Chapter 4
Results

This chapter we will apply Feynman path integral formulation to Bose-Einstein condensation in a system of a dilute weakly interacting bose gas trapped in an anisotropic magnetic field. By using variational method we can derived the approximated density matrix. This lead to the ground state energy and wavefunction. Our results are then compared to the mean field approach both analytically [51] and numerically [49].

It is known that magnetic trap can be approximated by harmonic oscillator potential. The interaction potential is approximated by a zero-range (hard sphere) potential in which the strength is given by the s-wave scattering length as mentioned in Chapter 3. Then the Lagrangian is

$$\mathcal{L} = \frac{m}{2}\sum_{i=1}^{N}\left[\dot{\mathbf{r}}_i^2 + \omega_\perp^2(x_i^2 + y_i^2) + \omega_z^2 z_i^2\right] + \frac{4\pi\hbar^2 a}{m}\sum_{i\neq j=1}^{N}\delta(\mathbf{r}_i - \mathbf{r}_j) \qquad (4.1)$$

where m is the atomic mass of the alkali gas. $\mathbf{r}_i$ and $\mathbf{r}_j$ are the coordinates of particles. $\omega_\perp$ and ω_z are the radial and axial frequencies and a is s-wave scattering length.

We know from chapter 2 that density matrix can be written as propagator in an imaginary time from Eq. (2.73)

$$\rho(\bar{\mathbf{r}}'', \beta; \bar{\mathbf{r}}', 0) = \int_a^b \exp\{-S(\dot{\bar{\mathbf{r}}}, \bar{\mathbf{r}}, u)/\hbar\}\, \mathcal{D}\,\bar{\mathbf{r}}(u) \qquad (4.2)$$

where

$$S = \int_0^{\beta\hbar} \mathcal{L}\, d\tau$$

with the conditions

$$u_a = 0 \quad , \quad u_b = \beta, \tag{4.3}$$

and $\bar{\mathbf{r}}'', \bar{\mathbf{r}}'$ denote an initial and final points in the configuration space $\mathbb{R}^{3N}$. β is $1/kT$, where k is the Boltzmann's constant and T is the absolute temperature. From now on $\hbar$ is set to unity and can be put back by dimensional investigation. Note that the action in imaginary time is a real number.

Since this density matrix can not be solved exactly, we seek to approximate it instead. One way to do this is the variational method in path integral formalism first introduced by Feynman in the polaron problem [56].

From the definition of the partition function

$$\mathcal{Z} = \int \rho(\bar{\mathbf{r}}, \beta; \bar{\mathbf{r}}, 0)\, d\bar{\mathbf{r}} \tag{4.4}$$

where the choice of the $3N-$coordinates $\bar{\mathbf{r}} \equiv (\mathrm{r}_1, \mathrm{r}_2, ..., \mathrm{r}_\mathrm{N})$ is arbitrary. We have neglected the permutation of the particle at the end points. However this is not actually correct when consider that the system is composed of identical particles. The effect of indistinguishability is the (anti)symmetry of density matrix when any pair of (fermionic)bosonic particles have been exchanged. The correct form of the partition function can then be written as

$$\mathcal{Z} = \int \frac{1}{N!} \sum_P \xi^P \rho_D(P\bar{\mathbf{r}}, \beta; \bar{\mathbf{r}}, 0)\, d\bar{\mathbf{r}} \tag{4.5}$$

where $\xi = +1$ for bosons and $\xi = -1$ for fermions. P denotes the permutation of the particles. ρ_D is density matrix for distinguishable system. It should be emphasized that P acts on the particle indices, not on the components of $\mathbf{r}$ separately.

However this is not the case in obtaining the ground state properties. Since it was derived from $T=0$ limit. Then we can write the density matrix as below

$$\lim_{\beta\to\infty}\rho(\bar{\mathbf{r}}'',\beta;\bar{\mathbf{r}}',0) \;=\; \Phi_0(\bar{\mathbf{r}}'')\Phi_0^*(\bar{\mathbf{r}}')e^{-\beta E_0}. \tag{4.6}$$

where Φ_0 is the ground state wavefunction.

