CHAPTER IV

THE COMMUTATOR TECHNIQUES FOR PATH PROPAGATORS

4.1 INTRODUCTION

In this chapter we show that the commutator techniques has manifested very useful and actually added a new method to path integral calculation. We find that the commutator method has changed the non - integrable problems in the path integral with the classical Lagrangian of the action has non - quadratic form to integrable problems in the partial differential form in the Schrödinger picture via the procedure of the Heisenberg picture, which we call this procedure the Baker - Hausdorff Rule.

4.2 QUANTUM MECHANICAL COMMUTATOR

In chapter III we have an equation of motion showing that the commutator $AB - BA$ of two operators plays a very prominent role in quantum mechanics. It is therefore appropriate to conclude in this chapter with the introduction of a simple and useful notation for commutators and a review of some rule of operator. All these rules apply, in particular, to the commutator method for propagators.

By definition, the ***commutator*** is written as a square bracket:

$$[A, B] = AB - BA \tag{4.2.1}$$

With this notation the following elementary rules of calculation are almost self - evident. All are easy to verify.

$$[A, B] + [B, A] = 0 \qquad (4.2.2)$$

$$[A, A] = 0 \qquad (4.2.3)$$

$$[A , B + C] = [A , B] + [A , C] \qquad (4.2.4)$$

$$[A + B, C] = [A , C] + [B,C] \qquad (4.2.5)$$

$$[A, BC] = [A, B]C + B[A, C] \qquad (4.2.6)$$

$$[AB, C] = [A, C]B + A[B , C] \qquad (4.2.7)$$

$$[A, [B, C]] + [C, [A,B]] + [B, [C, A]] = 0 \qquad (4.2.8)$$

$$[A, B^n] = nB^{n-1} [A, B] \qquad (4.2.9)$$

$$[A^n, B] = nA^{n-1} [A, B] \qquad (4.1.10)$$

Note the similarity of this process with differentiation. Apply to the some special case

$A = \hat{x}, B = \hat{p}_x$ and $C = \hat{x}$ or $\hat{p}_x$ as follows :

$$\left[\hat{x}_i , G(\hat{p})\right] = i\hbar \frac{\partial G}{\partial p_i} \qquad (4.2.11)$$

$$\left[\hat{p}_i, F(x)\right] = -i\hbar \frac{\partial F}{\partial x_i} \tag{4.2.12}$$

$$\left[\hat{x}, \hat{p}_x\right] = i\hbar \tag{4.2.13}$$

$$\left[\hat{x}^2, \hat{p}_x^2\right] = 2i\hbar\left[2\hat{x}\hat{p}_x - i\hbar\right] \tag{4.2.14}$$

$$\left[\hat{x}^2, \hat{p}_x\right] = 2i\hbar\hat{x} \tag{4.2.15}$$

$$\left[\hat{x}^3, \hat{p}_x\right] = 3i\hbar\hat{x}^2 \tag{4.2.16}$$

$$\left[\hat{p}^2, \hat{x}\hat{p}\right] = -2i\hbar\hat{p}^2 \tag{4.2.17}$$

$$\left[\hat{x}^2, \hat{x}\hat{p}\right] = 2i\hbar\hat{x}^2. \tag{4.2.18}$$

4.3 The Commutator Method For Propagator

The Schrödinger equation for finding the wave function $\psi(x,t)$ is

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t). \tag{4.3.1}$$

To solve this equation , we know that the Delta function and the time - evolution operator $U(t,0)$ shift coordinate and time respectively. So that we write the new wave function as

$$\psi(x,t) = \int_{-\infty}^{\infty} U(t,0)\delta(x-\xi)\psi(\xi,0)d\xi \tag{4.3.2}$$

and define

$$K(x,\xi,t) = \delta(x-\xi)U(t,0) \tag{4.3.3}$$

Substituting eq. (4.3.3) into eq. (4.3.2) , we obtain

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x,\xi,t)\psi(\xi,0)d\xi \tag{4.3.4}$$

This result can be stated in physical terms. The total amplitude to arrive at (x,t) [that is, $\psi(x,t)$] is the sum, or integral, over all possible values of ξ of the total amplitude at the point $(\xi,0)$ [that is, $\psi(\xi,0)$] multiplied by the amplitude to go from ξ to x [that is, $K(x,\xi,t)$]. This means that the effects of all the past history of a particle can be expressed in terms of a single function. If we forget everything we knew about a particle except its wave function at a particutar time, then we can calculate everything that can happen to that particle after that time. All history's effects upon the future of the universe could be obtained from a single gigantic wavefunction[6] .

