma_i$ and hence,

$$|f_k(x) - f(x)| < \varepsilon \quad . \#$$