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Common Factor Risk Models Based on Negative Binomial Time series

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นโยบายการประกันภัยมักมีปัจจัยที่เกี่ยวเนื่องกับความเสี่ยงที่มากขึ้นตามอัตราของการเอาประกันที่มีมูลค่าสูงขึ้น เพราะฉะนั้นการมีตัวแบบความเสี่ยงที่มีประสิทธิภาพมีความสำคัญอย่างมากต่อบริษัทประกันในการประมาณค่าความเสี่ยงของนโยบายการประกันภัย เครื่องมือทางสถิติและตัวแบบสำหรับการนับข้อมูลสามารถนำมาใช้วิเคราะห์หรือวัดระดับความเสี่ยงของบริษัทประกันได้ ในตัวแบบความเสี่ยงทั่วไปจำนวนการเอาประกันมักถูกกำหนดภายใต้ความเสี่ยงประเภทเดียวเท่านั้น อย่างไรก็ตาม ธุรกิจประกันภัยมักมีธุรกิจมากกว่าหนึ่งประเภท ในไม่กี่ปีมานี้นักวิจัยหลายท่านมีความสนใจในการศึกษาตัวแบบความเสี่ยงซึ่งจำนวนการเอาประกันถูกกำหนดโดยความเสี่ยงภายนอกที่มีผลกระทบมากกว่าหนึ่งประเภท ดังนั้นแนวคิดของตัวแบบความเสี่ยงที่มีปัจจัยร่วมจึงถูกนำเสนอขึ้นมา ในการศึกษาทั่วไปนั้น ตัวแบบความเสี่ยงที่มีปัจจัยร่วมมักจะศึกษาภายใต้ข้อสมมติของการแจกแจงปัวซง อย่างไรก็ตามคุณสมบัติของการมีค่าเฉลี่ยและความแปรปรวนเท่ากันของการแจกแจงปัวซงอาจไม่เหมาะสมในการประยุกต์บางอย่างโดยเฉพาะเมื่อข้อมูลมีการกระจายสูงมาก ดังนั้นในโครงงานนี้เราสร้างตัวแบบความเสี่ยงที่มีปัจจัยร่วมบนฐานของอนุกรมเวลาทวินามลบเพื่อศึกษาความเสี่ยงสำหรับธุรกิจการประกันที่มีจำนวนการเอาประกันสองประเภทซึ่งอาจไม่เป็นอิสระต่อกัน ในการศึกษานี้ เราได้ศึกษาคุณสมบัติความน่าจะเป็นบางประการและขอบเขตบนของความน่าจะเป็นในการล้มละลาย ยิ่งไปกว่านั้นเราได้แสดงผลการคำนวณเชิงตัวเลขของความน่าจะเป็นในการล้มละลายในกรณีต่างๆ กัน

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Insurance policies usually involve factors with greater risk of claims are charged at a higher rate. Therefore having efficient risk models is very important for insurers in estimating risk of insurance policies. Statistical methods and models for count data can be used to analyse or determine such insurance policy's risk levels. In classical risk model, the claim numbers are usually determined by the underling risk of only one class. However, insurance businesses generally have more than one class of businesses. In recent year, many researchers are interested in studying risk models that the claim number is determined by exterior risk which affects more than one class. Therefore, the concept of common shock risk model is introduced. The common shock risk models originally assume the Poisson distribution. However, the property of having equal mean and variance of the Poisson distributions may not hold in some applications in particular when the data is over dispersed. Therefore, in this project, we construct common shock risk models based on negative binomial time series to study risk of any business where the claim numbers in the two classes of business may not be independent. In this study, we derive some probabilistic properties and the upper bound of the ruin probability. Moreover, numerical calculations of ruin probability in different cases are provided.

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Contents

Page

Abstract in Thai **iv**

Abstract in English **v**

Acknowledgments **vi**

Contents **vii**

List of Tables **ix**

1 Introduction **1**

2 Preliminary **3**

2.1 Basic of probability theory 3

2.2 Compound random variable 7

2.3 Integer valued Moving Average models 9

3 Main work **13**

3.1 Introduction 13

3.1.1 Model for the number of claims 13

3.1.2 Risk model 14

3.1.3 Ruin probabilities 15

3.2 Common shock risk model based on the negative binomial MA(1) process (CNBMA(1)) 16

3.3 Common shock risk model based on negative binomial MA(q) (CNBMA(q)) 28

3.4 Numerical Examples and Simulations 43

3.4.1 CNBMA(1) model 43

3.4.2 CNBMA(q) model 45

4 Conclusion **48**

References **49**

Appendix Proposal **51**

Author Profile **56**

List of Tables

Page

3.1 Adjustment coefficient: claim effect of CNBMA(1) 43

3.2 Ruin probability $\psi(u)$: comparison time series of CNBMA(1) 44

3.3 Adjustment coefficient and ruin probability $\psi(u)$: time series effect of CNBMA(1) 45

3.4 Adjustment coefficient: claim effect of CNBMA(2) 46

3.5 Ruin probability $\psi(u)$: comparison time series of CNBMA(2) 46

3.6 Adjustment coefficient and ruin probability $\psi(u)$: time series effect of CNBMA(2) 47

Chapter 1

Introduction

A time series is a sequence of data that are collected at equally spaced points in time. Models for time series data can represent different stochastic processes. To describe the claim count process, broad classes of practical importance are the autoregressive (AR) models and the moving average (MA) models. Integer valued time series models in class of insurance business have innovations which are distributed on the set of non-negative integers. McKenZie (1985) examined the INAR(1) models which are interesting models for count data.

Cossette, Marceau and Deschamps (2010) studied risk models based on time series by considering the class of insurance business risk models based on poisson distributions. Later, Li (2012) introduced common factor or common shock risk models based on poisson time series. The common shock in risk models cooperate effects on insurance business having more than one class.

Poisson distribution is one of the most distributions studied and used in risk models. However, expectation and variance of poisson distributions are equal which may not hold in some applications. Therefore, many researchers proposed some alternative integer-valued time series models that generalize the Poisson INAR models for wider applications. For example, McKenzie (1985) proposed INMA models based on the negative binomial distributions for over-dispersed data. Later, Laphudomsakda and Suntornchost (2018) applied the Negative Binomial INMA model to construct the new class of business risk models based on negative binomial time series. Moreover, they derived some probabilistic properties and computed the ruin probability of their models.

In this project, we will extend the study of Laphudomsakda and Suntornchost (2018) to construct common shock risk models based on negative binomial time series and derive some probabilistic properties and also the upper bound of the ruin probability.

Chapter 2

Preliminary

In this chapter, we provide some basic concepts, definitions and theorems in probability theory used in our studies.

2.1 Basic of probability theory

Definition 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and X be random variable. The function $F_X : \mathbb{R} \to [0, 1]$ defined as

$$F_X(x) = P(\{\omega \in \Omega : X(\omega) \leq x\}) = P(X \leq x) \quad for\ x \in \mathbb{R}.$$

is called F_X the distribution function of X.

Definition 2. Let X be a random variable and $g : \mathbb{R} \to \mathbb{R}$. The expectation of $g(X)$, $E[g(X)]$, is defined by

$$E[g(X)] = \sum g(x)P(X = x) \quad \text{if } X \text{ is a discrete random variable}$$

and

$$E[g(X)] = \int_{-\infty}^{\infty} |g(x)|f(x)dx \quad \text{if } X \text{ is a continuous random variable .}$$

Theorem 1. *Let $a, b \in \mathbb{R}$ and X, Y be random variables whose expected values are finite. The following properties hold*

1. $E[a] = a$,
2. $E[aX] = aE[X]$,
3. $E[aX + b] = aE[X] + b$,
4. $E[X \pm Y] = E[X] \pm E[Y]$.

Definition 3. Let X be a random variable with finite second moment. The variance of X, $Var[X]$, is defined as

$$Var[X] = E[(X - E[X])^2].$$

Remarks 1. *An equivalent definition of the variance of X is*

$$Var[X] = E[X^2] - E^2[X].$$

Definition 4. Let X and Y be random variables. The covariance of X and Y, denote by $Cov(X, Y)$, is defined as

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])].$$

Corollary 1. *Let X,Y and Z be random variables. We have,*

1. $Cov(X, Y) = E[XY] - E[X]E[Y]$,
2. *If X, Y are independent, then $Cov(X, Y) = 0$,*
3. $Cov(X, X) = Var[X]$.
4. $Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$

Theorem 2. *Let $a, b \in \mathbb{R}$ and X, Y be random variables such that $E[X^2] < \infty$ and $E[Y^2] < \infty$. Then,*

1. $Var(aX + b) = a^2 Var(X)$,

2. $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$.

Definition 5. Let X be a random variable. For $t \in \mathbb{R}$, the generating function of X define by

$$G_X(t) = E[t^X]$$

where domain of G_X is the set of t for all $E[t^X] < \infty$.

Definition 6. Let X be a random variable. For $t \in \mathbb{R}$, the moment generating function of X define by

$$M_X(t) = E[e^{tX}],$$

where domain of M_X is the set of t for all $E[e^{tX}] < \infty$.

Definition 7. Let X and Y be discrete random variables such that $P(Y = y) > 0$, the conditional probability of X given Y define by

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}.$$

Theorem 3. *Let X and Y be random variables, then*

$$E[X] = E[E[X|Y]] \tag{2.1}$$

and

$$Var[X] = E[Var[X|Y]] + Var[E[X|Y]]. \tag{2.2}$$

If Y is a discrete random variable, then the formula is equivalent to

$$E[X] = \sum_y E[X|Y = y]P(Y = y).$$

Definition 8. The random variables $X_1, \ldots, X_n$ are said to be independent if and only if

$$F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) = F_{X_1}(x_1) \ldots F_{X_n}(x_n)$$

Definition 9. The random variables $X_1, \ldots, X_n$ are said to be identically if and only if

$$F_{X_1}(x) = F_{X_k}(x) \qquad \text{for all } k = 1, 2, \ldots, n$$

Definition 10. (Bernoulli distribution) A random variable X is said to have the Bernoulli distribution with parameter $p \in [0, 1]$ if

$$P(X = k) = p^k(1-p)^{1-k} \quad \text{for } k = 0, 1$$

We write X as $X \sim Ber(p)$.

Theorem 4. *Properties of Bernoulli distribution with parameter λ are as follows.*

1. $E[X] = p$
2. $Var[X] = p(1-p)$
3. $G_X(t) = (1-p) + pt$
4. $M_X(t) = (1-p) + pe^t$

Definition 11. (Poisson distribution) A random variable X is said to have the poisson distribution with parameter λ if

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for all } k = 0, 1, 2, \ldots$$

We write X as $X \sim Poi(\lambda)$.

Theorem 5. *Properties of poisson distribution with parameter* λ *are as follows.*

1. $E[X] = \lambda$
2. $Var[X] = \lambda$
3. $G_X(t) = e^{\lambda(t-1)}$
4. $M_X(t) = e^{\lambda(e^t-1)}$

Definition 12. (Negative Binomial distribution) A random variable X has the negative binomial distribution with parameters n and $p \in (0,1)$ if

$$P(X = k) = \binom{k+n-1}{k} p^k (1-p)^n \quad \text{for all } k = 0, 1, \ldots$$

We write X as $X \sim NB(n,p)$).

Theorem 6. *Properties of negative binomial distribution with parameter* $n,\ p \in [0,1]$ *are as follows.*

1. $E[X] = \dfrac{np}{q}$
2. $Var[X] = \dfrac{np}{q^2}$
3. $G_X(t) = \left(\dfrac{pe^t}{1-(1-p)t}\right)^n$
4. $M_X(t) = \left(\dfrac{pe^t}{1-(1-p)e^t}\right)^n$

2.2 Compound random variable

In this section, we introduce the definition of compound random variable and its properties.

Definition 13. (Compound random variable) Let N be a non-negative random variable and let $\{X_i, i = 1, 2, \ldots\}$ be a sequence of i.i.d. random variables, each with distribution function F and is independent of N. The random variable S_N defined as

$$S_N = \sum_{i=1}^{N} X_i$$

is called a compound random variable.

Note that $S_N = 0$ when $N = 0$.

Theorem 7. *Properties of compound random variable in Definition 13 is as follows.*

1. $E[S_N] = E[N]E[X]$
2. $Var[S_N] = E[N]Var[X] + Var[N]E^2[X]$
3. $G_{S_N}(t) = G_N(G_X)$

Proof. Since $S_N = \sum_{i=1}^{N} X_i$, we obtain that

1. From (2.1) and Theorem 1(2.), we have

$$\begin{aligned} E[S_N] &= E[\sum_{i=1}^{N} X_i] \\ &= E[E[\sum_{i=1}^{N} X_i | N]] \\ &= E[NE[X]] \\ &= E[N]E[X]. \end{aligned}$$

2. From (2.2) and Theorem 2, we have

$$\begin{aligned} Var[S_N] &= Var[\sum_{i=1}^{N} X_i] \\ &= E[Var[\sum_{i=1}^{N} X_i | N]] + Var[E[\sum_{i=1}^{N} X_i | N]] \\ &= E[NVar[X]] + Var[NE[X]] \\ &= E[N]Var[X] + Var[N]E^2[X]. \end{aligned}$$

3. From the definition of S_N,

$$\begin{aligned} G_{S_N}(t) &= G_{\sum_{i=1}^{N} X_i}(t) \\ &= E[t^{\sum_{i=1}^{N} X_i}] \\ &= E[E[t^{\sum_{i=1}^{N} X_i}|N]] \\ &= E[(E[t^X])^N] \\ &= E[(G_X(t))^N] \\ &= G_N(G_X(t)) \end{aligned}$$

□

2.3 Integer valued Moving Average models

In this section, we first give definition and properties of the binomial thinning operator. Then we discuss the definition of the integer value time series and provide applications of such models.

Definition 14. (Binomial thinning operator) Let M is a random variable, α is a parameter and δ_i is i.i.d. Bernoulli with mean α and independent of M define the operator "$\circ$"as

$$\alpha \circ M = \sum_{i=1}^{M} \delta_i.$$

Lemma 1. *Properties of thinning operator in Definition 14 is as following.*

1. $E[\alpha \circ M] = \alpha E[M]$.
2. $E[M(\alpha \circ M)] = \alpha E[M^2]$.
3. $E[(\beta \circ M)(\alpha \circ M)] = \beta\alpha E[M^2]$.
4. $Var[\alpha \circ M] = \alpha(1-\alpha)E[M] + \alpha^2 Var[M]$.

5. $Cov(\alpha \circ M, M) = \alpha Var[M]$.

6. $Cov(\beta \circ M, \alpha \circ M) = \beta\alpha Var[M]$.

7. $G_{\alpha \circ M} = G_M((1-\alpha) + \alpha t)$.

Proof. Throughout the proofs of this Lemma, we let $\alpha \circ M = \sum_{i=1}^{M} \delta_i$ where δ_i is i.i.d. $Ber(\alpha)$, and $\beta \circ M = \sum_{j=1}^{M} \eta_j$ where η_j is i.i.d. $Ber(\beta)$.