Thus, we can write K as

$$K(x,\xi,t) = \psi(x,t)\psi^*(\xi,0) \tag{4.3.5}$$

If we substitute eq. (4.3.5) into eq. (4.3.4), it becomes

$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x,t)\psi^*(\xi,0)\psi(\xi,0)d\xi \tag{4.3.6}$$

We then insert a completeness set of energy states into eq. (4.3.5), and obtain

$$K(x,\xi,t) = \sum_n \psi_n(x)\psi_n^*(\xi)e^{-\frac{i}{\hbar}E_n t} \tag{4.3.7}$$

where $\psi_n(x)$ and $\psi_n^*(\xi)$ are the eigenfunctions.

Form eq. (4.3.5), if we determine t = 0, it can be written as

$$K(x,\xi,0)=\delta(x-\xi) \tag{4.3.8}$$

and multipliy both sides by x, we obtain

$$xK(x,\xi,0)=x\delta(x-\xi). \tag{4.3.9}$$

From the property of delta function, we have

$$\delta(-y)=\delta(y). \tag{4.3.10}$$

Thus, eq. (4.3.9) can be written as

$$x\delta(x-\xi)=\xi(x-\xi)=\xi K(x,\xi,0) \tag{4.3.11}$$

If we let x in eq. (4.3.11) to be an operator, we can write

$$\hat{x}(0)K(x,\xi,0)=\xi K(x,\xi,0). \tag{4.3.12}$$

We can see that the function $K(x,\xi,0)$ is an eigenfunction of the operator $\hat{x}(0)$ with the eigenvalue ξ

Now, we consider that the time - dependent eigenfunction $K(x,\xi,t)$ has been operated by a coordinate operator in the Heisenberg picture.

For eq. (4.3.5) we can rewrite

$$K(x,\xi,t) = U(t,0)\psi(x,0)\psi^*(\xi,0)$$

$$= U(t,0)K(x,\xi,0). \tag{4.3.13}$$

Conversely, we write

$$K(x,\xi,0) = U^+(t,0)K(x,\xi,t) \tag{4.3.14}$$

and then we substitute eq. (4.3.14) into eq. (4.3.12), we obtain

$$\hat{x}(0)U^+(t,0)K(x,\xi,t) = \xi U^+(t,0)K(x,\xi,t). \tag{4.3.15}$$

Multiplying eq. (4.1.15) by $U(t,0)$, we obtain

$$U(t,0)\hat{x}(0)U^+(t,0)K(x,\xi,t) = \xi K(x,\xi,t). \tag{4.3.16}$$

Then, we compare eq. (4.3.16) with eq. (3.2.20), it can be written that

$$\hat{x}(-t)K(x,\xi,t) = \xi K(x,\xi,t). \tag{4.3.17}$$

where $\hat{x}(-t)$ is a coordinate operator in the Heisenberg picture.

4.3.1 The Propagator for Free Particle

From the result of eq. (3.2.22), we substitute $\hat{H}=\hat{p}^2/2m$ and due to $K(x,\xi,t)$ satisfying the Schrödinger equation. Using $\hat{x}(-t)$ has been operated on $K(x,\xi,t)$, we thus have

$$\hat{x}(-t)K(x,\xi,t)=\left(x+\frac{i\hbar t}{m}\frac{\partial}{\partial x}\right)K=\xi K(x,\xi,t). \tag{4.3.18}$$

The solution of this equation is of the form

$$K(x,\xi,t)\equiv K_0(t)\exp\left[\frac{-im}{2\hbar t}(x-\xi)^2\right] \tag{4.3.19}$$

where $K_0(t)$ is a time - dependent integration constant.

In order to find $K_0(t)$, we impose on (4.3.19) the requirement that it be a solution of the Schrödinger equation [8].

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)K=i\hbar\frac{\partial K}{\partial t}. \tag{4.3.20}$$

After calculation we obtain the equation

$$\left(\frac{\partial}{\partial t}-\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\right)K_0(t)\ \exp\left[\frac{-im}{2\hbar t}(x-\xi)^2\right]=0 \tag{4.3.21}$$

where the solution becomes

$$K_0(t)=C\sqrt{t}\ \exp\ \left(\frac{im}{\hbar t}(x-\xi)^2\right), \tag{4.3.22}$$

C being a normalization constant which can be found from substituting $K_0(t)$ in eq. (4.3.22) into eq. (4.3.19). We get

$$K(x,\xi,t)=C\ \sqrt{t}\ \exp\left(\frac{im}{2\hbar t}(x-\xi)^2\right) \tag{4.3.23}$$

and then, we find the normalization of the eq. (4.3.23)

$$C=\frac{1}{t}\sqrt{\frac{m}{2\pi i\hbar}}. \tag{4.3.24}$$

Thus , we obtain.

$$K(x,\xi,t) = \sqrt{\frac{m}{2\pi i\hbar}}\exp\left(\frac{im}{2\hbar t}(x-\xi)^2\right) \tag{4.3.25}$$

which is the propagator for free - particle in quantum mechanics.

