CHAPTER IV

THE CLASSICAL SEQUENCE SPACES

The classical sequence spaces are listed in Chapter I.In this chapter,convergence preserving matrix transformations,limit preserving matrix transformations, summability preserving matrix transformations and sum preserving matrix transformations from each of the classical sequence spaces into ℓ_∞ are characterized.

A necessary condition for an infinite matrix A to map a sequence space X into any sequence space is that each row of A is in the β-dual of X (see Chapter I, page 13), so we have seen that the β-duals of sequence spaces play an important role in studying matrix transformations. The first section of this chapter gives some properties of the β-duals of the classical sequence spaces which will be used for the remaining sections.

4.1 SOME PROPERTIES OF β-DUALS

All of the classical sequence spaces are BK-spaces containing all finite sequences.The norm of the β-dual of a BK-space which contains all finite sequences is given in Chapter I. Theorem 1.7 of Chapter I shows that the β-dual of each of the classical sequence space is one of the

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_p^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_p^{\beta}} < \infty$

and (ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$.

Theorem 5.1.2. For an infinite matrix A, $A:Ces_p \longrightarrow \ell_{\infty}$ $(1 < p < \infty)$ and A preserves limits if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_p^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_p^{\beta}} < \infty$ and

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$.

Theorem 5.1.3. For an infinite matrix A, $A:Ces_p \longrightarrow \ell_{\infty}$ $(1 < p < \infty)$ and A preserves summability if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_p^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_p^{\beta}} < \infty$,

(ii) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in (Ces_p \cap cs)^{\beta}$ for every $n \in \mathbb{N}$ and

$\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{(Ces_p \cap cs)^{\beta}} < \infty$ where the norm of $Ces_p \cap cs$ is $\max\{ \|.\|_{Ces_p} , \|.\|_{cs} \}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

Theorem 5.1.4. For an infinite matrix A, $A:Ces_p \longrightarrow \ell_{\infty}$ $(1 < p < \infty)$ and A preserves sums if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_p^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_p^{\beta}} < \infty$,

(ii) $(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty} \in (Ces_p \cap cs)^{\beta}$ for every $n \in \mathbb{N}$ and

$\sup_n \|(\sum_{i=1}^{n} A_{ik})_{k=1}^{\infty}\|_{(Ces_p \cap cs)^{\beta}} < \infty$ where the norm of $Ces_p \cap cs$ is $\max\{ \|.\|_{Ces_p} , \|.\|_{cs} \}$ and

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$.

5.2 THE SPACE Ces_∞

Recall that the space Ces_∞ is a BK-space containing all finite sequences but does not have the AK property.Then Lemma 4.2.2,Lemma 4.3.2, Lemma 4.4.6 and Lemma 4.5.4 can not be used for the study in this section. We shall use Lemma 4.2.1 for our study.

Theorem 5.2.1. For an infinite matrix A, $A:Ces_\infty \longrightarrow \ell_\infty$ and A preserves convergence if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_\infty^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_\infty^{\beta}} < \infty$,

(ii) $\lim_{n\to\infty} A_{nk}$ exists for every $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}$ exists .

Proof: Assume that $A : Ces_\infty \longrightarrow \ell_\infty$ and A preserves convergence, we have from Lemma 4.2.1 that the condition (i) holds. Since $e^{(k)} \in Ces_\infty$ for all $k \in \mathbb{N}$ and $e \in Ces_\infty$, (ii) and (iii) are obtained by Proposition 2.3(i) and (iii).

Conversely, assume that (i), (ii), and (iii) hold. It follows by Lemma 4.2.1 that (i) implies $A:Ces_\infty \longrightarrow \ell_\infty$. By (i), there exists $M > 0$ such that $\sup_n |\sum_{k=1}^{\infty} A_{nk}x_k| < M$ for all $n \in \mathbb{N}$ and all $x \in Ces_\infty$ with $\|x\|_{Ces_\infty} \leq 1$. Fix $n \in \mathbb{N}$ and let y be the sequence such that $y_k = \frac{|A_{nk}|}{A_{nk}}$ if $A_{nk} \neq 0$ and 0 otherwise. Then $y \in Ces_\infty$ and $\|y\|_{Ces_\infty} \leq 1$. Hence $\sum_{k=1}^{\infty} |A_{nk}| = |\sum_{k=1}^{\infty} A_{nk}y_k| \leq \sup |\sum_{k=1}^{\infty} A_{nk}x_k| < M$. This holds for all $n \in \mathbb{N}$, and therefore $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$. By Theorem 1.13, $A(c) \subseteq c$. Hence $A:Ces_\infty \longrightarrow \ell_\infty$ and A preserves convergence.#

Theorem 5.2.2. For an infinite matrix A, A : $Ces_\infty \longrightarrow \ell_\infty$ and A preserves limits if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_\infty^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_\infty^{\beta}} < \infty$,

(ii) $\lim_{n\to\infty} A_{nk} = 0$ for every $k \in \mathbb{N}$ and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk} = 1$.

