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# Appendices

# Appendix A:
# Trotter Product Formula

To prove the Trotter product we first show that the two operator functions

$$\begin{aligned} \hat{F}(\alpha) &= e^{-\alpha(\hat{T}+\hat{V})} \qquad \text{and} \\ \hat{G}(\alpha) &= e^{-\alpha\hat{T}}e^{-\alpha\hat{V}} \qquad \text{with} \quad \alpha = \frac{\lambda}{N} \end{aligned} \tag{A.1}$$

which differ only by commutation terms, and vanish in the limit $N \to \infty$.

An operator function is defined by its Taylor series, e.g.

$$\hat{G}(\alpha) = \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} \left( \frac{d^n \hat{G}}{d\alpha^n} \right) \Bigg|_{\alpha=0} . \tag{A.2}$$

In the following, a useful operator identity will be applied:

$$\hat{K}(\alpha) = e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} [\hat{A}, \hat{B}]_{(n)}, \tag{A.3}$$

with $[\hat{A}, \hat{B}]_{(0)} = \hat{B}, [\hat{A}, \hat{B}]_{(1)} = [\hat{A}, \hat{B}], [\hat{A}, \hat{B}]_{(2)} = [\hat{A}, [\hat{A}, \hat{B}]], \ldots$. For the proof of Eq. (A.3) the coefficients $(d^n\hat{K}/d\alpha^n)|_{\alpha=0}$ of the Taylor series have to be calculated:

$$\hat{K}(\alpha) = \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} \left( \frac{d^n \hat{K}}{d\alpha^n} \right) \Bigg|_{\alpha=0} . \tag{A.4}$$

Thus:

$n = 0$

$$\hat{K}(0) = [\hat{A}, \hat{B}]_{(0)} = \hat{B}; \tag{A.5}$$

$n = 1$

$$\begin{aligned} \frac{d\hat{K}}{d\alpha} &= \hat{A}e^{+\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} - e^{\alpha\hat{A}}\hat{B}\hat{A}e^{-\alpha\hat{A}} \\ &= e^{\alpha\hat{A}}[\hat{A}, \hat{B}]e^{-\alpha\hat{A}} \end{aligned} \tag{A.6}$$

$$\frac{d\hat{K}}{d\alpha}\Bigg|_{\alpha=0} = [\hat{A}, \hat{B}]_{(1)} = [\hat{A}, \hat{B}]; \tag{A.7}$$

$n = 2$

$$\frac{d^2\hat{K}}{d\alpha^2} = \hat{A}e^{\alpha\hat{A}}[\hat{A},\hat{B}]e^{-\alpha\hat{A}} - e^{\alpha\hat{A}}[\hat{A},\hat{B}]\hat{A}e^{-\alpha\hat{A}}$$
$$= e^{\alpha\hat{A}}[\hat{A},[\hat{A},\hat{B}]]e^{-\alpha\hat{A}}, \tag{A.8}$$
$$\left.\frac{d^2\hat{K}}{d\alpha^2}\right|_{\alpha=0} = [\hat{A},[\hat{A},\hat{B}]] = [\hat{A},\hat{B}]_{(2)}. \tag{A.9}$$

For any $n$ one has

$$\frac{d^n\hat{K}}{d\alpha^n}|_{\alpha=0} = \underbrace{[\hat{A},\cdots[\hat{A},\hat{B}]]}_{n\,\text{times}} = [\hat{A},\hat{B}]_{(n)}. \tag{A.10}$$

Inserting Eq. (A.10) into the Taylor series for $\hat{K}(\alpha)$ yields the identity Eq. (A.3).