1. Form Theorem 7 (1), we have

$$\begin{aligned} E[\alpha \circ M] &= E[\sum_{i=1}^{M} \delta_i] \\ &= E[M]E[\delta] \\ &= \alpha E[M]. \end{aligned}$$

2.

$$\begin{aligned} E[M(\alpha \circ M)] &= E[M \sum_{i=1}^{M} \delta_i] \\ &= E[E[M \sum_{i=1}^{M} \delta_i \big| M]] \\ &= E[M^2 E[\delta]] \\ &= E[M^2]E[\delta] \\ &= \alpha E[M^2]. \end{aligned}$$

3.

$$\begin{aligned} E[(\beta \circ M)(\alpha \circ M)] &= E[\sum_{j=1}^{M} \eta_j \sum_{i=1}^{M} \delta_i] \\ &= E[E[\sum_{j=1}^{M} \eta_j \sum_{i=1}^{M} \delta_i \big| M]] \\ &= E[E[\sum_{j=1}^{M} \eta_j \big| M]E[\sum_{i=1}^{M} \delta_i \big| M]] \\ &= E[M^2 E[\eta]E[\delta]] \\ &= E[M^2]E[\eta]E[\delta] \\ &= \beta\alpha E[M^2]. \end{aligned}$$

4. From Theorem 7 (2), we have

$$\begin{aligned} Var[\alpha \circ M] &= Var[\sum_{i=1}^{M} \delta_i] \\ &= E[M]Var[\delta] + Var[M]E^2[\delta] \\ &= \alpha(1-\alpha)E[M] + \alpha^2 Var[M]. \end{aligned}$$

5. From 1. and 2. and the definition of covariance, we have

$$\begin{aligned} Cov(\alpha \circ M, M) &= E[(\alpha \circ M)M] - E[\alpha \circ M]E[M] \\ &= \alpha E[M^2] - \alpha E[M]E[M]E \\ &= \alpha Var[M]. \end{aligned}$$

6. From 1. and 3. and the definition of covariance, we obtain

$$\begin{aligned} Cov(\beta \circ M, \alpha \circ M) &= E[(\beta \circ M)(\alpha \circ M)] - E[\beta \circ M]E[\alpha \circ M] \\ &= E[M^2]E[\eta]E[\delta] - E[M]E[\eta]E[M]E[\delta] \\ &= E[\eta]E[\delta]Var[M] \\ &= \beta\alpha Var[M]. \end{aligned}$$

7. From Theorem 7 (3), we have

$$\begin{aligned} G_{\alpha \circ M}(t) &= E[t^{\sum_{i=1}^{M} \delta_i}] \\ &= G_M(G_\delta(t)) \\ &= G_M((1-\alpha) + \alpha t). \end{aligned}$$

□

Definition 15. (INMA model) First-order Integer-valued Moving Average INMA(1) is constructed by thinning operator to model integer-value time series defined by

$$N_t = \alpha \circ \varepsilon_{t-1} + \varepsilon_t,$$

and the Integer-Valued moving average model of order q INMA(q) is defined as

$$N_t = \alpha_1 \circ \epsilon_{t-1} + \alpha_2 \circ \epsilon_{t-2} + \cdots + \alpha_q \circ \epsilon_{t-q} + \epsilon_t,$$

where $\{\varepsilon_t, t = 0, 1, \ldots\}$ is a sequence of i.i.d random variables which are independent of N_t for all $t \in \mathbb{N}$.

Example of application 1. *For* N_t *is MA(1) model we let* ε_t *be the new claims during in period* t*,* $\beta \circ \varepsilon_{t-1}$ *is the claims from period* $t-1$ *and*

$$\beta \circ \varepsilon_{t-1} = \sum_{j=1}^{\varepsilon_{t-1}} \delta_{t-1,j}.$$

Therefore, N_t *is the number of claims in period* t

Chapter 3

Main work

3.1 Introduction

In this chapter, we consider discrete-time risk models with common shock.

Let $N_t^{(11)}, N_t^{(22)}$ and $N_t^{(12)}$ be independent random variables for count. The claim-number processes for two dependent classes of business is defined by

$$N_t^{(1)} = N_t^{(11)} + N_t^{(12)},$$
$$N_t^{(2)} = N_t^{(22)} + N_t^{(12)}.$$

Cossette and Marceau (2000) assumed that $N_t^{(11)}, N_t^{(22)}$ and $N_t^{(12)}$ are independent poisson processes with parameters $\lambda_{11}, \lambda_{22}$ and λ_{12} respectively.

Risk model with common shock risk model extend $N_t^{(11)}, N_t^{(22)}$ and $N_t^{(12)}$ to poisson integer-value time series process by Li (2012).

In our common shock risk model for number of claims, $N_t^{(11)}, N_t^{(22)}$ and $N_t^{(12)}$ are assumed to be independent negative binomial time series process.

3.1.1 Model for the number of claims

In risk model with common shock, the common shock components have effect on more than one business class.

For $i = 1, 2$, let $N_t^{(i)}$ be the number of claims of ith classes in the period t. Define

$$N_t^{(i)} = N_t^{(ii)} + N_t^{(12)}, \quad i = 1, 2. \tag{3.1}$$

Then $N_t^{(i)}$ is the common shock risk model and

$$N_t^{(1)} = N_t^{(11)} + N_t^{(12)}, \tag{3.2}$$

$$N_t^{(2)} = N_t^{(22)} + N_t^{(12)}, \tag{3.3}$$

where $N_t^{(ii)}$ for $i = 1, 2$ is the number of claims independent of another class, and $N_t^{(12)}$ is the common component of the two classes.

Assume that $N_t^{(11)}$, $N_t^{(22)}$ and $N_t^{(12)}$ are independent negative binomial MA(1) processes which have the form

$$N_t^{(ij)} = \alpha_{ij} \circ \varepsilon_{t-1}^{(ij)} + \varepsilon_t^{(ij)}, \quad i, j = 1, 2;\ i \leq j,$$

where $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. negative binomial random variables with parameters (θ_{ij}, γ). The thinning operator "$\alpha\circ$" is defined as

$$\alpha \circ \varepsilon = \sum_{i=1}^{\varepsilon} \delta_i,$$

where $\{\delta_i; i = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables following the Bernoulli distribution with mean α and is independent of ε.

3.1.2 Risk model

Let U_n be the surplus process at time $n \in \mathbb{N}$ given by

$$U_n = u + n\pi - S_n, \tag{3.4}$$

where u is the initial surplus, π is the premium income rate.

Define W_t as the aggregate claim amount in period t. Then we have the accumulate aggregate claim amount

$$S_n = \sum_{t=1}^{n} W_t. \tag{3.5}$$

We also define $W_t^{(i)}$ be the aggregate claim amount of claims of ith class for $i = 1, 2$ in the period t. Then the aggregate claim amount W_t can be express as

$$W_t = W_t^{(1)} + W_t^{(2)} = \sum_{j=1}^{N_t^{(1)}} X_{1,t,j} + \sum_{j=1}^{N_t^{(2)}} X_{2,t,j}. \tag{3.6}$$

where $X_{i,t,1}, X_{i,t,2}, \ldots,$ is a sequence claim sizes which are i.i.d. random variables with distribution F_{X_i} and is independent of $N_t^{(i)}$.

Remarks 2. *If $E[n\pi - S_n] > 0$, then we satisfies the usual solvency condition $n\pi > E[S_n]$.*

3.1.3 Ruin probabilities

Let T be the time of ruin, the first time that the surplus becomes negative, defined as

$$T = \inf\{n \in \mathbb{N}^+ | U_n \leq 0, \}.$$

Then the infinite time ruin probability is given by

$$\psi(u) = P\{T < \infty | U_0 = u\},$$

and we have the asymptotic Lundberg-type result

$$\lim_{u \to \infty} \frac{-\ln(\psi(u))}{u} = R,$$

where R is the Lundberg adjustment coefficient. Based on this asymptotic result and for large values of u, $\psi(u)$ can be approximated by

$$\psi(u) \simeq e^{-Ru}.$$

Nyrhinen (1998) and Müller and Pflug (2001) defined the convex function

$$c_n(r) = \frac{1}{n} \ln(E[e^{r(S_n - n\pi)}]). \tag{3.7}$$

The adjustment coefficient function $c(r)$ defined as

$$c(r) = \lim_{n \to \infty} c_n(r). \tag{3.8}$$

For $r_0 > 0$, if the adjustment coefficient function $c(r)$ for all $0 < r < r_0$ exists, there also exists $r \in (0, r_0)$ such that $c(r) = 0$ and the positive zero-root r is the adjustment coefficient R.

3.2 Common shock risk model based on the negative binomial MA(1) process (CNBMA(1))

Definition 16. Assume that $N_t^{(11)}$, $N_t^{(22)}$ and $N_t^{(12)}$ are independent negative binomial MA(1) processes which have the form

$$N_t^{(ij)} = \alpha_{ij} \circ \varepsilon_{t-1}^{(ij)} + \varepsilon_t^{(ij)}, \quad i, j = 1, 2; i \leq j,$$

where $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. random variables having negative binomial with parameter (θ_{ij}, γ) and the probability mass function defined as

$$f_Y(y) = \binom{y + \theta_{ij} - 1}{y} \gamma^y (\overline{\gamma})^{\theta_{ij}}.$$

where $\overline{\gamma} = 1 - \gamma$.

Theorem 8. *(Laphudomsakda and Suntornchost, 2018) Let* $\{N_t^{(ij)}, t \in \mathbb{N}$ *and* $i, j = 1, 2\}$ *defined in Definition 16, then* $\{N_t^{(ij)}, t \in \mathbb{N}$ *and* $i, j = 1, 2\}$ *has the following properties.*

1. $\{N_t^{(ij)}, t \in \mathbb{N}\}$ *is a stationary process.*
2. $E[N_t^{(ij)}] = \dfrac{\theta_{ij}\gamma}{\overline{\gamma}}(1 + \alpha_{ij}).$
3. $Var[N_t^{(ij)}] = \dfrac{\theta_{ij}\gamma}{\overline{\gamma}^2}(\alpha_{ij}^2\gamma + \alpha_{ij}\overline{\gamma} + 1).$
4. $Cov(N_t^{(ij)}, N_{t-l}^{(ij)}) = \begin{cases} \dfrac{\theta_{ij}\alpha_{ij}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1. \end{cases}$

Theorem 9. *The count process* $N_t^{(i)}$ $(t > 1)$ *and* N_t *defined in Definition 16 have the following properties. For* $i = 1, 2$,

1. $E[N_t^{(i)}] = \dfrac{\theta_{ii}\gamma}{\overline{\gamma}}(1 + \alpha_{ii}) + \dfrac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12}),$
2. $Var[N_t^{(i)}] = \dfrac{\theta_{ii}\gamma}{\overline{\gamma}^2}(\alpha_{ii}^2\gamma + \alpha_{ii}\overline{\gamma} + 1) + \dfrac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1),$

3. $Cov(N_t^{(i)}, N_{t-l}^{(i)}) = \begin{cases} \dfrac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}} + \dfrac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1, \end{cases}$

4. $Cov(N_t, N_{t-l}) = \begin{cases} \dfrac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}} + \dfrac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}} + \dfrac{4\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1. \end{cases}$

Proof. For $i, j = 1, 2$ and $i \leq j$.
From $N_t^{(i)} = N_t^{(ii)} + N_t^{(12)}$ we obtain

1. From Theorem 1 (2),

$$\begin{aligned} E[N_t^{(i)}] &= E[N_t^{(ii)} + N_t^{(12)}] \\ &= E[N_t^{(ii)}] + E[N_t^{(12)}] \\ &= \frac{\theta_{ii}\gamma}{\overline{\gamma}}(1 + \alpha_{ii}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12}). \end{aligned}$$

2. Since $N_t^{(ii)}$ and $N_t^{(12)}$ are independent and Theorem 8 (3), we have

$$\begin{aligned} Var[N_t^{(i)}] &= Var[N_t^{(ii)} + N_t^{(12)}] \\ &= Var[N_t^{(ii)}] + Var[N_t^{(12)}] \\ &= \frac{\theta_{ii}\gamma}{\overline{\gamma}^2}(\alpha_{ii}^2\gamma + \alpha_{ii}\overline{\gamma} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1). \end{aligned}$$

3. Since $N_t^{(ii)}$ and $N_t^{(12)}$ are independent and Theorem 8 (4), we have

$$\begin{aligned} Cov(N_t^{(i)}, N_{t-l}^{(i)}) &= Cov(N_t^{(ii)} + N_t^{(12)}, N_{t-l}^{(ii)} + N_{t-l}^{(12)}) \\ &= Cov(N_t^{(ii)}, N_{t-l}^{(ii)}) + Cov(N_t^{(12)}, N_{t-l}^{(12)}) \qquad (3.9) \\ &= \begin{cases} \dfrac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}} + \dfrac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1. \end{cases} \end{aligned}$$

4. Form $N_t = N_t^{(1)} + N_t^{(2)}$ and since $N_t^{(ii)}$ and $N_t^{(12)}$ are independent and Theorem 8 (4), we have

$$Cov(N_t, N_{t-l}) = Cov(N_t^{(11)} + N_t^{(22)} + 2N_t^{(12)}, N_{t-l}^{(11)} + N_{t-l}^{(22)} + 2N_{t-l}^{(12)})$$

$$= Cov(N_t^{(11)}, N_{t-l}^{(11)}) + Cov(N_t^{(22)}, N_{t-l}^{(22)}) + 4Cov(N_t^{(12)}, N_{t-l}^{(12)}) \tag{3.10}$$

$$= \begin{cases} \frac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}} + \frac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}} + \frac{4\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1. \end{cases}$$

Where we use property 3 in Theorem 8 to proof (3.9) and(3.10)

□

Theorem 10. *Properties of $W_t^{(i)}$ for $i = 1, 2$ and W_t defined in equation* (3.6) *are as follows.*

1. $E[W_t^{(i)}] = \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(1 + \alpha_{ii}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12})\right) E[X_i].$

2. $$Var[W_t^{(i)}] = \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(1 + \alpha_{ii}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12})\right) Var[X_i] + \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}(\alpha_{ii}^2\gamma + \alpha_{ii}\overline{\gamma} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1)\right) E^2[X_i].$$

3. $$Cov(W_t^{(i)}, W_{t-l}^{(i)}) = \begin{cases} E[X_i]^2 \left(\frac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}} + \frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}\right), & l = 1, \\ 0, & l > 1. \end{cases}$$

4. $$Cov(W_t^{(1)}, W_t^{(2)}) = \frac{\theta_{12}\gamma(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1)}{\overline{\gamma}^2} E[X_1]E[X_2].$$

5. $$E[W_t] = \left(\frac{\theta_{11}\gamma}{\overline{\gamma}}(1 + \alpha_{11}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12})\right) E[X_1] + \left(\frac{\theta_{22}\gamma}{\overline{\gamma}}(1 + \alpha_{22}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12})\right) E[X_2].$$

6. $$Var[W_t] = \frac{\theta_{11}\gamma}{\overline{\gamma}}(1 + \alpha_{11})Var[X_1] + \frac{\theta_{22}\gamma}{\overline{\gamma}}(1 + \alpha_{22})Var[X_2] + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{12})Var[X_1 + X_2] + \frac{\theta_{11}\gamma}{\overline{\gamma}^2}(\alpha_{11}^2\gamma + \alpha_{11}\overline{\gamma} + 1)E^2[X_1] + \frac{\theta_{22}\gamma}{\overline{\gamma}^2}(\alpha_{22}^2\gamma + \alpha_{22}\overline{\gamma} + 1)E^2[X_2] + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1)(E[X_1 + X_2])^2.$$

7. $$Cov(W_t, W_{t-l}) = \begin{cases} E[X_1]^2\frac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}} + E[X_2]^2\frac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}} + E[X_1 + X_2]^2\frac{4\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1. \end{cases}$$

Proof. For $i, j = 1, 2$ and $i \leq j$. From 3.1.2, we have $W_t = W_t^{(1)} + W_t^{(2)}$ and $W_t^{(i)} = \sum_{j=1}^{N_t^{(i)}} X_{i,t,j}$.

