Chapter 2

Classical Uniform Distribution

In this chapter, we discuss briefly the theory of uniform distribution of sequences in the classical case. We introduce the basic concepts of *uniform distribution modulo 1* and *uniform distribution modulo m* and some of their applications. Most of these results can be found in Kuipers and Niederreiter [8].

2.1 Uniform Distribution Modulo 1

This section covers basic definitions, the Weyl criterion and properties of uniform distribution modulo 1.

For a real number x, let $[x]$ denote the integral part of x, that is, the greatest integer $\leq x$ and $\{x\} = x - [x]$ the fractional part of x.

Definition 2.1.1. A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is *uniformly distributed modulo 1* (abbreviated u.d.mod 1) if and only if for all subintervals $[a, b)$ of $[0, 1)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : \{x_n\} \in [a, b)\}| = b - a.$$

Remark 2.1.2. (1) A definition equivalent to Definition 2.1.1 is the following: A sequence $(x_n)_{n=1}^{\infty}$ of real numbers is u.d.mod 1 if and only if for all subintervals $[0, c)$ of $[0, 1)$ we have

$$\lim_{N\to\infty} \frac{1}{N}|\{n \leq N : \{x_n\} \in [0, c)\}| = c.$$

(2) If a real sequence $(x_n)_{n=1}^{\infty}$ is u.d.mod 1, then the sequence $(\{x_n\})_{n=1}^{\infty}$ of fractional parts is everywhere dense in $[0, 1)$.

(3) If a real sequence $(x_n)_{n=1}^{\infty}$ is u.d.mod 1, then $\{\{x_n\} : n \in \mathbb{N}\}$ is infinite.

Example 2.1.3. The sequence $(r_n)_{n=1}^{\infty} = (\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots)$ is u.d.mod 1. To show this, let $c \in (0, 1]$. In each block with denominator q, we want to find all nonnegative integers p such that $0 \leq \frac{p}{q} < c$, equivalently $0 \leq p < cq$; note that for a fixed q, the number of such p's is $[cq]$ or $[cq] + 1$. Now, let N be any positive integer. Then, there is a positive integer n such that $\frac{(n-1)n}{2} \leq N \leq \frac{n(n+1)}{2}$. Thus,

$$\frac{cn^2 - (c+2)n + 2}{n^2 + n} = \frac{\sum_{q=1}^{n-1}(cq - 1)}{\frac{n(n+1)}{2}} \leq \frac{\sum_{q=1}^{n-1}[cq]}{\frac{n(n+1)}{2}} \leq \frac{|\{n \leq N : r_n \in [0, c)\}|}{N}$$

$$\leq \frac{\sum_{q=1}^{n}([cq] + 1)}{\frac{(n-1)n}{2}} \leq \frac{\sum_{q=1}^{n}(cq + 1)}{\frac{(n-1)n}{2}} = \frac{cn^2 + (c+2)n}{n^2 - n}.$$

Then

$$\begin{aligned} c = \lim_{n\to\infty} \frac{cn^2 - (c+2)n + 2}{n^2 + n} &\leq \liminf_{N\to\infty} \frac{1}{N}|\{n \leq N : r_n \in [0, c)\}| \\ &\leq \limsup_{N\to\infty} \frac{1}{N}|\{n \leq N : r_n \in [0, c)\}| \\ &\leq \lim_{n\to\infty} \frac{cn^2 + (c+2)n}{n^2 - n} \\ &= c. \end{aligned}$$

Therefore,

$$\lim_{N\to\infty} \frac{1}{N}|\{n \le N : r_n \in [0,c)\}| = c.$$

Hence, $(r_n)_{n=1}^{\infty}$ is u.d.mod 1.

The following theorem and corollary were proved by Hermann Weyl.

Theorem 2.1.4. *The real sequence $(x_n)_{n=1}^{\infty}$ is u.d.mod 1 if and only if*

$$\forall f \in \Re[0,1),\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)dx,$$

where, $\Re[0,1)$ denotes the space of Riemann integrable functions on $[0,1)$.

Proof. See Theorem 1.1 and Corollary 1.1 of Chapter 1 in [8]. □

Corollary 2.1.5. *The real sequence $(x_n)_{n=1}^{\infty}$ is u.d.mod 1 if and only if for every complex-valued continuous function f on $\mathbb{R}$ with period 1 we have*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx.$$

Proof. See corollary 1.2 of Chapter 1 in [8]. □

The fundamental result in the theory of uniform distribution modulo 1 is Hermann Weyl's uniform distribution criterion.

