CHAPTER III

THEORY

3.1 Plantwide Control

3.1.1 Introduction

A chemical plant may have thousands of measurements and control loops. By the term *plantwide control* it is *not* meant the tuning and behavior of each of these loops, but rather the *control philosophy* of the overall plant with emphasis on the *structural decisions*. The structural decision include the selection/ placement of manipulators and measurements as well the *decomposition* of the overall problem into smaller subproblems (the control configuration).

Thus, a very important (if not the most important) problem in plantwide control is the issue of determining the *control structure*:

- Which "boxes" (controllers; usually consisting of a data handling and/or decision making part) should we have and what information should be send between them?

Note that that we are here not interested in what should be inside the boxes (which is the controller design or tuning problem). More precisely, **control structure design** is defined as the *structural decisions* involved in control system design, including the following tasks

1. *Selection of controlled outputs* c (variables with setpoints)
2. *Selection of manipulated inputs* m
3. *Selection of measurements* v (for control purposes including stabilization)
4. *Selection of control* **configuration** (a structure interconnecting measurements /setpoints and manipulated variables, i.e. the structure of the

controller K which interconnects the variables c_s and v (controller inputs) with the variables m)

5. *Selection of controller type* (control law specification, e.g., PID, decoupler, LQG, etc.).

In most cases the control structure is solved by a mixture between a top-down consideration of control objectives and which degrees of freedom are available to meet these (tasks 1 and 2), combined with a bottom-up design of the control system, starting with the stabilization of the process (tasks 3,4 and 5). In most cases the problem is solved without the use of any theoretical tools. In fact, the industrial approach to plantwide control is still very much along the lines described by Page Buckley in his book from 1964.

Of course, the control field has made many advances over these years, for example, in methods for and applications of on-line optimization and predictive control. Advances has also been made in control theory and in the formulation of tools for analyzing the controllability of a plant. These latter tools can be most helpful in screening alternative control structures. However, a systematic method for generating promising alternative structures has been lacking. This is related to the fact the plantwide control problem itself has not been well understood or even defined.

The control structure design problem is difficult to define mathematically, both because of the size of the problem, and the large cost involved in making a precise problem definition, which would include, for example, a detailed dynamic and steady-state model. An alternative to this is to develop heuristic rules based on experience and process understanding. This is what will be referred to as the *process oriented approach.*

The realization that the field of control structure design is underdeveloped is not new. In the 1970's several "critique" articles where written on the gap between theory and practice in the area of process control. The most famous is the one of (Foss 1973) who made the observation that in many areas application was *ahead* of theory, and he stated that

> The central issue to be resolved by the new theories is the determination of the control system structure. Which variables should be measured, which inputs should be manipulated and which links should be made between the two sets. ... The gap is present indeed, but contrary to the views of many, it is the theoretician who must close it.

Many authors point out that the need for a plant-wide perspective on control is mainly due to changes in the way plants are designed – with more heat integration and recycle and less inventory. Indeed, these factors lead to more interactions and therefore the need for a perspective beyond individual units. However, we would like to point out that even without any integration there is still a need for a plant-wide perspective as a chemical plant consists of a string of units connected in series, and one unit will act as a disturbance to the next, for example, all units must have the same through-put at steady-state.

3.1.2 General Reviews and Books on Plantwide Control

Morari (1982) presents a well-written review on plantwide control, where he discusses why modern control techniques were not (at that time) in widespread use in the process industry. The four main reasons were believed to be

1. Large scale system aspects.
2. Sensitivity (robustness).
3. Fundamental limitations to control quality.
4. Education.

He then proceeds to look at how two ways of decompose the problem:

1. Multi-layer (vertical), where the difference between the layers are in the frequency of adjustment of the input.
2. Horizontal decomposition, where the system is divided into noninteracting parts.

Stephanopoulos (1982) states that the synthesis of a control system for a chemical plant is still to a large extent an art. He asks: "Which variables should be measured in order to monitor completely the operation of a plant? Which input should be manipulated for effective control? How should measurements be paired with the manipulations to form the control structure, and finally, what the control laws are?"

He notes that the problem of plantwide control is "multi-objective" and "There is a need for a systematic and organized approach which will identify all necessary control objectives". The article is comprehensive,and discusses many of the problems in the synthesis of control systems for chemical plants.

Rinard and Downs (1992) review much of the relevant work in the area of plantwide control, and they also refer to important papers that we have not referenced. They conclude the review by stating that "the problem probably never will be solved in the sense that a set of algorithms will lead to the complete design of a plantwide control system". They suggests that more work should be done on the following items: (1) A way of answering whether or not the control system will meet all the objectives, (2) Sensor selection and location (where they indicate that theory on partial control may be useful), (3) Processes with recycle. The also welcome computer-aided tools, better education and good new test problems.

The book by Balchen and Mumme (1988) attempts to combine process and control knowledge, and to use this to design control systems for some common unit operations and also consider plantwide control. The book provides many practical examples, but there is little in terms of analysis tools or a systematic framework.

The book "Integrated process control and automation" by Rijnsdorp (1991), contains several subjects that are relevant here. Part II in the book is on optimal operation. He distinguishes between two situations, sellers marked (maximize production) and buyers marked (produce a given amount at lowest possible cost). He also have a procedure for design of a optimizing control system.

Loe (1994) presents a systematic way of looking at plants with the focus is on functions. The author covers "qualitative" dynamics and control of important unit operations.

Van deWal and de Jager (1995) lists several criteria for evaluation of control structure design methods: generality, applicable to nonlinear control systems, controller-independent, direct, quantitative, efficient, effective, simple and

theoretically well developed. After reviewing they concludes that such a method does not exist.

The book by Skogestad and Postlethwaite (1996) has two chapters on controllability analysis and one chapter on control structure design. Many of the ideas presented in this paper are based on this work.

The coming monograph by Ng and Stephanopoulos (1998*a*) deals almost exclusively with plantwide control.

There also exists a large body of system-theoretic literature within the field of large scale systems, but most of it has little relevance to plantwide control. One important exception is the book by Findeisen et al. (1980) on "Control and coordination in hierarchical systems" which probably deserves to be studied more carefully by the process control community.

3.1.3 Decomposition of The Problem

The task of designing a control system for complete plants is a large and difficult task. Therefore most methods will try to decompose the problem into manageable parts. Four common ways of decomposing the problem are

1. Decomposition based on process units
2. Decomposition based on process structure
3. Decomposition based on control objectives (material balance, energy balance, quality, etc.)
4. Decomposition based on time scale

The first is a horizontal (decentralized) decomposition whereas the three latter latter three provide hierarchical decompositions. Most practical approaches contain elements from several categories.

Many of the methods described below perform the optimization at the end of the procedure after checking if there degrees of freedom left. However, as discussed above, it should be possible to identify the steady-state degrees of freedom initially,

and make a preliminary choice on controlled outputs (cc's) before getting into the detailed design.