1. Since $X_{i,t,j}, j \geq 1$ is i.i.d, and from Theorem 7 (1) and Theorem 9 (1), we have

$$
\begin{aligned}
E[W_t^{(i)}] &= E[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j}] \\
&= E[N_t^{(i)}]E[X_i] \\
&= \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(1+\alpha_{ii}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12}) \right) E[X_i].
\end{aligned}
$$

2. Since $X_{i,t,j}, j \geq 1$, is i.i.d, and from Theorem 7 (2) and Theorem 9 (1,2), we have

$$
\begin{aligned}
Var[W_t^{(i)}] &= Var[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j}] \\
&= E[N_t^{(i)}]Var[X_i] + Var[N_t^{(i)}]E^2[X_i] \\
&= \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(1+\alpha_{ii}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12}) \right) Var[X_i] \\
&\quad + \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}(\alpha_{ii}^2\gamma + \alpha_{ii}\overline{\gamma} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1) \right) E^2[X_i].
\end{aligned}
$$

3. Note that $Cov(W_t^{(i)}, W_{t-l}^{(i)}) = E[W_t^{(i)}W_{t-l}^{(i)}] - E[W_t^{(i)}]E[W_{t-l}^{(i)}]$.
From Theorem 7 (1),

$$
E[W_t^{(i)}]E[W_{t-l}^{(i)}] = E^2[X_i]E[N_t^{(i)}]E[N_{t-l}^{(i)}]. \tag{3.11}
$$

Since $X_{1,t,j}$ and $X_{1,t-l,j}$ are independent,

$$
\begin{aligned}
E[W_t^{(i)}W_{t-l}^{(i)}] &= E[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j} \sum_{j=1}^{N_{t-l}^{(i)}} X_{i,t-l,j}] \\
&= E[E[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j} \sum_{j=1}^{N_{t-l}^{(i)}} X_{i,t-l,j} \big| N_t^{(i)} N_{t-l}^{(i)}]] \\
&= E[E[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j} \big| N_t^{(i)}]E[\sum_{j=1}^{N_{t-l}^{(i)}} X_{i,t-l,j} \big| N_{t-l}^{(i)}]] \\
&= E[N_t^{(i)}E[X_i]N_{t-l}^{(i)}E[X_i]]
\end{aligned}
$$

$$= E^2[X_i]E[N_t^{(i)}N_{t-l}^{(i)}]. \tag{3.12}$$

Therefore, from (3.11) and (3.12) we have

$$\begin{aligned}
Cov(W_t^{(i)}, W_{t-l}^{(i)}) &= E^2[X_i]E[N_t^{(i)}N_{t-l}^{(i)}] + E^2[X_i]E[N_t^{(i)}]E[N_{t-l}^{(i)}] \\
&= E^2[X_i]Cov(N_t^{(i)}, N_{t-l}^{(i)}) \\
&= \begin{cases} E[X_i]^2(\frac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}} + \frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}), & l = 1, \\ 0, & l > 1. \end{cases}
\end{aligned}$$

4. Note that $Cov(W_t^{(1)}, W_t^{(2)}) = E[W_t^{(1)}W_t^{(2)}] - E[W_t^{(1)}]E[W_t^{(2)}]$

   From Theorem 7 (1),

$$E[W_t^{(1)}]E[W_t^{(2)}] = E[X_1]E[N_t^{(1)}]E[X_2]E[N_t^{(2)}]. \tag{3.13}$$

   Since $X_{1,t,j}$ and $X_{2,t,j}$ are independent,

$$\begin{aligned}
E[W_t^{(1)}W_t^{(2)}] &= E[\sum_{j=1}^{N_t^{(1)}} X_{1,t,j} \sum_{j=1}^{N_t^{(2)}} X_{2,t,j}] \\
&= E[E[\sum_{j=1}^{N_t^{(1)}} X_{1,t,j} \sum_{j=1}^{N_t^{(2)}} X_{2,t,j} | N_t^{(1)}, N_t^{(2)}]] \\
&= E[E[\sum_{j=1}^{N_t^{(1)}} X_{1,t,j} | N_t^{(1)}]E[\sum_{j=1}^{N_t^{(2)}} X_{2,t,j} | N_t^{(2)}]] \\
&= E[N_t^{(1)}E[X_1]N_t^{(2)}E[X_2]] \\
&= E[N_t^{(1)}N_t^{(2)}]E[X_1]E[X_2]. \qquad (3.14)
\end{aligned}$$

   Then from (3.13) and (3.14), we have

$$\begin{aligned}
Cov(W_t^{(1)}, W_t^{(2)}) &= E[N_t^{(1)}N_t^{(2)}]E[X_1]E[X_2] - E[N_t^{(1)}]E[X_1]E[N_t^{(2)}]E[X_2] \\
&= E[X_1]E[X_2](Var[N_t^{(1)} + N_t^{(2)}] - Var[N_t^{(1)}] - Var[N_t^{(2)}])/2 \\
&= E[X_1]E[X_2](Var[N_t^{(11)}] + Var[N_t^{(22)}] + 4Var[N_t^{(12)}] \\
&\quad - Var[N_t^{(11)}] - Var[N_t^{(22)}] - 2Var[N_t^{(12)}])/2 \\
&= E[X_1]E[X_2]Var[N_t^{(12)}] \\
&= \frac{\theta_{12}\gamma(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma} + 1)}{\overline{\gamma}^2}E[X_1]E[X_2],
\end{aligned}$$

   where we use Theorem 8(3) to obtain the last equality.

5. From Theorem 1., we have

$$\begin{aligned}E[W_t] &= E[W_t^{(1)}] + E[W_t^{(2)}] \\ &= \left(\frac{\theta_{11}\gamma}{\overline{\gamma}}(1+\alpha_{11}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12})\right) E[X_1] \\ &\quad + \left(\frac{\theta_{22}\gamma}{\overline{\gamma}}(1+\alpha_{22}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12})\right) E[X_2].\end{aligned}$$

6. From 2. and 4. ,we have

$$\begin{aligned}Var[W_t] &= Var[W_t^{(1)} + W_t^{(2)}] \\ &= Var[W_t^{(1)}] + Var[W_t^{(2)}] + 2Cov(W_t^{(1)}, W_t^{(2)}) \\ &= \left(\frac{\theta_{11}\gamma}{\overline{\gamma}}(1+\alpha_{11}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12})\right) Var[X_1] \\ &\quad + \left(\frac{\theta_{11}\gamma}{\overline{\gamma}^2}(\alpha_{11}^2\gamma + \alpha_{11}\overline{\gamma}+1) + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma}+1)\right) E^2[X_1]. \\ &\quad + \left(\frac{\theta_{22}\gamma}{\overline{\gamma}}(1+\alpha_{22}) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12})\right) Var[X_2] \\ &\quad + \left(\frac{\theta_{22}\gamma}{\overline{\gamma}^2}(\alpha_{22}^2\gamma + \alpha_{22}\overline{\gamma}+1) + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma}+1)\right) E^2[X_2]. \\ &\quad + 2\frac{\theta_{12}\gamma(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma}+1)}{\overline{\gamma}^2} E[X_1]E[X_2] \\ &= \frac{\theta_{11}\gamma}{\overline{\gamma}}(1+\alpha_{11})Var[X_1] + \frac{\theta_{22}\gamma}{\overline{\gamma}}(1+\alpha_{22})Var[X_2] \\ &\quad + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1+\alpha_{12})Var[X_1+X_2] + \frac{\theta_{11}\gamma}{\overline{\gamma}^2}(\alpha_{11}^2\gamma + \alpha_{11}\overline{\gamma}+1)E^2[X_1] \\ &\quad + \frac{\theta_{22}\gamma}{\overline{\gamma}^2}(\alpha_{22}^2\gamma + \alpha_{22}\overline{\gamma}+1)E^2[X_2] + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}(\alpha_{12}^2\gamma + \alpha_{12}\overline{\gamma}+1)(E[X_1+X_2])^2\end{aligned}$$

7. Note that $Cov(W_t, W_{t-l}) = E[W_tW_{t-l}] - E[W_t]E[W_{t-l}]$

   Since $\{W_t, t \in \mathbb{N}\}$ is a stationary process and from 5.,

$$\begin{aligned}E[W_t]E[W_{t-l}] &= E^2[W_t] \\ &= (E[W_t^{(1)}] + E[W_t^{(2)}])^2 \\ &= (E[X_1]E[N_t^{(1)}] + E[X_2]E[N_t^{(2)}])^2 \\ &= E^2[X_1]E[N_t^{(1)}]E[N_t^{(1)}] + 2E[X_1]E[X_2]E[N_t^{(1)}]E[N_t^{(2)}] \\ &\quad + E^2[X_2]E[N_t^{(2)}]E[N_t^{(2)}]. \qquad (3.15)\end{aligned}$$

Consider $E[W_t W_{t-l}]$,

$$\begin{aligned} E[W_t W_{t-l}] &= E[(W_t^{(1)} + W_t^{(2)})(W_{t-l}^{(1)} + W_{t-l}^{(2)})] \\ &= E[W_t^{(1)} W_{t-l}^{(1)}] + E[W_t^{(1)} W_{t-l}^{(2)}] + E[W_{t-l}^{(1)} W_t^{(2)}] + E[W_t^{(2)} W_{t-l}^{(2)}] \end{aligned}$$

Similarly to (3.14), we have

$$\begin{aligned} E[W_t^{(1)} W_{t-l}^{(1)}] &= E^2[X_1]E[N_t^{(1)} N_{t-l}^{(1)}], \\ E[W_t^{(1)} W_{t-l}^{(2)}] &= E[X_1]E[X_2]E[N_t^{(1)}]E[N_{t-l}^{(2)}], \\ E[W_{t-l}^{(1)} W_t^{(2)}] &= E[X_1]E[N_{t-l}^{(1)}]E[X_2]E[N_t^{(2)}], \\ E[W_t^{(2)} W_{t-l}^{(2)}] &= E^2[X_2]E[N_t^{(2)} N_{t-l}^{(2)}]. \end{aligned}$$

Therefore,

$$\begin{aligned} E[W_t W_{t-l}] = E^2[X_1]E[N_t^{(1)} N_{t-l}^{(1)}] &+ E[X_1]E[X_2]E[N_t^{(1)}]E[N_{t-l}^{(2)}] \\ &+ E[X_1]E[N_{t-l}^{(1)}]E[X_2]E[N_t^{(2)}] + E^2[X_2]E[N_t^{(2)} N_{t-l}^{(2)}], \end{aligned} \tag{3.16}$$

Then from (3.15) and (3.16) we have

$$\begin{aligned} Cov(W_t, W_{t-l}) &= E^2[X_1](E[N_t^{(1)} N_{t-l}^{(1)}] - E[N_t^{(1)}]E[N_t^{(1)}]) \\ &\quad + E^2[X_2](E[N_t^{(2)} N_{t-l}^{(2)}] - E[N_t^{(2)}]E[N_t^{(2)}]) \\ &= E^2[X_1]Cov(N_t^{(1)}, N_{t-l}^{(1)}) + E^2[X_2]Cov(N_t^{(2)}, N_{t-l}^{(2)}) \\ &= \begin{cases} E[X_1]^2 \dfrac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}} + E[X_2]^2 \dfrac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}} + E[X_1 + X_2]^2 \dfrac{4\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}}, & l = 1, \\ 0, & l > 1. \end{cases} \end{aligned}$$

□

Let $S_n^{(12)}$ be the aggregate claim amount with claim-number process $N_t^{(12)}$ and claim-size distribution $F_{X_1+X_2}$, and let $S_n^{(ii)}$ $(i = 1, 2)$ be the aggregate claim amount with claim-number process $N_t^{(ii)}$ and claim-size distribution F_{X_i}. We write $S_n^{(12)} = \sum_{t=1}^{n} N_t^{(12)}$, $S_n^{(11)} = \sum_{t=1}^{n} N_t^{(11)}$ and $S_n^{(11)} = \sum_{t=1}^{n} N_t^{(11)}$. Then we can obtain the moment generating function of S_n as the following theorem.

Theorem 11. *The moment generating function of S_n can be expressed as*

$$E[e^{rS_n}] = E[e^{rS_n^{(11)}}]E[e^{rS_n^{(22)}}]E[e^{rS_n^{(12)}}]. \tag{3.17}$$

Proof. With (3.5) we have

$$\begin{aligned}
S_n &= W_1 + W_2 + \ldots + W_n \\
&= \sum_{t=1}^{n} \left(W_t^{(1)} + W_t^{(2)} \right) \\
&= \sum_{t=1}^{n} \left(\sum_{j=1}^{N_t^{(1)}} X_{1,t,j} + \sum_{j=1}^{N_t^{(2)}} X_{2,t,j} \right) \\
&= \sum_{t=1}^{n} \left(\sum_{j=1}^{N_t^{(11)}+N_t^{(12)}} X_{1,t,j} + \sum_{j=1}^{N_t^{(22)}+N_t^{(12)}} X_{2,t,j} \right) \\
&= \sum_{t=1}^{n} \left(\sum_{j=1}^{N_t^{(12)}} (X_{1,t,j} + X_{2,t,j}) + \sum_{j=1}^{N_t^{(11)}} X_{1,t,j} + \sum_{j=1}^{N_t^{(22)}} X_{2,t,j} \right) \\
&= \sum_{t=1}^{n} \sum_{j=1}^{N_t^{(12)}} (X_{1,t,j} + X_{2,t,j}) + \sum_{t=1}^{n} \sum_{j=1}^{N_t^{(11)}} X_{1,t,j} + \sum_{t=1}^{n} \sum_{j=1}^{N_t^{(22)}} X_{2,t,j}.
\end{aligned}$$

Then

$$\begin{aligned}
E[e^{rS_n}] &= E[\exp\{r(\sum_{t=1}^{n} \sum_{j=1}^{N_t^{(12)}} (X_{1,t,j} + X_{2,t,j}) + \sum_{t=1}^{n} \sum_{j=1}^{N_t^{(11)}} X_{1,t,j} + \sum_{t=1}^{n} \sum_{j=1}^{N_t^{(22)}} X_{2,t,j})\}] \\
&= E[(M_{X_1+X_2}(r))^{\sum_{t=1}^{n} N_t^{(12)}} (M_{X_1}(r))^{\sum_{t=1}^{n} N_t^{(11)}} (M_{X_2}(r))^{\sum_{t=1}^{n} N_t^{(22)}}] \\
&= E[(M_{X_1+X_2}(r))^{\sum_{t=1}^{n} N_t^{(12)}}]E[(M_{X_1}(r))^{\sum_{t=1}^{n} N_t^{(11)}}]E[(M_{X_2}(r))^{\sum_{t=1}^{n} N_t^{(22)}}] \qquad (3.18)
\end{aligned}$$

Hence we may consider the aggregate-claim-amount process S_n as the sum of three independent univariate aggregate-claim-amount process $S_n^{(12)}$, $S_n^{(11)}$ and $S_n^{(22)}$.

Then (3.18) becomes

$$E[e^{rS_n}] = E[e^{rS_n^{(11)}}]E[e^{rS_n^{(22)}}]E[e^{rS_n^{(12)}}].$$

□

Theorem 12. *The adjustment coefficient function* $c(r)$ *of* U_n *where* $N_t^{(i)}$ *be a CNBMA(1) is*

$$c(r) = \theta_{11}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{11}M_{X_1}(r)+\alpha_{11}M_{X_1}^2(r))}\right) + \theta_{22}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M_{X_2}^2(r))}\right) + \theta_{12}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M_{X_1+X_2}^2(r))}\right) - r\pi,$$

for $r \in \mathbb{R}^+$ *such that* $1-\gamma(\overline{\alpha}_{ij}M_{X_1}(r)+\alpha_{ij}M_{X_1}^2(r)) > 0$, $1-\gamma(\overline{\alpha}_{ij}M_{X_2}(r)+\alpha_{ij}M_{X_2}^2(r)) > 0$, *and* $1-\gamma(\overline{\alpha}_{ij}M_{X_1+X_2}(r)+\alpha_{ij}M_{X_1+X_2}^2(r)) > 0$.