Turning to the operator function $\hat{G}(\alpha) = e^{-\alpha\hat{T}}e^{-\alpha\hat{V}}$ and calculating explicitly the first terms of its Taylor series, we obtain:

$n = 0$

$$\hat{G}(\alpha)|_{\alpha=0} = \hat{\mathbf{1}}; \tag{A.11}$$

$n = 1$

$$\frac{d\hat{G}}{d\alpha} = (-)\hat{T}\hat{G}(\alpha) + (-)e^{-\alpha\hat{T}}\hat{V}e^{-\alpha\hat{V}}$$
$$= (-)\hat{T}\hat{G}(\alpha) + (-)e^{-\alpha\hat{T}}\hat{V}e^{\alpha\hat{T}}e^{-\alpha\hat{T}}e^{-\alpha\hat{V}}$$
$$= (-)\hat{T}\hat{G}(\alpha) + (-)\left(\hat{V} + \sum_{m=1}^{\infty}\frac{(-\alpha)^m}{m!}[\hat{T},\hat{V}]_{(m)}\right)\hat{G}(\alpha),$$
$$= (-)(\hat{T}+\hat{V})\hat{G}(\alpha) + (-)\sum_{m=1}^{\infty}\frac{(-\alpha)^m}{m!}[\hat{T},\hat{V}]_{(m)}\hat{G}(\alpha), \tag{A.12}$$
$$\left.\frac{d\hat{G}}{d\alpha}\right|_{\alpha=0} = (-)(\hat{T}+\hat{V}); \tag{A.13}$$

$n = 2$

$$\frac{d^2\hat{G}}{d\alpha^2} = \left((-1)(\hat{T}+\hat{V}) + (-1)\sum_{m=1}^{\infty}\frac{(-\alpha)^m}{m!}[\hat{T},\hat{V}]_{(m)}\right)\frac{d\hat{G}}{d\alpha}$$
$$+(-1)^2\sum_{m=1}^{\infty}\frac{(-\alpha)^{m-1}}{(m-1)!}[\hat{T},\hat{V}]_{(m)}\hat{G}(\alpha) \tag{A.14}$$
$$\left.\frac{d^2\hat{G}}{d\alpha^2}\right|_{\alpha=0} = (-1)^2(\hat{T}+\hat{V})^2 + (-1)^2[\hat{T},\hat{V}]. \tag{A.15}$$

In the way indicated, all higher derivatives can be determined. Then one gets

$$\left.\frac{d^n\hat{G}}{d\alpha^n}\right|_{\alpha=0} = (-1)^n(\hat{T}+\hat{V})^n + \text{commutator terms}. \tag{A.16}$$

Inserting this into the Taylor expansion Eq. (A.2) and performing the summation one obtains

$$\hat{G}(\alpha) = \hat{F}(\alpha) + \frac{\alpha^2}{2}[\hat{T},\hat{V}] + O(\alpha^3). \tag{A.17}$$

Hence we find

$$[\hat{F}(\alpha)]^N - \left(\hat{G}(\alpha)\right)^N = O(\alpha^2), \tag{A.18}$$

i.e. the above difference is at least proportional to $\alpha^2 = \lambda^2/N^2$. In the limit $N \to \infty$ the right-hand side of Eq. (A.18) vanishes, which proves the validity of Trotter's formula.

# Appendix B: Pseudopotential

In this appendix, we give some details of deriving the pseudopotential. The argument follows Huang [62, 63].

We consider the two-body problem. Each particle has the mass, $m$, and an inter-particle potential, $v(\mathbf{r})$, is the "hard-sphere" one,

$$v(\mathbf{r}) = \begin{cases} 0 & (r > a) \\ \infty & (r \leq a), \end{cases} \tag{B.1}$$

where $a$ is the hard-sphere diameter with $\mathbf{r}$ the relative position vector between two particles and $r = |\mathbf{r}|$. The Schrödinger equation in the center-of-mass system is

$$\frac{\hbar^2}{2\mu} \left( \nabla^2 + k^2 \right) \psi(\mathbf{r}) = v(\mathbf{r})\, \psi(\mathbf{r}), \tag{B.2}$$

where $\mu$ means the reduced mass,

$$\mu = m/2. \tag{B.3}$$

Obviously, $\psi(\mathbf{r})$ is the wavefunction in the center-of-mass coordinate system, and $(\hbar k)^2/(2\mu)$ is the energy of the relative motion. Substituting (B.1) into (B.2), we have

$$\begin{aligned} \left( \nabla^2 + k^2 \right) \psi(\mathbf{r}) &= 0 \quad (r > a), \\ \psi(\mathbf{r}) &= 0 \quad (r \leq a). \end{aligned} \tag{B.4}$$

