CHAPTER I

PRELIMINARIES

Throughout this research our scalar field is either the field of real numbers or the field of complex numbers. Let $\mathbb{N}$ denote the set of all natural numbers.

Let x be a sequence and A an infinite matrix. For $k \in \mathbb{N}$, the k^{th} term of the sequence x is denoted by x_k, and for n, $k \in \mathbb{N}$, the element of A in the n^{th} row and k^{th} column is denoted by A_{nk}. If $\sum_{k=1}^{\infty} A_{nk}x_k$ converges for every $n \in \mathbb{N}$, we say that Ax exists and let Ax be the sequence with $\sum_{k=1}^{\infty} A_{nk}x_k$ as its n^{th} term for every $n \in \mathbb{N}$, so Ax = $(\sum_{k=1}^{\infty} A_{nk}x_k)_{n=1}^{\infty}$. Let e be the sequence with $e_k = 1$ for every $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $e^{(k)}$ be the sequence such that

$$e_n^{(k)} = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

If A and B are infinite matrices such that $\sum_{i=1}^{\infty} A_{ni}B_{ik}$ converges for all $n,k \in \mathbb{N}$, then we say that AB exists and it is defined to be the infinite matrix C with $C_{nk} = \sum_{i=1}^{\infty} A_{ni}B_{ik}$ for all $n,k \in \mathbb{N}$.

The series $\sum_{k=1}^{\infty} A_{nk}$ is said to <u>converge</u> <u>uniformly</u> on n = 1,2,3,... if for every $\epsilon > 0$, there exists $k_o \in \mathbb{N}$ such that $|\sum_{k=k_o+1}^{k_o+p} A_{nk}| < \epsilon$ for every $p,n \in \mathbb{N}$.

By the Cesaro matrix we mean the infinite matirx C such that

$$C_{nk} = \begin{cases} \frac{1}{n} & \text{if } 1 \le k \le n , \\ 0 & \text{if } k > n , \end{cases}$$

for all $n,k \in \mathbb{N}$, that is ,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix} .$$

An infinite matrix A is said to be row-bounded if there exists $k_o \in \mathbb{N}$ such that $A_{nk} = 0$ for all $n \in \mathbb{N}$, $k > k_o$. A column-bounded matrix is defined similarly. A finite matrix is an infinite matrix which is both row-bounded and column-bounded. It is clearly seen that if A is the Cesaro matrix or a row-bounded matrix, then $\sum_{k=1}^{\infty} |A_{nk}-A_{n(k+1)}|$ converges uniformly on n = 1,2,3,...

By the Borel matrix we mean the matrix A such that $A_{nk}= \frac{n^{(k-1)}}{(k-1)!\ e^n}$ for all $n, k \in \mathbb{N}$. If A is the Borel matrix, then $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1 = \sup_n \sum_{k=1}^{\infty} |A_{nk}|$.

A Norlund matrix is an infinite matrix A defined by

$$A_{nk} = \begin{cases} \dfrac{p_{n-k+1}}{\sum_{i=1}^{n} p_i} & \text{if } 1 \le k \le n, \\ 0 & \text{if } k > n , \end{cases}$$

where (p_n) is a sequence of positive real numbers. Then if A is a Norlund matrix, then $\sum_{k=1}^{\infty} A_{nk} = 1$ for every $n \in \mathbb{N}$.

The space of all sequences is denoted by W and let Φ denote the space of all finite sequences, that is,

$$\Phi = \left\{ (x_k) \mid x_k = 0 \text{ for all but a finite number of } k \right\}.$$

The list of all the classical sequence spaces with their norms is as follows:

$$\ell_\infty = \text{the space of all bounded sequences ,}$$

$$\|x\|_{\ell_\infty} = \sup_k |x_k| \ ,$$

$$c = \text{the space of all convergent sequences ,}$$

$$\|x\|_c = \sup_k |x_k| \ ,$$

$$c_o = \text{the space of all null sequences ,}$$

$$= \left\{ (x_k) \mid \lim_{k\to\infty} x_k = 0 \right\} ,$$

$$\|x\|_{c_o} = \sup_k |x_k| \ ,$$

$$\ell_p = \text{the space of all sequences } x = (x_k) \text{ such that } \sum_{k=1}^{\infty} |x_k|^p < \infty$$