It is also interesting to see how the methods differ in terms of how important inventory (level) control is considered. Some regard inventory control as the most important (as is probably correct when viewed purely from a control point of view) whereas Ponton (1994) states that "inventory should normally be regarded as the least important of all variables to be regulated" (which is correct when viewed from a design point of view). We feel that there is a need to integrate the viewpoints of the control and design people.

3.1.3.1 The Unit Based Approach

The unit-based approach, suggested by Umeda *et al.* (1978), proposes to

1. Decompose the plant into individual unit of operations
2. Generate the best control structure for each unit
3. Combine all these structures to form a complete one for the entire plant.
4. Eliminate conflicts among the individual control structures through mutual adjustments.

This approach has always been widely used in industry, and has its main advantage that many effective control schemes have been established over the years for individual units (e.g. Shinskey (1988)). However, with an increasing use of material recycle, heat integration and the desire to reduce buffer volumes between units, this approach may result in too many conflicts and become impractical.

As a result, one has to shift to plant-wide methods, where a hierarchical decomposition is used. The first such approach was Buckley's (1964) division of the control system into material balance control and product quality control, and three plantwide approaches partly based on his ideas are described in the following.

3.1.3.2 Hierarchical Decomposition Based on Process Structure

The hierarchy given in Douglas (1988) for process design starts at a crude representation and gets more detailed:

Level 1 Batch vs continuous

Level 2 Input-output structure

Level 3 Recycle structure

Level 4 General structure of separation system

Level 5 Energy interaction

Fisher *et al.* (1988) propose to use this hierarchy when performing controllability analysis, and Ponton and Liang (1993) point out that this hierarchy, (e.g. level 2 to level 5) could also be used foı control system design. This framework enables parallel development for the process and the control system. Within each of the levels above any design method might be applied.

Douglas (1988) present a different hierarchy for control system design. In this hierarchy the view point is not one the flowsheet but on steady-state, normal dynamic response and abnormal dynamic operation.

Ng and Stephanopoulos (1998*b*) propose to use a similar hierarchy for control structure design. The difference between Douglas (1988) and Ng and Stephanopoulos (1998*b*)'s hierarchy is that level 1 is replaced by a preliminary analysis and level 4 and on is replaced by more and more detailed structures. At each step the objectives identified at an earlier step is translated to this level and new objectives are identified. The focus is on construction of mass and energy balance control. The method is applied to the Tennessee Eastman case.

All these methods have in common that at each level a key point is to check if there are enough manipulative variables to meet the constraints and to optimize operation. The methods are easy to follow and gives a good process understanding, and the concept of a hierarchical view is possible to combine with almost any design method.

3.1.3.3 Hierarchical Decomposition Based on Control Objectives

The hierarchy based on control objectives is sometimes called the tiered procedure. This bottom-up procedure focuses on the tasks that the controller has to perform. Normally one starts by stabilizing the plant, which mainly involves placing inventory (mass and energy) controllers.

Price *et al.* (1993) build on the ideas that was introduced by Buckley (1964) and they introduce a tiered framework. The framework is divided into four different tasks:

I Inventory and production rate control.

II Product specification control

III Equipment & operating constraints

IV Economic performance enhancement.

Their paper does not discuss points III or IV. They perform a large number (318) of simulations with different control structures, controllers (P or PI), and tunings on a simple process consisting of a reactor, separator and recycle of unreacted reactant. The configurations are ranked based on integrated absolute error of the product composition for steps in the disturbance. From these simulation they propose some guidelines for selecting the through-put manipulator and inventory controls. (1) Prefer internal flows as through-put manipulator. (2) the through-put manipulator and inventory controls should be self-consistent(self-consistency is fulfilled when a change in the through-put propagates through the process by "itself" and does not depend on composition controllers). They apply their ideas on the Tennessee Eastman problem (Price *et al.* 1994).

Ricker (1996) comments upon the work of Price *et al.* (1994). Ricker points out that plants are often run at full capacity, corresponding to constraints in one or several variables. If a manipulated variable that is used for level control saturates, one looses a degree of freedom for maximum production. This should be considered when choosing a through-put manipulator.

Luyben *et al.* (1997) point out three limitations of the approach of Buckley. First, he did not explicitly discuss energy management. Second, he did not look at recycle. Third, he placed emphasis on inventory control before quality control. Their plantwide control design procedure is listed below:

1. Establish control objectives.
2. Determine the control degrees of freedom by counting the number of independent valves.
3. Establish energy inventory control, for removing the heats of reactions and to prevent propagation of thermal disturbances.
4. Set production rate. The production rate can only be increased by increasing the reaction rate in the reactor. One recommendation is to use the input to the separation section.
5. Product quality and safety control. Here they recommend the usual "control close"-rule.
6. Inventory control. Fix a flow in all liquid recycle loops. They state that all liquid levels and gas pressures should be controlled.
7. Check component balances. (After this point it might bee necessary to go back to item 4.)
8. Unit operations control.
9. Optimize economics or improve dynamic controllability.

Step 3 comes before determining the throughput manipulator, since the reactor is typically the heart of the process and the methods for heat removal are intrinsically part of the reactor design. In order to avoid recycling of disturbances they suggest to set a flowrate in all recycles loops; this is discussed more in section 6. They suggest in step 6 to control all inventories, but this may not be necessary in all cases; e.g. it may be optimal to let the pressure float (Shinskey 1988). They apply their procedure on several test problems; the vinyl acetate monomer process, the Tennessee Eastman process, and the HDA process.

3.1.3.4 Hierarchical Decomposition Based on Time Scales

Buckley (1964) proposed to design the quality control system as high-pass filters for disturbances and to design the mass balance control system will as low pass filters. If the

resonance frequency of the quality control system is designed to be an order of magnitude higher than the break frequency of the mass balance system then the two loops will be non-interacting.

McAvoy and Ye (1994) divide their method into four stages:

1. Design of inner cascade loops.
2. Design of basic decentralized loops, except those associated with quality and production rate.
3. Production rate and quality controls.
4. Higher layer controls.

The decomposition in stages 1-3 is based on the speed of the loops. In stage 1 the idea is to locally reduce the effect of disturbances. In stage 2 there generally are a large number of alternatives configurations. These may be screened using simple controllability tools, such as the RGA. One problem of selecting outputs based on a controllability analysis is that one may end up with the outputs that are easy to control, rather than the ones that are important to control. The method is applied to the Tennessee Eastman test problem.

3.2 Interaction Analysis and Multiple Single Loop Designs

This section is concerned with discussing how interaction analysis is carried out and how the results of such analysis are used for multiple single-loop controller designs. The mail questions regarding *when* to use this control strategy and *how* to design the multiple single-loop controllers will be answered in this section.