Now we apply the variational method to our problem. We choose the trial action to be that of harmonic oscillator with transverse and axial frequency $\Omega_\perp, \Omega_z$ as variational parameters

$$S' \;=\; \frac{m}{2}\sum_{i=1}^{N}\int_0^\beta \left[\dot{\mathbf{r}}_i^2+\Omega_\perp^2(x_i^2+y_i^2)+\Omega_z^2 z_i^2\right]\,du \tag{4.7}$$

and the trial density matrix can be readily evaluated by transform the propagator of harmonic oscillator in real time to negative imaginary time. The result is

$$\begin{aligned}\rho'(\bar{\mathbf{r}}',\beta;\bar{\mathbf{r}},0) \;=\; & \left(\frac{m\Omega_\perp}{2\pi\hbar\sinh(\Omega_\perp\beta)}\right)^N\left(\frac{m\Omega_z}{2\pi\hbar\sinh(\Omega_z\beta)}\right)^{N/2}\\ & \times\exp\left\{-\frac{Nm\Omega_\perp}{2\hbar\sinh(\Omega_\perp\beta)}[(x'^2+x^2)\cosh(\Omega_\perp\beta)-2xx']\right\}\\ & \times\exp\left\{-\frac{Nm\Omega_\perp}{2\hbar\sinh(\Omega_\perp\beta)}[(y'^2+y^2)\cosh(\Omega_\perp\beta)-2yy']\right\}\\ & \times\exp\left\{-\frac{Nm\Omega_z}{2\hbar\sinh(\Omega_z\beta)}[(z'^2+z^2)\cosh(\Omega_z\beta)-2zz']\right\}\end{aligned} \tag{4.8}$$

Consider the average terms. Since the kinetic term in S and S' always cancel each other, the argument then

$$\begin{aligned}\langle S-S'\rangle_{S'} \;=\; & \frac{m}{2}\sum_{i=1}^{N}\int_0^\beta du[(\omega_\perp^2-\Omega_\perp^2)\left\langle x_i^2+y_i^2\right\rangle_{S'}+(\omega_z^2-\Omega_z^2)\left\langle z_i^2\right\rangle_{S'}]\\ & +\frac{4\pi\hbar^2 a}{m}\sum_{i,j=1}^{N}\int_0^\beta du\,\langle\delta(\mathbf{r}_i-\mathbf{r}_j)\rangle_{S'}\,.\end{aligned} \tag{4.9}$$

Since there is no coupling between each coordinate then

$$\langle x_i^2 + y_i^2 \rangle_{S'} = \langle x_i^2 \rangle_{S'} + \langle y_i^2 \rangle_{S'} . \tag{4.10}$$

First evaluate $\langle x_i^2 \rangle_{S'}$, using generating function. Consider the system of forced harmonic oscillator in one dimension, the generating function in imaginary time is as follow:

$$\begin{aligned} \left\langle e^{\int f(u)x(u)\,du} \right\rangle_{S'(x)} &= \frac{K_{\text{force}}(x',\beta;x,0)}{K_{\text{ho}}(x',\beta;x,0)} \\ &= e^{(S'_{cl}-S_{cl})} \end{aligned} \tag{4.11}$$

where $f(u)$ is the force function and can be chosen arbitrarily. S_{cl} and S'_{cl} is the classical action in an imaginary time of the forced harmonic oscillator and harmonic oscillator respectively.

Both classical action can be evaluated easily and the difference between S_{cl} and S'_{cl} is in the force-dependent terms. So we get

$$\begin{aligned} (S'_{cl} - S_{cl}) &= x'_i \int_0^\beta du f(u) \frac{\sinh \Omega_\perp u}{\sinh \Omega_\perp \beta} + x_i \int_0^\beta du f(u) \frac{\sinh \Omega_\perp (\beta - u)}{\sinh \Omega_\perp \beta} \\ &+ \frac{1}{m\Omega_\perp} \int_0^\beta du \int_0^u du' f(u) f(u') \frac{\sinh \Omega_\perp (\beta - u) \sinh \Omega_\perp u'}{\sinh \Omega_\perp \beta}. \end{aligned} \tag{4.12}$$