Proof. With (3.17), we write $c_n(r)$ as

$$\begin{aligned} c_n(r) &= \frac{1}{n}\log(E[e^{r(S_n-n\pi)}]) \\ &= \frac{1}{n}\log(E[e^{rS_n}]) - r\pi \\ &= \frac{1}{n}\log(E[e^{rS_n^{(11)}}]E[e^{rS_n^{(22)}}]E[e^{rS_n^{(12)}}]) - r\pi \\ &= \frac{1}{n}\left(\log(E[e^{rS_n^{(11)}}]) + \log(E[e^{rS_n^{(22)}}]) + \log(E[e^{rS_n^{(12)}}])\right) - r\pi. \end{aligned}$$

Thus we have

$$\begin{aligned} c(r) &= \lim_{n\to\infty} c_n(r) \\ &= \lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(11)}}]) + \lim_{n\to\infty}\frac{1}{n}\log(E[e^{rS_n^{(22)}}]) + \lim_{n\to\infty}\frac{1}{n}\log(E[e^{rS_n^{(12)}}])) - r\pi. \end{aligned}$$

Referring to Laphudomsakda and Suntornchost (2018), for $i,j = 1,2$ we have

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(ij)}}]) = \theta_{ij}\log\left(\frac{q}{1-p(\overline{\alpha}_{ij}M_{Z_{ij}}(r)+\alpha_{ij}M_{Z_{ij}}^2(r))}\right),$$

for $r \in \mathbb{R}^+$ such that $1-p(\overline{\alpha}_{ij}M_{Z_{ij}}(r)+\alpha_{ij}M_{Z_{ij}}^2(r) > 0$ and $i,j = 1,2; i \le j$, where $Z_{11} \overset{d}{=} X_1, Z_{22} \overset{d}{=} X_2$ and $Z_{12} \overset{d}{=} X_1 + X_2$.

Therefore $c(r)$ can be rewritten as

$$c(r) = \theta_{11}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{11}M_{X_1}(r)+\alpha_{11}M_{X_1}^2(r))}\right) + \theta_{22}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M_{X_2}^2(r))}\right) + \theta_{12}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M_{X_1+X_2}^2(r))}\right) - r\pi.$$

for $r \in \mathbb{R}^+$ such that $1 - p(\overline{\alpha}_{ij} M_{X_1}(r) + \alpha_{ij} M^2_{X_1}(r)) > 0$, $1 - p(\overline{\alpha}_{ij} M_{X_2}(r) + \alpha_{ij} M^2_{X_2}(r)) > 0$, $1 - p(\overline{\alpha}_{ij} M_{X_1+X_2}(r) + \alpha_{ij} M^2_{X_1+X_2}(r)) > 0$.

□

Remarks 3. *The first two derivatives of the adjustment coefficient function $c(r)$ in Theorem 8 are given as follows*

1.

$$
\begin{aligned}
\frac{dc(r)}{dr} = & \frac{\theta_{11}\gamma(\overline{\alpha}_{11} M'_{X_1}(r) + 2\alpha_{11} M_{X_1}(r) M'_{X_1}(r))}{1 - \gamma(\overline{\alpha}_{11} M_{X_1}(r) + \alpha_{11} M^2_{X_1}(r))} \\
& + \frac{\theta_{22}\gamma(\overline{\alpha}_{22} M'_{X_2}(r) + 2\alpha_{22} M_{X_2}(r) M'_{X_2}(r))}{1 - \gamma(\overline{\alpha}_{22} M_{X_2}(r) + \alpha_{22} M^2_{X_2}(r))} \\
& + \frac{\theta_{12}\gamma(\overline{\alpha}_{12} M'_{X_1+X_2}(r) + 2\alpha_{12} M_{X_1+X_2}(r) M'_{X_1+X_2}(r))}{1 - \gamma(\overline{\alpha}_{12} M_{X_1+X_2}(r) + \alpha_{12} M^2_{X_1+X_2}(r))} - \pi
\end{aligned}
$$

2.

$$
\begin{aligned}
\frac{d^2c(r)}{dr^2} = & \frac{\theta_{11}\gamma(\overline{\alpha}_{11} M''_{X_1}(r) + 2\alpha_{11} M_{X_1}(r) M''_{X_1}(r) + 2\alpha_{11}(M'_{X_1}(r))^2)}{1 - \gamma(\overline{\alpha}_{11} M_{X_1}(r) + \alpha_{11} M^2_{X_1}(r))} \\
& + \frac{\theta_{22}\gamma(\overline{\alpha}_{22} M''_{X_2}(r) + 2\alpha_{22} M_{X_2}(r) M''_{X_2}(r) + 2\alpha_{22}(M'_{X_2}(r))^2)}{1 - \gamma(\overline{\alpha}_{22} M_{X_2}(r) + \alpha_{22} M^2_{X_2}(r))} \\
& + \frac{\theta_{12}\gamma(\overline{\alpha}_{12} M''_{X_1+X_2}(r) + 2\alpha_{12} M_{X_1+X_2}(r) M''_{X_1+X_2}(r) + 2\alpha_{12}(M'_{X_1+X_2}(r))^2)}{1 - \gamma(\overline{\alpha}_{12} M_{X_1+X_2}(r) + \alpha_{12} M^2_{X_1+X_2}(r))} \\
& + \frac{\alpha\gamma(\overline{\alpha}_{11} M'_{X_1}(r) + 2\alpha_{11} M'_{X_1}(r))^2}{(1 - \gamma(\overline{\alpha}_{11} M_{X_1}(r) + \alpha_{11} M^2_{X_1}(r))^2} + \frac{\alpha\gamma(\overline{\alpha}_{22} M'_{X_2}(r) + 2\alpha_{22} M'_{X_2}(r))^2}{(1 - \gamma(\overline{\alpha}_{22} M_{X_2}(r) + \alpha_{22} M^2_{X_2}(r))^2} \\
& + \frac{\alpha\gamma(\overline{\alpha}_{12} M'_{X_1+X_2}(r) + 2\alpha_{12} M'_{X_1+X_2}(r))^2}{(1 - \gamma(\overline{\alpha}_{12} M_{X_1+X_2}(r) + \alpha_{12} M^2_{X_1+X_2}(r))^2}.
\end{aligned}
$$

For $r \in \mathbb{R}^+$ such that $1 - p(\overline{\alpha}_{11} M_{X_1}(r) + \alpha_{11} M^2_{X_1}(r)) > 0$, $1 - p(\overline{\alpha}_{22} M_{X_2}(r) + \alpha_{22} M^2_{X_2}(r)) > 0$, $1 - p(\overline{\alpha}_{12} M_{X_1+X_2}(r) + \alpha_{12} M^2_{X_1+X_2}(r)) > 0$.

Lemma 2. *From the expression for adjustment coefficient function of common shock risk model based on Negative Binomial MA(1), the equation $c(r) = 0$ has the unique solution.*

Proof. To prove that $c(r)$ has a unique positive solution, it is sufficient to show that

1. $\frac{dc(r)}{dr}|_{r=0} < 0$,

2. $\dfrac{d^2c(r)}{dr^2} > 0,$

3. There exists r^* for $M_Z(r) \le \dfrac{1}{\gamma}, 1-\overline{\gamma}(\overline{\alpha}_{ij}M_Z(r)+\alpha_{ij}M_Z(r) \ge 0, 1-\gamma(\overline{\alpha}_{ij}M_Z(r)+\alpha_{ij}M_Z^2(r) \ge 0$ such that $\lim_{r\to r^{*-}} c(r) = +\infty$ for $i, j = 1, 2$.

1. Evaluating $\dfrac{dc(r)}{dr}$ at $r = 0$, we obtain.

$$\begin{aligned}\frac{dc(r)}{dr}|_{r=0} &= \frac{\theta_{11}}{\overline{\gamma}}\gamma(1+\alpha_{11})E(X_1) + \frac{\theta_{22}}{\overline{\gamma}}\gamma(1+\alpha_{22})E(X_2) \\ &\quad + \frac{\theta_{12}}{\overline{\gamma}}\gamma(1+\alpha_{12})E(X_1+X_2) - \pi \\ &= E[W] - \pi < 0,\end{aligned}$$

by using the fact that $E[W] < \pi$.

2. Since X_i is claim size, $X_i > 0$.

   Consequently $M_{Z_{ij}}(r) > 0, M'_{Z_{ij}}(r) > 0$ and $M_{Z_{ij}}''(r) > 0$.

   Since $1 - \gamma(\overline{\alpha}_{ij}M_{Z_{ij}}(r) + \alpha_{ij}M^2_{Z_{ij}}(r)) > 0$, $\dfrac{d^2c(r)}{dr^2} > 0$.

3. Consider $1 - \gamma(\overline{\alpha}_{11}M_{X_1}(r) + \alpha_{11}M^2_{X_1}(r))$.

   Since $M_{X_1}(r)$ is increasing to infinity and continuous function on $[0, \infty]$, we obtain that

$$1 - \gamma(\overline{\alpha}_{11}M_{X_1}(r) + \alpha_{11}M^2_{X_1}(r))$$

is decreasing and continuous function.

Then there exists r_1^* such that

$$\lim_{r\to r_1^{*-}} 1 - \gamma(\overline{\alpha}_{11}M_{X_1}(r) + \alpha_{11}M^2_{X_1}(r)) = 0$$

and

$$1 - \gamma(\overline{\alpha}_{11}M_{X_1}(r) + \alpha_{11}M^2_{X_1}(r)) \ge 0 \text{ for all } 0 \le r \le r_1^*.$$

Hence,

$$\lim_{r\to r_1^{*-}} \theta_{11}\log\left(\frac{\overline{\gamma}}{1 - \gamma(\overline{\alpha}_{11}M_{X_1}(r) + \alpha_{11}M^2_{X_1}(r))}\right) = +\infty.$$

Consider $1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M^2_{X_2}(r))$.

Since $M_{X_2}(r)$ is increasing to infinity and continuous function on $[0,\infty]$, we obtain that

$$1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M^2_{X_2}(r))$$

is decreasing and continuous function.

Then there exists r_2^* such that

$$\lim_{r\to r_2^{*-}} 1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M^2_{X_2}(r))=0$$

and

$$1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M^2_{X_2}(r))\geq 0 \text{ for all } 0\leq r\leq r_2^*.$$

Hence,

$$\lim_{r\to r_2^{*-}} \theta_{22}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{22}M_{X_2}(r)+\alpha_{22}M^2_{X_2}(r))}\right)=+\infty.$$

Consider $1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M^2_{X_1+X_2}(r))$.

Since $M_{X_1+X_2}(r)$ is increasing to infinity and continuous function on $[0,\infty]$, we obtain that

$$1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M^2_{X_1+X_2}(r))$$

is decreasing and continuous function.

Then there exists r_3^* such that

$$\lim_{r\to r_3^{*-}} 1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M^2_{X_1+X_2}(r))=0$$

and

$$1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M^2_{X_1+X_2}(r))\geq 0 \text{ for all } 0\leq r\leq r_3^*.$$

Hence,

$$\lim_{r\to r_3^{*-}} \theta_{12}\log\left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{12}M_{X_1+X_2}(r)+\alpha_{12}M^2_{X_1+X_2}(r))}\right)=+\infty.$$

Therefore, $\lim\limits_{r\to r^{*-}} c(r)=+\infty$ where $r^*=\min\{r_1^*,r_2^*,r_3^*\}$.

□

3.3 Common shock risk model based on negative binomial MA(q) (CNBMA(q))

In this section, we introduce the structure and properties of CNBMA(q) process.

Definition 17. Assume that $N_t^{(11)}$, $N_t^{(22)}$ and $N_t^{(12)}$ are independent negative binomial MA(1) processes which have the form

$$N_t^{(ij)} = \alpha_{1ij} \circ \varepsilon_{t-1}^{(ij)} + \alpha_{2ij} \circ \varepsilon_{t-2}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}, \quad i, j = 1, 2; i \le j,$$

where $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. negative binomial random variables with parameter (θ_{ij}, γ) and probability mass function defined as

$$f_Y(y) = \binom{y + \theta_{ij} - 1}{y} \gamma^y (\overline{\gamma})^{\theta_{ij}},$$

where $\overline{\gamma} = 1 - \gamma$.

Theorem 13. *The sequence of* $\{N_t^{(ij)}, t \in \mathbb{N}$ *and* $i, j = 1, 2\}$ *defined in Definition 17 have the following properties.*

1. $\{N_t^{(ij)}, t \in \mathbb{N}\}$ *is a stationary process.*
2. $E[N_t^{(ij)}] = \dfrac{\theta_{ij}\gamma}{\overline{\gamma}}(\alpha_{1ij} + \cdots + \alpha_{qij} + 1)$.
3. $Var[N_t^{(ij)}] = \dfrac{\theta_{ij}\gamma}{\overline{\gamma}^2}[(\alpha_{1ij}^2 + \cdots + \alpha_{qij}^2)\gamma + (\alpha_{1ij} + \cdots + \alpha_{qij})\overline{\gamma} + 1]$.
4. $Cov(N_t^{(ij)}, N_{t-l}^{(ij)}) = \begin{cases} \dfrac{\theta_{ij}\gamma}{\overline{\gamma}^2} B_{lq}^{(ij)}, & l < q, \\ \dfrac{\theta_{ij}\alpha_{ij}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q, \end{cases}$

   where $B_{lq}^{(ij)} = \alpha_{lij} + \alpha_{(l+1)ij}\alpha_{1ij} + \cdots + \alpha_{qij}\alpha_{(q-l)ij}$.

Proof. For $i, j = 1, 2$ and $i \le j$.

1. Since $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. $NB(\theta_{ij}, \gamma)$ random variables,

$$\begin{aligned}
G_{N_t^{(ij)}}(r) &= E[r^{N_t^{(ij)}}] \\
&= E[r^{\alpha_{1ij}\circ\varepsilon_{t-1}^{(ij)}+\alpha_{2ij}\circ\varepsilon_{t-2}^{(ij)}+\cdots+\alpha_{qij}\circ\varepsilon_{t-q}^{(ij)}+\varepsilon_t^{(ij)}}] \\
&= E[r^{\alpha_{1ij}\circ\varepsilon_{t-1}^{(ij)}}]\cdots E[r^{\alpha_{qij}\circ\varepsilon_{t-q}^{(ij)}}]E[r^{\varepsilon_t^{(ij)}}] \\
&= G_{\varepsilon_{t-1}^{(ij)}}((1-\alpha)+\alpha r)\cdots G_{\varepsilon_{t-q}^{(ij)}}((1-\alpha)+\alpha r)G_{\varepsilon_t^{(ij)}}(r) \qquad (3.19) \\
&= G_{\varepsilon_t^{(ij)}}(r)\prod_{k=1}^{q} G_{\varepsilon_t^{(ij)}}((1-\alpha)+\alpha r),
\end{aligned}$$

where we use Lemma 1 (7) to prove equation (3.19).

We can observe that $G_{N_t^{(ij)}}(r)$ does not depend on t, then $\{N_t^{(ij)}, t \in \mathbb{N}\}$ is a stationary process.