3.2.1 Preliminary Considerations of Interaction Analysis and Loop Pairing

3.2.1.1 A Measure of Control Loop Interactions

Consider an example 2 x 2 system (see Figure 3.1) with two output variables, y_1 and y_2, and two input variables that we will now refer to as m_1 and m_2 because we do not yet know which one will be paired with which output variable.

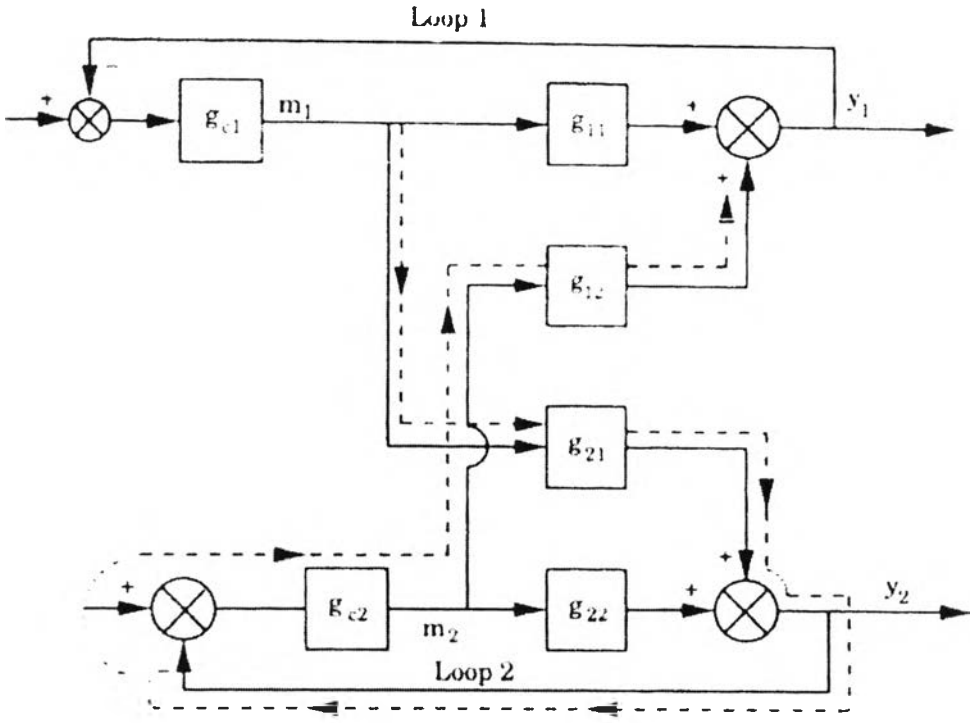

Figure 3.1 Loop interactions for a 2 x 2 system.

Note that each loop can be opened or closed. When both loops are open, then m_1 and m_2 can be manipulated independently and the effect of each of these inputs on each of the outputs is represented by the following transfer function model:

$$y_1(s) = g_{11}(s)m_1(s) + g_{12}(s)m_2(s) \qquad (3.1)$$

$$y_2(s) = g_{21}(s)m_1(s) + g_{22}(s)m_2(s) \qquad (3.2)$$

where each transfer function element consists of an unspecified dynamic portion, and a steady-state gain term K_{ij}.

Let us now consider m_1 as a candidate input variable to pair with y_1. In order to evaluate this choice against the alternative of using m_2 instead, we will perform two "experiments" on the system.

- **Experiment 1:** ***Unit step change in m_1 with all loops open.***

When a unit step change is made in the input variable m_1, with all the loops open, the output variable y_1 will change, and so will y_2, but for now we are primarily interested in the response in y_1.

After steady state has been achieved, let the change observed in y_1 as a result of this change in m_1 be referred to as Δy_{1m} this represents the *main effect* of m_1 on y_1.Observe from Eq. (3.1) that:

$$\Delta y_{1m} = K_{11} \tag{3.3}$$

Keep in mind that no other input variable apart from m_1 has been changed, and all the control loops are opened so that there is no feedback control involved.

- **Experiment 2:** ***Unit step change in m_1 with loops 2 closed.***

In this "experiment," the same unit step change implemented in m_1 in the first "experiment" will be implemented; however, this time, Loop 2 is closed, and the controller g_{c2} is charged with the responsibility of using the other input variable m_2 to ward off any upset in y_2 occurring as a result of this step change in m_1. Note that Figure 3.1 shows that m_1 has both a direct influence on y_1 and an indirect influence with the path given by the dotted line. In particular the following things happen to the process due to the change in m_1:

1. Obviously y_1 changes because of g_{11}, but because of interactions via the g_{21} element, so does y_2.
2. Under feedback control, Loop 2 wards off this interaction effect on y_2 by manipulating m_2 until y_2 is restored to its initial state before the occurrence of the "disturbance."
3. The changes in m_2 now return to affect y_1 via the g_{12} transfer function element.

Thus, the changes observed in y_1 are from two different sources: (1) the direct influence of m_1 on y_1 (which we already know from "experiment" 1 to be Δy_{1m}), and (2) the additional, *indirect* influence, precipitated by the retaliatory action from the Loop 2 controller in warding off the interaction effect of m_1 on y_2, say, Δy_{1r}. Note that this quantity, Δy_{1r} is indicative of how much interaction m_1 is able to provoke from the other control loop in its attempt to control y_2.

After the dynamic transients die away and steady state has been achieved, there will be a net change observed in y_1, Δy_1^* given by:

$$\Delta y_1^* = \Delta y_{1m} + \Delta y_{1r}$$

a sum of the *main effect* of m_1 on y_1 and that of the interactive effect provoked by m_1 interacting with the other loop.

As we will show below, the quantity Δy_1^* will be given by, say:

$$\Delta y_1^* = K_{11}\left(1 - \frac{K_{12}K_{21}}{K_{11}K_{22}}\right) = K_{11}^* \tag{3.4}$$

Observe now that a good measure of how well the process can be controlled if m_1 is used to control y_1 is:

$$\lambda_{11} = \frac{\Delta y_{1m}}{\Delta y_1^*}$$

or

$$\lambda_{11} = \frac{\Delta y_{1m}}{\Delta y_{1m} + \Delta y_{1r}} \tag{3.5}$$

This quantity is a measure of the main effect of m_1 on y_1 compared to the total effect including the effect it provokes from the other controller, since it cannot control y_1 without upsetting y_2.Thus the quantity, λ_{11}, provides a measure of the extent of interaction in using m_1 to control y_1 while using m_2 to control y_2.

Observe that we can now perform a similar set of "experiments" that, this time, investigate the candidacy of m_2 as the input variable to use in controlling y_1. By properly interpreting the measures calculated from Eq. (3.5), we can quantify the degree of steady-state interaction involved with each control configuration, and thus determine which configuration minimizes steady-state interaction.