Functional derivative of Eq. (4.11) with respect to $f(u)$ gives

$$\left\langle x(u)\, e^{\int f(u)x(u)\,du} \right\rangle_{S'(x)} = \frac{\delta(S'_{cl} - S_{cl})}{\delta f(u)} e^{(S'_{cl}-S_{cl})}. \tag{4.13}$$

If one sets $f = 0$ gives

$$\langle x(u) \rangle_{S'(x)} = \left. \frac{\delta(S'_{cl} - S_{cl})}{\delta f(u)} \right|_{f \equiv 0} . \tag{4.14}$$

Continue the procedure, obtains

$$\langle x^2(u) \rangle_{S'(x)} = \left. \frac{\delta^2(S'_{cl} - S_{cl})}{\delta f(u) \delta f(u')} \right|_{f \equiv 0} + \left. \frac{\delta(S'_{cl} - S_{cl})}{\delta f(u)} \frac{\delta(S'_{cl} - S_{cl})}{\delta f(u')} \right|_{f \equiv 0} . \tag{4.15}$$

Substituting Eq. (4.12) into Eq. (4.15) and ignoring the coordinate dependent terms gives

$$\left\langle x^2(u)\right\rangle_{S'(x)} = \frac{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}{m\Omega_\perp\sinh\Omega_\perp\beta}. \qquad (4.16)$$

Apply this method to N-body system. Since other coordinates are not coupled to x_i and are cancelled out by the denominator. Resulting in

$$\begin{aligned}\left\langle x_i^2(u)\right\rangle_{S'} &= \frac{\displaystyle\int e^{S'}\, x_i^2\, \mathcal{D}\,\bar{\mathbf{r}}(u)}{\displaystyle\int e^{S'}\, \mathcal{D}\,\bar{\mathbf{r}}(u)} \\ &= \frac{\displaystyle\int e^{S'(x_i)} x_i^2\, \mathcal{D}\, x_i(u)}{\displaystyle\int e^{S'(x_i)}\, \mathcal{D}x_i(u)} \\ &= \left\langle x_i^2(u)\right\rangle_{S'(x_i)}. \end{aligned} \qquad (4.17)$$

Also this argument is valid for $\langle y_i^2\rangle_{S'}$ and $\langle z_i^2\rangle_{S'}$. In summary, we get

$$\begin{aligned}\left\langle x_i^2\right\rangle_{S'} = \left\langle y_i^2\right\rangle_{S'} &= \frac{1}{m\Omega_\perp}\left(\frac{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}{\sinh\Omega_\perp\beta}\right) \\ \left\langle z_i^2\right\rangle_{S'} &= \frac{1}{m\Omega_z}\left(\frac{\sinh\Omega_z(\beta-u)\sinh\Omega_z u}{\sinh\Omega_z\beta}\right). \end{aligned} \qquad (4.18)$$

Next consider the average of delta function, using Fourier transformation

$$\begin{aligned}\langle\delta(\mathbf{r}_i-\mathbf{r}_j)\rangle_{S'} &= \int_{-\infty}^{\infty}\frac{(dk_\perp)^2}{(2\pi)^2}\left\langle e^{ik_\perp(x_i-x_j)}\right\rangle_{S'}\left\langle e^{ik_\perp(y_i-y_j)}\right\rangle_{S'} \\ &\times\int_{-\infty}^{\infty}\frac{dk_z}{2\pi}\left\langle e^{ik_z(z_i-z_j)}\right\rangle_{S'} \end{aligned} \qquad (4.19)$$

where $k_\perp, k_z$ are the wave number correspond to $\Omega_\perp, \Omega_z$. In the case of x coordinate, let us define

$$f^{\pm}(u) = \pm ik_\perp\delta(u-u') \qquad (4.20)$$

then

$$\left\langle e^{ik_\perp(x_i-x_j)}\right\rangle_{S'} = \left\langle e^{\int f^+(u)x_i(u)\,du}\right\rangle_{S'}\left\langle e^{\int f^-(u)x_j(u)\,du}\right\rangle_{S'}, \tag{4.21}$$