2. Since $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. $NB(\theta_{ij}, \gamma)$ random variables and by Lemma 1 (1),

$$\begin{aligned}
E[N_t^{(ij)}] &= E[\alpha_{1ij}\circ\varepsilon_{t-1}^{(ij)} + \cdots + \alpha_{qij}\circ\varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}] \\
&= E[\alpha_{1ij}\circ\varepsilon_{t-1}^{(ij)}] + \cdots + E[\alpha_{qij}\circ\varepsilon_{t-q}^{(ij)}] + E[\varepsilon_t^{(ij)}] \\
&= \alpha_{1ij}E[\varepsilon_{t-1}^{(ij)}] + \cdots + \alpha_{qij}E[\varepsilon_{t-q}^{(ij)}] + E[\varepsilon_t^{(ij)}] \\
&= \frac{\theta_{ij}\gamma}{\overline{\gamma}}(\alpha_{1ij} + \cdots + \alpha_{qij} + 1).
\end{aligned}$$

3. Since $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. $NB(\theta_{ij}, \gamma)$ random variables and by Lemma 1 (4),

$$\begin{aligned}
Var[N_t^{(ij)}] &= Var[\alpha_{1ij}\circ\varepsilon_{t-1}^{(ij)} + \cdots + \alpha_{qij}\circ\varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}] \\
&= Var[\alpha_{1ij}\circ\varepsilon_{t-1}^{(ij)}] + \cdots + Var[\alpha_{qij}\circ\varepsilon_{t-q}^{(ij)}] + Var[\varepsilon_t^{(ij)}] \\
&= E[\varepsilon_{t-1}^{(ij)}]Var[\alpha_{1ij}] + Var[\varepsilon_{t-1}^{(ij)}]E^2[\alpha_{1ij}] + \cdots \\
&\quad + E[\varepsilon_{t-q}^{(ij)}]Var[\alpha_{qij}] + Var[\varepsilon_{t-q}^{(ij)}]E^2[\alpha_{qij}] + Var[\varepsilon_t^{(ij)}] \\
&= \left(\frac{\theta_{ij}\gamma}{\overline{\gamma}}(\alpha_{1ij})(1-\alpha_{1ij}) + \frac{\theta_{ij}\gamma}{\overline{\gamma}^2}\alpha_{1ij}^2\right) + \cdots \\
&\quad + \left(\frac{\theta_{ij}\gamma}{\overline{\gamma}}(\alpha_{qij})(1-\alpha_{qij}) + \frac{\theta_{ij}\gamma}{\overline{\gamma}^2}\alpha_{qij}^2\right) + \frac{\theta_{ij}\gamma}{\overline{\gamma}^2} \\
&= \frac{\theta_{ij}\gamma}{\overline{\gamma}^2}\left((\alpha_{1ij}^2 + \cdots + \alpha_{qij}^2)\gamma + (\alpha_{1ij} + \cdots + \alpha_{qij})\overline{\gamma} + 1\right)
\end{aligned}$$

4. To prove (4) we use the fact that $\{\varepsilon_t^{(ij)}, t = 0, 1, 2, \ldots\}$ is a sequence of i.i.d. $NB(\theta_{ij}, \gamma)$ random variables having variance $Var[\varepsilon_t^{(ij)}] = \dfrac{\theta_{ij}\gamma}{\overline{\gamma}^2}$.

   We divide the proof into three cases; $l < q$, $l = q$, and $l > q$.

   For $l < q$, we first show calculation for $l = 1$ and then the calculation for any $l < q$.

   For $l = 1$, we have

$$
\begin{aligned}
&Cov(N_t^{(ij)}, N_{t-1}^{(ij)})\\
&= Cov(\alpha_{1ij} \circ \varepsilon_{t-1}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}, \alpha_{1ij} \circ \varepsilon_{t-2}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q-1}^{(ij)} + \varepsilon_{t-1}^{(ij)})\\
&= Cov(\alpha_{1ij} \circ \varepsilon_{t-1}^{(ij)}, \varepsilon_{t-1}^{(ij)}) + Cov(\alpha_{2ij} \circ \varepsilon_{t-2}^{(ij)}, \alpha_{1ij} \circ \varepsilon_{t-2}^{(ij)}) + \cdots + Cov(\alpha_{qij} \circ \varepsilon_{t-q}^{(ij)}, \alpha_{(q-1)ij} \circ \varepsilon_{t-q}^{(ij)})\\
&= \alpha_{1ij} Var[\varepsilon_{t-1}^{(ij)}] + \alpha_{2ij}\alpha_{1ij} Var[\varepsilon_{t-2}^{(ij)}] + \cdots + \alpha_{qij}\alpha_{(q-1)ij} Var[\varepsilon_{t-q}^{(ij)}]\\
&= \frac{\theta_{ij}\gamma}{\overline{\gamma}^2}[\alpha_{1ij} + \alpha_{2ij}\alpha_{1ij} + \cdots + \alpha_{qij}\alpha_{(q-1)ij}].
\end{aligned}
$$

   For $l < q$, we have

$$
\begin{aligned}
&Cov(N_t^{(ij)}, N_{t-3}^{(ij)})\\
&= Cov(\alpha_{1ij} \circ \varepsilon_{t-1}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}, \alpha_{1ij} \circ \varepsilon_{t-1-l}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q-l}^{(ij)} + \varepsilon_{t-l}^{(ij)})\\
&= Cov(\alpha_{lij} \circ \varepsilon_{t-l}^{(ij)}, \varepsilon_{t-l}^{(ij)}) + Cov(\alpha_{(l+1)ij} \circ \varepsilon_{t-(l+1)}^{(ij)}, \alpha_{1ij} \circ \varepsilon_{t-(l+1)}^{(ij)}) + \cdots\\
&\quad + Cov(\alpha_{qij} \circ \varepsilon_{t-q}^{(ij)}, \alpha_{(q-l)ij} \circ \varepsilon_{t-q}^{(ij)})\\
&= \alpha_{lij} Var[\varepsilon_{t-l}^{(ij)}] + \alpha_{(l+1)ij}\alpha_{1ij} Var[\varepsilon_{t-(l+1)}^{(ij)}] + \cdots + \alpha_{qij}\alpha_{(q-l)ij} Var[\varepsilon_{t-q}^{(ij)}]\\
&= \frac{\theta_{ij}\gamma}{\overline{\gamma}^2}[\alpha_{lij} + \alpha_{(l+1)ij}\alpha_{1ij} + \cdots + \alpha_{qij}\alpha_{(q-l)ij}],
\end{aligned}
$$

   For $l = q$, we have

$$
\begin{aligned}
&Cov(N_t^{(ij)}, N_{t-q}^{(ij)})\\
&= Cov(\alpha_{1ij} \circ \varepsilon_{t-1}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}, \alpha_{1ij} \circ \varepsilon_{t-q-1}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q-q}^{(ij)} + \varepsilon_{t-q}^{(ij)})\\
&= Cov(\alpha_{qij} \circ \varepsilon_{t-q}^{(ij)}, \varepsilon_{t-q}^{(ij)})\\
&= \frac{\theta_{ij}\gamma}{\overline{\gamma}^2}\alpha_{qij},
\end{aligned}
$$

   where we use Lemma 1 (6) to prove the last equality.

   For $l > q$, then $t - l - 1 < t - q$.

   Therefore,

$$
Cov(N_t^{(ij)}, N_{t-l}^{(ij)})
$$

$$= Cov(\alpha_{1ij} \circ \varepsilon_{t-1}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q}^{(ij)} + \varepsilon_t^{(ij)}, \alpha_{1ij} \circ \varepsilon_{t-l-1}^{(ij)} + \cdots + \alpha_{qij} \circ \varepsilon_{t-q-l}^{(ij)} + \varepsilon_{t-l}^{(ij)}) \tag{3.20}$$

$$= 0,$$

where $\varepsilon_k^{(ij)}$ in (3.20) are all different.

Hence, $Cov(N_t^{(ij)}, N_{t-l}^{(ij)}) = \begin{cases} \dfrac{\theta_{ij}\gamma}{\overline{\gamma}^2} B_{lq}^{(ij)}, & l < q, \\ \dfrac{\theta_{ij}\alpha_{ij}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q, \end{cases}$

□

Theorem 14. *Properties of $N_t^{(i)}$ for $i = 1, 2$ and N_t defined in Definition 16 are as follows.*

1. $E[N_t^{(i)}] = \left(\dfrac{\theta_{ii}\gamma}{\overline{\gamma}}(\alpha_{1ii} + \cdots + \alpha_{qii} + 1)\right) + \left(\dfrac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{12} + \cdots + \alpha_{q12} + 1)\right).$

2. $$Var[N_t^{(i)}] = \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}\left((\alpha_{1ii}^2 + \cdots + \alpha_{qii}^2)\gamma + (\alpha_{1ii} + \cdots + \alpha_{qii})\overline{\gamma} + 1\right)\right) + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}^2}\left((\alpha_{112}^2 + \cdots + \alpha_{qij}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1\right)\right).$$

3. $$Cov(N_t^{(i)}, N_{t-l}^{(i)}) = \begin{cases} \dfrac{\theta_{ii}\gamma}{\overline{\gamma}^2} B_{lq}^{(ii)} + \dfrac{\theta_{12}\gamma}{\overline{\gamma}^2} B_{lq}^{(12)}, & l < q, \\ \dfrac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}^2} + \dfrac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q. \end{cases}$$

4. $$Cov(N_t, N_{t-l}) = \begin{cases} \dfrac{\theta_{11}\gamma}{\overline{\gamma}^2} B_{lq}^{(11)} + \dfrac{\theta_{22}\gamma}{\overline{\gamma}^2} B_{lq}^{(22)} + \dfrac{4\theta_{12}\gamma}{\overline{\gamma}^2} B_{lq}^{(12)}, & l < q, \\ \dfrac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}^2} + \dfrac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}^2} + \dfrac{4\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q. \end{cases}$$

where $B_{lq}^{(ij)} = \alpha_{lij} + \alpha_{(l+1)ij}\alpha_{1ij} + \cdots + \alpha_{qij}\alpha_{(q-l)ij}$.

Proof. For $i, j = 1, 2$ and $i \leq j$. From $N_t^{(i)} = N_t^{(ii)} + N_t^{(12)}$ and Definition 17, we obtain the following poofs.

1. From Theorem 13 (2), we have

$$\begin{aligned} E[N_t^{(i)}] &= E[N_t^{(ii)} + N_t^{(12)}] \\ &= E[N_t^{(ii)}] + E[N_t^{(12)}] \\ &= \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(\alpha_{1ii} + \cdots + \alpha_{qii} + 1)\right) + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)\right). \end{aligned}$$

2. Since $N_t^{(ii)}$, $N_t^{(12)}$ are independent and by Theorem 13 (3), we have

$$\begin{aligned} Var[N_t^{(i)}] &= Var[N_t^{(ii)} + N_t^{(12)}] \\ &= Var[N_t^{(ii)}] + Var[N_t^{(12)}] \\ &= \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}[(\alpha_{1ii}^2 + \cdots + \alpha_{qii}^2)\gamma + (\alpha_{1ii} + \cdots + \alpha_{qii})\overline{\gamma} + 1]\right) \\ &\quad + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}^2}[(\alpha_{112}^2 + \cdots + \alpha_{q12}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1]\right). \end{aligned}$$

3. Since $N_t^{(ii)}$, $N_t^{(12)}$ are independent and by Theorem 13 (4), we have

$$\begin{aligned} Cov(N_t^{(i)}, N_{t-l}^{(i)}) &= Cov(N_t^{(ii)} + N_t^{(12)}, N_{t-l}^{(ii)} + N_{t-l}^{(12)}) \\ &= Cov(N_t^{(ii)}, N_{t-l}^{(ii)}) + Cov(N_t^{(12)}, N_{t-l}^{(12)}) \\ &= \begin{cases} \frac{\theta_{ii}\gamma}{\overline{\gamma}^2}B_{lq}^{(ii)} + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}B_{lq}^{(12)}, & l < q, \\ \frac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}^2} + \frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q. \end{cases} \end{aligned}$$

4. Note that $N_t = N_t^{(1)} + N_t^{(2)}$. Therefore,

$$\begin{aligned} Cov(N_t, N_{t-l}) &= Cov(N_t^{(11)} + N_t^{(22)} + 2N_t^{(12)}, N_{t-l}^{(11)} + N_{t-l}^{(22)} + 2N_{t-l}^{(12)}) \\ &= Cov(N_t^{(11)}, N_{t-l}^{(11)}) + Cov(N_t^{(22)}, N_{t-l}^{(22)}) + 4Cov(N_t^{(12)}, N_{t-l}^{(12)}) \\ &= \begin{cases} \frac{\theta_{11}\gamma}{\overline{\gamma}^2}B_{lq}^{(11)} + \frac{\theta_{22}\gamma}{\overline{\gamma}^2}B_{lq}^{(22)} + \frac{4\theta_{12}\gamma}{\overline{\gamma}^2}B_{lq}^{(12)}, & l < q, \\ \frac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}^2} + \frac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}^2} + \frac{4\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q. \end{cases} \end{aligned}$$

□

Theorem 15. *Properties of $W_t^{(i)}$ for $i = 1, 2$ and W_t defined in 3.1.2 are as follows.*

1. $E[W_t^{(i)}] = \left(\left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(\alpha_{1ii} + \cdots + \alpha_{qii} + 1)\right) + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)\right)\right) E[X_i].$

2. $$\begin{aligned} Var[W_t^{(i)}] = &\left(\left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(\alpha_{1ii} + \cdots + \alpha_{qii} + 1)\right) + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)\right)\right) Var[X_i] \\ &+ \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}[(\alpha_{1ii}^2 + \cdots + \alpha_{qii}^2)\gamma + (\alpha_{1ii} + \cdots + \alpha_{qii})\overline{\gamma} + 1]\right) E^2[X_i] \\ &+ \left(\frac{\theta_{12}\gamma}{\overline{\gamma}^2}[(\alpha_{112}^2 + \cdots + \alpha_{qij}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1]\right) E^2[X_i]. \end{aligned}$$

3. $Cov(W_t^{(i)}, W_{t-l}^{(i)})$
$$= \begin{cases} E^2[X_i]\left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}B_{lq}^{(ii)} + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}B_{lq}^{(12)}\right), & l < q, \\ E^2[X_i]\left(\frac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}^2} + \frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}\right), & l = q, \\ 0, & l > q. \end{cases}$$

4. $Cov(W_t^{(1)}, W_t^{(2)}) = \frac{\theta_{12}\gamma}{\overline{\gamma}^2}((\alpha_{112}^2 + \cdots + \alpha_{q12}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1)E[X_1]E[X_2].$

5. $$\begin{aligned} E[W_t] = &\left(\frac{\theta_{11}\gamma}{\overline{\gamma}}(\alpha_{111} + \cdots + \alpha_{q11} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)\right) E[X_1] \\ &+ \left(\frac{\theta_{22}\gamma}{\overline{\gamma}}(\alpha_{122} + \cdots + \alpha_{q22} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{112} + \cdots + \alpha_{q12} + 1)\right) E[X_2]. \end{aligned}$$

6. $$\begin{aligned} Var[W_t] = &\frac{\theta_{11}\gamma}{\overline{\gamma}}(\alpha_{111} + \cdots + \alpha_{q11} + 1)Var[X_1] + \frac{\theta_{22}\gamma}{\overline{\gamma}}(\alpha_{122} + \cdots + \alpha_{q22} + 1)Var[X_2] \\ &+ \frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)(Var[X_1] + Var[X_2]) \\ &+ \frac{\theta_{11}\gamma}{\overline{\gamma}^2}((\alpha_{111}^2 + \cdots + \alpha_{q11}^2)\gamma + (\alpha_{111} + \cdots + \alpha_{q11})\overline{\gamma} + 1)E^2[X_1] \\ &+ \frac{\theta_{22}\gamma}{\overline{\gamma}^2}((\alpha_{122}^2 + \cdots + \alpha_{q22}^2)\gamma + (\alpha_{122} + \cdots + \alpha_{q22})\overline{\gamma} + 1)E^2[X_2] \\ &+ \frac{\theta_{12}\gamma}{\overline{\gamma}^2}((\alpha_{112}^2 + \cdots + \alpha_{q12}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1)(E[X_1] + E[X_2])^2. \end{aligned}$$

7. $Cov(W_t, W_{t-l})$
$$= \begin{cases} E^2[X_1]\frac{\theta_{11}\gamma}{\overline{\gamma}^2}B_{lq}^{(11)} + E^2[X_2]\frac{\theta_{22}\gamma}{\overline{\gamma}^2}B_{lq}^{(22)} + E^2[X_1 + X_2]\frac{\theta_{12}\gamma}{\overline{\gamma}^2}B_{lq}^{(12)}, & l < q, \\ E^2[X_1]\frac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}^2} + E^2[X_2]\frac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}^2} + E^2[X_1 + X_2]\frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q. \end{cases}$$

where $B_{lq}^{(ij)} = \alpha_{lij} + \alpha_{(l+1)ij}\alpha_{1ij} + \cdots + \alpha_{qij}\alpha_{(q-l)ij}$.