3.2.1.2 Loop Pairing on The Basis of Interaction Analysis

Let us return to the results of the "experiments" performed on the example 2 x 2 system of the previous subsection and now interpret the quantity λ_{11} as a measure of loop interaction. We will do this by evaluating the consequences of having various value λ_{11},

1. The case when $\lambda_{11} = 0$

This can happen only if Δy_{1r} is zero; in physical terms, it means that the main effect of m_1 on y_1, measured when all the loops are opened, and the total effect, measured when the other loop is closed, *are identical.*

This will be the case only if:

- m_1 does not affect y_2, in which case there is no retaliatory control action from m_2, or
- m_1 does affect y_2, but the retaliatory control action from m_2 does not cause any changes in y_1 because m_2 does not affect y_1.

Under such circumstances, observe that m_1 is the perfect input variable to use in controlling y_1, because we will then have *no* interaction problems.

2. The case when $\lambda_{11} = 1$

This indicates a condition in which m_1 has no effect on y_1, for only then will Δy_{1m} be zero in response to a change in m_1. Observe that under these circumstances, m_1 is the perfect input variable for controlling *not* y_1 but y_2: since m_1 does not affect y_1, we can therefore control y_2 with m_1 without interacting with y_1 at all.

3. The case when $0 < \lambda_{11} < 1$

This corresponds to the condition in which the direction of the interaction effect is the same as that of the main effect. In this case the total effect. In this case the total effect is greater than the main effect. For $\lambda_{11} > 0.5$, the main effect contributes more to the total effect than the interaction effect, and as the main effect contribution increases, λ_{11} becomes closer to 1. For $\lambda_{11} < 0.5$, the contribution from

the interaction effect dominates, and as this contribution increases, λ_{11} moves closer to zero. For $\lambda_{11} = 0.5$, the contributions from the main effect and the interactive effect are exactly equal.

4. The case when $\lambda_{11} > 1$

This corresponds to the condition where Δy_{1r} (the interaction effect) is opposite in sign to Δy_{1m} (the main effect), but smaller in absolute value. In this case the total effect, Δy_1^*, is less than the main effect, Δy_{1m}, and thus a larger controller action m_1 is required to achieve a given change in y_1 in the closed loop than in the open loop. Obviously for λ_{11} very large and positive, the interactive effect almost cancels the main effect and closed-loop control of y_1 by m_1 will be very difficult to achieve.

5. The case when $\lambda_{11} < 0$

This case arises when Δy_{1r} (the interaction effect) is not only opposite in sign to Δy_{1m} (the main effect), but is larger in absolute value. The pairing of m_1 with y_1 in this case is not very desirable because the direction of the effect of m_1 on y_1 in the open loop is opposite to the direction in the closed loop. The consequences of using such a pairing could be catastrophic as we shall demonstrate below.

With these preliminary ideas in mind, it is possible to generalize the interaction analysis to multivariable systems of arbitrary dimension.

3.2.2 The Relative Gain Array (RGA)

The quantity λ_{11} introduced in the last section is known as the *relative gain* between output y_1 and input m_1, and as we have seen, it provides a measure of the extent of the influence of process interactions when m_1 is used to control y_1. Even though we introduced this quantity in reference to a 2 x 2 system, it can be generalized to any other multivariable system of arbitrary dimension.

Let us define λ_{ij}, the relative gain between output variable y_i and input variable m_j, as the ratio of two steady-state gains:

$$\lambda_{ij} = \frac{\left(\frac{\partial y_i}{\partial m_j}\right)_{\text{all loops open}}}{\left(\frac{\partial y_i}{\partial m_j}\right)_{\text{all loops closed except for the } m_j \text{ loop}}} \tag{3.6}$$

$$= \left(\frac{\text{open - loop gain}}{\text{closed - loop gain}}\right) \text{ for loop } i \text{ under the control of } m_j$$

When the relative gain is calculated for all the input/output combinations of a multivariable system, and the results are presented in an array of the form shown below:

$$\Lambda = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} \tag{3.7}$$

the result is what is known as the *relative gain array* (RGA) or the *Bristol array*. The RGA was first introduced by Bristol [1] and has become the most widely used measure of interaction.

3.2.2.1 Properties of the RGA

The following are some of the most important properties of the RGA; they provide the bases for its utility in studying and quantifying interactions among the control loops of a multivariable system.

1. The elements of the RGA across any row, or down any column, sum up to 1, i.e.:

$$\sum_{i=1}^{n} \lambda_{ij} = \sum_{j=1}^{n} \lambda_{ij} = 1 \tag{3.8}$$

2. λ_{ij} is dimensionless; therefore, neither the units, nor the absolute values actually taken by the variables m_j, or y_i, affect it.
3. The value of λ_{ij} is a measure of the *steady-state* interaction expected in the *i*th loop of the multivariable system if its output y_i is paired with m_j; in particular, $\lambda_{ij} = 1$ implies that m_j affects y_i without interacting with, and/or eliciting interaction from, the other control loops. By the same token, $\lambda_{ij} = 0$ implies that m_j has absolutely no effect on y_i.
4. Let K_{ij}* represent the loop *I* steady-state gain when all the other loops – excepting this one – are closed, (i.e., a generalization of Eq. (3.4) for the example 2 x 2 system), whereas K_{ij} represents the normal, *open-loop* gain. By the definition of λ_{ij}, observe that:

$$K_{ij}^* = \frac{1}{\lambda_{ij}} K_{ij} \tag{3.9}$$

With the following, very important implication: $1/\lambda_{ij}$ tells us by what factor the *open-loop gain* between output y_i and input m_j will be altered when the other loops are closed.

5. When λ_{ij} is negative, it is indicative of a situation in which loop *i*, with all loops open, will produce a change in y_i in response to a change in m_j totally opposite in direction to that when the other loops are closed. Such input/output pairings are potentially unstable and should be avoided.

Before discussing the general principles of how the RGA is actually used for loop pairing, it seems appropriate first to discuss the methods by which it is computed.

3.2.2.2 Interpreting The RGA Elements

It is important to know how to interpret the RGA elements properly before we can use the RGA for loop pairing.

For the purpose of properly interpreting the RGA element λ_{ij}, it is convenient to classify all the possible values it can take into five categories, and in each case,

investigate the implications for loop interaction, and what such implications suggest regarding input/output pairing.

1. $\lambda_{ij} = 1$, indicating that the open-loop gain between y_i and m_j is *identical* to the closed-loop gain:

- *Implication for loop interactions*: loop i will *not* be subject to retaliatory action from other control loops when they are closed, therefore m_j can control y_i without interference from the other control loops. If, however, any of the g_{ik} elements in the transfer function matrix are nonzero, then the ith loop will experience some disturbance from control actions taken in the other loops, but these are not provoked by control actions taken in the ith loop.
- *Pairing recommendation*: pairing y_i with m_j will therefore be ideal.