both terms on the right-hand side are the propagator of forced harmonic oscillator divided by that of harmonic oscillator. In addition, since their prefactors are identical, the only difference lies in the exponential term,

$$\begin{aligned}\left\langle e^{\int f^+(u)x_i(u)\,du}\right\rangle_{S'} &= \exp\left\{\tfrac{1}{2m\Omega_\perp}\int_0^\beta du\int_0^\beta du' f^+(u)f^+(u)\tfrac{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u'}{\sinh\Omega_\perp\beta}\right.\\ &\left.+\int_0^\beta du f^+(u)\left(x_i'\tfrac{\sinh\Omega_\perp u}{\sinh\Omega_\perp\beta}+x_i\tfrac{\sinh\Omega_\perp(\beta-u)}{\sinh\Omega_\perp\beta}\right)\right\}\end{aligned} \tag{4.22}$$

Again the end-point terms have no contribution to the energy, we neglect them and substitute $f^+(u)$ into Eq. (4.22) to get

$$\left\langle e^{\int f^+(u)x_i(u)\,du}\right\rangle_{S'} = \exp\left(-\frac{k_\perp^2\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}{m\Omega_\perp\sinh\Omega_\perp\beta}\right) \tag{4.23}$$

for $f^-(u)$ the result is the same,

$$\begin{aligned}\int\frac{dk_\perp}{2\pi}\left\langle e^{ik_\perp(x_i-x_j)}\right\rangle_{S'} &= \int\frac{dk_\perp}{2\pi}\exp\left(-\frac{k_\perp^2\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}{m\Omega_\perp\sinh\Omega_\perp\beta}\right)\\ &= \frac{1}{2}\left(\frac{m\Omega_\perp\sinh\Omega_\perp\beta}{\pi\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}\right)^{1/2}.\end{aligned} \tag{4.24}$$

Calculation for y and z coordinates can be done in the same manner. Replacing this result in Eq. (4.19) gives

$$\langle\delta(\mathbf{r}_i-\mathbf{r}_j)\rangle_{S'} = \frac{m\Omega_\perp\sinh\Omega_\perp\beta}{4\pi\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}\left(\frac{m\Omega_z\sinh\Omega_z\beta}{4\pi\sinh\Omega_z(\beta-u)\sinh\Omega_z u}\right)^{1/2} \tag{4.25}$$

Finally substituting Eq. (4.18) and Eq. (4.25) into Eq. (4.9), we get

$$\begin{aligned}\langle S-S'\rangle_{S'} &= Nm(\omega_\perp^2-\Omega_\perp^2)\int_0^\beta du\tfrac{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}{m\Omega_\perp\sinh\Omega_\perp\beta}\\ &+\tfrac{Nm}{2}(\omega_z^2-\Omega_z^2)\int_0^\beta du\tfrac{\sinh\Omega_z(\beta-u)\sinh\Omega_z u}{m\Omega_z\sinh\Omega_z\beta}\\ &+\tfrac{N^2 2\pi a}{m}\int_0^\beta du\tfrac{m\Omega_\perp\sinh\Omega_\perp\beta}{4\pi\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}\sqrt{\tfrac{m\Omega_z\sinh\Omega_z\beta}{4\pi\sinh\Omega_z(\beta-u)\sinh\Omega_z u}}.\end{aligned} \tag{4.26}$$

Consider the first term on the right-hand side of Eq. (4.26), the integrand is evaluated as follow

$$\begin{aligned}
\int_0^\beta \frac{\sinh\Omega_\perp u \sinh\Omega_\perp(\beta-u)}{\sinh\Omega_\perp\beta}du &= \int_0^\beta (\sinh\Omega_\perp u\cosh\Omega_\perp u - \coth\Omega_\perp\beta\sinh^2\Omega_\perp u)du \\
&= \frac{\sinh^2\Omega_\perp\beta}{2\Omega_\perp} - \coth\Omega_\perp\beta\left(\frac{\sinh\Omega_\perp\beta\cosh\Omega_\perp\beta}{2\Omega_\perp} - \frac{\beta}{2}\right) \\
&= \frac{\sinh^2\Omega_\perp\beta - \cosh^2\Omega_\perp\beta}{2\Omega_\perp} + \frac{\beta\coth\Omega_\perp\beta}{2} \\
&= -\frac{1}{2\Omega_\perp} + \frac{\beta\coth\Omega_\perp\beta}{2}. \qquad (4.27)
\end{aligned}$$