4.3.2 The Propagator for a Simple Harmonic Oscillator

We can find the propagator for the harmonic oscillator by performing similarly to that of the free particle. From the coordinate operator $\hat{x}(t)$ in eq. (3.2.25). We use $\hat{x}(-t)$ to operate on $K(x,\xi,t)$, it can be written that

$$\hat{x}(-t)K = \left(x\ \cos(\omega t) + \frac{i\hbar\ \sin(\omega t)}{m\omega}\frac{\partial}{\partial x}\right)K = \xi K \tag{4.3.26}$$

and then , the solution is

$$K(x,\xi,t) = K_0(t)\exp\left(\frac{im\omega}{2\hbar\ \sin(\omega t)}\left(x^2\cos(\omega t) - 2x\xi\right)\right) \tag{4.3.27}$$

To find $K_0(t)$ using the Schrödinger equation[9]

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{m\omega^2x^2}{2}\right)K=i\hbar\frac{\partial K}{\partial t}. \tag{4.3.28}$$

After substituting $K(x,\xi,t)$ from eq. (4.3.12) into eq. (4.3.28), we can see that[9]

$$\frac{dK_0(t)}{dt}-\left(\frac{\omega}{2}\cot(\omega t)+\frac{im\omega^2\xi^2}{2\hbar\,\sin^2(\omega t)}\right)K_0(t)=0 \tag{4.3.29}$$

where the solution becomes

$$K_0(t)=\frac{C}{\sqrt{\sin(\omega t)}}\exp\left(\frac{im\omega\xi^2}{2\hbar}\cot(\omega t)\right) \tag{4.3.30}$$

Thus, the eq. (4.3.27) can be rewritten as

$$K(x,\xi,t)=\frac{C}{\sqrt{\sin(\omega t)}}\exp\left(\frac{im\omega}{2\hbar\ \sin(\omega t)}\left\{(x^2+\xi^2)\cos(\omega t)-2x\xi\right\}\right).$$

(4.3.31)

We can find C from the initial condition $(t\to 0)$ by comparing eq. (4.3.31) with the free - particle propagator (4.3.25), we obtain

$$C=\sqrt{\frac{-im\omega}{2\pi\hbar}}.$$

(4.3.32)

Finally, we have that

$$K(x,\xi,t)=\sqrt{\frac{m\omega}{2\pi i\hbar\ \sin(\omega t)}}\exp\left(\frac{im\omega}{2\hbar\ \sin(\omega t)}\left\{(x^2+\xi^2)\cos(\omega t)-2x\xi\right\}\right).$$

(4.3.33)

Then, this is the simple harmonic oscillator propagator.

4.3.3 The Propagator for Potential of an Oscillator with Quadratic and Cubic Terms

Now, we have the potential of system in the form

$$V(x)=ax^2+bx^3$$

and the ***Hamiltonian*** of the system can be written as

$$\hat{H} = \frac{\hat{p}^2}{2m} + ax^2 + bx^3, \tag{4.3.34}$$

where a and b are arbitrary constants, and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ momentum operator in coordinate representation in Schrödinger picture. Thus, we use the ***Hamiltonian*** in eq. (4.3.34) to substitute into eq. (3.2.22), we obtain the coordinate operator in the ***Heisenberg*** picture which to be equal to the left - hand side operator in the Schrödinger picture, by using Baker - Hausdroff lemma

$$\hat{x}(t) = \hat{x}(0) + \left(\frac{it}{\hbar}\right)\left[\frac{\hat{p}^2}{2m} + ax^2 + bx^3, x\right] + \left(\frac{i^2t^2}{2!\hbar^2}\right)\left[\frac{\hat{p}^2}{2m} + ax^2 + bx^3, \left[\frac{\hat{p}^2}{2m} + ax^2 + bx^2, x\right]\right]$$

$$+\left(\frac{i^3t^3}{3!\hbar^3}\right)\left[\frac{\hat{p}^2}{2m} + ax^2 + bx^2, \left[\frac{\hat{p}^2}{2m} + ax^2 + bx^3, \left[\frac{\hat{p}^2}{2m} + ax^2 + bx^3, x\right]\right]\right] + \ldots \tag{4.3.35}$$

where the commutator sequences in eq. (4.3.35) have been calculated by programming in ***Mathematica*** (see Appendix A) as follows:

H = p ** p / (2 m) + a x ** x + b x ** x ** x

$$\frac{p ** p}{2\ m} + a\ x ** x + b\ x ** x ** x$$

[H, x] =

$$-\frac{I\ hb\ p}{m} \tag{4.3.36a}$$

[H, [H, x]] =

$$\frac{2\ a\ hb^2\ x}{m} + \frac{3\ b\ hb^2\ x ** x}{m} \tag{4.3.36b}$$

[H, [H, [H, x]]] =

$$-\frac{2\ I\ a\ hb^3\ p}{m^2} - \frac{3\ b\ hb^4\ U}{m^2} - \frac{6\ I\ b\ hb^3\ x ** p}{m^2} \tag{4.3.36c}$$

[H, [H, [H, [H, x]]]] =

$$\frac{4\ a^2\ hb^4\ x}{m^2} - \frac{6\ b\ hb^4\ p ** p}{m^3} + \frac{18\ a\ b\ hb^4\ x ** x}{m^2} + \frac{18\ b^2\ hb^4\ x ** x ** x}{m^2} \tag{4.3.36d}$$

[H, [H, [H, [H, [H, x]]]]] =

$$-\frac{4\ I\ a^2\ hb^5\ p}{m^3} - \frac{30\ a\ b\ hb^6\ U}{m^3} - \frac{90\ b^2\ hb^6\ x}{m^3} - \frac{60\ I\ a\ b\ hb^5\ x ** p}{m^3} - \frac{90\ I\ b^2\ hb^5\ x ** x ** p}{m^3} \tag{4.3.36e}$$

[H, [H, [H, [H, [H, [H, x]]]]]] =

$$\frac{180\ I\ b^2\ hb^7\ p}{m^4} + \frac{8\ a^3\ hb^6\ x}{m^3} - \frac{60\ a\ b\ hb^6\ p ** p}{m^4} + \frac{132\ a^2\ b\ hb^6\ x ** x}{m^3} - \frac{180\ b^2\ hb^6\ x ** p ** p}{m^4} + \frac{360\ a\ b^2\ hb^6\ x ** x ** x}{m^3} + \frac{270\ b^3\ hb^6\ x ** x ** x ** x}{m^3} \tag{4.3.36f}$$

$$[H, [H, [H, [H, [H, [H, [H, x]]]]]]] =$$

$$-\frac{8\, I\, a^{3}\, hb^{7}\, p}{m^{4}} - \frac{252\, a^{2}\, b\, hb^{8}\, U}{m^{4}} - \frac{2160\, a\, b^{2}\, hb^{8}\, x}{m^{4}} - \frac{504\, I\, a^{2}\, b\, hb^{7}\, x \ast\ast p}{m^{4}} - \frac{3240\, b^{3}\, hb^{8}\, x \ast\ast x}{m^{4}} + \frac{180\, I\, b^{2}\, hb^{7}\, p \ast\ast p \ast\ast p}{m^{5}} - \frac{2160\, I\, a\, b^{2}\, hb^{7}\, x \ast\ast x \ast\ast p}{m^{4}} - \frac{2160\, I\, b^{3}\, hb^{7}\, x \ast\ast x \ast\ast x \ast\ast p}{m^{4}} \qquad (4.3.36g)$$

$$[H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]] =$$

$$\frac{5400\, I\, a\, b^{2}\, hb^{9}\, p}{m^{5}} + \frac{4320\, b^{3}\, hb^{10}\, U}{m^{5}} + \frac{16\, a^{4}\, hb^{8}\, x}{m^{4}} - \frac{504\, a^{2}\, b\, hb^{8}\, p \ast\ast p}{m^{5}} + \frac{16200\, I\, b^{3}\, hb^{9}\, x \ast\ast p}{m^{5}} + \frac{1032\, a^{3}\, b\, hb^{8}\, x \ast\ast x}{m^{4}} - \frac{5400\, a\, b^{2}\, hb^{8}\, x \ast\ast p \ast\ast p}{m^{5}} + \frac{5832\, a^{2}\, b^{2}\, hb^{8}\, x \ast\ast x \ast\ast x}{m^{4}} - \frac{8100\, b^{3}\, hb^{8}\, x \ast\ast x \ast\ast p \ast\ast p}{m^{5}} + \frac{10800\, a\, b^{3}\, hb^{8}\, x \ast\ast x \ast\ast x \ast\ast x}{m^{4}} + \frac{6480\, b^{4}\, hb^{8}\, x \ast\ast x \ast\ast x \ast\ast x \ast\ast x}{m^{4}} \qquad (4.3.36h)$$