2. $\lambda_{ij} = 0$, indicating that the open-loop gain between y_i and m_j is *zero*:

- *Implication for loop interactions*: m_j has no direct influence on y_i (even though it may affect other output variables).
- *Pairing recommendation*: do *not* pair y_i with m_j; pairing m_j with some other output will however be advantageous, since we are sure that at least y_i will be immune to interaction from this loop.

3. $0 < \lambda_{ij} < 1$, indicating that the open-loop gain between y_i and m_j is *smaller* than the closed-loop gain:

- *Implication for loop interactions*: since the closed-loop gain is the sum of the open-loop gain *and* the retaliatory effect from the other loops:
  a) The loops are definitely interacting, but
  b) They do so in such a way that the retaliatory effect from the other loops is in the save direction as the main effect of m_j on y_i.

Thus the loop interactions "assist" m_j in controlling y_i. The exten of the "assistance" from other loops is indicated by how close λ_{ij} is to 0.5.

When $\lambda_{ij} = 0.5$, the main effect of m_j on y_i is exactly identical to the retaliatory (but complementary) effect it provokes from other loops; when $\lambda_{ij} > 0.5$ (but still less than 1) this retaliatory effect from the other interacting loop

is lower than the main effect of m_j on y_i; if, however, $0 < \lambda_{ij} < 0.5$, then the retaliatory effect is actually more substantial than the main effect.

- *Pairing recommendation*: If possible, avoid pairing y_i with m_j whenever $\lambda_{ij} \leq 0.5$.

4. $\lambda_{ij} > 1$, indicating that the open-loop gain between y_i and m_j is *larger* than the closed-loop gain:

- *Implication for loop interactions*: the loops interact, and the retaliatory effect from the other loops acts in *opposition* to the main effect of m_j on y_i (thus reducing the loop gain when the other loops are closed); but the main effect is still dominant, otherwise λ_{ij} will be negative. For large λ_{ij} values, the controller gain for loop i will have to be chosen much larger than when all the other loops are open. This could cause loop i to become unstable when the other loops are open.
- *Pairing recommendation*: The higher the value of λ_{ij} the greater the opposition m_j experiences from the other control loops in trying to control y_{ij}; therefore, where possible, do not pair m_j with y_i if λ_{ij} takes a very high value.

5. $\lambda_{ij} < 0$, indicating that the open-loop and closed-loop gains between y_i and m_j have opposite signs:

- *Implication for loop interactions*: the loops interact, and the retaliatory effect from the other loops is not only in *opposition* to the main effect of m_j on y_i, it is also the more dominant of the two effects. This is a potentially dangerous situation because opening the other loops will likely cause loop i to become unstable.
- *Pairing recommendation*: because the retaliatory effect that m_j provokes from the other loops acts in opposition to, and in fact dominates, its main effect on y_i, avoid pairing m_j with y_i.

3.2.3 Singular Value Analysis

The previous section contained a number of quantitative guideline for selecting controlled and manipulated variables. In this section we consider a powerful

analytical technique, singular value analysis (SVA), which can be used to solve the following important control problems:

1. Selection of controlled and manipulated variables.
2. Evaluating of the robustness of a proposed control strategy.
3. Determination of the best multiloop control configuration.

Singular value analysis and its extensions, including singular value decomposition (SVD), also have many uses in numerical analysis and the design of multivariable control systems which are beyond the scope of this book In this section we provide a brief introduction to SVA that is based on an analysis of steady-state process models.

Consider a process that has n controlled variables and n manipulated variables. We assume that a steady-state process model is available and that it has been linearized to give

$$\boldsymbol{Y} = \boldsymbol{K}\,\boldsymbol{M} \qquad (3.10)$$

Where $\boldsymbol{Y}$ is the vector of n controlled variables, $\boldsymbol{M}$ is the vector of n manipulated variables, and $\boldsymbol{K}$ is the steady-state gain matrix. The elements of $\boldsymbol{Y}$ and $\boldsymbol{M}$ are expressed as deviation variables. One desirable property of $\mathbf{K}$ is that the n linear equation in n unknowns represented by (3.10) be linearly independent. In contrast, if the equations are dependent, then not all of the n controlled variables can be independently regulated. This characteristic property of linear independence can be checked by several methods . For example, if the determinant of $\boldsymbol{K}$ is zero, the matrix is singular and the n equation is (3.10) are not linearly independent.

Another way to check for linear for linear independence is to calculate one of the most important properties of a matrix-its eigenvalues. The eigenvalue of marix K are the root of the equation:

$$|\boldsymbol{K} - \alpha\mathbf{I}| = \mathbf{0} \qquad (3.11)$$

Where $|\boldsymbol{K} - \alpha\mathbf{I}|$ denotes the determinant of matrix $\boldsymbol{K} - \alpha\mathbf{I}$, and $\mathbf{I}$ is the n x n identity matrix. The n eigenvalues of $\boldsymbol{K}$ will be denoted by $\sigma_1, \alpha_2, \ldots, \alpha_n$. If any of the

eigenvalues is zero, matrix $\boldsymbol{K}$ is singular, and difficulties will be encountered in controlling the process, as noted above. If one eigenvalues is very small compared to the others, then large changes in one or more manipulated variables will be required to control the process, as is show at the end of this section.

Another important property of $\boldsymbol{K}$ is its singular value, $\sigma_1, \sigma_2, \ldots, \alpha_n$ .The singular values are nonnegative numbers that are defined as the positive square root of the eigenvalues of $\boldsymbol{K}^T\boldsymbol{K}$. The first r singular values are positive numbers where r is the rank of matrix $\boldsymbol{K}^T\boldsymbol{K}$. The remaining n -r singular values are zero. Usually the nonzero singular values are ordered with σ_1 denoting the largest and σr denoting the smallest.

The final matrix property of interest here is the condition number, CN. Assume that $\boldsymbol{K}$ is nonsingular. Then the condition number of $\boldsymbol{K}$ is defined as the ratio of the largest and smallest nonzero singular value:

$$CN = \frac{\sigma_1}{\sigma_r} \tag{3.12}$$

If $\boldsymbol{K}$ is singular, then it is ill-conditioned and by convection CN = ∞. The concept of a condition number can also be extended to non-square matrices .

The condition number is a positive number that provides a measure of how Ill-conditioned the gain matrix is. It also provides useful information on the sensitivity of the matrix properties to variations in the elements of the matrices. This important topic, which is related to control system robustness, will be considered later in this section. But first we consider a simple example.