In terms of the spherical coordinate,

$$\mathbf{r} = \left( r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta \right), \tag{B.5}$$

the solution of Eq. (B.4) for $r > a$ can be written as

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\theta, \phi) A_{lm} \left( j_l(kr) - \tan \eta_l \, n_l(kr) \right), \tag{B.6}$$

with the boundary condition,

$$\psi(\mathbf{r})|_{r=a} = 0. \tag{B.7}$$

Here $Y_{lm}(\theta, \phi)$ is a normalized spherical harmonic function, $j_l(x)$ and $n_l(x)$ the spherical Bessel and Neumann functions respectively, and $A_{lm}$ and $\eta_l$ constants. We note that the constant $\eta_l$ is determined by the condition (B.7) as

$$\tan \eta_l = j_l(ka)/n_l(ka). \tag{B.8}$$

The scattering length $a_l$ for the partial $l$-wave is defined by

$$a_l \equiv -\lim_{k \to 0} \tan \eta_l(k)/k. \tag{B.9}$$

In what follows, we assume that the energy of the relative motion $(\hbar k)^2/(2\mu)$ is sufficiently small, and thus we consider a spherically symmetric ($s$-wave) solution,

$$\psi(\mathbf{r}) = A \left( j_0(kr) - \tan \eta_0 \, n_0(kr) \right), \tag{B.10}$$

where

$$j_0(x) = \sin x/x, \tag{B.11}$$

$$n_0(x) = -\cos x/x, \tag{B.12}$$

$$A \equiv A_{00}/\sqrt{4\pi}. \tag{B.13}$$

From (B.8), (B.11) and (B.12), we have

$$\tan \eta_0 = -\tan(ka), \tag{B.14}$$

leading to

$$\eta_0 = -ka. \tag{B.15}$$

Thus, $a$ is identified with the $s$-wave scattering length.

An idea of the pseudopotential is as follows: we find an equation with some "potential" such that (B.10) is the solution everywhere. For sufficiently small $x$, $j_0(x)$ and $n_0(x)$ behave like

$$j_0(x) \approx 1, \quad n_0(x) \approx -1/x \quad (x \ll 1). \tag{B.16}$$

Thus, from (B.10), for sufficiently small $kr$, we get

$$r\psi(\mathbf{r}) = A\left(r + \frac{\tan\eta_0}{k}\right), \tag{B.17}$$

which gives

$$A = \frac{\partial}{\partial r}\left(r\psi(\mathbf{r})\right). \tag{B.18}$$

We remark that the relation (B.18) is used only at $\mathbf{r} = 0$. Because $j_0(x)$ is regular at $x = 0$, $j_0(kr)$ satisfies

$$(\nabla^2 + k^2) j_0(kr) = 0, \tag{B.19}$$

for all $r$. On the other hand, $n_0(x)$ is singular at $x = 0$. Then, we calculate

$$F_0(r) \equiv (\nabla^2 + k^2) n_0(kr), \tag{B.20}$$

with greater care. We integrate $F_0(r)$ over a sphere $V$ of radius $\epsilon$ about the origin. From (B.20), we have

$$\int_V \mathrm{d}^3\mathbf{r}\, F_0(r) = \int_V \mathrm{d}^3\mathbf{r}\, \nabla^2 n_0(kr) + k^2 \int_V \mathrm{d}^3\mathbf{r}\, n_0(kr). \tag{B.21}$$

By applying the divergence theorem to the first term in the right-hand side of Eq. (B.21), we get

$$\begin{aligned}\int_V \mathrm{d}^3\mathbf{r}\,\nabla^2 n_0(kr) &= \int_{\partial V} \mathrm{d}\mathbf{S}\cdot\nabla n_0(kr)\\ &= 4\pi\epsilon^2 \frac{\partial}{\partial r} n_0(kr)\bigg|_{r=\epsilon}\\ &= 4\pi\epsilon\sin(k\epsilon) + \frac{4\pi}{k}\cos(k\epsilon).\end{aligned} \tag{B.22}$$

The second term in Eq. (B.21) gives

$$\begin{aligned}k^2\int_V \mathrm{d}^3\mathbf{r}\,n_0(kr) &= 4\pi\int_0^\epsilon r^2\mathrm{d}r\left(\frac{-\cos(kr)}{kr}\right)\\ &= -4\pi\epsilon\sin(k\epsilon) - \frac{4\pi}{k}\cos(k\epsilon) + \frac{4\pi}{k}.\end{aligned} \tag{B.23}$$