$$[H, [H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]]] =$$

$$-\frac{16\, I\, a^{4}\, hb^{9}\, p}{m^{5}} - \frac{2040\, a^{3}\, b\, hb^{10}\, U}{m^{5}} - \frac{42120\, a^{2}\, b^{2}\, hb^{10}\, x}{m^{5}} + \frac{24300\, b^{3}\, hb^{10}\, p \ast\ast p}{m^{6}} - \frac{4080\, I\, a^{3}\, b\, hb^{9}\, x \ast\ast p}{m^{5}} - \frac{162000\, a\, b^{3}\, hb^{10}\, x \ast\ast x}{m^{5}} + \frac{5400\, I\, a\, b^{2}\, hb^{9}\, p \ast\ast p \ast\ast p}{m^{6}} - \frac{42120\, I\, a^{2}\, b^{2}\, hb^{9}\, x \ast\ast x \ast\ast p}{m^{5}} - \frac{162000\, b^{4}\, hb^{10}\, x \ast\ast x \ast\ast x}{m^{5}} + \frac{16200\, I\, b^{3}\, hb^{9}\, x \ast\ast p \ast\ast p \ast\ast p}{m^{6}} - \frac{108000\, I\, a\, b^{3}\, hb^{9}\, x \ast\ast x \ast\ast x \ast\ast p}{m^{5}} - \frac{81000\, I\, b^{4}\, hb^{9}\, x \ast\ast x \ast\ast x \ast\ast x \ast\ast p}{m^{5}} \qquad (4.3.36i)$$

$$[H, [H, [H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]]]] =$$

$$\frac{116640\, I\, a^2\, b^2\, hb^{11}\, p}{m^6} + \frac{243000\, a\, b^3\, hb^{12}\, U}{m^6} + \frac{729000\, b^4\, hb^{12}\, x}{m^6} + \frac{32\, a^5\, hb^{10}\, x}{m^5} - \frac{4080\, a^3\, b\, hb^{10}\, p ** p}{m^6} + \frac{939600\, I\, a\, b^3\, hb^{11}\, x ** p}{m^6} + \frac{8208\, a^4\, b\, hb^{10}\, x ** x}{m^5} - \frac{116640\, a^2\, b^2\, hb^{10}\, x ** p ** p}{m^6} + \frac{1409400\, I\, b^4\, hb^{11}\, x ** x ** p}{m^6} + \frac{96480\, a^3\, b^2\, hb^{10}\, x ** x ** x}{m^5} + \frac{16200\, b^3\, hb^{10}\, p ** p ** p ** p}{m^7} - \frac{469800\, a\, b^3\, hb^{10}\, x ** x ** p ** p}{m^6} + \frac{342360\, a^2\, b^3\, hb^{10}\, x ** x ** x ** x}{m^5} - \frac{469800\, b^4\, hb^{10}\, x ** x ** x ** p ** p}{m^6} + \frac{486000\, a\, b^4\, hb^{10}\, x ** x ** x ** x ** x}{m^5} + \frac{243000\, b^5\, hb^{10}\, x ** x ** x ** x ** x ** x}{m^5}$$

(4.3.36j)

$$[H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]]]]] =$$

$$-\frac{2527200\, I\, b^4\, hb^{13}\, p}{m^7} - \frac{32\, I\, a^5\, hb^{11}\, p}{m^6} - \frac{16368\, a^4\, b\, hb^{12}\, U}{m^6} - \frac{780480\, a^3\, b^2\, hb^{12}\, x}{m^6} + \frac{1603800\, a\, b^3\, hb^{12}\, p ** p}{m^7} - \frac{32736\, I\, a^4\, b\, hb^{11}\, x ** p}{m^6} - \frac{5922720\, a^2\, b^3\, hb^{12}\, x ** x}{m^6} + \frac{116640\, I\, a^2\, b^2\, hb^{11}\, p ** p ** p}{m^7} + \frac{4811400\, b^4\, hb^{12}\, x ** p ** p}{m^7} - \frac{780480\, I\, a^3\, b^2\, hb^{11}\, x ** x ** p}{m^6} - \frac{14256000\, a\, b^4\, hb^{12}\, x ** x ** x}{m^6} + \frac{1069200\, I\, a\, b^3\, hb^{11}\, x ** p ** p ** p}{m^7} - \frac{3948480\, I\, a^2\, b^3\, hb^{11}\, x ** x ** x ** p}{m^6} - \frac{10692000\, b^5\, hb^{12}\, x ** x ** x ** x}{m^6} + \frac{1603800\, I\, b^4\, hb^{11}\, x ** x ** p ** p ** p}{m^7} - \frac{7128000\, I\, a\, b^4\, hb^{11}\, x ** x ** x ** x ** p}{m^6} - \frac{4276800\, I\, b^5\, hb^{11}\, x ** x ** x ** x ** x ** p}{m^6}$$