For the limit $\beta \to \infty, \coth\Omega_\perp\beta = 1$. The first term on the right-hand side of Eq. (4.27) is smaller than the second one, then we ignore it. Also this is true in the z-axis. Now evaluate the last integrand of Eq. (4.26) using the limit above

$$\begin{aligned}
\frac{\sinh\Omega_\perp\beta}{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u} &= \frac{1}{\sinh\Omega_\perp u\cosh\Omega_\perp u - \sinh^2\Omega_\perp u} \\
&= \frac{4}{(e^{\Omega_\perp u} - e^{-\Omega_\perp u})(e^{\Omega_\perp u} + e^{-\Omega_\perp u}) - (e^{\Omega_\perp u} - e^{-\Omega_\perp u})^2} \\
&= \frac{2}{1 - e^{-2\Omega_\perp u}}. \qquad (4.28)
\end{aligned}$$

Then the last term of Eq. (4.26) is

$$\begin{aligned}
\int_0^\beta du\frac{\sinh\Omega_\perp\beta}{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}\sqrt{\frac{\sinh\Omega_z\beta}{\sinh\Omega_z(\beta-u)\sinh\Omega_z u}} \\
= \int_0^\beta du\frac{2^{3/2}}{(1-e^{-2\Omega_\perp u})(1-e^{-2\Omega_z u})^{1/2}} \qquad (4.29)
\end{aligned}$$

We approximate this term by neglecting the exponential terms. Then

$$\int_0^\beta du\frac{\sinh\Omega_\perp\beta}{\sinh\Omega_\perp(\beta-u)\sinh\Omega_\perp u}\sqrt{\frac{\sinh\Omega_z\beta}{\sinh\Omega_z(\beta-u)\sinh\Omega_z u}} = 2^{3/2}\beta \qquad (4.30)$$

$$\begin{aligned}
\langle S - S'\rangle_{S'} &= \frac{N\beta}{2}\left(\frac{\omega_\perp^2 - \Omega_\perp^2}{\Omega_\perp}\right) + \frac{N\beta}{4}\left(\frac{\omega_z^2 - \Omega_z^2}{\Omega_z}\right) + \frac{N^2 a\Omega_\perp\beta 2^{2/3}}{4}\left(\frac{m\Omega_z}{\pi}\right)^{1/2} \\
&= \beta N\hbar\left[\frac{\omega_\perp^2}{2\Omega_\perp} - \frac{\Omega_\perp}{2} + \frac{\omega_z^2}{4\Omega_z} - \frac{\Omega_z}{4} + Na\Omega_\perp\Omega_z^{1/2}\left(\frac{m}{2\pi\hbar}\right)^{1/2}\right] \qquad (4.31)
\end{aligned}$$

Now the approximated density matrix can be written as

$$
\begin{aligned}
\rho(\bar{\mathbf{r}}',\beta;\bar{\mathbf{r}},0) &= \rho_0(\bar{\mathbf{r}}',\beta;\bar{\mathbf{r}},0)\exp\left\{-\frac{\langle S-S'\rangle_{S'}}{\hbar}\right\} \\
&= \left(\frac{m\Omega_\perp}{2\pi\hbar\sinh(\Omega_\perp\beta)}\right)^N\left(\frac{m\Omega_z}{2\pi\hbar\sinh(\Omega_z\beta)}\right)^{N/2} \\
&\quad\times\exp\left\{-\frac{Nm\Omega_\perp}{2\hbar\sinh(\Omega_\perp\beta)}[(x'^2+x^2+y'^2+y^2)\cosh(\Omega_\perp\beta)-2xx'-2yy']\right\} \\
&\quad\times\exp\left\{-\frac{Nm\Omega_z}{2\hbar\sinh(\Omega_z\beta)}[(z'^2+z^2)\cosh(\Omega_z\beta)-2zz']\right\} \\
&\quad\times\exp\left\{-N\hbar\beta\left(\frac{-\Omega_\perp}{2}+\frac{\omega_\perp^2}{2\Omega_\perp}-\frac{\Omega_z}{4}+\frac{\omega_z^2}{4\Omega_z}+\left(\frac{m}{2\pi\hbar}\right)^{1/2}Na\Omega_\perp\Omega_z^{1/2}\right)\right\}.
\end{aligned}
$$