Theorem 2.1.6 (Weyl Criterion). *The real sequence $(x_n)_{n=1}^{\infty}$ is u.d.mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \textit{ for all integers } h \ne 0.$$

Proof. See Theorem 2.1 of Chapter 1 in [8]. □

Theorem 2.1.7. *Let the sequence $(x_n)_{n=1}^{\infty}$ be u.d.mod 1. Then*

(i) the sequence $(x_n + \alpha)_{n=1}^{\infty}$ is u.d.mod 1, for every real constant α,

(ii) if $(y_n)_{n=1}^{\infty}$ is a sequence with the property

$$\lim_{n\to\infty} (x_n - y_n) = \alpha,$$

where α is a real constant, then $(y_n)_{n=1}^{\infty}$ is u.d.mod 1,

(iii) $(mx_n)_{n=1}^{\infty}$ is u.d.mod 1 for every nonzero integer m.

Proof. See Lemma 1.1, Theorem 1.2 and Exercise 2.4 of Chapter 1 in [8]. □

Example 2.1.8. The sequence $(n\alpha)_{n=1}^{\infty}$ is u.d.mod 1 if and only if α is an irrational number. If α is an irrational number, then

$$\begin{aligned} \left|\sum_{n=1}^{N} e^{2\pi i h n\alpha}\right| &= \frac{|e^{2\pi i h N\alpha} - 1|}{|e^{2\pi i h\alpha} - 1|} \\ &= \frac{\sqrt{1 - \cos 2\pi h N\alpha}}{\sqrt{1 - \cos 2\pi h\alpha}} \\ &\leq \frac{\sqrt{2}}{\sqrt{2\sin^2 \pi h\alpha}} \\ &\leq \frac{1}{|\sin \pi h\alpha|} \neq 0 \end{aligned}$$

for all integers $h \neq 0$; hence $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i h n\alpha} \to 0$ as $N \to \infty$ since $\sin \pi h\alpha \neq 0$ for all integers $h \neq 0$. If α is a rational number, say $\alpha = \frac{a}{b}$ where a and b are relatively prime, then $\{\{\frac{na}{b}\} : n \in \mathbb{N}\} = \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b}\}$, which is finite, and so $(n\alpha)_{n=1}^{\infty}$ cannot be u.d.mod 1 by (3) of Remark 2.1.2.

Example 2.1.9. The converse of Remark 2.1.2 (2) is not necessarily true. The sequence $(\log n)_{n=1}^{\infty}$ is not u.d.mod 1, but the sequence $(\{\log n\})_{n=1}^{\infty}$ is dense in $[0,1)$. Note that for each nonnegative integer h,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ih\log n} &= \frac{1}{N}\sum_{n=1}^{N}(e^{\log n})^{2\pi ih}\\ &= \frac{1}{N}\sum_{n=1}^{N} n^{2\pi ih}\\ &= \frac{N^{2\pi ih}}{N}\sum_{n=1}^{N}\left(\frac{n}{N}\right)^{2\pi ih}\\ &\sim N^{2\pi ih}\int_0^1 x^{2\pi ih}dx\\ &= \frac{N^{2\pi ih}}{1+2\pi ih}\text{ , by the theory of Riemann integral.}\end{aligned}$$

Thus, $\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i\log n}$ does not tend to 0, and so the sequence $(\log n)_{n=1}^{\infty}$ is not u.d.mod 1. However, we observe that the sequence $(\{\log n\})_{n=1}^{\infty}$ is dense in $[0,1)$. To see this, let $0 \leq a < b \leq 1$. Since $e^n(e^b - e^a) \to \infty$ as $n \to \infty$, there is an integer k such that $e^{a+k} - e^{b+k} > 1$. Thus, there is an integer n such that $e^{a+k} < n < e^{b+k}$. That is $a+k < \log n < b+k$. Hence, $a \leq \{\log n\} < b$.

Next , we introduce the Van der Corput's Difference Theorem.

Lemma 2.1.10 (Van der Corput's Fundamental Inequality). *Let $u_1, \ldots, u_N$ be complex numbers, and H be an integer with $1 \leq H \leq N$. Then*

$$H^2\left|\sum_{n=1}^{N} u_n\right|^2 \leq H(N+H-1)\sum_{n=1}^{N}|u_n|^2 + 2(N+H-1)\sum_{h=1}^{H-1}(H-h)Re\sum_{n=1}^{N-h} u_n\overline{u}_{n+h},$$

where Re z denotes the real part of $z \in \mathbb{C}$.