Substituting (B.22) and (B.23) into (B.21), we obtain

$$\int_V \mathrm{d}^3\mathbf{r}\,F_0(r) = \frac{4\pi}{k}. \tag{B.24}$$

Noting that $F_0(r)$ is identically equal to zero for $r \neq 0$, we conclude from (B.24) that

$$F_0(r) = (\nabla^2 + k^2)n_0(kr) = \frac{4\pi}{k}\delta(\mathbf{r}). \tag{B.25}$$

Using (B.14), (B.18), (B.19) and (B.25) in Eq. (B.10), we have an equation that the solution (B.10) satisfies everywhere,

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = \frac{4\pi}{k}\tan(ka)\,\delta(\mathbf{r})\,\frac{\partial}{\partial r}\left(r\psi(\mathbf{r})\right). \tag{B.26}$$

For sufficiently small $ka$, we can replace $\tan(ka)$ by $ka$. Then, by dividing both sides of Eq. (B.26) by $\hbar^2/(2\mu)$, we finally arrive at

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{r}) + \bar{v}(\mathbf{r})\,\psi(\mathbf{r}) = \frac{\hbar^2}{2\mu}k^2\psi(\mathbf{r}), \tag{B.27}$$

where

$$\tilde{v}(\mathbf{r}) \equiv \frac{4\pi\hbar^2 a}{m}\delta(\mathbf{r})\frac{\partial}{\partial r}(r\,\cdot). \tag{B.28}$$

The operator $\tilde{v}(\mathbf{r})$ (B.28) is known as the pseudopotential [62, 63]. We note that $\partial/\partial r\,(r\,\cdot)$ appearing in (B.28) is not a hermitian operator. But, if $\psi(\mathbf{r})$ is well behaved, namely differentiable at the origin, we can replace $\partial/\partial r\,(r\,\cdot)$ by unity. So far, we have considered $a$ to be positive. In general, however, the "diameter" of the hard-sphere $a$ can be extended to be negative. This occurs when we may replace the low energy scattering from an attractive inter-particle potential of finite range by that from a hard-sphere one, known as the "shape-independent approximation."

# Appendix C: Gross-Pitaevskii Equation

The alternative way to derive the GP equation is presented. We first write the energy functional as

$$E[\Phi(\mathrm{r},t)] = \int d^3\mathrm{r} \left[\frac{\hbar^2}{2\mathrm{m}}|\nabla\Phi(\mathrm{r},\mathrm{t})|^2 + \mathrm{V_{ho}}(\mathrm{r})|\Phi(\mathrm{r},\mathrm{t})|^2 + \frac{\mathrm{g}}{2}\,|\Phi(\mathrm{r},\mathrm{t})|^4\right]. \tag{C.1}$$

where terms on the right-hand side denote the kinetic, harmonic and interaction energies. First we consider the kinetic term. Applying the divergence theorem, we can write in explicit form as

$$\frac{\hbar^2}{2m}\int d^3\mathrm{r}\;|\nabla\Phi(\mathrm{r},t)|^2 = \frac{-\hbar^2}{2m}\int d^3\mathrm{r}\;\Phi^*(\mathrm{r},t)\nabla^2\,\Phi(\mathrm{r},t). \tag{C.2}$$

Using the identity

$$\frac{\delta f(r)}{\delta f(r')} = \delta(r-r'). \tag{C.3}$$

The functional derivative of kinetic term gives

$$\begin{aligned}\frac{-\hbar^2}{2m}\frac{\delta}{\delta\Phi^*(\mathrm{r}',t)}\int d^3\mathrm{r}\;\Phi^*(\mathrm{r},t)\;\nabla^2\,\Phi(\mathrm{r},t) &= \frac{-\hbar^2}{2m}\int d^3\mathrm{r}\;\delta(\mathrm{r}-\mathrm{r}')\;\nabla^2\,\Phi(\mathrm{r},\mathrm{t}) \\ &= \frac{-\hbar^2}{2m}\;\nabla^2\,\Phi(\mathrm{r}',t). \end{aligned}\tag{C.4}$$