3.2.4 Condition Number

Two measures (Condition number and RGA) which are used to quantify the degree of directionality and The level of (two-way) interactions in MIMO systems, are the condition number and the relative gain array (RGA), respectively

We define the condition number of a matrix as the ratio between the maximum and minimum singular values,

$$\gamma(G) \triangleq \bar{\sigma}(G)/\underline{\sigma}(G) \qquad (3.13)$$

A matrix with a large condition number is said to be ill-conditioned, For a non singular (square) $\underline{\sigma}(G) = 1/\bar{\sigma}(G^{-1})$, so $\gamma(G) = \bar{\sigma}(G)\bar{\sigma}(G^{-1})$ .that the condition number is large if both G and G^{-1} have large elements.

The condition number has been used as an input – output controllability measure, and in particular it has been postulated that a large condition number indicates sensitivity to uncertainty. This is not true in general, but the reverse holds; if the condition number is small, then the multivariable effects of uncertainty are not likely to be serious .

If the condition number is large (say, larger than 10), then this may indicate control problems:

1. A large condition number $\gamma(G) \triangleq \bar{\sigma}(G)/\underline{\sigma}(G)$ may be caused by a small value of $\underline{\sigma}(G)$, which is generally undesirable (on the other hand, a large value of $\bar{\sigma}(G)$ need not necessarily be a problem).
2. A large condition number may mean that the plant has a large minimized condition number, or equivalently, it has large RGA-elements which indicate fundamental control problems.
3. A large condition number does imply that the system is sensitive to 'unstructured' (full-block) input uncertainty, but this kind of uncertainty often does not occur in practice. Therefore we cannot generally conclude that a plant with a large condition number is sensitive to uncertainty.

3.2.5 Scaling

Scaling is very important in practical applications as it makes model analysis and controller design (weight selection) much simpler. It requires the engineer to make a judgement at the start of the design process about the required performance of the system. To do, decisions are made on the expected magnitudes of disturbances and reference changes, on the allowed magnitude of each input signal, and on the allowed deviation of each output.

Let the unscaled (or originally scaled) linear model of the process in deviation variables be

$$\hat{y} = \hat{G}\hat{u} + \hat{G}_d\hat{d}; \qquad \hat{e} = \hat{y} - \hat{r} \tag{3.14}$$

where a hat ($\hat{}$) is used to show that the variables are in their unscaled units. A useful approach for scaling is to make the variables less than one in magnitude. This is done by dividing each variable by its maximum expected or allowed change. For disturbances and manipulated inputs, we use the scaled variables.

$$d = \hat{d} / \hat{d}_{\max}, \quad u = \hat{u} / \hat{u}_{\max} \tag{3.15}$$

where:

- $\hat{d}_{\max}$ --- largest expected change in disturbance
- $\hat{u}_{\max}$ --- Largest allowed input change

The maximum deviation from a norminal value should be chosen by thinking of the maximum value one can expect, or allow, as a function of time.

The variable $\hat{y}$, $\hat{e}$ and $\hat{r}$ are in the same units, so the same scaling factor should be applied to each. Two alternatives are possible:

- $\hat{e}_{\max}$ --- largest allowed control error
- $\hat{r}_{\max}$ --- largest expected change in reference value

Since a major objective of control is to minimize the control error $\hat{e}$, we here usually choose to scale with respect to the maximum control error:

$$y = \hat{y} / \hat{e}_{\max}, r = \hat{r} / \hat{e}_{\max}, e = \hat{e} / \hat{e}_{\max} \tag{3.16}$$

Then the following model in terms of scaled variables

$$y = Gu + G_d d; e = y - r \tag{3.17}$$

3.3 Decentralized Feedback Control

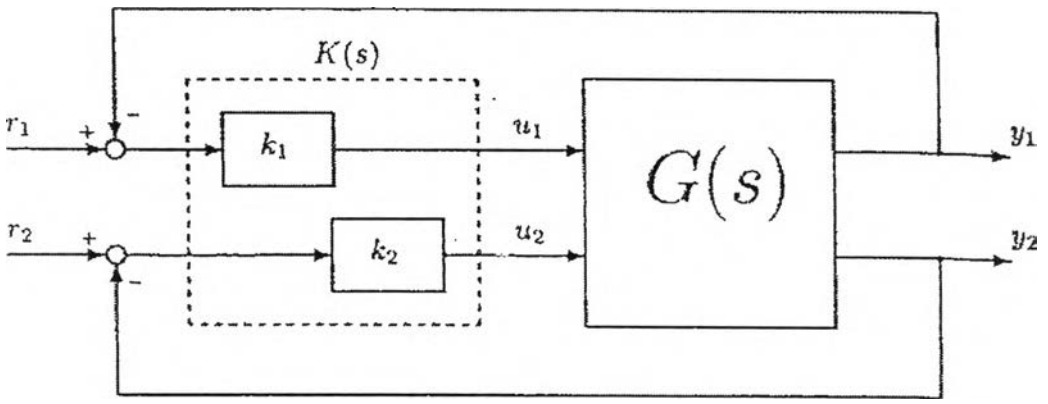

Figure 3.2 Decentralized diagonal control of a 2x2 plant

In this section, G(s) is a square plant which is to be controlled using a diagonal controller (see Figure 3.2)

$$K(s) \triangleq diag\{k_i(s)\} = \begin{bmatrix} k_1(s) & & & \\ & k_2(s) & & \\ & & \cdots & \\ & & & k_m(s) \end{bmatrix} \tag{3.18}$$

This is the problem of decentralized diagonal feedback control. The design of decentralized control systems involves two steps :

1. The choice of pairings (control configuration selection)
2. The design (turning) of each controller, $k_i(s)$.

The optimal solution to this problem is very difficult mathematically, because the optimal controller is in general of infinite order and may be non-unique. Rather we aim at providing simple tools for pairing selections (step 1) and for analyzing the achievable performance (controllability) of diagonally controlled plants (which may assist in step 2).

Remark. The treatment in this section may be generalized to block-diagonal controllers .

Notation for decentralized diagonal control. G(s) denotes a square m x m plant with elements g_{ij}. $G^{ij}(s)$ denotes the remaining (m – 1) x (m – 1) plant obtained by removing row i and column j in *G(s).* With a particular choice of pairing we can rearrange the columns or rows of *G(s)* such that the paired elements are along the diagonal of *G(s).* We then have that the controller *K (s)* is diagonal *(diag* $\{k_i\}$*),* and we also introduce

$$\tilde{G}(s) \triangleq diag\{G_{ii}(s)\} = \begin{bmatrix} g_{11}(s) & & & \\ & g_{22}(s) & & \\ & & \cdots & \\ & & & g_{mm}(s) \end{bmatrix} \qquad (3.19)$$

as the matrix consisting of the diagonal element of G. The loop transfer function in loop I is denoted $L_i = g_{ij}k_i$, which is also equal to the i'th diagonal element of $L = GK$.