Proof. See Lemma 3.1 of Chapter 1 in [8]. □

Theorem 2.1.11 (Van der Corput's Difference Theorem). *Let (x_n) be a given sequence of real numbers. If for every positive integer h the sequence $(x_{n+h} - x_n)_{n=1}^{\infty}$ is u.d.mod 1, then (x_n) is u.d.mod 1.*

Proof. See Theorem 3.1 of Chapter 1 in [8]. □

This theorem yields an important sufficient condition for u.d.mod 1, but not a necessary one, as is seen by considering the sequence $(n\alpha)_{n=1}^{\infty}$ with α irrational. One of the many applications of Theorem 2.1.11 is to sequences of polynomial values.

Theorem 2.1.12. *Let $p(x) = \alpha_m x^m + \alpha_{m-1}x^{m-1} + \ldots + \alpha_0$, $m \geq 1$, be a polynomial with real coefficients and let at least one of the coefficients α_j with $j > 0$ be irrational. Then the sequence $(p(n))_{n=1}^{\infty}$ is u.d.mod 1.*

Proof. See Theorem 3.2 of Chapter 1 in [8]. □

2.2 Applications

In this section, we present some results in the theory of power series which are deduced from the fact that sequences $(n\alpha)_{n=1}^{\infty}$ with irrational α are u.d.mod 1. The next two theorems are slight extensions of Theorem 1 and 2 of Newman [16].

Theorem 2.2.1. *Let α and β be real numbers, and let g be a polynomial over $\mathbb{C}$ of positive degree. Define*

$$G(x) = \sum_{n=0}^{\infty} g([n\alpha + \beta])x^n.$$

Then $G(x)$ is a rational function if and only if α is a rational number.

Proof. The proof is based on the following auxiliary result: Let α be an irrational number, and let S be a finite set of nonintegral real numbers. Then there are infinitely many positive integers m such that

$$[\{m\alpha+\beta\}+\eta]=[\eta] \quad \text{for all } \eta \in S \tag{2.2.1}$$

and also infinitely many positive integers n such that

$$[\{n\alpha+\beta\}+\eta]=1+[\eta] \quad \text{for all } \eta \in S. \tag{2.2.2}$$

Observe that (2.2.1) is equivalent to

$$0 \leq \{m\alpha+\beta\}+\{\eta\}<1 \quad \text{for all } \eta \in S,$$

and that (2.2.2) is equivalent to

$$0 \leq \{n\alpha+\beta\}+\{\eta\}-1<1 \quad \text{for all } \eta \in S.$$

These relations follow easily from the fact that the sequence $(n\alpha+\beta)_{n=1}^{\infty}$ is u.d.mod 1 or in fact from the property that the sequence $(\{n\alpha+\beta\})_{n=1}^{\infty}$ is everywhere dense in $[0,1)$.

Now we turn to the proof of the theorem. Let α be irrational. If $G(x)$ were rational, then polynomials $A(x)$ and $B(x)$, of degrees $a \geq 1$ and b, respectively, would exist such that $G(x)=B(x)/A(x)$. Assume that

$$A(x)=x^a-c_1x^{a-1}-\cdots-c_{a-1}x-c_a.$$

From $A(x)G(x)=B(x)$ it follows, by equating corresponding coefficients of x^{n+a}, that

$$g([n\alpha+\beta])=\sum_{r=1}^{a} g([n\alpha+\beta+r\alpha])c_r \quad \text{for } n \geq \max\{0,b-a+1\}. \tag{2.2.3}$$

Since g is a polynomial of degree $p \geq 1$, we have

$$\lim_{n\to\infty} \frac{g([n\alpha+\beta+r\alpha])}{g([n\alpha+\beta])} = \lim_{n\to\infty} \frac{[n\alpha+\beta+r\alpha]^p}{[n\alpha+\beta]^p} = 1,$$

so that (2.2.3) implies

$$c_1 + c_2 + \cdots + c_a = 1. \tag{2.2.4}$$

Moreover, (2.2.3) and (2.2.4) imply

$$\sum_{r=1}^{a} (g([n\alpha+\beta+r\alpha]) - g([n\alpha+\beta]))c_r = 0. \tag{2.2.5}$$

We have $[n\alpha+\beta+r\alpha] = [\{n\alpha+\beta\}+r\alpha] + [n\alpha+\beta]$, and so

$$g([n\alpha+\beta+r\alpha]) - g([n\alpha+\beta]) = \sum_{k=1}^{p} \frac{g^{(k)}([n\alpha+\beta])}{k!}[\{n\alpha+\beta\}+r\alpha]^k.$$