Proof: Assume that A : $Ces_\infty \longrightarrow \ell_\infty$ and A preserves limits. By Lemma 4.2.1, we have that (i) holds. Since $e^{(k)} \in Ces_\infty$ for all $k \in \mathbb{N}$ and $e \in Ces_\infty$, it follows from Proposition 2.3 (ii) and (iv) that (ii) and (iii) hold.

Conversely, assume that (i), (ii) and (iii) hold. By theorem 5.2.1, A : $Ces_\infty \longrightarrow \ell_\infty$ and $A(c) \subseteq c$. By Theorem 1.13 we have that $\sup_n \sum_{k=1}^{\infty} |A_{nk}| < \infty$. Hence it follows from Theorem 1.16 that A preserves limits on Ces_∞. #

Since $cs \subseteq Ces_\infty$, we have by Theorem 1.14 and Theorem 1.17 that

(a) for an infinite matrix A, A: $Ces_\infty \longrightarrow \ell_\infty$ and A preserves summability if and only if A : $Ces_\infty \longrightarrow \ell_\infty$ and $\sup_n \sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)} \right| < \infty$ and $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$ and

(b) for an infinite matrix A, A: $Ces_\infty \longrightarrow \ell_\infty$ and A preserves sums if and only if A : $Ces_\infty \longrightarrow \ell_\infty$ and $\sup_n \sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)} \right| < \infty$ and $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$.

Hence by Lemma 4.2.1 and the above facts, we have the following two Theorems.

Theorm 5.2.3. For an infinite matrix A, A : $Ces_\infty \longrightarrow \ell_\infty$ and A preserves summability if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_\infty^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_\infty^{\beta}} < \infty$,

(ii) $\sup_n \sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)} \right| < \infty$ and

(iii) $\sum_{n=1}^{\infty} A_{nk}$ converges for every $k \in \mathbb{N}$.

Theorm 5.2.4. For an infinite matrix A, A:$Ces_\infty \longrightarrow \ell_\infty$ and A preserves sums if and only if

(i) $(A_{nk})_{k=1}^{\infty} \in Ces_\infty^{\beta}$ for every $n \in \mathbb{N}$ and $\sup_n \|(A_{nk})_{k=1}^{\infty}\|_{Ces_\infty^{\beta}} < \infty$,

(ii) $\sup_n \sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} A_{ik} - \sum_{i=1}^{n} A_{i(k+1)} \right| < \infty$ and

(iii) $\sum_{n=1}^{\infty} A_{nk} = 1$ for every $k \in \mathbb{N}$.

We end this chapter by giving a remark on the Cesaro matrix C that if X = Ces_p or Ces_∞, then C : X $\longrightarrow \ell_\infty$ and C preserves limits but does not preserve summability. First, we note that for $(x_n) \in X$, $\|(x_n)\|_X \le 1$ implies $\frac{1}{n}\sum_{k=1}^{n} |x_k| \le 1$ for every $n \in \mathbb{N}$. Since $(C_{nk})_{k=1}^{\infty}$ is a finite sequence, we have $(C_{nk})_{k=1}^{\infty} \in X^{\beta}$ for all $n \in \mathbb{N}$. If $(x_n) \in X$ is such that $\|(x_n)\|_X \le 1$, then $\left| \sum_{k=1}^{\infty} C_{nk}x_k \right| = \left| \frac{1}{n}\sum_{k=1}^{n} x_k \right| \le \frac{1}{n}\sum_{k=1}^{n} |x_k| \le 1$ for all $n \in \mathbb{N}$. Thus $\|(C_{nk})_{k=1}^{\infty}\|_{X^{\beta}} \le 1$ for every $n \in \mathbb{N}$ and hence $\sup_n \|(C_{nk})_{k=1}^{\infty}\|_{X^{\beta}} \le 1$. By the properties of C, $\lim_{n\to\infty} C_{nk} = 0$ for every $k \in \mathbb{N}$, $\lim_{n\to\infty} \sum_{k=1}^{\infty} C_{nk} = 1$ and $\sum_{n=1}^{\infty} C_{n1}$ diverges. Hence by Theorem 5.1.2, Theorem 5.1.3, Theorem 5.2.2 and Theorem 5.2.3, we have that C:X$\longrightarrow \ell_\infty$ and C preserves limits but does not preserve summability.