For the harmonic term, it is easily obtained

$$\begin{aligned}\frac{\delta}{\delta\Phi^*(\mathrm{r},t)}\int d^3\mathrm{r}\;V_{\mathrm{ho}}(\mathrm{r})\;\Phi^*(\mathrm{r},t)\;\Phi(\mathrm{r},t) &= \int d^3\mathrm{r}\;\delta(\mathrm{r}-\mathrm{r}')\;\mathrm{V_{ho}}(\mathrm{r})\;\Phi(\mathrm{r},\mathrm{t}) \\ &= V_{\mathrm{ho}}(\mathrm{r}')\;\Phi(\mathrm{r}',t). \end{aligned}\tag{C.5}$$

The interaction term can be obtained in the same manner

$$
\begin{aligned}
\frac{g}{2}\frac{\delta}{\delta\Phi^*(\mathrm{r}',t)}\int d^3\mathrm{r}\ |\Phi(\mathrm{r},t)|^4 &= \frac{g}{2}\ \frac{\delta}{\delta\Phi^*(\mathrm{r}',t)}\int d^3\mathrm{r}\ \Phi^*(\mathrm{r},t)\ \Phi^*(\mathrm{r},t)\ \Phi(\mathrm{r},t)\ \Phi(\mathrm{r},t) \\
&= \frac{g}{2}\int d^3\mathrm{r}\ \delta(\mathrm{r}-\mathrm{r}')\ 2\ |\Phi(\mathrm{r},\mathrm{t})|^2\ \Phi(\mathrm{r},\mathrm{t}) \\
&= g\ |\Phi(\mathrm{r}',t)|^2\ \Phi(\mathrm{r}',t). \qquad \text{(C.6)}
\end{aligned}
$$

Finally we get

$$
i\hbar\frac{\partial}{\partial t}\Phi(\mathrm{r},t) = \left(-\frac{\hbar\nabla^2}{2m} + V_{\mathrm{ext}}(\mathrm{r}) + g\ |\Phi(\mathrm{r},t)|^2\right)\Phi(\mathrm{r},t). \qquad \text{(C.7)}
$$

# Appendix D: Baym's Approach

Now we show the detail calculation of the ground state energy by the mean field approach in Baym's work [51]. The energy functional is written as

$$E[\Phi(\mathrm{r})] = \int d^3\mathrm{r} \left[\frac{\hbar^2}{2m}|\nabla\Phi(\mathrm{r})|^2 + \frac{m}{2}(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2)|\Phi(\mathrm{r})|^2 + \frac{g}{2}|\Phi(\mathrm{r})|^2\right]. \quad \text{(D.1)}$$

where $r_\perp = x\ \hat{i} + y\ \hat{j}$. They chose the trial wavefunction in the form of the Gaussian

$$\Phi(\mathrm{r}) = N^{1/2}\Omega_\perp^{1/2}\Omega_z^{1/4}\left(\frac{m}{\pi\hbar}\right)^{3/4} e^{-m(\Omega_\perp r_\perp^2 + \Omega_z z^2)/2\hbar}. \quad \text{(D.2)}$$

Putting the trial wavefunction into the kinetic term, then we get

$$\begin{aligned}
\frac{\hbar^2}{2m}\int d^3\mathrm{r}|\nabla\Phi(\mathrm{r})|^2 &= \left(\frac{m}{\pi\hbar}\right)^{3/2}\frac{N\Omega_\perp\Omega_z^{1/2}\hbar^2}{2m} \\
&\quad \times \int d^3\mathrm{r}|\nabla e^{-m(\Omega_\perp r_\perp^2 + \Omega_z z^2)/2\hbar}|^2 \\
&= \left(\frac{m}{\pi\hbar}\right)^{3/2}\frac{N\Omega_\perp\Omega_z^{1/2}\hbar^2}{2m} \\
&\quad \times \frac{m^2}{\hbar^2}\int d^3\mathrm{r}\left(\Omega_\perp^2 r_\perp^2 + \Omega_z^2 z^2\right) e^{-m(\Omega_\perp r_\perp^2 + \Omega_z z^2)/\hbar} \quad \text{(D.3)}
\end{aligned}$$