Therefore, after multiplying both sides of this last equality by c_r and summing from $r=1$ to $r=a$, for large n one obtains using (2.2.5),

$$\sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r + \sum_{r=1}^{a}\sum_{k=2}^{p} \frac{g^{(k)}([n\alpha+\beta])}{k!g'([n\alpha+\beta])}[\{n\alpha+\beta\}+r\alpha]^k c_r = 0. \tag{2.2.6}$$

For $p=1$ the last sum on the left of (2.2.6) is empty, and if $p \geqslant 2$, we have

$$\lim_{n\to\infty} \frac{g^{(k)}([n\alpha+\beta])}{g'([n\alpha+\beta]}[\{n\alpha+\beta\}+r\alpha]^k = 0 \qquad \text{for } 2 \leq k \leq p \text{ and } 1 \leq r \leq a.$$

So we have

$$\lim_{n\to\infty} \sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r = 0. \tag{2.2.7}$$

The numbers $r\alpha$ in (2.2.7) are not integers. Thus, according to the auxiliary result and (2.2.7) we can find integers m and n such that the expressions

$$\sum_{r=1}^{a}[\{m\alpha+\beta\}+r\alpha]c_r = \sum_{r=1}^{a}[r\alpha]c_r$$

and

$$\sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r=\sum_{r=1}^{a}(1+[r\alpha])c_r$$

differ from 0 as little as we please, which contradicts (2.2.4). In this way, it is shown that if α is irrational, $G(x)$ is not a rational function.

Now assume that α is rational. Set $\alpha = c/d$, where c and d are integers with $d > 0$. Applying the division algorithm, we have $n = md + r$ with $0 \leq r \leq d-1$, and so

$$n\alpha+\beta=\frac{nc}{d}+\beta=\frac{(md+r)c}{d}+\beta=mc+\frac{rc}{d}+\beta$$

so that $[n\alpha+\beta]=mc+[\frac{rc}{d}+\beta]$. Then

$$\begin{aligned}G(x)&=\sum_{n=0}^{\infty}g([n\alpha+\beta])x^n\\&=\sum_{r=0}^{d-1}\sum_{m=0}^{\infty}g\left(mc+\left[\frac{rc}{d}+\beta\right]\right)x^{md+r}\\&=\sum_{r=0}^{d-1}\sum_{m=0}^{\infty}\sum_{k=0}^{p}\frac{g^{(k)}\left(\left[\frac{rc}{d}+\beta\right]\right)}{k!}(mc)^k x^{md+r}\\&=\sum_{r=0}^{d-1}\sum_{k=0}^{p}\frac{g^{(k)}\left(\left[\frac{rc}{d}+\beta\right]\right)}{k!}c^k x^r\sum_{m=0}^{\infty}m^k x^{md}.\end{aligned}$$

Now

$$\sum_{m=0}^{\infty}m^k x^m=\left(x\frac{d}{dx}\right)^k(1-x)^{-1}$$

is rational, and so it is shown that $G(x)$ is rational. □

Remark 2.2.2. There is another result which is given by Meijer [10]. He proved that if $\alpha \in \mathbb{R}, k \in \mathbb{Z}^+$ and $g(x)$ is a polynomial over $\mathbb{C}$, then the series

$$\sum_{n=0}^{\infty}g([\alpha n^k])x^n$$

represents a rational function of x if and only if α is a rational number.

Theorem 2.2.3. *Let $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$. Let*

$$F(x) = \sum_{t=1}^{\infty} x^{[t\alpha+\beta]}.$$

Then $F(x)$ is a rational function if and only if α is rational.

Proof. Since $\alpha \in \mathbb{R}^+$, there is a positive integer t_0 such that $t\alpha+\beta \geq 0$ for all positive integers $t \geq t_0$. Thus

$$F(x) = \sum_{t=1}^{t_0-1} x^{[t\alpha+\beta]} + \sum_{t=t_0}^{\infty} x^{[t\alpha+\beta]}.$$

Now $F(x)$ is rational if and only if $\sum_{t=t_0}^{\infty} x^{[t\alpha+\beta]}$ is rational. Therefore, without loss of generality, we may assume that $t\alpha + \beta \geq 0$ for every positive integer t.

($\Rightarrow$) Suppose that α is irrational. Let $X(n)$ be the number of solutions of $n = [t\alpha+\beta]$ in positive integers t. Then $F(x) = \sum_{n=0}^{\infty} X(n)x^n$.

Case 1. $\forall t \in \mathbb{Z}^+,\ t\alpha + \beta \notin \mathbb{Z}$.

Let N be a nonnegative integer such that $N \geq \beta$.