where $1 < p < \infty$,

$$\|x\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} ,$$

$$\ell = \text{the space of all sequences } x = (x_k) \text{ such that } \sum_{k=1}^{\infty} |x_k| < \infty ,$$

$$\|x\|_\ell = \sum_{k=1}^{\infty} |x_k| \ ,$$

$$bv = \text{the space of all sequences of bounded variation,}$$

$$= \left\{ (x_k) \mid \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty \right\} ,$$

$$\|x\|_{bv} = \sum_{k=1}^{\infty} |x_k - x_{k+1}| + \lim_{k\to\infty} |x_k| ,$$

$$bv_o = bv \cap c_o ,$$

$$\|x\|_{bv_o} = \sum_{k=1}^{\infty} |x_k - x_{k+1}| ,$$

bs = the space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} x_k$ is a bounded series,

$$\|x\|_{bs} = \sup_n \left| \sum_{k=1}^{n} x_k \right| ,$$

cs = the space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} x_k$ is a convergent series,

and $\|x\|_{cs} = \sup_n \left| \sum_{k=1}^{n} x_k \right| .$

The Cesaro sequence spaces are as follows:

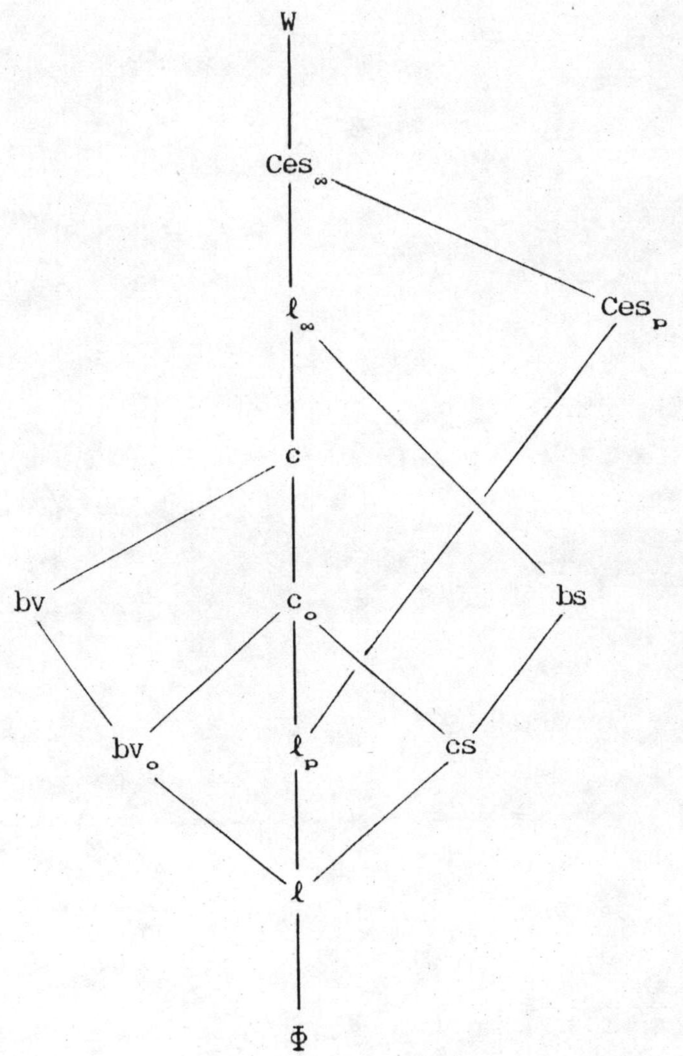
Ces_p = the space of all sequences $x = (x_k)$ such that $\left(\frac{1}{n}\sum_{k=1}^{n} |x_k| \right)_{n=1}^{\infty} \in \ell_p$,

$$\|x\|_{Ces_p} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} \text{ where } 1 < p < \infty$$

Ces_∞ = the space of all sequences $x = (x_k)$ such that $\left(\frac{1}{n}\sum_{k=1}^{n} |x_k| \right)_{n=1}^{\infty} \in \ell_\infty$ and

$$\|x\|_{Ces_\infty} = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k| .$$

The following diagram shows the relationships under set inclusion among the sequence spaces mentioned above:



Note that the following statements hold :

(1) c_o is a closed normed linear subspace of c and $c = c_o + \langle e \rangle$ $= \left\{ x + \alpha e \mid x \in c_o , \alpha \in F \right\}$ where F is the scalar field.