The magnitude of the off-diagonal elements in G (the interactions) relative to its diagonal elements are given by the matrix

$$E \triangleq (G - \tilde{G})\tilde{G}^{-1} \qquad (3.20)$$

A very important relationship for decentralized control is given by the following factorization of the return difference operator :

$$\underbrace{(I + GK)}_{overall} = \underbrace{(I + E\tilde{T})}_{interactions} \underbrace{(I + \tilde{G}K)}_{individualloops} \qquad (3.21)$$

or equivalently in terms of the sensitivity function $S = (I + GK)^{-1}$,

$$s = \tilde{s}(I + E\tilde{T})^{-1} \qquad (3.22)$$

Here

$$\tilde{S} \triangleq (I + \tilde{G}K)^{-1} = diag\left\{\frac{1}{1 + g_{ii}k_i}\right\} \text{ and } \tilde{T} = I - \tilde{S} \qquad (3.23)$$

contain the sensitivity and complementary sensitivity functions for the individual loops. Note that $\tilde{S}$ is not equal to the matrix of diagonal element of S. (3.22) with $G = \tilde{G}$ and $G' = G$. The reader is encouraged to confirm that (3.22) is correct, because most of the important results for stability and performance using decentralized control may be derived from this expression. An alternative factorization is

$$s = (I + \tilde{s}(\Gamma - I))^{-1}\tilde{s}\Gamma \tag{3.24}$$

where Γ is the *Performance Relative Gain Array (PRGA)*

$$\Gamma(s) \triangleq \tilde{G}(s)G^{-1}(s) \tag{3.25}$$

Which is a scaled inverse of the plant. Note that $E = \Gamma^{-1} - I$. At frequencies where feedback is effective ($\tilde{S} \approx 0$), (3.24) yields $S \approx \tilde{S}\Gamma$ which shows that Γ is important when evaluating performance with decentralized control. The diagonal elements of the PRGA-matrix are equal to the diagonal elements of the RGA, $\gamma_{ii} = \lambda_{ii}$, and this is the reason for its name. Note that the off-diagonal elements of the PRGA depend on the relative scaling on the outputs, whereas the RGA is scaling independent. On the other hand, the PRGA measures also one-way interaction, whereas the RGA only measures two-way interaction.

We also will make use of the related *Closed-Loop Disturbance Gain (CLDG)* matrix, defined as

$$\tilde{G}_d(s) \triangleq \Gamma(s)G_d(s) = \tilde{G}(s)G^{-1}(s)G_d(s) \tag{3.26}$$

The CLDG depends on both output and disturbance scaling.

3.3.1 RGA as Interaction Measure for Decentralized Control

We here follow Bristol (1966), and show that the RGA provides a measure of the interactions caused by decentralized diagonal control. Let U_j and γ_i denote a particular input and output for the multivariable plant G(s), and assume that our task is to use U_j to control γ_i. Bristol argued that there will be two extreme cases:

- Other loops open: All other inputs are constant, i.e. $u_k = 0, \forall k \neq j$
- Other loops closed: All other outputs are constant, i.e. $y_k = 0, \forall k \neq i$

In the latter case, it is assumed that the other loops are closed with perfect control. Prefect control is only possible at steady-state, but it is a good approximation at frequencies within the bandwidth of each loop. We now evaluate the effect $\partial y_i / \partial u_j$ of "our" given input u_j on "our" given output y_i for the two extreme cases. We get

$$\text{Other loops open: } \left(\frac{\partial y_i}{\partial u_j} \right)_{u_k = 0, k \neq j} = g_{ij} \tag{3.27}$$

$$\text{Other loops closed: } \left(\frac{\partial y_i}{\partial u_j} \right)_{y_k = 0, k \neq i} \triangleq \hat{g}_{ij} \tag{3.28}$$

Here $g_{ij} = [G]_{ij}$ is the ij'th element of G, whereas $\hat{g}_{ij}$ is the inverse of the ji'th element of G^{-1}

$$\hat{g}_{ij} = 1 / [G^{-1}]_{ji} \tag{3.29}$$

To derive (3.29) note that

$$y = Gu \Rightarrow \left(\frac{\partial y_i}{\partial u_j} \right)_{u_k = 0, k \neq j} = [G]_{ij} \tag{3.30}$$

and interchange the roles of G and G^{-1}, of u and y, and of i and j to get

$$u = G^{-1} y \Rightarrow \left(\frac{\partial u_j}{\partial y_i} \right)_{y_k = 0, k \neq i} = [G^{-1}]_{ji} \tag{3.31}$$

and (3.29) follows. Bristol argued that the ratio between the gains in (3.27) and (3.28), corresponding to the two extreme cases, is a useful measure of interactions, and he introduced the term, ij' th relative gain defined as

$$\lambda_{ij} \triangleq \frac{g_{ij}}{\hat{g}_{ij}} = [G]_{ij} [G^{-1}]_{ji} \tag{3.32}$$

The Relative Gain Array (RGA) is the corresponding matrix of relative gains. From (3.32) we get $\Lambda(G) = G \times (G^{-1})^T$ where x denotes element-by-element multiplication (the Schur product). This is identical to our definition of the RGA matrix.

Clearly, we would like to pair variables u_j to y_i so that λ_{ij} is close to 1, because this means that the gain from u_j to y_i is unaffected by closing the other loops. On the other hand, a pairing corresponding to $\lambda_{ij}(0) < 0$ is clearly undesirable, because it means that the steady-state gain in "our" given loop changes sign when the other loops are closed.

3.3.2 Performance of Decentralized Control System

In the following, we consider performance in terms of the control error

$$e = y\text{-}r = Gu + G_d d - r \qquad (3.33)$$

Support the system has been scaled , such that at each frequency:

1. Each disturbance is less than 1 in magnitude, $|d_k| < 1$.
2. Each reference change is less than the corresponding diagonal element in R, $|r_j| < R_j$.
3. For each output the acceptable control error is less than 1, $|e_i| < 1$.

For SISO systems, we found that in terms of scaled variables we must at all frequencies require

$$|1+L| > |G_d| \text{ and } |1+L| > |R| \qquad (3.34)$$

for acceptable disturbance rejection and command tracking, respectively. Note that L, G_d and R are all scalars in this case. For decentralized control these requirements may be directly generalized by introducing the PRGA-matrix. $\Gamma = \tilde{G} G^{-1}$, in (3.26) and the CLDG-matrix, $\tilde{G}_d = \Gamma G_d$, in (3.27) These generalizations will be presented and discussed next, and subsequently proved.