(4.3.36k)

$$[\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, [\text{H}, \text{x}]]]]]]]]]]]] =$$

$$\frac{2260800\, I\, a^3\, b^2\, hb^{13}\, p}{m^7} + \frac{9830160\, a^2\, b^3\, hb^{14}\, U}{m^7} + \frac{73483200\, a\, b^4\, hb^{14}\, x}{m^7} +$$

$$\frac{64\, a^6\, hb^{12}\, x}{m^6} - \frac{32736\, a^4\, b\, hb^{12}\, p \ast\ast p}{m^7} + \frac{38620800\, I\, a^2\, b^3\, hb^{13}\, x \ast\ast p}{m^7} +$$

$$\frac{110224800\, b^5\, hb^{14}\, x \ast\ast x}{m^7} + \frac{65568\, a^5\, b\, hb^{12}\, x \ast\ast x}{m^6} - \frac{6415200\, I\, b^4\, hb^{13}\, p \ast\ast p \ast\ast p}{m^8} -$$

$$\frac{2260800\, a^3\, b^2\, hb^{12}\, x \ast\ast p \ast\ast p}{m^7} + \frac{143272800\, I\, a\, b^4\, hb^{13}\, x \ast\ast x \ast\ast p}{m^7} +$$

$$\frac{1659168\, a^4\, b^2\, hb^{12}\, x \ast\ast x \ast\ast x}{m^6} + \frac{1069200\, a\, b^3\, hb^{12}\, p \ast\ast p \ast\ast p \ast\ast p}{m^8} -$$

$$\frac{19310400\, a^2\, b^3\, hb^{12}\, x \ast\ast x \ast\ast p \ast\ast p}{m^7} + \frac{143272800\, I\, b^5\, hb^{13}\, x \ast\ast x \ast\ast x \ast\ast p}{m^7} +$$

$$\frac{10238400\, a^3\, b^3\, hb^{12}\, x \ast\ast x \ast\ast x \ast\ast x}{m^6} + \frac{3207600\, b^4\, hb^{12}\, x \ast\ast p \ast\ast p \ast\ast p \ast\ast p}{m^8} -$$

$$\frac{47757600\, a\, b^4\, hb^{12}\, x \ast\ast x \ast\ast x \ast\ast p \ast\ast p}{m^7} + \frac{26101440\, a^2\, b^4\, hb^{12}\, x \ast\ast x \ast\ast x \ast\ast x \ast\ast x}{m^6} -$$

$$\frac{35818200\, b^5\, hb^{12}\, x \ast\ast x \ast\ast x \ast\ast x \ast\ast p \ast\ast p}{m^7} +$$

$$\frac{29937600\, a\, b^5\, hb^{12}\, x \ast\ast x \ast\ast x \ast\ast x \ast\ast x \ast\ast x}{m^6} +$$

$$\frac{12830400\, b^6\, hb^{12}\, x \ast\ast x \ast\ast x \ast\ast x \ast\ast x \ast\ast x \ast\ast x}{m^6} \qquad (4.3.361)$$

[H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]]]]]]] =

$$-\frac{280908000\,I\,a\,b^4\,hb^{15}\,p}{m^8}-\frac{64\,I\,a^6\,hb^{13}\,p}{m^7}-\frac{148716000\,b^5\,hb^{16}\,U}{m^8}-$$

$$\frac{131040\,a^5\,b\,hb^{14}\,U}{m^7}-\frac{14217120\,a^4\,b^2\,hb^{14}\,x}{m^7}+\frac{70761600\,a^2\,b^3\,hb^{14}\,p**p}{m^8}-$$

$$\frac{842724000\,I\,b^5\,hb^{15}\,x**p}{m^8}-\frac{262080\,I\,a^5\,b\,hb^{13}\,x**p}{m^7}-\frac{197640000\,a^3\,b^3\,hb^{14}\,x**x}{m^7}+$$

$$\frac{2260800\,I\,a^3\,b^2\,hb^{13}\,p**p**p}{m^8}+\frac{545292000\,a\,b^4\,hb^{14}\,x**p**p}{m^8}-$$

$$\frac{14217120\,I\,a^4\,b^2\,hb^{13}\,x**x**p}{m^7}-\frac{874800000\,a^2\,b^4\,hb^{14}\,x**x**x}{m^7}+$$

$$\frac{47174400\,I\,a^2\,b^3\,hb^{13}\,x**p**p**p}{m^8}+\frac{817938000\,b^5\,hb^{14}\,x**x**p**p}{m^8}-$$

$$\frac{131760000\,I\,a^3\,b^3\,hb^{13}\,x**x**x**p}{m^7}-\frac{1523610000\,a\,b^5\,hb^{14}\,x**x**x**x}{m^7}-$$

$$\frac{3207600\,I\,b^4\,hb^{13}\,p**p**p**p**p}{m^9}+\frac{181764000\,I\,a\,b^4\,hb^{13}\,x**x**p**p**p}{m^8}-$$