Then for $n \geq N$, $X(n)$ is the number of integers t satisfying $n - \beta \leq t\alpha < n + 1 - \beta$, and since $\forall t \in \mathbb{Z}^+,\ t\alpha + \beta \notin \mathbb{Z}$, $X(n) = [\frac{n+1-\beta}{\alpha}] - [\frac{n-\beta}{\alpha}]$, and therefore

$$F(x) = \sum_{n=0}^{N-1} X(n)x^n + \sum_{n=N}^{\infty} \left(\left[\frac{n+1-\beta}{\alpha}\right] - \left[\frac{n-\beta}{\alpha}\right] \right) x^n.$$

Case 2. $\exists k \in \mathbb{Z}^+,\ k\alpha + \beta = l$ where $l \in \mathbb{Z}^+ \cup \{0\}$.

Then $\beta = l - k\alpha$. Thus, $\forall t \in \mathbb{Z}^+ \setminus \{k\}$, $t\alpha + \beta = t\alpha + l - k\alpha = (t-k)\alpha + l \notin \mathbb{Z}$. This implies that k is the only positive integer such that $k\alpha + \beta \in \mathbb{Z}$.

Now, let M be a positive integer such that $M > \max\{\beta, k\alpha + \beta\}$. Then for $n \geq M$, $X(n)$ is the number of integers t satisfying $n - \beta < t\alpha < n + 1 - \beta$; hence, $X(n) = [\frac{n+1-\beta}{\alpha}] - [\frac{n-\beta}{\alpha}]$, and therefore

$$F(x) = \sum_{n=0}^{M-1} X(n)x^n + \sum_{n=M}^{\infty} \left(\left[\frac{n+1-\beta}{\alpha}\right] - \left[\frac{n-\beta}{\alpha}\right] \right) x^n.$$

In any case

$$F(x)=\sum_{n=0}^{K-1}X(n)x^n+\sum_{n=K}^{\infty}\left(\left\lceil\frac{n+1-\beta}{\alpha}\right\rceil-\left\lceil\frac{n-\beta}{\alpha}\right\rceil\right)x^n \quad \text{for some } K\in\mathbb{Z}.$$

Note that

$$\sum_{n=K}^{\infty}\left(\left\lceil\frac{n+1-\beta}{\alpha}\right\rceil-\left\lceil\frac{n-\beta}{\alpha}\right\rceil\right)x^n=$$

$$\left(\frac{1-x}{x}\right)\cdot\left\{\sum_{n=K+1}^{\infty}\left\lceil n\left(\frac{1}{\alpha}\right)-\frac{\beta}{\alpha}\right\rceil x^n\right\}-\left\lceil\frac{K-\beta}{\alpha}\right\rceil x^K.$$

Now,

$$F(x)=$$

$$\sum_{n=0}^{K-1}X(n)x^n+\left(\frac{1-x}{x}\right)\cdot\left\{\sum_{n=0}^{\infty}\left\lceil n\left(\frac{1}{\alpha}\right)-\frac{\beta}{\alpha}\right\rceil x^n-\sum_{n=0}^{K}\left\lceil\frac{n-\beta}{\alpha}\right\rceil x^n\right\}-\left\lceil\frac{K-\beta}{\alpha}\right\rceil x^K.$$

According to theorem 2.2.1, $\sum_{n=0}^{\infty}\lceil n(\frac{1}{\alpha})-\frac{\beta}{\alpha}\rceil x^n$ is not a rational function, and hence $F(x)$ is not a rational function.

($\Leftarrow$) Suppose that α is rational. Write $\alpha=c/d$ with positive integers c and d. Then, using $t=md+r$ with $0\le r\le d-1$, we have

$$\begin{aligned}
x^{[\beta]}+F(x)&=\sum_{t=0}^{\infty}x^{[t\alpha+\beta]}\\
&=\sum_{r=0}^{d-1}\sum_{m=0}^{\infty}x^{[mc+rc/d+\beta]}\\
&=\sum_{r=0}^{d-1}\sum_{m=0}^{\infty}x^{mc+[rc/d+\beta]}\\
&=\sum_{r=0}^{d-1}x^{[rc/d+\beta]}\cdot\sum_{m=0}^{\infty}(x^c)^m\\
&=\sum_{r=0}^{d-1}x^{[rc/d+\beta]}\cdot(1-x^c)^{-1}\\
&=(1-x^c)^{-1}\cdot\sum_{r=0}^{d-1}x^{[rc/d+\beta]},
\end{aligned}$$

so that $F(x)$ is rational. $\square$

Next, we give and prove another result.