(2) c is a closed normed linear subspace of ℓ_∞ .

(3) bv_o is a closed normed linear subspace of bv, $bv = bv_o + \langle e \rangle$ $= \left\{ x + \alpha e \mid x \in bv_o, \alpha \in F \right\}$ and $\|x + \alpha e\|_{bv} = \|x\|_{bv_o} + |\alpha|$ for all $x \in bv_o$, $\alpha \in F$.

(4) cs is a closed normed linear subspace of bs.

For $k \in \mathbb{N}$, by the k^{th} coordinate mapping we mean the mapping p_k defined by $p_k(x) = x_k$ for each $x \in W$. A topological sequence space X is said to be a K-space if each coordinate mapping is continuous on X. By a BK-space we mean a Banach sequence space which is a K-space. It is easy to see that if X is one of the classical sequence spaces and the Cesaro sequence spaces, then for each $k \in \mathbb{N}$, there exists $\alpha_k > 0$ such that $|x_k| \leq \alpha_k \|x\|$ for every $x \in X$. Therefore all of the classical sequence spaces and the Cesaro sequence spaces are K-spaces. It is known that all of the classical sequence spaces are Banach spaces. Hence we have the following theorem.

Theorem 1.1. The sequence spaces ℓ_∞, c, c_o, $\ell_p (1 < p < \infty)$, ℓ, bv, bv_o, bs and cs are BK-spaces.

A proof that the Cesaro sequence spaces are Banach spaces has been given by Leibowitz in [8]. Therefore the following theorem is obtained.

Theorem 1.2. The sequence spaces $Ces_p (1 < p < \infty)$ and Ces_∞ are BK-spaces.

A metric d on a vector space X is said to be invariant if $d(x,y) = d(x-y, 0)$ for all $x, y \in X$. A topological vector space X is said to be an F-space if its topology is induced by a complete invariant metric. By an FK-space we mean a topological sequence space which is both an F-space and a K-space. Hence every BK-space is an FK-space. Therefore, by Theorem 1.1 and Theorem 1.2, all of the classical sequence spaces and the Cesaro sequence spaces are FK-spaces.

The following theorem of FK-spaces is known.

<u>Theorem 1.3</u>. ([10]) Let X and Y be FK-spaces. If X is a subset of Y, then the inclusion mapping from X into Y is continuous.

A topological sequence space X is said to have the <u>AK property</u> if X contains all finite sequences and for each $x \in X$, $x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)}$ in X, that is, $\lim_{n\to\infty} (x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots) = (x_1, x_2, x_3, \ldots)$, or equivalently $\lim_{n\to\infty} (x - \sum_{k=1}^{n} x_k e^{(k)}) = \lim_{n\to\infty} (0,0,0,\ldots,x_{n+1}, x_{n+2}, \ldots)$ $= (0,0,0,\ldots)$.

Therefore a normed sequence space X has the AK property if and only if $\Phi \subseteq X$ and $\lim_{n\to\infty} \| x - \sum_{k=1}^{n} x_k e^{(k)} \|_X = 0$.

Not all of the classical sequence spaces have the AK property. It is known that c_o, ℓ, ℓ_p $(1 < p < \infty)$, bv_o and cs have the AK property and it will be shown that the rest of them do not have the AK property.

<u>Theorem 1.4</u>. The sequence spaces c_o, ℓ , $\ell_p (1 < p < \infty)$, bv_o and cs have AK property.

It is easily seen that for every $n \in \mathbb{N}$, $\|e - \sum_{k=1}^{n} e^{(k)}\|_{\ell_\infty} = 1 = \|e - \sum_{k=1}^{n} e^{(k)}\|_{c}$ and $\|e - \sum_{k=1}^{n} e^{(k)}\|_{bv} = 2$. It follows that ℓ_∞, c and bv do not have the AK property. Since $((-1)^n)_{n=1}^{\infty} \in bs$ and $\|((-1)^n)_{n=1}^{\infty} - \sum_{k=1}^{m} (-1)^k e^{(k)}\|_{bs} = 1$ for every $m \in \mathbb{N}$, we have that bs does not have the AK property.

Let X be a Hausdorff topological vector space. A sequence (x_n) in X is said to form a <u>basis</u> of X if for each $y \in X$ there is a unique sequence (λ_n) of scalars such that $y = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda_k x_k$ $(= \sum_{k=1}^{\infty} \lambda_k x_k)$ in X. The following theorem has been proved by Kwang in [7].