$$\frac{437400000\,I\,a^2\,b^4\,hb^{13}\,x**x**x**x**p}{m^7}-$$

$$\frac{914166000\,b^6\,hb^{14}\,x**x**x**x**x}{m^7}+$$

$$\frac{181764000\,I\,b^5\,hb^{13}\,x**x**x**p**p**p}{m^8}-$$

$$\frac{609444000\,I\,a\,b^5\,hb^{13}\,x**x**x**x**x**p}{m^7}-$$

$$\frac{304722000\,I\,b^6\,hb^{13}\,x**x**x**x**x**x**p}{m^7} \qquad (4.3.36m)$$

[H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]]]]]]]] =

$$\frac{41999040\, I\, a^4\, b^2\, hb^{15}\, p}{m^8} + \frac{352728000\, a^3\, b^3\, hb^{16}\, U}{m^8} + \frac{4984416000\, a^2\, b^4\, hb^{16}\, x}{m^8} +$$

$$\frac{128\, a^7\, hb^{14}\, x}{m^7} - \frac{1853118000\, b^5\, hb^{16}\, p ** p}{m^9} - \frac{262080\, a^5\, b\, hb^{14}\, p ** p}{m^8} +$$

$$\frac{1397347200\, I\, a^3\, b^3\, hb^{15}\, x ** p}{m^8} + \frac{17668044000\, a\, b^5\, hb^{16}\, x ** x}{m^8} +$$

$$\frac{524352\, a^6\, b\, hb^{14}\, x ** x}{m^7} - \frac{791208000\, I\, a\, b^4\, hb^{15}\, p ** p ** p}{m^9} -$$

$$\frac{41999040\, a^4\, b^2\, hb^{14}\, x ** p ** p}{m^8} + \frac{9794260800\, I\, a^2\, b^4\, hb^{15}\, x ** x ** p}{m^8} +$$

$$\frac{17668044000\, b^6\, hb^{16}\, x ** x ** x}{m^8} + \frac{29220480\, a^5\, b^2\, hb^{14}\, x ** x ** x}{m^7} +$$

$$\frac{47174400\, a^2\, b^3\, hb^{14}\, p ** p ** p ** p}{m^9} - \frac{2373624000\, I\, b^5\, hb^{15}\, x ** p ** p ** p}{m^9} -$$

$$\frac{698673600\, a^3\, b^3\, hb^{14}\, x ** x ** p ** p}{m^8} + \frac{23094720000\, I\, a\, b^5\, hb^{15}\, x ** x ** x ** p}{m^8} +$$

$$\frac{306171360\, a^4\, b^3\, hb^{14}\, x ** x ** x ** x}{m^7} + \frac{395604000\, a\, b^4\, hb^{14}\, x ** p ** p ** p ** p}{m^9} -$$

$$\frac{3264753600\, a^2\, b^4\, hb^{14}\, x ** x ** x ** p ** p}{m^8} +$$

$$\frac{17321040000\, I\, b^6\, hb^{15}\, x ** x ** x ** x ** p}{m^8} +$$

$$\frac{1270080000\, a^3\, b^4\, hb^{14}\, x ** x ** x ** x ** x}{m^7} +$$

$$\frac{593406000\, b^5\, hb^{14}\, x ** x ** p ** p ** p ** p}{m^9} -$$

$$\frac{5773680000\, a\, b^5\, hb^{14}\, x ** x ** x ** x ** p ** p}{m^8} +$$

$$\frac{2531088000\, a^2\, b^5\, hb^{14}\, x ** x ** x ** x ** x ** x}{m^7} -$$

$$\frac{3464208000\, b^6\, hb^{14}\, x ** x ** x ** x ** x ** p ** p}{m^8} +$$

$$\frac{2437776000\, a\, b^6\, hb^{14}\, x ** x ** x ** x ** x ** x ** x}{m^7} +$$

$$\frac{914166000\, b^7\, hb^{14}\, x ** x ** x ** x ** x ** x ** x ** x}{m^7} \qquad (4.3.36n)$$

$$[H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, [H, x]]]]]]]]]]]]]]] =$$

$$-\frac{20658110400\, I\, a^2\, b^4\, hb^{17}\, p}{m^9} - \frac{128\, I\, a^7\, hb^{15}\, p}{m^8} - \frac{26121528000\, a\, b^5\, hb^{18}\, U}{m^9} -$$

$$\frac{1048512\, a^6\, b\, hb^{16}\, U}{m^8} - \frac{78364584000\, b^6\, hb^{18}\, x}{m^9} - \frac{257230080\, a^5\, b^2\, hb^{16}\, x}{m^8} +$$

$$\frac{2662113600\, a^3\, b^3\, hb^{16}\, p ** p}{m^9} - \frac{150010704000\, I\, a\, b^5\, hb^{17}\, x ** p}{m^9} -$$