Now we ready to evaluate the ground state energy and wavefunction. First we approximate the prefactor

$$
\begin{aligned}
\left(\frac{m\Omega_\perp}{2\pi\hbar\sinh(\Omega_\perp\beta)}\right)^N &= \left(\frac{m\Omega_\perp}{\pi\hbar}\right)^N\left(\frac{1}{e^{\Omega_\perp\beta}-e^{-\Omega_\perp\beta}}\right)^N \\
&= \left(\frac{m\Omega_\perp}{\pi\hbar}\right)^N\left(\frac{e^{-\Omega_\perp\beta}}{1-e^{-2\Omega_\perp\beta}}\right)^N. \qquad (4.32)
\end{aligned}
$$

Again $\beta\to\infty$, then $e^{-2\Omega_\perp\beta}=0$. This gives

$$
\left(\frac{m\Omega_\perp}{2\pi\hbar\sinh(\Omega_\perp\beta)}\right)^N = \left(\frac{m\Omega_\perp}{\pi\hbar}\right)^N e^{-N\Omega_\perp\beta}. \qquad (4.33)
$$

Consequently the approximated density matrix is written as

$$
\begin{aligned}
\rho(\bar{\mathbf{r}}',\beta;\bar{\mathbf{r}},0) &= \left(\frac{m\Omega_\perp}{\pi\hbar}\right)^N\left(\frac{m\Omega_z}{\pi\hbar}\right)^{N/2}\exp\left\{-\frac{Nm\Omega_z}{2\hbar\sinh(\Omega_z\beta)}[(z'^2+z^2)\cosh(\Omega_z\beta)-2zz']\right\} \\
&\quad\times\exp\left\{-\frac{Nm\Omega_\perp}{2\hbar\sinh(\Omega_\perp\beta)}[(x'^2+x^2+y'^2+y^2)\cosh(\Omega_\perp\beta)-2xx'-2yy']\right\} \\
&\quad\times\exp\left\{-N\hbar\beta\left(\frac{\Omega_\perp}{2}+\frac{\omega_\perp^2}{2\Omega_\perp}+\frac{\Omega_z}{4}+\frac{\omega_z^2}{4\Omega_z}+\left(\frac{m}{2\pi\hbar}\right)^{1/2}Na\Omega_\perp\Omega_z^{1/2}\right)\right\}.
\end{aligned}
\qquad (4.34)
$$

We can see that the ground state energy is

$$
E = N\hbar\left(\frac{\Omega_\perp}{2}+\frac{\omega_\perp^2}{2\Omega_\perp}+\frac{\Omega_z}{4}+\frac{\omega_z^2}{4\Omega_z}+\left(\frac{m}{2\pi\hbar}\right)^{1/2}Na\Omega_\perp\Omega_z^{1/2}\right). \qquad (4.35)
$$

This is exactly the same as Baym's work Eq. (3.44) which derive from the mean field Gross-Pitaevskii equation (see Appendix D). Minimizing the energy with respect to $\Omega_\perp$ we get the relation

$$\Omega_\perp = \frac{\omega_\perp}{\triangle} \tag{4.36}$$

where

$$\triangle = \left(1 + \frac{\zeta^5}{(32\pi^3)^{1/2}}\left(\frac{\Omega_z}{\omega_\perp}\right)^{1/2}\right)^{1/2}$$

and

$$\zeta = \left(\frac{8\pi Na}{a_{\mathrm{ho}}}\right)^{1/5}$$

Now Eq. (4.35) is arranged to be

$$E(\Omega_z) = N\hbar\left[\omega_\perp\triangle + \frac{\Omega_z}{4} + \frac{\omega_z^2}{4\Omega_z}\right]. \tag{4.37}$$