Using formulae

$$\begin{aligned}
\int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx &= \frac{1}{2a}\sqrt{\frac{\pi}{a}} \\
\int_{-\infty}^{\infty} e^{-ax^2}\, dx &= \sqrt{\frac{\pi}{a}}, \quad \text{(D.4)}
\end{aligned}$$

then the $r_\perp$ component is integrated to be

$$\int d^3\mathrm{r}\ r_\perp^2 e^{-m(\Omega_\perp r_\perp^2 + \Omega_z z^2)/\hbar} = \pi^{3/2}\left(\frac{\hbar}{m\Omega_\perp}\right)^2\sqrt{\frac{\hbar}{m\Omega_z}}, \quad \text{(D.5)}$$

and the $z$ component can be obtained to be

$$\int d^3\mathrm{r}\; z^2 e^{-m(\Omega_\perp r_\perp^2+\Omega_z z^2)/\hbar} = \pi^{3/2}\left(\frac{\hbar}{m\Omega_z}\right)^{3/2}\frac{\hbar}{2m\Omega_\perp}. \tag{D.6}$$

Finally the kinetic energy term is

$$\begin{aligned}\frac{\hbar^2}{2m}\int d^3\mathrm{r}|\nabla\Phi(\mathrm{r})|^2 &= \left(\frac{m}{\pi\hbar}\right)^{3/2}\frac{N\Omega_\perp\Omega_z^{1/2}\hbar^2\pi^{3/2}}{2m}\\ &\times\left[\sqrt{\frac{\hbar}{m\Omega_z}}+\frac{1}{2\Omega_\perp}\sqrt{\frac{\hbar\Omega_z}{m}}\right]\\ &= N\hbar\left[\frac{\Omega_\perp}{2}+\frac{\Omega_z}{4}\right]. \end{aligned}\tag{D.7}$$

Next consider the harmonic term

$$\begin{aligned}\int d^3\mathrm{r}\frac{m}{2}(\omega_\perp^2 r_\perp^2+\omega_z^2 z^2)|\Phi(\mathrm{r})|^2 &= \left(\frac{m}{\pi\hbar}\right)^{3/2}\frac{N\Omega_\perp\Omega_z^{1/2}m}{2}\\ &\times\int d^3\mathrm{r}\left(\omega_\perp^2 r_\perp^2+\omega_z^2 z^2\right)e^{-m(\Omega_\perp r_\perp^2+\Omega_z z^2)/\hbar}\end{aligned}\tag{D.8}$$

then

$$\int d^3\mathrm{r}\;\omega_\perp^2 r_\perp^2\; e^{-m(\Omega_\perp r_\perp^2+\Omega_z z^2)/\hbar} = \omega_\perp^2\left(\frac{\pi\hbar}{m\Omega_\perp}\right)^2\sqrt{\frac{\pi\hbar}{m\Omega_z}} \tag{D.9}$$

and

$$\int d^3\mathrm{r}\;\omega_z^2 z^2\; e^{-m(\Omega_\perp r_\perp^2+\Omega_z z^2)/\hbar} = \omega_z^2\left(\frac{\pi\hbar}{2m\Omega_\perp}\right)\left(\frac{\pi\hbar}{m\Omega_z}\right)^{3/2}. \tag{D.10}$$

Substituting Eq. (D.9) and Eq. (D.10) into Eq. (D.8), we get

$$\int d^3\mathrm{r}\frac{m}{2}(\omega_\perp^2 r_\perp^2+\omega_z^2 z^2)|\Phi(\mathrm{r})|^2 = N\hbar\left[\frac{\omega_\perp^2}{2\Omega_\perp}+\frac{\omega_z^2}{4\Omega_z}\right]. \tag{D.11}$$

The interaction is straightforwardly evaluated

$$\begin{aligned}\frac{g}{2}\int d^3\mathrm{r}\;|\Phi(\mathrm{r})|^2 &= \frac{gN^2\Omega_\perp^2\Omega_z}{2}\left(\frac{m}{\pi\hbar}\right)^3\int d^3\mathrm{r}\; e^{-2m(\Omega_\perp r_\perp^2+\Omega_z z^2)/\hbar}\\ &= \frac{gN^2\Omega_\perp^2\Omega_z}{2}\left(\frac{m}{\pi\hbar}\right)^3\frac{\pi\hbar}{2m\Omega_\perp}\sqrt{\frac{\pi\hbar}{2m\Omega_z}}\\ &= N\hbar\left[Na\Omega_\perp\Omega_z^{1/2}\sqrt{\frac{m}{2\pi\hbar}}\right]. \end{aligned}\tag{D.12}$$