Single disturbance. Consider a single disturbance, in which case G_d is a vector, and let g_{di} denote the i'th element of G_d. Let $L_i = g_{ii}k_i$ denote the loop transfer function in loop i. Consider frequencies where feedback is effective so $\tilde{S}\,\Gamma$ is small. Then for acceptable disturbance reject ($|e_i| < 1$) we must with decentralized control require for each loop i,

$$|1+L_i| > |\tilde{g}_{di}| \;\forall i \qquad (3.35)$$

which is the same as the SISO-condition except that G_d is replace by the CLDG, $\tilde{g}_{di}$ In words, $\tilde{g}_{di}$ give the "apparent" disturbance gain as seen from loop I when the system is controlled using decentralized control.

Single reference change. Similarly, consider a change in reference for output j of magnitude R_j. Consider frequencies where feedback is effective . Then for acceptable reference tracking($|e_i| < 1$) we must require for each loop i.

$$|1+L_i| > |\gamma ij| \bullet |R_j| \quad \forall j \qquad (3.36)$$

which is the same as the SISO-condition except for the PRGA-factor, $|\gamma_{ij}|$. In other words, when the other loop are closed the response in loop i gets slower by a factor $|\gamma_{ij}|$. Consequently, for *performance* it is desirable to have *small* elements in Γ, at least at frequencies where feed back is effective. However, at frequencies close to crossover, stability is the main issue, and since the diagonal elements of the PRGA and RGA are equal, we usually prefer to have γ_{ii} close to 1

Proofs of (3.35) and (3.36): At frequencies where feedback is effective, S is small, so

$$I + \tilde{S}(\Gamma - I) \approx I \qquad (3.37)$$

and from (3.25) we have

$$S \approx \tilde{S}\Gamma \qquad (3.38)$$

The closed-loop response then becomes

$$e = SG_d d - S_r \;\approx\; \tilde{S}\tilde{G}_d d - \tilde{S}\Gamma_r \qquad (3.39)$$

and the response in output i to a single disturbance d_k and a single reference change r_j is

$$e_i \approx \tilde{s}_i \tilde{g}_{dik} d_k - \tilde{s}_i \gamma_{ik} r_k \tag{3.40}$$

where $\tilde{s}_i = 1/(1+ g_{ii}k_i)$ is the sensitivity function for loop I by itself. Thus, to achieve $|e_i| < 1$ for $|d_k| = 1$ we must require $|\tilde{s}_i \tilde{g}_{dik}| < 1$ and (3.35) follows. Similarly, to achieve $|e_i| < 1$ for $|r_j| = |R_j|$ we must require $|s_i \gamma_{ik} R_j| < 1$ and (3.36) follows. Also note that $|s_i \gamma_{ik}| < 1$ will imply that assumption (3.37) is valid. Since R usually has all of its elements larger than 1, in most case (3.37) will be automatically satisfied if (3.36) is satisfied, so we normally need check assumption (3.37).

Remark 1 (3.38) may also be derived from (3.22) by assuming $\tilde{T} \approx I$ which yields $(I+E\tilde{T})^{-1} \approx (I+E)^{-1} = \Gamma$.

Remark 2 Consider a particular disturbance with model g_d. Its effect on output i with no control is g_{di} , and the ratio between $\tilde{g}_{di}$ (the CLDG) and g_{di} is the *relative disturbance gain* (RDG) (β_i) of Stanley et al. (1985) (see also Skogestad and Morari (1987b):

$$\beta_i \triangleq \tilde{g}_{di} / g_{di} = \left[\tilde{G}G^{-1}g_d\right]_i / \left[G_d\right]_i \tag{3.41}$$

Thus, β_i which is scaling independent, gives the change in the effect of disturbance caused by decentralized control. It is desirable to have β_i small, as this means that the interactions are such that they reduce the apparent effect of the disturbance, such that one does not need high gains $|L_i|$ in the individual loops.

3.3.3 Summary: Controllability Analysis for Decentralized Control

When considering decentralized control of a plant, one should first check that the plant is controllable with any controller. If the plant is unstable, then as usual the unstable modes must be controllable and observable. In addition, the unstable modes must not be *decentralized fixed modes*, otherwise the plant cannot be stabilized with a diagonal controller

The next step is to compute the RGA-matrix as a function of frequency, and to determine if one can find a good set of input-output pairs bearing in mind the following:

1. Prefer pairings which have the RA-matrix close to identity at frequencies around crossover, i.e. the RGA-number $\|\Lambda(j\omega)-I\|$ should be small. This rule is to ensure that interactions from other loops do not cause instability
2. Avoid a pairing *ij* with negative steady-state RGA element, $\lambda_{ij}(G(0))$.
3. Prefer a pairing *ij* where g_{ij} puts minimal restrictions on the achievable bandwidth. Specifically, the frequency ω_{uij} where $\angle g_{ij}(j\omega_{uij}) = -180^{o}$ should be as large as possible.

   This rule favours pairing on variables "close to each other", which makes it easier to satisfy (3.35) and (3.36) physically while at the same time achieving stability. It is also consistent with the desire and perform a controllability analysis.

   When a reasonable choice of pairings has been made, one should rearrange G to have the paired elements along the diagonal and perform a controllability analysis.
4. Compute the CLDG and PRGA, and plot these as a function of frequency.
5. For systems with many loops, it is best to perform the analysis one loop at the time, that is, for each loop i, plot $|\tilde{g}_{dik}|$ for each disturbance *k* and plot $|\gamma_{ij}|$ for each reference *j* (assuming here for simplicity that each reference is of unit magnitude).

   For performance ,we need $|1+L_i|$ to be larger than each of these

$$\text{Performance}: |1+L_i| > \max_{k,j}\{|\tilde{g}_{dik}|, |\gamma_{ij}|\} \qquad (3.42)$$

   To achieve stability of the individual loops one must analyze $g_{ii}(s)$ to ensure that the bandwidth required by (3.42) is achievable.
6. As already mentioned one may check for constraints by considering the elements of $G^{-1}G_d$ and making sure that they do not exceed one in magnitude within the frequency range where control is needed. Equivalently, one may for each loop *I* plot $|g_{ii}|$,and the requirement is then that

$$\text{To avoid input constraints: } |g_{ii}| > |\tilde{g}_{dik}|, \forall k \qquad (3.43)$$

At frequencies where $|\tilde{g}_{dik}|$ is larger than 1 (this follow since $\tilde{G}_d = \tilde{G}G^{-1}G_d$). This provides a direct generalization of the requirement $|G| > |G_d|$ for SISO systems. The advantage of (3.43) compared to using $G^{-1}G_d$ is that we can limit ourselves to frequencies where control is needed to reject the disturbance (where $|\tilde{g}_{dik}|>1$)

If the plant is not controllable, then one may consider another choice of pairings and go back to step 4. If one still cannot find any pairings which are controllable, then one should consider multivariable control

7. If the chosen pairing is controllable then the analysis base on (3.42) tells us directly how large $|L_i| = |g_{ii}k_i|$ must be, and can be used as a basis foe designing the controller $k_i(s)$ for loop i.