Minimizing the ground state energy with respect to Ω_z, we get

$$1 + \frac{\zeta^5}{\triangle}\left(\frac{\omega_\perp}{32\pi^3\Omega_z}\right)^{1/2} = \frac{\omega_z^2}{\Omega_z^2}. \tag{4.38}$$

Solving Ω_z for each N, we can determine $\Omega_\perp, \Omega_z$ and E (see Appendix E). In order to compare with the work of Dalfovo and Stringari, we use the same scaling factors such that

$$\begin{aligned} E &= \hbar\omega_\perp E_1, \\ r &= a_\perp r_1 \end{aligned} \tag{4.39}$$

where $a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}$, $\omega_z/2\pi = 220$ Hz, the asymmetry parameter of the trap is $\lambda = \omega_z/\omega_\perp$, and $a = 100a_0$, where a_0 is the Bohr radius. The ground state energy are then shown in Table 4.1.

N	$\Omega_\perp$	Ω_z	E_1/N	$(E_1/N)_{\rm kin}$	$(E_1/N)_{\rm ho}$	$(E_1/N)_{\rm int}$
100	391.488	1277.31	2.66	1.05	1.38	0.223
200	338.276	1210.59	2.87	0.96	1.52	0.376
500	258.375	1088.83	3.34	0.82	1.84	0.681
1000	201.549	975.417	3.92	0.70	2.21	1.00
2000	153.723	849.439	4.76	0.59	2.74	1.43
5000	105.863	676.050	6.40	0.45	3.75	2.20
10000	79.6350	550.615	8.19	0.36	4.84	2.98
15000	67.4479	482.923	9.51	0.31	5.64	3.55
20000	59.9710	438.161	10.6	0.28	6.30	4.01

Table 4.1: The ground state energies are evaluated from Eq. (4.35). Energy are in units of $\hbar\omega_\perp$ in accord with [49].

The ground state wavefunction is determined from the coordinate-dependent term in the trial propagator. This is similar to the Rayleigh-Ritz variation method in term of trial wavefunction. We then derive the expression for the condensation wavefunction as

$$\begin{aligned}\Phi_0(\mathbf{r}')\Phi_0^*(\mathbf{r}) &= \left(\frac{m\Omega_\perp}{\pi\hbar}\right)^N\left(\frac{m\Omega_z}{\pi\hbar}\right)^{N/2}\exp\left\{-\frac{Nm\Omega_\perp}{2\hbar\sinh(\Omega_\perp\beta)}\right.\\ &\quad\left.\left[(x'^2+x^2+y'^2+y^2)\cosh(\Omega_\perp\beta)-2xx'-2yy'\right]\right\}\\ &\quad\times\exp\left\{-\frac{Nm\Omega_z}{2\hbar\sinh(\Omega_z\beta)}[(z'^2+z^2)\cosh(\Omega_z\beta)-2zz']\right\}. \qquad (4.40)\end{aligned}$$

In the limit $\beta\to\infty$

$$\frac{1}{\sinh(\Omega_\perp\beta)}=0 \quad \text{and} \quad \frac{\cosh(\Omega_\perp\beta)}{\sinh(\Omega_\perp\beta)}=1, \qquad (4.41)$$

then

$$\begin{aligned}\Phi_0(\mathbf{r}') &= \prod_{i=1}^{N}\phi_0(\mathbf{r}'_i)\\ &= \prod_{i=1}^{N}\left(\frac{m\Omega_\perp}{\pi\hbar}\right)\left(\frac{m\Omega_z}{\pi\hbar}\right)^{1/2}\exp\left\{-\frac{m\Omega_\perp}{2\hbar}(x_i'^2+y_i'^2)-\frac{m\Omega_z}{2\hbar}z_i'^2\right\}\\ &= \left(\frac{m\Omega_\perp}{\pi\hbar}\right)^N\left(\frac{m\Omega_z}{\pi\hbar}\right)^{N/2}\exp\left\{-\frac{mN\Omega_\perp}{2\hbar}(x'^2+y'^2)-\frac{mN\Omega_z}{2\hbar}z'^2\right\}. \qquad (4.42)\end{aligned}$$