Putting Eq. (D.7), Eq. (D.11) and Eq. (D.12) into Eq. (D.1), we finally get

$$E(\Omega_\perp, \Omega_z) = N\hbar\left(\frac{\Omega_\perp}{2} + \frac{\omega_\perp^2}{2\Omega_\perp} + \frac{\Omega_z}{4} + \frac{\omega_z^2}{4\Omega_z} + Na\Omega_\perp\Omega_z^{1/2}\left(\frac{m}{2\pi\hbar}\right)^{1/2}\right). \quad \text{(D.13)}$$

# Appendix E: Numerical Program

The numerical results shown in Chapter 4 is done on *Mathematica*. Here we give the commands to obtain them. First we declare the constants in the calculation as follows: subscript 1 denotes $\perp$ and 2 denotes $z$.

In[1]:= $\omega 2 = 2$ Pi 220

In[2]:= $\omega 1 = \frac{\omega 2}{\sqrt{8}}$

In[3]:= $\mathrm{a} = 0.529 \times 10^{-8}$

In[4]:= $\mathrm{a1} = 1.222 \times 10^{-6}$

In[5]:= $\mathrm{hbar} = 1.055 \times 10^{-34}$

In each loop, we need to clear the remaining value in each function:

In[6]:= Clear[n, k, delta, $\Omega 1$, $\Omega 2$, en, en1, enkin, enho, enint]

In[7]:= n =*%put the number of particles%*

In[8]:= $\mathrm{k} = \frac{8\ \mathrm{Pi\ n\ a}}{\mathrm{a1}}$

In[9]:= Solve[$\mathrm{k} \times \sqrt{\frac{\omega 1}{32\times \mathrm{Pi}^3\ \Omega 2}} \times \left(1 + \mathrm{k} \times \sqrt{\frac{\Omega 2}{32\times\mathrm{Pi}^3\ \omega 1}}\right)^{-1/2} + 1 - \left(\frac{\omega 2}{\Omega 2}\right)^2 == 0$]

In[10]:= $\Omega 2 =$ *%put the value from the above command%*

In[11]:= $\mathrm{delta} = \sqrt{1 + \mathrm{k} \left(\frac{\Omega 2}{32\times\mathrm{Pi}^3\ \omega 1}\right)^{1/2}}$

In[12]:= $\Omega 1 = \frac{\omega 1}{\mathrm{delta}}$

In[13]:= $\mathrm{en} = \mathrm{n\ hbar} \left(\omega 1\ \mathrm{delta} + \frac{\Omega 2}{4} + \frac{\omega 2^2}{4\Omega 2}\right)$

In[14]:= $\mathrm{en1} = \frac{\mathrm{en}}{\mathrm{n\ hbar}\ \omega 1}$

In[15]:= $\mathrm{enkin} = \frac{\Omega 1}{2\omega 1} + \frac{\Omega 2}{4\omega 1}$

In[16]:= $\mathrm{enint} = \frac{\mathrm{n}\ 4.33\times 10^{-3}\ \Omega 1\ \Omega 2^{1/2}}{2\ \mathrm{Pi}\ \omega 1^3}$

# Vitae



Mr.Natthapon Nakpathomkun is born on Dec 31, 1973 in Bangkok. He is the sixth child of Mr.Wirat Nakpathomkun and Mrs.Nualjun Nakpathomkun. He received the bachelor degree of Science (physics) from King Mongkut's University of Technology Thonburi (KMUTT) in 1996.

After graduate from KMUTT he had been hired by Alpha-TI, the first wafer fabrication company in Thailand. His position is quality control engineer. He is one of a team member that had been sent by Alpha-TI to Texas Instrument, Dallas, in order to transfer 64 MDRAM technology. During graduate study he received the teaching assistantship from Chulalongkorn University in 1999.