Proof. For $i, j = 1, 2$ and $i \leq j$. From 3.1.2 we have $W_t = W_t^{(1)} + W_t^{(2)}$ and $W_t^{(i)} = \sum_{j=1}^{N_t^{(i)}} X_{i,t,j}$,

1. From Theorem 7 (1) and of Theorem 9 (1), we have

$$\begin{aligned} E[W_t^{(i)}] &= E[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j}] \\ &= E[N_t^{(i)}]E[X_i] \\ &= \left(\left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(\alpha_{1ii} + \cdots + \alpha_{qii} + 1) \right) + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1) \right) \right) E[X_i]. \end{aligned}$$

2. From Theorem 7 (2) and Theorem 9 (1,2), we have

$$\begin{aligned} Var[W_t^{(i)}] &= Var[\sum_{j=1}^{N_t^{(i)}} X_{i,t,j}] \\ &= E[N_t^{(i)}]Var[X_i] + Var[N_t^{(i)}]E^2[X_i] \\ &= \left(\left(\frac{\theta_{ii}\gamma}{\overline{\gamma}}(\alpha_{1ii} + \cdots + \alpha_{qii} + 1) \right) + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1) \right) \right) Var[X_i] \\ &\quad + \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2}[(\alpha_{1ii}^2 + \cdots + \alpha_{qii}^2)\gamma + (\alpha_{1ii} + \cdots + \alpha_{qii})\overline{\gamma} + 1] \right) E^2[X_i] \\ &\quad + \left(\frac{\theta_{12}\gamma}{\overline{\gamma}^2}[(\alpha_{112}^2 + \cdots + \alpha_{qij}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1] \right) E^2[X_i]. \end{aligned}$$

3. Similar to Theorem 10 (3), we have

$$\begin{aligned} Cov(W_t^{(i)}, W_{t-l}^{(i)}) &= E[W_t^{(i)}W_{t-l}^{(i)}] - E[W_t^{(i)}]E[W_{t-l}^{(i)}] \\ &= E^2[X_i]Cov(N_t^{(i)}, N_{t-l}^{(i)}) \\ &= \begin{cases} E^2[X_i] \left(\frac{\theta_{ii}\gamma}{\overline{\gamma}^2} B_{lq}^{(ii)} + \frac{\theta_{12}\gamma}{\overline{\gamma}^2} B_{lq}^{(12)} \right), & l < q, \\ E^2[X_i] \left(\frac{\theta_{ii}\alpha_{ii}\gamma}{\overline{\gamma}^2} + \frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2} \right), & l = q, \\ 0, & l > q. \end{cases} \end{aligned}$$

4. Similar to Theorem 10 (4), we have

$$\begin{aligned} Cov(W_t^{(1)}, W_t^{(2)}) &= E[W_t^{(1)}W_t^{(2)}] - E[W_t^{(1)}]E[W_t^{(2)}] \\ &= E[N_t^{(1)}N_t^{(2)}]E[X_1]E[X_2] - E[N_t^{(1)}]E[X_1]E[N_t^{(2)}]E[X_2] \end{aligned}$$

$$= E[X_1]E[X_2]Var[N_t^{(12)}]$$
$$= \frac{\theta_{12}\gamma}{\overline{\gamma}^2}((\alpha_{112}^2 + \cdots + \alpha_{q12}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1)E[X_1]E[X_2]$$

5. From Theorem 9 (1), we have

$$\begin{aligned} E[W_t] &= E[W_t^{(1)}] + E[W_t^{(2)}] \\ &= \left(\frac{\theta_{11}\gamma}{\overline{\gamma}}(\alpha_{111} + \cdots + \alpha_{q11} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)\right) E[X_1] \\ &\quad + \left(\frac{\theta_{22}\gamma}{\overline{\gamma}}(\alpha_{122} + \cdots + \alpha_{q22} + 1) + \frac{\theta_{12}\gamma}{\overline{\gamma}}(1 + \alpha_{112} + \cdots + \alpha_{q12} + 1)\right) E[X_2]. \end{aligned}$$

6. From 2. and 4. we have

$$\begin{aligned} Var[W_t] &= Var[W_t^{(1)} + W_t^{(2)}] \\ &= Var[W_t^{(1)}] + Var[W_t^{(2)}] + 2Cov(W_t^{(1)}, W_t^{(2)}) \\ &= \frac{\theta_{11}\gamma}{\overline{\gamma}}(\alpha_{111} + \cdots + \alpha_{q11} + 1)Var[X_1] + \frac{\theta_{22}\gamma}{\overline{\gamma}}(\alpha_{122} + \cdots + \alpha_{q22} + 1)Var[X_2] \\ &\quad + \frac{\theta_{12}\gamma}{\overline{\gamma}}(\alpha_{112} + \cdots + \alpha_{q12} + 1)Var[X_1] + Var[X_2] \\ &\quad + \frac{\theta_{11}\gamma}{\overline{\gamma}^2}((\alpha_{111}^2 + \cdots + \alpha_{q11}^2)\gamma + (\alpha_{111} + \cdots + \alpha_{q11})\overline{\gamma} + 1)E^2[X_1] \\ &\quad + \frac{\theta_{22}\gamma}{\overline{\gamma}^2}((\alpha_{122}^2 + \cdots + \alpha_{q22}^2)\gamma + (\alpha_{122} + \cdots + \alpha_{q22})\overline{\gamma} + 1)E^2[X_2] \\ &\quad + \frac{\theta_{12}\gamma}{\overline{\gamma}^2}((\alpha_{112}^2 + \cdots + \alpha_{q12}^2)\gamma + (\alpha_{112} + \cdots + \alpha_{q12})\overline{\gamma} + 1)(E[X_1] + E[X_2])^2 \end{aligned}$$

7. Similar to Theorem 10 (7), we have

$$\begin{aligned} &Cov(W_t, W_{t-l}) \\ &= E[W_t W_{t-l}] - E[W_t]E[W_{t-l}] \\ &= E[(W_t^{(1)} + W_t^{(2)})(W_{t-l}^{(1)} + W_{t-l}^{(2)})] - E^2[W_t] \\ &= E^2[X_1]Cov(N_t^{(1)}, N_{t-l}^{(1)}) + E^2[X_2]Cov(N_t^{(2)}, N_{t-l}^{(2)}) \\ &= \begin{cases} E^2[X_1]\frac{\theta_{11}\gamma}{\overline{\gamma}^2}B_{lq}^{(11)} + E^2[X_2]\frac{\theta_{22}\gamma}{\overline{\gamma}^2}B_{lq}^{(22)} + E^2[X_1 + X_2]\frac{\theta_{12}\gamma}{\overline{\gamma}^2}B_{lq}^{(12)}, & l < q, \\ E^2[X_1]\frac{\theta_{11}\alpha_{11}\gamma}{\overline{\gamma}^2} + E^2[X_2]\frac{\theta_{22}\alpha_{22}\gamma}{\overline{\gamma}^2} + E^2[X_1 + X_2]\frac{\theta_{12}\alpha_{12}\gamma}{\overline{\gamma}^2}, & l = q, \\ 0, & l > q. \end{cases} \end{aligned}$$

□

Lemma 3. *For $N_t^{(12)}$, $N_t^{(11)}$ and $N_t^{(22)}$ be a NBMA(q) defined in Definition 17, we have*

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(ij)}}]) = \theta_{12}\log\left(\frac{\overline{\gamma}}{1-\gamma M_{X_1+X_2}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}(r))}\right)$$

where $r \in \{r \in \mathbb{R}^+ | M_{X_1+X_2}\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}) < \frac{1}{\gamma}\}$,

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(11)}}]) = \theta_{11}\log\left(\frac{\overline{\gamma}}{1-\gamma M_{X_1}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i11}+\alpha_{i11}M_{X_1}(r))}\right)$$

where $r \in \{r \in \mathbb{R}^+ | M_{X_1}\prod_{i=1}^{q}(\overline{\alpha}_{i11}+\alpha_{i11}M_{X_1}) < \frac{1}{\gamma}\}$,

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(22)}}]) = \theta_{22}\log\left(\frac{\overline{\gamma}}{1-\gamma M_{X_2}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i22}+\alpha_{i22}M_{X_2}(r))}\right)$$

where $r \in \{r \in \mathbb{R}^+ | M_{X_2}\prod_{i=1}^{q}(\overline{\alpha}_{i22}+\alpha_{i22}M_{X_2}) < \frac{1}{\gamma}\}$.

Proof. We rewrite $S_n^{(12)} = \sum_{t=1}^{n}\sum_{j=1}^{N_t^{(12)}} X_{1,t,j} + X_{2,t,j}$ as $S_n^{(12)} = \sum_{k=1}^{N_{(n)}^{(12)}} X_{1k} + X_{2k}$.
Assume that $\{X_{1k} + X_{2k}, k = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with m.g.f. $M_{X_1+X_2}$.
Then the m.g.f. of $S_n^{(12)}$ can be expressed as

$$\begin{aligned}
E[e^{rS_n^{(12)}}] &= E[e^{r\sum_{l=1}^{N_{(n)}^{(12)}} X_{1k}+X_{2k}}] \\
&= E[E[e^{r\sum_{l=1}^{N_{(n)}^{(12)}} X_{1k}+X_{2k}} | N_{(n)}^{(12)}]] \\
&= E[\prod_{k=1}^{N_{(n)}^{(12)}} E[e^{rX_{1k}+X_{2k}}]] \\
&= E[M_{X_1+X_2}^{N_{(n)}^{(12)}}(r)] \\
&= G_{N_{(n)}^{(12)}}(M_{X_1+X_2}(r)), \qquad (3.21)
\end{aligned}$$

where $G_{N_{(n)}^{(12)}}(r)$ is the generating function of $N_{(n)}^{(12)}$. Then

$$G_{N_{(n)}^{(12)}}(r) = E[r^{N_1^{(12)}+N_2^{(12)}+\cdots+N_n^{(12)}}]$$

$$= E[r^{(\varepsilon_1^{(12)}+\alpha_{112}\circ\varepsilon_{1-1}^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_{1-q}^{(12)})+(\varepsilon_2^{(12)}+\alpha_{112}\circ\varepsilon_{2-1}^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_{2-q}^{(12)})+\cdots+(\alpha_{112}\circ\varepsilon_{n-1}^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_{n-q}^{(12)}+\varepsilon_n^{(12)})}]$$

$$= E[r^{\alpha_{q12}\circ\varepsilon_{1-q}^{(12)}}]E[r^{\alpha_{(q-1)12}\circ\varepsilon_{2-q}^{(12)}+\alpha_{q12}\circ\varepsilon_{2-q}^{(12)}}]\times\cdots\times E[r^{\alpha_{112}\circ\varepsilon_0^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_0^{(12)}}]$$

$$\times\prod_{m=1}^{n-q}E[r^{\alpha_{112}\circ\varepsilon_m^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_m^{(12)}+\varepsilon_m^{(12)}}]E[r^{\alpha_{112}\circ\varepsilon_{n-q-1}^{(12)}+\cdots+\alpha_{(q-1)12}\circ\varepsilon_{n-q-1}^{(12)}+\varepsilon_{n-q-1}^{(12)}}]$$

$$\times\cdots\times E[r^{\alpha_{112}\circ\varepsilon_{n-1}^{(12)}+\varepsilon_{n-1}^{(12)}}]E[r^{\varepsilon_n^{(12)}}] \tag{3.22}$$

where $t, q \in \mathbb{N}$.

Since $\{\delta_{li12}\}$ is a sequence of i.i.d. Bernoulli random variables with mean α_{i12} and $\{\varepsilon_t^{(12)}, t = 0, 1, 2, ...\}$ is a sequence of i.i.d. Negative Binomial random variable with parameters (θ_{12}, γ).

Consider

$$\begin{aligned}
E[r^{\alpha_{q12}\circ\varepsilon_{1-q}^{(12)}}] &= E[E[r^{\sum_{l=1}^{\varepsilon_{1-q}^{(12)}}\delta_{lq12}}|\varepsilon_{1-q}^{(12)}]] \\
&= E\left[\left(\prod_{l=1}^{\varepsilon_{1-q}^{(12)}}E[r^{\delta_{lq12}}]\right)\right] \\
&= E\left[(\overline{\alpha}_{q12}+\alpha_{q12}r)^{\varepsilon_{1-q}^{(12)}}\right] \\
&= \left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{q12}+\alpha_{q12}r)}\right)^{\theta_{12}},
\end{aligned} \tag{3.23}$$

for $\overline{\alpha}_{q12}+\alpha_{q12}r < \dfrac{1}{\gamma}$.