In the scaled units mentioned previously the ground state wavefunction then be

$$\phi_0(\mathbf{r}) = \Omega_\perp^{1/2}\Omega_z^{1/4}\left(\frac{1}{\pi\omega_\perp}\right)^{3/4}\exp\left\{-\frac{\Omega_\perp}{2\omega_\perp}r_\perp^2-\frac{\Omega_z}{4\omega_z}z^2\right\} \quad (4.43)$$

the single-particle ground state wavefunction is plotted and displayed in Figure 4.1.

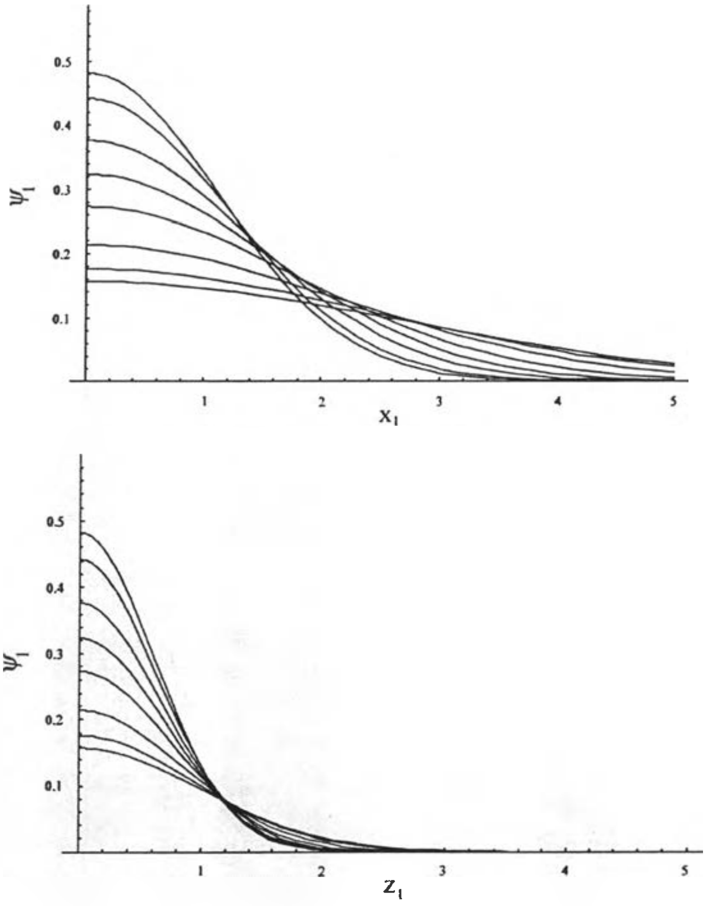

Figure 4.1: The ground state wavefunctions are obtained from Eq. (4.42). Distances are in units of $a_\perp$ in accord with [49]. Each line corresponds to $N = 100, 200, 500, 1000, 2000, 5000, 10^4, 1.5\times10^4$, in descending order of central density.

From Eq. (3.5), the width of the condensate cloud is corresponding to the width of the Gaussian wavefunction. From the Figure 4.1 we see that the radius of the cloud in $x-y$ plane is larger than the z direction. This is due to the trapping is weaker in the $x - y$ plane. The anisotropy provides the evidence for Bose condensate. Because if it was not condensed the distribution should be identical due to the equipartition theorem. The increasing in N means the strength of interaction is increased. This causes the condensate cloud to expand. So the interaction plays an important role in the phenomena. Without interaction the condensate cloud is not affected by the different kind of atomic gas and by the increasing of N. Our result is similar to the work of Dalfovo and Stringari [49]. This fact confirms that the variational path integral is also applicable to BEC problem.