Theorem 2.2.4. *Let α, β be real numbers and f, g polynomials over $\mathbb{C}$ of positive degrees. Define*

$$G(x) = \sum_{n=0}^{\infty} \left(\frac{f}{g}\right)([n\alpha+\beta])x^n \qquad (g([n\alpha+\beta]) \neq 0 \ \textit{for all}\ n \in \mathbb{Z}^+ \cup \{0\}).$$

If $G(x)$ is a rational function, then α is a rational number.

Proof. Let α be irrational. If $G(x)$ were rational, then polynomials $A(x)$ and $B(x)$, of degrees $a \geq 1$ and b, respectively, would exist such that $G(x) = B(x)/A(x)$. Assume that $A(x) = x^a - c_1x^{a-1} - \ldots - c_{a-1}x - c_a$. From $A(x)G(x) = B(x)$ it follows, by equating corresponding coefficients of x^{n+a} where $n > a+b$, that

$$\left(\frac{f}{g}\right)([n\alpha+\beta]) = \sum_{r=1}^{a} \left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha])c_r. \tag{2.2.8}$$

Since f, g are polynomials of positive degrees ,

$$\lim_{n\to\infty} \frac{f([n\alpha+\beta+r\alpha])}{f([n\alpha+\beta])} = \lim_{n\to\infty} \frac{g([n\alpha+\beta])}{g([n\alpha+\beta+r\alpha])} = 1.$$

Hence

$$\lim_{n\to\infty} \frac{(\frac{f}{g})([n\alpha+\beta+r\alpha])}{(\frac{f}{g})([n\alpha+\beta])} = \lim_{n\to\infty} \frac{f([n\alpha+\beta+r\alpha])g([n\alpha+\beta])}{f([n\alpha+\beta])g([n\alpha+\beta+r\alpha])}$$

$$= 1 \quad \text{for each } r = 1, \ldots, a,$$

so that (2.2.8) implies

$$c_1 + c_2 + \ldots + c_a = 1. \tag{2.2.9}$$

Moreover, (2.2.8) and (2.2.9) implies

$$\sum_{r=1}^{a}\left(\left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha])-\left(\frac{f}{g}\right)([n\alpha+\beta])\right)c_r=0. \tag{2.2.10}$$

Note that $[n\alpha+\beta+r\alpha]=[\{n\alpha+\beta\}+r\alpha]+[n\alpha+\beta]$. By Taylor's Theorem, for each large integer n, there is a real number $c_{n,r}$ between $[n\alpha+\beta]$ and $[n\alpha+\beta+r\alpha]$ such that

$$\left(\frac{f}{g}\right)([n\alpha+\beta+r\alpha])-\left(\frac{f}{g}\right)([n\alpha+\beta])$$
$$=\frac{\left(\frac{f}{g}\right)'([n\alpha+\beta])}{1!}\cdot[\{n\alpha+\beta\}+r\alpha]+\frac{\left(\frac{f}{g}\right)''(c_{n,r})}{2!}\cdot[\{n\alpha+\beta\}+r\alpha]^2.$$

Therefore, after multiplying both sides of this last equality by c_r and summing from $r=1$ to a , for large n one obtains using (2.2.10),

$$0=\sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r+\sum_{r=1}^{a}\frac{\left(\frac{f}{g}\right)''(c_{n,r})}{2!\left(\frac{f}{g}\right)'([n\alpha+\beta])}\cdot[\{n\alpha+\beta\}+r\alpha]^2c_r. \tag{2.2.11}$$

Note that for each $r=1,\ldots,a$,

$$\lim_{n\to\infty}\frac{\left(\frac{f}{g}\right)''(c_{n,r})}{\left(\frac{f}{g}\right)'([n\alpha+\beta])}=0$$

since $c_{n,r}$ is between $[n\alpha+\beta]$ and $[n\alpha+\beta+r\alpha]$ (we see that $[n\alpha+\beta+r\alpha]-[n\alpha+\beta]=[\{n\alpha+\beta\}+r\alpha]$) and $\frac{\left(\frac{f}{g}\right)''(x)}{\left(\frac{f}{g}\right)'(x)}$ is in the form $\frac{p(x)}{q(x)}$ where $p(x)$ has degree $\leq 4j+l-1$ and $q(x)$ has degree $4j+l$ where j is the degree of g and l is degree of the numerator polynomial of $\left(\frac{f}{g}\right)'$. Now,

$$\lim_{n\to\infty}\frac{\left(\frac{f}{g}\right)''(c_{n,r})}{2!\left(\frac{f}{g}\right)'([n\alpha+\beta])}\cdot[\{n\alpha+\beta\}+r\alpha]^2=0 \quad \text{for} \quad r=1,2,\ldots,a\ .$$