<u>Theorem 1.5</u>. ([7]) The sequence $(e^{(k)})_{k=1}^{\infty}$ forms a basis of Ces_p where $1 < p < \infty$.

If a Hausdorff K-space X has $(e^{(k)})_{k=1}^{\infty}$ as a basis then for each $x \in X$, $x = \lim_{n\to\infty} \sum_{k=1}^{n} x_k e^{(k)}$ which implies that X has the AK property. Since Ces_p $(1 < p < \infty)$ is a K-space, the following theorem is obtained by Theorem 1.5.

<u>Theorem 1.6</u>. The sequence space $Ces_p (1 < p < \infty)$ has the AK property.

The β-dual of a sequence space X is defined to be

$$X^\beta = \left\{ (y_k) \mid \sum_{k=1}^{\infty} x_k y_k \text{ converges for all } (x_k) \in X \right\}$$

Observe that X^β is a subspace of W, $\Phi \subseteq X^\beta$, $\Phi^\beta = W$ and $W^\beta = \Phi$. In general if Y is a subspace of X, then $X^\beta \subseteq Y^\beta$.

The following theorem is known.

Theorem 1.7. (1) $\ell_\infty^\beta = \ell$,

(2) $c^\beta = \ell$,

(3) $c_o^\beta = \ell$,

(4) $\ell_p^\beta = \ell_q$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

(5) $\ell^\beta = \ell_\infty$,

(6) $bv^\beta = cs$,

(7) $bv_o^\beta = bs$,

(8) $bs^\beta = bv_o$ and

(9) $cs^\beta = bv$.

If X is a BK-space containing all finite sequences, then X^β is a normed sequence space with a norm defined by

$$\|(y_k)\|_{X^\beta} = \sup\left\{ \left|\sum_{k=1}^{\infty} x_k y_k\right| \;\middle|\; (x_k) \in X,\ \|(x_k)\|_X \le 1 \right\}$$

It is clearly seen that if X and Y are BK-spaces containing all finite sequences and Y is a normed linear subspace of X, then X^β is a vector subspace of Y^β and $\|.\|_{X^\beta} \ge \|.\|_{Y^\beta}$ on X since for $(y_k) \in X^\beta$,

$$\begin{aligned} \|(y_k)\|_{X^\beta} &= \sup\left\{ \left|\sum_{k=1}^{\infty} x_k y_k\right| \;\middle|\; (x_k)\in X,\ \|(x_k)\|_X \le 1 \right\} \\ &\ge \sup\left\{ \left|\sum_{k=1}^{\infty} x_k y_k\right| \;\middle|\; (x_k)\in Y,\ \|(x_k)\|_X \le 1 \right\} \\ &= \sup\left\{ \left|\sum_{k=1}^{\infty} x_k y_k\right| \;\middle|\; (x_k)\in Y,\ \|(x_k)\|_Y \le 1 \right\} \\ &= \|(y_k)\|_{Y^\beta} . \end{aligned}$$

Having the AK property is a sufficient condition for a BK-space to have its β-dual be a BK-space.

Theorem 1.8.([10]) If X is a BK-space with AK property, then X^{β} is a BK-space.

An infinite matrix A is said to map a sequence space X into a sequence space Y, written as $A : X \longrightarrow Y$ if Ax exists and $Ax \in Y$ for all $x \in X$, that is, for every $x = (x_k) \in X$, $\sum_{k=1}^{\infty} A_{nk}x_k$ converges for all $n \in \mathbb{N}$ and $(\sum_{k=1}^{\infty} A_{nk}x_k)_{n=1}^{\infty} \in Y$. Then for any sequence space X and for any infinite matrix A, $A : X \longrightarrow W$ if and only if each row of A belongs to X^{β}, that is, $(A_{nk})_{k=1}^{\infty} \in X^{\beta}$ for all $n \in \mathbb{N}$.

In general, matrix transformations between topological sequence spaces need not be continuous. It is well-known that matrix transformations between FK-spaces are always continuous.

Theorem 1.9.([10]) Let X and Y be topological sequence spaces and A an infinite matrix such that $A : X \longrightarrow Y$. If X and Y are FK-spaces, then A is continuous on X.