Consider

$$\begin{aligned}
E[r^{\alpha_{(q-1)12}\circ\varepsilon_{2-q}^{(12)}+\alpha_{q12}\circ\varepsilon_{2-q}^{(12)}}] &= E[E[r^{\sum_{l=1}^{\varepsilon_{2-q}^{(12)}}\delta_{l(q-1)12}+\sum_{l=1}^{\varepsilon_{2-q}^{(12)}}\delta_{lq12}}|\varepsilon_{2-q}^{(12)}]] \\
&= E\left[E\left[r^{\sum_{l=1}^{\varepsilon_{2-q}^{(12)}}\delta_{l(q-1)12j}}|\varepsilon_{2-q}^{(12)}\right]E\left[r^{\sum_{l=1}^{\varepsilon_{2-q}^{(12)}}\delta_{lq12j}}|\varepsilon_{2-q}^{(12)}\right]\right] \\
&= E\left[\left(\prod_{l=1}^{\varepsilon_{2-q}^{(12)}}E[r^{\delta_{l(q-1)12}}]\right)\left(\prod_{l=1}^{\varepsilon_{2-q}^{(12)}}E[r^{\delta_{lq12}}]\right)\right] \\
&= E\left[(\overline{\alpha}_{(q-1)12}+\alpha_{(q-1)12}r)^{\varepsilon_{2-q}^{(12)}}(\overline{\alpha}_{q12}+\alpha_{q12}r)^{\varepsilon_{2-q}^{(12)}}\right] \\
&= E\left[\left((\overline{\alpha}_{(q-1)12}+\alpha_{(q-1)12}r)(\overline{\alpha}_{q12}+\alpha_{q12}r)\right)^{\varepsilon_{2-q}^{(12)}}\right] \\
&= \left(\frac{\overline{\gamma}}{1-\gamma\prod_{i=q-1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}},
\end{aligned} \tag{3.24}$$

for $\prod_{i=q-1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)<\frac{1}{\gamma}$.
Consider

$$
\begin{aligned}
E[r^{\alpha_{112}\circ\varepsilon_0^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_0^{(12)}}] &= E[E[r^{\sum_{l=1}^{\varepsilon_0^{(12)}}\delta_{l112}+\cdots+\sum_{l=1}^{\varepsilon_0^{(12)}}\delta_{lq12}}|\varepsilon_0^{(12)}]] \\
&= E\left[\prod_{i=1}^{q} E\left[r^{\sum_{l=1}^{\varepsilon_0^{(12)}}\delta_{li12j}}|\varepsilon_0^{(12)}\right]\right] \\
&= E\left[\prod_{i=1}^{q}\left(\prod_{l=1}^{\varepsilon_0^{(12)}} E[r^{\delta_{li12}}]\right)\right] \\
&= E\left[\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)^{\varepsilon_0^{(12)}}\right] \\
&= E\left[\left(\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)\right)^{\varepsilon_0^{(12)}}\right] \\
&= \left(\frac{\overline{\gamma}}{1-\gamma\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}}, \qquad (3.25)
\end{aligned}
$$

for $\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)<\frac{1}{\gamma}$.
For $m=1,2,\ldots,n-q$,

$$
\begin{aligned}
E[r^{\alpha_{112}\circ\varepsilon_m^{(12)}+\cdots+\alpha_{q12}\circ\varepsilon_m^{(12)}+\varepsilon_m^{(12)}}] &= E[E[r^{\sum_{l=1}^{\varepsilon_m^{(12)}}\delta_{l112}+\cdots+\sum_{l=1}^{\varepsilon_m^{(12)}}\delta_{lq12}+\varepsilon_m^{(12)}}|\varepsilon_m^{(12)}]] \\
&= E\left[r^{\varepsilon_m^{(12)}}\prod_{i=1}^{q} E\left[r^{\sum_{l=1}^{\varepsilon_m^{(12)}}\delta_{li12}}|\varepsilon_m^{(12)}\right]\right] \\
&= E\left[r^{\varepsilon_m^{(12)}}\prod_{i=1}^{q}\left(\prod_{l=1}^{\varepsilon_m^{(12)}} E[r^{\delta_{li12}}]\right)\right] \\
&= E\left[r^{\varepsilon_m^{(12)}}\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)^{\varepsilon_m^{(12)}}\right] \\
&= E\left[\left(r\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)\right)^{\varepsilon_m^{(12)}}\right] \\
&= \left(\frac{\overline{\gamma}}{1-\gamma r\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}}, \qquad (3.26)
\end{aligned}
$$

for $r\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)<\frac{1}{\gamma}$.

Consider

$$\begin{aligned}
E[r^{\alpha_{112}\circ\varepsilon^{(12)}_{n-q-1}+\cdots+\alpha_{(q-1)12}\circ\varepsilon^{(12)}_{n-q-1}+\varepsilon^{(12)}_{n-q-1}}] &= E[E[r^{\sum_{l=1}^{\varepsilon^{(12)}_{n-q-1}}\delta_{l112}+\cdots+\sum_{l=1}^{\varepsilon^{(12)}_{n-q-1}}\delta_{l(q-1)12}+\varepsilon^{(12)}_{n-q-1}}|\varepsilon^{(12)}_{n-q-1}]] \\
&= E\left[r^{\varepsilon^{(12)}_{n-q-1}}\prod_{i=1}^{q-1}E\left[r^{\sum_{l=1}^{\varepsilon^{(12)}_{n-q-1}}\delta_{li12}}|\varepsilon^{(12)}_{n-q-1}\right]\right] \\
&= E\left[r^{\varepsilon^{(12)}_{n-q-1}}\prod_{i=1}^{q-1}\left(\prod_{l=1}^{\varepsilon^{(12)}_{n-q-1}}E[r^{\delta_{li12}}]\right)\right] \\
&= E\left[r^{\varepsilon^{(12)}_{n-q-1}}\prod_{i=1}^{q-1}(\overline{\alpha}_{i12}+\alpha_{i12}r)^{\varepsilon^{(12)}_{n-q-1}}\right] \\
&= E\left[\left(r\prod_{i=1}^{q-1}(\overline{\alpha}_{i12}+\alpha_{i12}r)\right)^{\varepsilon^{(12)}_{n-q-1}}\right] \\
&= \left(\frac{\overline{\gamma}}{1-\gamma r\prod_{i=1}^{q-1}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}}, \qquad (3.27)
\end{aligned}$$

for $r\prod_{i=1}^{q-1}(\overline{\alpha}_{i12}+\alpha_{i12}r) < \dfrac{1}{\gamma}$.
Consider

$$\begin{aligned}
E[r^{\alpha_{112}\circ\varepsilon^{(12)}_{n-1}+\varepsilon^{(12)}_{n-1}}] &= E[E[r^{\sum_{l=1}^{\varepsilon^{(12)}_{n-1}}\delta_{l112}+\varepsilon^{(12)}_{n-1}}|\varepsilon^{(12)}_{n-1}]] \\
&= E\left[r^{\varepsilon^{(12)}_{n-1}}E\left[r^{\sum_{l=1}^{\varepsilon^{(12)}_{n-1}}\delta_{l112}}|\varepsilon^{(12)}_{n-1}\right]\right] \\
&= E\left[r^{\varepsilon^{(12)}_{n-1}}\left(\prod_{l=1}^{\varepsilon^{(12)}_{n-1}}E[r^{\delta_{l112}}]\right)\right] \\
&= E\left[r^{\varepsilon^{(12)}_{n-1}}(\overline{\alpha}_{112}+\alpha_{112}r)^{\varepsilon^{(12)}_{n-1}}\right] \\
&= E\left[\left(\overline{\alpha}_{112}r+\alpha_{112}r^2\right)^{\varepsilon^{(12)}_{n-1}}\right] \\
&= \left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{112}r+\alpha_{112}r^2)}\right)^{\theta_{12}}, \qquad (3.28)
\end{aligned}$$

for $\overline{\alpha}_{112}r+\alpha_{112}r^2 < \dfrac{1}{\gamma}$.
and

$$E[r^{\varepsilon^{(12)}_n}] = \left(\frac{\overline{\gamma}}{1-\gamma r}\right)^{\theta_{12}} \qquad (3.29)$$

for $0 < r < \dfrac{1}{\gamma}$.
Substituting equations (3.23) - (3.29) into (3.22), we obtain

$$
\begin{aligned}
G_{N_{(n)}^{(ij)}}(r) = & \left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{q12}+\alpha_{q12}r)}\right)^{\theta_{12}} \times \left(\frac{\overline{\gamma}}{1-\gamma\prod_{i=q-1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}} \times \cdots \\
& \times \left(\frac{\overline{\gamma}}{1-\gamma\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}} \times \left(\frac{\overline{\gamma}}{1-\gamma r\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}(n-q)} \\
& \times \left(\frac{\overline{\gamma}}{1-\gamma r\prod_{i=1}^{q-1}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}} \times \cdots \times \left(\frac{\overline{\gamma}}{1-\gamma(\overline{\alpha}_{112}r+\alpha_{112}r^2)}\right)^{\theta_{12}} \times \left(\frac{\overline{\gamma}}{1-\gamma r}\right)^{\theta_{12}} \\
= & \prod_{s=1}^{q}\left(\frac{\overline{\gamma}}{1-\gamma\prod_{i=s}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}} \times \left(\frac{\overline{\gamma}}{1-\gamma r\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}(n-q)} \\
& \times \prod_{s=1}^{q-1}\left(\frac{\overline{\gamma}}{1-\gamma r\prod_{i=1}^{s}(\overline{\alpha}_{i12}+\alpha_{i12}r)}\right)^{\theta_{12}} \times \left(\frac{\overline{\gamma}}{1-\gamma r}\right)^{\theta_{12}} \qquad (3.30)
\end{aligned}
$$

where
$r \in \{r \in \mathbb{R}^+ | max\{\prod_{i=s}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r), r\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}r), r\prod_{i=1}^{s}(\overline{\alpha}_{i12}+\alpha_{i12}r), r\} < \dfrac{1}{\gamma}\}$.
Combining equation (3.21), (3.22) and (3.30), we have

$$
\begin{aligned}
E[e^{rS_n^{(12)}}] = & \prod_{s=1}^{q}\left(\frac{\overline{\gamma}}{1-\gamma\prod_{i=s}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}(r))}\right)^{\theta_{12}} \\
& \times \left(\frac{\overline{\gamma}}{1-\gamma M_{X_1+X_2}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}(r))}\right)^{\theta_{12}(n-q)} \\
& \times \prod_{s=1}^{q-1}\left(\frac{\overline{\gamma}}{1-\gamma M_{X_1+X_2}(r)\prod_{i=1}^{s}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}(r))}\right)^{\theta_{12}} \\
& \times \left(\frac{\overline{\gamma}}{1-\gamma M_{X_1+X_2}(r)}\right)^{\theta_{12}} \qquad (3.31)
\end{aligned}
$$

where $r \in \{r \in \mathbb{R}^+ | \prod_{i=s}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}), M_{X_1+X_2}\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}), M_{X_1+X_2}\prod_{i=1}^{s}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}), M_{X_1+X_2} < \frac{1}{\gamma}\}$.

Hence,

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(12)}}]) = \theta_{12}\log\left(\frac{\overline{\gamma}}{1-\gamma M_{X_1+X_2}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}(r))}\right)$$

where $r \in \{r \in \mathbb{R}^+ | M_{X_1+X_2}\prod_{i=1}^{q}(\overline{\alpha}_{i12}+\alpha_{i12}M_{X_1+X_2}) < \frac{1}{\gamma}\}$.

We rewrite $S_n^{(11)} = \sum_{t=1}^{n}\sum_{j=1}^{N_t^{(11)}} X_{1,t,j}$ as $S_n^{(11)} = \sum_{k=1}^{N_{(n)}^{(12)}} X_{1k}$.

Assume that $\{X_{1k}, k = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with m.g.f. M_{X_1}.

Similarly to $S_n^{(12)}$ we can derive

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(11)}}]) = \theta_{11}\log\left(\frac{\overline{\gamma}}{1-\gamma M_{X_1}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i11}+\alpha_{i11}M_{X_1}(r))}\right)$$

where $r \in \{r \in \mathbb{R}^+ | M_{X_1}\prod_{i=1}^{q}(\overline{\alpha}_{i11}+\alpha_{i11}M_{X_1}) < \frac{1}{\gamma}\}$.

We rewrite $S_n^{(22)} = \sum_{t=1}^{n}\sum_{j=1}^{N_t^{(22)}} X_{2,t,j}$ as $S_n^{(22)} = \sum_{k=1}^{N_{(n)}^{(22)}} X_{2k}$.

Assume that $\{X_{2k}, k = 1, 2, \ldots\}$ is a sequence of i.i.d. random variables with m.g.f. M_{X_2}.

Similarly to $S_n^{(12)}$ we can derive

$$\lim_{n\to\infty}\frac{1}{n}(\log(E[e^{rS_n^{(22)}}]) = \theta_{22}\log\left(\frac{\overline{\gamma}}{1-\gamma M_{X_2}(r)\prod_{i=1}^{q}(\overline{\alpha}_{i22}+\alpha_{i22}M_{X_2}(r))}\right)$$

where $r \in \{r \in \mathbb{R}^+ | M_{X_2}\prod_{i=1}^{q}(\overline{\alpha}_{i22}+\alpha_{i22}M_{X_2}) < \frac{1}{\gamma}\}$.

□

Theorem 16. *The adjustment coefficient function* $c(r)$ *of* U_n *where* $N_t^{(i)}$ $(i = 1, 2)$ *be a CNBMA(q) is*

$$\begin{aligned} c(r) =& \theta_{12} \log \left(\frac{\overline{\gamma}}{1 - \gamma M_{X_1+X_2}(r) \prod_{i=1}^{q} (\overline{\alpha}_{i12} + \alpha_{i12} M_{X_1+X_2}(r))} \right) \\ &+ \theta_{11} \log \left(\frac{\overline{\gamma}}{1 - \gamma M_{X_1}(r) \prod_{i=1}^{q} (\overline{\alpha}_{i11} + \alpha_{i11} M_{X_1}(r))} \right) \\ &+ \theta_{22} \log \left(\frac{\overline{\gamma}}{1 - \gamma M_{X_2}(r) \prod_{i=1}^{q} (\overline{\alpha}_{i22} + \alpha_{i22} M_{X_2}(r))} \right) - r\pi, \end{aligned}$$

where $r \in \{r \in \mathbb{R}^+ | max\{M_{X_1+X_2} \prod_{i=1}^{q} (\overline{\alpha}_{i12} + \alpha_{i12} M_{X_1+X_2}), M_{X_1} \prod_{i=1}^{q} (\overline{\alpha}_{i11} + \alpha_{i11} M_{X_1}),$
$M_{X_2} \prod_{i=1}^{q} (\overline{\alpha}_{i22} + \alpha_{i22} M_{X_2})\} < \frac{1}{\gamma}\}.$

Proof. Similarly to Theorem 12, we have

$$c(r) = \lim_{n\to\infty} \frac{1}{n} (\log(E[e^{rS_n^{(11)}}]) + \lim_{n\to\infty} \frac{1}{n} \log(E[e^{rS_n^{(22)}}]) + \lim_{n\to\infty} \frac{1}{n} \log(E[e^{rS_n^{(12)}}])) - r\pi.$$

Therefore, by Lemma 3 c(r) can be rewritten as

$$\begin{aligned} c(r) =& \theta_{12} \log \left(\frac{\overline{\gamma}}{1 - \gamma M_{X_1+X_2}(r) \prod_{i=1}^{q} (\overline{\alpha}_{i12} + \alpha_{i12} M_{X_1+X_2}(r))} \right) \\ &+ \theta_{11} \log \left(\frac{\overline{\gamma}}{1 - \gamma M_{X_1}(r) \prod_{i=1}^{q} (\overline{\alpha}_{i11} + \alpha_{i11} M_{X_1}(r))} \right) \\ &+ \theta_{22} \log \left(\frac{\overline{\gamma}}{1 - \gamma M_{X_2}(r) \prod_{i=1}^{q} (\overline{\alpha}_{i22} + \alpha_{i22} M_{X_2}(r))} \right) - r\pi, \end{aligned}$$

where $r \in \{r \in \mathbb{R}^+ | max\{M_{X_1+X_2} \prod_{i=1}^{q} (\overline{\alpha}_{i12} + \alpha_{i12} M_{X_1+X_2}), M_{X_1} \prod_{i=1}^{q} (\overline{\alpha}_{i11} + \alpha_{i11} M_{X_1}),$
$M_{X_2} \prod_{i=1}^{q} (\overline{\alpha}_{i22} + \alpha_{i22} M_{X_2})\} < \frac{1}{\gamma}\}.$ □

3.4 Numerical Examples and Simulations

In this section, numerical studies are carried out to assess the adjustment coefficient and ruin probability.

We write the premium rate π as

$$\pi = (1+\eta)E[W_t],$$

where $\eta > 0$ is regarded as the relative security loading.