Thus, by (2.2.11) we have

$$\lim_{n\to\infty}\sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r=0. \tag{2.2.12}$$

The numbers $r\alpha$ in (2.2.12) are not integers. Thus, according to (2.2.12) and the fact that the sequence $(n\alpha+\beta)_{n=1}^{\infty}$ is u.d.mod 1, we can find integers m and n such that the expressions

$$\sum_{r=1}^{a}[\{m\alpha+\beta\}+r\alpha]c_r=\sum_{r=1}^{a}[r\alpha]c_r \quad \text{and} \quad \sum_{r=1}^{a}[\{n\alpha+\beta\}+r\alpha]c_r=\sum_{r=1}^{a}(1+[r\alpha])c_r$$

differ from 0 as little as we please, which contradicts (2.2.9). In this way, it is shown that if α is irrational, $G(x)$ is not a rational function. $\square$

2.3 The Multidimensional Case

In this section, we discuss the concept of uniform distribution modulo 1 in multidimensional case. All of the following results can be found in [8] .

Definition 2.3.1. Let m be a positive integer. Let $(\mathrm{x}_n)_{n=1}^{\infty}=((x_1(n),x_2(n),\ldots,x_m(n)))_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^m$. The sequence $(\mathrm{x}_n)_{n=1}^{\infty}$ is said to be *uniformly distributed modulo 1* (abbreviated u.d.mod 1) in $\mathbb{R}^m$ if and only if $\forall[a_1,b_1)\subseteq[0,1)\forall[a_2,b_2)\subseteq[0,1)\ldots\forall[a_m,b_m)\subseteq[0,1)$,

$$\lim_{N\to\infty}\frac{1}{N}\cdot|\{n\leq N:\{x_i(n)\}\in[a_i,b_i)\text{ for all }i=1,2,\ldots,m\}|=\prod_{i=1}^{m}(b_i-a_i).$$

We also have the Weyl criterion in the multidimensional case.

Theorem 2.3.2 (Weyl Criterion). *A sequence* $(\boldsymbol{x}_n)_{n=1}^{\infty} = ((x_1(n), x_2(n), \ldots, x_m(n)))_{n=1}^{\infty}$ *is u.d.mod 1 in* $\mathbb{R}^m$ *if and only if for every* $(h_1, \ldots, h_m) \in \mathbb{Z}^m$, $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i(h_1 x_1(n)+\ldots+h_m x_m(n))} = 0.$$

Proof. See Theorem 6.2 of Chapter 1 in [8]. □

Corollary 2.3.3. *A sequence* $(\boldsymbol{x}_n)_{n=1}^{\infty} = ((x_1(n), x_2(n), \ldots, x_m(n)))_{n=1}^{\infty}$ *is u.d.mod 1 in* $\mathbb{R}^m$ *if and only if for every* $(h_1, \ldots, h_m) \in \mathbb{Z}^m$, $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$, *the sequence of real number* $(h_1 x_1(n) + \ldots + h_m x_m(n))_{n=1}^{\infty}$ *is u.d.mod 1.*

Proof. See Theorem 6.3 of Chapter 1 in [8]. □

Theorem 2.3.4. *Let* $1, \theta_1, \ldots, \theta_m$ *are linearly independent over the rational numbers, then the sequence* $((n\theta_1, n\theta_2, \ldots, n\theta_m))_{n=1}^{\infty}$ *is u.d.mod 1 in* $\mathbb{R}^m$.

Proof. See Example 6.1 of Chapter 1 in [8]. □

Theorem 2.3.5. *Let* $\mathbf{p}(x) = (p_1(x), \ldots, p_m(x))$, *where all* $p_i(x)$ *are real polynomials, and suppose* $\mathbf{p}(x)$ *has the property that for each* $(h_1, h_2, \ldots, h_m) \in \mathbb{Z}^m$, $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$, *the polynomial* $h_1 p_1(x) + h_2 p_2(x) + \ldots + h_m p_m(x)$ *has at least one nonconstant term with irrational coefficient. Then the sequence* $(\mathbf{p}(x))_{n=1}^{\infty} = ((p_1(x), \ldots, p_m(x)))_{n=1}^{\infty}$ *is u.d.mod 1 in* $\mathbb{R}^m$.

Proof. See Theorem 6.4 of Chapter 1 in [8]. □

2.4 Uniform Distribution of Integers

In this section, we introduce the concept of uniform distribution of integers.