In particular, if X and Y are BK-spaces, then A is a continuous linear transformation, or equivalently,

$$\|A\| = \sup \left\{ \|Ax\|_Y \mid x \in X, \|x\|_X \le 1 \right\} < \infty.$$

The following known characterizations of infinite matrices mapping between some sequence spaces mentioned previously will be referred in this research.

Theorem 1.10. ([2]) For an infinite matrix A, $A : W \longrightarrow \ell_\infty$ if and only if

(i) A is row-bounded and

(ii) $\sup_n |A_{nk}| < \infty$ for every $k \in \mathbb{N}$.

Theorem 1.11. ([2]) For an infinite matrix A, $A : W \longrightarrow c$ if and only if

(i) A is row-bounded and

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

Theorem 1.12. ([10]) Let $X = \ell_\infty$, c or c_o and A an infinite matrix. Then $A : X \longrightarrow \ell_\infty$ if and only if $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$.

Theorem 1.13. (Kojima - Schur Theorem, [10]) For an infinite matrix A, $A : c \longrightarrow c$ if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$,

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists .

An infinite matrix A which satisfies the condition (i) of Theorem 1.13 is called a K_r- matrix, while if A satisfies the conditions (i), (ii) and (iii) of Theorem 1.13, A is called a Kojima matrix. Then a Kojima matrix is a K_r-matrix. Note that all finite matrices, all scalar matrices, the Cesaro matrix and the Borel matrix are Kojima matrices, so all of them are K_r-matrices. A Norlund matrix is a K_r-matrix but it is not necessarily a Kojima matrix.

Theorem 1.14.([10]) For an infinite matrix A, A : cs ⟶ cs if and only if

(i) $\sup_n \sum_{k=1}^{\infty} |\sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)}| < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for all $k \in \mathbb{N}$.

Theorem 1.15.([10]) For an infinite matrix A, A : ℓ ⟶ cs if and only if

(i) $\sup_{n,k} |\sum_{i=1}^{n} A_{ik}| < \infty$ and

(ii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

Let X and Y be sequence spaces and let A be an infinite matrix such that A : X ⟶ Y. The matrix A is said to

(1) preserve convergence if for $x = (x_n) \in X$, (x_n) converges implies $Ax \left(= ((Ax)_n)_{n=1}^{\infty} \right)$ converges ,

(2) preserve limits if for $x = (x_n) \in X$, (x_n) converges implies Ax converges and $\lim_{n\to\infty} (Ax)_n = \lim_{n\to\infty} x_n$,

(3) preserve summability if for $x = (x_n) \in X$, $\sum_{n=1}^{\infty} x_n$ converges implies $\sum_{n=1}^{\infty} (Ax)_n$ converges and

(4) preserve sums if for $x = (x_n) \in X$, $\sum_{k=1}^{\infty} x_n$ converges implies $\sum_{n=1}^{\infty} (Ax)_n$ converges and $\sum_{n=1}^{\infty} (Ax)_n = \sum_{n=1}^{\infty} x_n$.

The following two theorems of limit preserving matrix transformations and sum preserving matrix transformations between some certain classical sequence spaces are well-known.

Theorem 1.16. (Silverman - Toeplitz Theorem, [10]) If A is an infinite matrix, then $A : c \longrightarrow c$ and A preserves limits if and only if

(i) $\sup\limits_{n} \sum\limits_{k=1}^{\infty} |A_{nk}| < \infty$,

(ii) $\lim\limits_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$ and

(iii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} A_{nk} = 1$.

An infinite matrix A which satisfies the conditions (i), (ii) and (iii) of Theorem 1.16 is called a Toeplitz matrix. Then every Toeplitz matrix is a Kojima matrix, so it is a K_r-matrix. The Cesaro matrix and the Borel matrix are also Toeplitz matrices. However, a finite matrix cannot be a Toeplitz matrix. The identity matrix is the only scalar matrix which is a Toeplitz matrix.

Theorem 1.17. ([10]) If A is an infinite matrix, then $A : cs \longrightarrow cs$ and A preserves sums if and only if

(i) $\sup\limits_{n} \sum\limits_{k=1}^{\infty} \left| \sum\limits_{i=1}^{n} A_{ik} - \sum\limits_{i=1}^{n} A_{i(k+1)} \right| < \infty$ and

(ii) $\sum\limits_{n=1}^{\infty} A_{nk} = 1$ for all $k \in \mathbb{N}$.