3.4.1 CNBMA(1) model

For CNBMA(1) model we study the effect of the Negative Binomial MA(1) dependence parameters on the adjustment coefficient R. In the CNBMA(1) model, we have

$$\pi = (1+\eta)\left(\left(\frac{\theta_{11}\gamma}{\bar{\gamma}}(1+\alpha_{11}) + \frac{\theta_{12}\gamma}{\bar{\gamma}}(1+\alpha_{12})\right)E[X_1] + \left(\frac{\theta_{22}\gamma}{\bar{\gamma}}(1+\alpha_{22}) + \frac{\theta_{12}\gamma}{\bar{\gamma}}(1+\alpha_{12})\right)E[X_2]\right).$$

Example 1. *We set $\theta_{11} = 0.6$, $\theta_{22} = 0.9$, $\theta_{12} = 0.2$, and consider claim-size distributions in the two classes are $X_1 \sim Exp(\beta_1)$, $X_2 \sim Exp(\beta_2)$, where the mean parameters $\beta_1 = 1$ and $\beta_2 \in \{0.5, 1, 1.5, 2\}$.*
We set relative security loading $\eta = 0.2$ and initial surplus $u = 10$. For the time series dependence parameters $(\alpha_{11}, \alpha_{22}, \alpha_{12})$, we use six sets of values $(0,0,0)$, $(0.8,0,0)$, $(0,0.4,0)$, $(0,0,0.5)$, $(0.8,0.4,0.5)$ and $(1,1,1)$.

		$(\alpha_{11}, \alpha_{22}, \alpha_{12})$					
Claims X_1	Claims X_2	(0,0,0)	(0.8,0,0)	(0,0.4,0)	(0,0,0.5)	(0.8,0.4,0.5)	(1,1,1)
Exp(1)	Exp(0.5)	0.07804	0.07745	0.06388	0.06568	0.05904	0.04968
Exp(1)	Exp(1)	0.13224	0.11319	0.11749	0.10684	0.09507	0.08329
Exp(1)	Exp(1.5)	0.15672	0.12169	0.14898	0.12630	0.10919	0.09885
Exp(1)	Exp(2)	0.16605	0.12274	0.16422	0.13508	0.11402	0.10522

Table 3.1: Adjustment coefficient: claim effect of CNBMA(1)

Results in Table 3.1 show that the ordering of the adjustment coefficient R is given by $R_{0,0,0} > (R_{0.8,0,0}, R_{0,0.4,0}, R_{0,0,0.5}) > R_{0.8,0.4,0.5} > R_{1,1,1}$ *where the ranking of* $R_{0.8,0,0}$, $R_{0,0.4,0}$, *and* $R_{0,0,0.5}$ *in the bracket is not stable. In our example, we observe that the ranking of* $R_{0.8,0,0}$, $R_{0,0.4,0}$, *and* $R_{0,0,0.5}$ *change if* β_2 *change. We have an impression that the parameter for the common shock* α_{12} *affects on adjustment coefficient more than* α_{11} *and* α_{22}*. Hence we can confirm that if other things are equal, the greater the negative binomial MA(1) dependence parameter* α_{12} *gives the smaller the adjustment coefficient R.*

		$(\alpha_{11}, \alpha_{22}, \alpha_{12})$					
Claims X_1	Claims X_2	(0,0,0)	(0.8,0,0)	(0,0.4,0)	(0,0,0.5)	(0.8,0.4,0.5)	(1,1,1)
Exp(1)	Exp(0.5)	0.4582	0.4609	0.5278	0.5184	0.5540	0.6084
Exp(1)	Exp(1)	0.2664	0.3224	0.3088	0.3432	0.3864	0.4347
Exp(1)	Exp(1.5)	0.2086	0.2961	0.2254	0.2827	0.3355	0.3721
Exp(1)	Exp(2)	0.1900	0.2930	0.1935	0.2590	0.3197	0.3491

Table 3.2: Ruin probability $\psi(u)$: comparison time series of CNBMA(1)

Results in Table 3.2 show that the ranking of the ruin probability is given by $\psi_{0,0,0}(u) < (\psi_{0.8,0,0}(u), \psi_{0,0.4,0}(u), \psi_{0,0,0.5}(u)) < \psi_{0.8,0.4,0.5}(u) < \psi_{1,1,1}(u)$. *We notice that the probability of ruin increases as the adjustment coefficient decreases. We also observe that* $\psi_{0.8,0,0}(u) < \psi_{0,0,0.5}(u) < \psi_{0,0.4,0}(u)$ *if* $\beta_2 = 0.5$ *the ranking of the ruin probability is not consistent with the ranking of the adjustment coefficient. This may be due to the computational errors in approximating the ultimate ruin probabilities.*

Example 2. *We set* $\theta_{11} = 0.6$, $\theta_{22} = 0.9$, $\theta_{12} = 0.2$, *and consider claim-size distributions in the two classes as* $X_1 \sim Exp(\beta_1)$ *and* $X_2 \sim Exp(\beta_2)$, *where the mean parameters* $\beta_1 = 1$ *and* $\beta_2 = 0.5$.
We set relative security loading $\eta = 0.2$ *and initial surplus* $u = 10$. *For the time series dependence parameters we set* $\alpha_{11} = 0.8$, $\alpha_{22} = 0.4$ *and we change* α_{12} *to six sets of values* 0, 0.2, 0.4, 0.6, 0.8 *and* 1.

	α_{12}					
	0	0.2	0.4	0.6	0.8	1
R	0.06482	0.06236	0.06010	0.05801	0.05607	0.05427
$\psi(u)$	0.52296	0.53595	0.54822	0.55980	0.57076	0.58114

Table 3.3: Adjustment coefficient and ruin probability $\psi(u)$: time series effect of CNBMA(1)

Results in Table 3.3 clearly confirms that the adjustment coefficient R decreases as the dependence parameter α_{12} increases. As the adjustment coefficient decreases, the upper bound $\psi(u)$ increases. Therefore, if the dependence claim number random variables is at a high level then probability that business classes will be more risky to be ruined.

3.4.2 CNBMA(q) model

For CNBMA(q) model we study the effect of the Negative Binomial MA(q) dependence parameters on the adjustment coefficient R. In this subsection we will show numerical examples of CNBMA(q) in the case $q = 2$.

In the CNBMA(2) model, we have

$$\pi = (1+\eta)\left(\frac{\theta_{11}\gamma}{\bar{\gamma}}(\alpha_{111}+\alpha_{211}+1) + \frac{\theta_{12}\gamma}{\bar{\gamma}}(\alpha_{112}+\alpha_{212}+1)\right)E[X_1] + \left(\frac{\theta_{22}\gamma}{\bar{\gamma}}(\alpha_{122}+\alpha_{222}+1) + \frac{\theta_{12}\gamma}{\bar{\gamma}}(\alpha_{112}+\alpha_{212}+1)\right)E[X_2].$$

Example 3. *We set $\theta_{11} = 0.6$, $\theta_{22} = 0.9$, $\theta_{12} = 0.2$, and consider claim-size distributions in the two classes as $X_1 \sim Exp(\beta_1)$ and $X_2 \sim Exp(\beta_2)$, where the mean parameters $\beta_1 = 1$ and $\beta_2 \in \{0.5, 1, 1.5, 2\}$.*
We set relative security loading $\eta = 0.2$ and initial surplus $u = 10$. For simplicity, the time series dependence parameters we assume that $(\alpha_{i11}, \alpha_{i22}, \alpha_{i12}) = (\alpha_1, \alpha_2, \alpha_3)$ for $i = 1, 2$ and consider six sets of values: $(0,0,0)$, $(0.8,0,0)$, $(0,0.4,0)$, $(0,0,0.5)$, $(0.8,0.4,0.5)$ *and* $(1,1,1)$.

		$(\alpha_1, \alpha_2, \alpha_3)$					
Claims X_1	Claims X_2	(0,0,0)	(0.8,0,0)	(0,0.4,0)	(0,0,0.5)	(0.8,0.4,0.5)	(1,1,1)
Exp(1)	Exp(0.5)	0.07804	0.07310	0.05394	0.05469	0.04825	0.03640
Exp(1)	Exp(1)	0.13224	0.09329	0.10390	0.08651	0.07533	0.06073
Exp(1)	Exp(1.5)	0.15672	0.09509	0.13876	0.10216	0.08507	0.07214
Exp(1)	Exp(2)	0.16605	0.09405	0.15890	0.10999	0.08816	0.07694

Table 3.4: Adjustment coefficient: claim effect of CNBMA(2)

Similar to Example 1, results in Table 3.4 show that the ordering of the adjustment coefficient R is given by $R_{0,0,0} > (R_{0.8,0,0}, R_{0,0.4,0}, R_{0,0,0.5}) > R_{0.8,0.4,0.5} > R_{1,1,1}$ *where the ranking of* $R_{0.8,0,0}$, $R_{0,0.4,0}$, *and* $R_{0,0,0.5}$ *in the bracket is not consistent. Hence we can confirm that if other things is equal, the greater the Negative binomial MA(1) dependence parameter* α_{12} *give the smaller the adjustment coefficient R.*

		$(\alpha_1, \alpha_2, \alpha_3)$					
Claims X_1	Claims X_2	(0,0,0)	(0.8,0,0)	(0,0.4,0)	(0,0,0.5)	(0.8,0.4,0.5)	(1,1,1)
Exp(1)	Exp(0.5)	0.4582	0.4814	0.5830	0.5787	0.6171	0.6948
Exp(1)	Exp(1)	0.2664	0.3934	0.3537	0.4209	0.4707	0.5447
Exp(1)	Exp(1.5)	0.2086	0.3863	0.2496	0.3600	0.4270	0.4860
Exp(1)	Exp(2)	0.1900	0.3904	0.2041	0.3328	0.4141	0.4632

Table 3.5: Ruin probability $\psi(u)$: comparison time series of CNBMA(2)

Similar to Example 1, results in Table 3.2 show that as usual the ranking of the ruin probability is given by $\psi_{0,0,0}(u) < (\psi_{0.8,0,0}(u), \psi_{0,0.4,0}(u), \psi_{0,0,0.5}(u)) < \psi_{0.8,0.4,0.5}(u) < \psi_{1,1,1}(u)$, *we see that the probability of ruin increases as the adjustment coefficient decreases.*

Example 4. *We set* $\theta_{11} = 0.6$, $\theta_{22} = 0.9$, $\theta_{12} = 0.2$, *and consider claim-size distributions in the two classes* $X_1 \sim Exp(\beta_1)$, $X_2 \sim Exp(\beta_2)$, *where the mean parameters* $\beta_1 = 1$ *and* $\beta_2 = 0.5$.

We set relative security loading $\eta = 0.2$ *and initial surplus* $u = 10$. *For simplicity,* *the time series dependence parameters we assume that* $(\alpha_{i11}, \alpha_{i22}, \alpha_{i12}) = (\alpha_1, \alpha_2, \alpha_3)$ *for* $i = 1, 2$. *We set* $\alpha_1 = 0.8$, $\alpha_2 = 0.4$ *and* $\alpha_3 \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$.

	α_3					
	0	0.2	0.4	0.6	0.8	1
R	0.05500	0.05246	0.04968	0.04682	0.04401	0.04133
$\psi(u)$	0.57690	0.59174	0.60844	0.62608	0.64391	0.66143

Table 3.6: Adjustment coefficient and ruin probability $\psi(u)$: time series effect of CNBMA(2)

Similar to Example 2, results in Table 3.3 clearly confirm that the adjustment coefficient R *decreases as the dependence parameter* α_{12} *increases. As the adjustment coefficient decreases, the upper bound* $\psi(u)$ *increases. Therefore, if the dependence claim number random variables is at a high level then probability that business classes will be more risky to be ruined.*

Chapter 4

Conclusion

In this project we have constructed and derived probabilistic properties of common shock risk models based on negative binomial moving average models. In our study, we first constructed the common shock risk model based on INBMA(1) in Section 3.2 and then generalized the model to a more general risk model based on INBMA(q) modes in Section 3.3. Numerical results of the cases when $q = 1$ and $q = 2$ were shown in Section 3.4. The numerical results suggest that the level of the ruin probability depend on the level of the adjustment coefficient but the level of the ruin probability in CNBMA(q) model is greater than CNBMA(1) model. These results suggest that the surplus process is more risky when the dependency between the two classes of business is higher.

Our study can be extended in many different directions such as to extend the model to n classes of business or to change the moving average model to the autoregressive model.

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The Project Proposal of Course 2301399 Project Proposal Academic Year 2018

Project Tittle (Thai)	ตัวแบบความเสี่ยงที่มีปัจจัยร่วมบนฐานของอนุกรมเวลาทวินามลบ
Project Tittle (English)	Common Factor Risk Models Based on Negative Binomial Time series
Advisor	Assistant Prof. Jiraphan Suntornchost, Ph.D.
By	Thanasaporn Thanasirithanakorn ID 5833521223 Mathematics, Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University

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Background and Rationale

A time series is a sequence of data that are collected at equally spaced points in time. Models for time series data can represent different stochastic process. To describe the claim count process, broad classes of practical importance are the autoregressive (AR) models and the moving average (MA) models. Integer valued time series models in class of insurance business have innovations which are distributed on the set of non-negative integers. McKenZie (1985) examined the INAR(1) models which are interesting models for count data.

Cossette, Marceau and Deschamps (2010) studied risk models based on time series by considering the class of insurance business risk models based on poisson distributions. Later, Li (2012) introduced common factor or common shock risk models based on poisson time series. The common shock in risk models cooperate effects on insurance business having more than one class.

Poisson distribution is one of the most distributions studied and used in risk models. However, expectation and varience of poisson distributions are equal which may not hold in some applications. Therefore, there are many researchers proposed some alternative integer-valued time series models that generalize the Poisson INAR models for wider applications. For example, McKenzie (1985) proposed INMA models based on the negative binomial distributions for over-dispersed data. Later, Laphudomsakda and Suntornchost (2018) applied the Negative Binomial INMA model to construct the new class of business risk models based on negative binomial time series. Moreover, they derived some probabilistic properties and computed the ruin probability of their models.

In this project, we will extend the study of Laphudomsakda and Suntornchost (2018) to construct common shock risk models based on negative binomial time series and derive some probabilistic properties and also the upper bound of the ruin probability.

Objectives

To extend the risk models based on negative binomial time series to common shock risk models based on negative binomial time series.

Scope

The integer-valued time series consider in this project is based on the negative binomial distributions. The probabilistic properties considered in this project are expectation, variance, covariance, adjustment coefficient and infinite-time ruin probability.

Project Activities

1. Study fundamental concepts of probability theory and risk models.
2. Study basic properties of common shock risk models based on poisson time series.
3. Study basic properties of risk models based on negative binomial time series.
4. Construct common shock risk models based on negative binomial time series.
5. Study properties of common shock risk models based on negative binomial time series.
6. Summarize and write the report.

Scheduled Operations

Procedures	Months								
	Aug.	Sep.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.
1. Study fundamental concepts of probability theory and risk models.	■	■							
2. Study common shock risk models based on poisson time series.		■	■	■	■				
3. Study risk models based on negative binomial time series.		■	■	■	■				
4. Construct common shock risk models based on negative binomial time series.			■	■	■				
5. Study properties of common shock risk models based on negative binomial time series.					■	■			
6. Summarize and write the report.						■	■	■	■

Benefits

The benefits for student who implement this project.

1. To learn properties and applications of risk models in actuarial science.
2. To gain knowledge in probability theory and apply the models to suitable applications.

The benefits for users of the project.

To have a general thinning risk model based on negative binomial time series for wider applications.

Equipments

Software

1. Beamer
2. Adobe PDF
3. Latex
4. R
5. Mathematica

Hardware

1. Printer
2. Computer

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