Definition 2.4.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of rational integers and m a positive integer≥ 2 . The sequence $(a_n)_{n=1}^{\infty}$ is said to be *uniformly distributed modulo m* (u.d.mod m) if and only if for each $j = 0, 1, 2, \ldots, m-1$,

$$\lim_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : a_n \equiv j (mod\ m)\}| = \frac{1}{m} \quad ,$$

and $(a_n)_{n=1}^{\infty}$ is said to be *uniformly distributed* in $\mathbb{Z}$ (u.d. in $\mathbb{Z}$) if $(a_n)_{n=1}^{\infty}$ is u.d.mod m for every integer $m \geq 2$.

Example 2.4.2. Let m be a positive integer greater than 1. The sequence $(x_n)_{n=1}^{\infty} = 0, 1, \ldots, m-1, 0, 1, \ldots, m-1, \ldots$ is u.d.mod m. To see this, let $j \in \{0, 1, \ldots, m-1\}$. Let N be sufficiently large integer. Write $N = am+b$ where $a \in \mathbb{Z}^+$ and $0 \leq b \leq m-1$. Then

$$\frac{N-b}{mN} = \frac{a}{N} \leq \frac{1}{N} \cdot |\{n \leq N : x_n \equiv j (mod\ m)\}| \leq \frac{a+1}{N} = \frac{N-b+m}{mN} \ .$$

Therefore

$$\begin{aligned}
\frac{1}{m} &= \lim_{N\to\infty} \frac{N-b}{mN} \\
&\leq \liminf_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : x_n \equiv j (mod\ m)\}| \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : x_n \equiv j (mod\ m)\}| \\
&\leq \lim_{N\to\infty} \frac{N-b+m}{mN} \\
&= \frac{1}{m} \ .
\end{aligned}$$

Then

$$\lim_{N\to\infty} \frac{1}{N} \cdot |\{n \leq N : x_n \equiv j (mod\ m)\}| = \frac{1}{m} \ .$$

Hence, $(x_n)_{n=1}^{\infty}$ is u.d. mod m.

Moreover, the sequence $(y_n)_{n=1}^{\infty} = 0, 1, 2, 3, 4, \ldots$ is u.d. in $\mathbb{Z}$ since for each positive integer $m > 1$, $x_n \equiv y_n (mod\ m)$ for every positive integer n.

The following theorem is a Weyl Criterion for u.d.mod m. This Theorem was first proved by Uchiyama [18].

Theorem 2.4.3. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of integers. A necessary and suffiecient condition that $(a_n)_{n=1}^{\infty}$ be u.d.mod m is that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h a_n/m} = 0 \quad \textit{for all } h = 1, 2, \ldots, m-1.$$

Proof. See Theorem 1.2 of Chapter 5 in [8]. □

Corollary 2.4.4. *A necessary and sufficient condition that $(a_n)_{n=1}^{\infty}$ be u.d.mod $\mathbb{Z}$ is that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i t a_n} = 0 \quad \textit{for all rational numbers } t \in \mathbb{Z}.$$

Proof. See Corollary 1.1 of Chapter 5 in [8]. □

Theorem 2.4.5. *If a sequence of integers is u.d.mod m and if $k|m$ and , $k \geq 2$, then the sequence is also u.d.mod k.*

Proof. See Exercise 1.1 of Chapter 5 in [8]. □

Theorem 2.4.6. *Let* $(x_n)_{n=1}^{\infty}$ *be a sequence of real numbers such that the sequence* $(x_n/m)_{n=1}^{\infty}$ *is u.d.mod 1 for all integers* $m \geq 2$. *Then the sequence* $([x_n])_{n=1}^{\infty}$ *of integral parts is u.d. in* $\mathbb{Z}$.

Proof. See Theorem 1.4 of Chapter 5 in [8]. □

Theorem 2.4.7. *Let* $f(x) = \alpha_k x^k + \alpha_{k-1}x^{k-1} + \ldots + \alpha_1 x + \alpha_0$ *be a polynomial over* $\mathbb{R}$ *with at least one of the coefficients* α_i, $i \geq 1$, *being irrational. Then the sequence* $([f(n)])_{n=1}^{\infty}$ *is u.d. in* $\mathbb{Z}$.

Proof. See Example 1.1 of Chapter 5 in [8]. □

We end this section by presenting the close relation between u.d.mod 1 and u.d. of integers.

Theorem 2.4.8. *The sequence* $(x_n)_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is u.d.mod 1 if and only if the sequence* $([mx_n])_{n=1}^{\infty}$ *is u.d.mod* m *for all integers* $m \geq 2$.

Proof. See Theorem 1.6 of Chapter 5 in [8]. □