$$\frac{2097024\, I\, a^6\, b\, hb^{15}\, x ** p}{m^8} - \frac{6407061120\, a^4\, b^3\, hb^{16}\, x ** x}{m^8} +$$

$$\frac{41999040\, I\, a^4\, b^2\, hb^{15}\, p ** p ** p}{m^9} + \frac{40575556800\, a^2\, b^4\, hb^{16}\, x ** p ** p}{m^9} -$$

$$\frac{225016056000\, I\, b^6\, hb^{17}\, x ** x ** p}{m^9} - \frac{257230080\, I\, a^5\, b^2\, hb^{15}\, x ** x ** p}{m^8} -$$

$$\frac{47202912000\, a^3\, b^4\, hb^{16}\, x ** x ** x}{m^8} - \frac{2967030000\, b^5\, hb^{16}\, p ** p ** p ** p}{m^{10}} +$$

$$\frac{1774742400\, I\, a^3\, b^3\, hb^{15}\, x ** p ** p ** p}{m^9} + \frac{146651472000\, a\, b^5\, hb^{16}\, x ** x ** p ** p}{m^9} -$$

$$\frac{4271374080\, I\, a^4\, b^3\, hb^{15}\, x ** x ** x ** p}{m^8} - \frac{144674424000\, a^2\, b^5\, hb^{16}\, x ** x ** x ** x}{m^8} -$$

$$\frac{395604000\, I\, a\, b^4\, hb^{15}\, p ** p ** p ** p ** p}{m^{10}} +$$

$$\frac{13525185600\, I\, a^2\, b^4\, hb^{15}\, x ** x ** p ** p ** p}{m^9} +$$

$$\frac{146651472000\, b^6\, hb^{16}\, x ** x ** x ** p ** p}{m^9} -$$

$$\frac{23601456000\, I\, a^3\, b^4\, hb^{15}\, x ** x ** x ** x ** p}{m^8} -$$

$$\frac{196690032000\, a\, b^6\, hb^{16}\, x ** x ** x ** x ** x}{m^8} -$$

$$\frac{1186812000\, I\, b^5\, hb^{15}\, x ** p ** p ** p ** p ** p}{m^{10}} +$$

$$\frac{32589216000\, I\, a\, b^5\, hb^{15}\, x ** x ** x ** p ** p ** p}{m^9} -$$

$$\frac{57869769600\, I\, a^2\, b^5\, hb^{15}\, x ** x ** x ** x ** x ** p}{m^8} -$$

$$\frac{98345016000\, b^7\, hb^{16}\, x ** x ** x ** x ** x ** x}{m^8} +$$

$$\frac{24441912000\ I\ b^6\ hb^{15}\ x ** x ** x ** x ** p ** p ** p}{m^9} -$$

$$\frac{65563344000\ I\ a\ b^6\ hb^{15}\ x ** x ** x ** x ** x ** x ** p}{m^8} -$$

$$\frac{28098576000\ I\ b^7\ hb^{15}\ x ** x ** x ** x ** x ** x ** x ** p}{m^8} \qquad (4.3.36p)$$

Form this, in eq. (4.3.36a - 4.3.36p), we consider approximately to select specifically x, x^2, xp, p and U terms, and first-order of arbitrary constant b terms. Thus, the right-hand side in eq. (4.3.35) can be separated to five parts and rearranged that

$$\left\{1-\frac{2at^2}{2!m}+\frac{4a^2}{4!m^2}-\frac{8a^3t^6}{6!m^3}+\frac{16a^4t^8}{8!m^4}-\frac{32a^5t^{10}}{10!m^5}+\frac{64a^6t^{12}}{12!m^6}-\frac{128a^7t^{14}}{14!m^7}+\ldots\ldots\right\}x$$

$$=\left\{1-\frac{\omega^2t^2}{2!}+\frac{\omega^4t^4}{4!}-\frac{\omega^6t^6}{6!}+\frac{\omega^8t^8}{8!}-\frac{\omega^{10}t^{10}}{10!}+\frac{\omega^{12}t^{12}}{12!}-\frac{\omega^{14}t^{14}}{14!}\ldots\ldots\right\}x$$

$$=x\ \cos(\omega t), \tag{4.3.37}$$

$$\left\{\frac{-3bt^2}{2!m}+\frac{18abt^4}{4!m^2}-\frac{132a^2bt^6}{6!m^3}+\frac{1032a^3bt^8}{8!m^4}-\frac{8208a^4bt^{10}}{10!m^5}+\frac{65568a^5bt^{12}}{12!m^6}-\ldots\ldots\right\}x^2$$

$$=\left\{\frac{-3t^2}{2!}+\frac{9\omega^2t^4}{4!}-\frac{33\omega^4t^6}{6!}+\frac{129\omega^6t^8}{8!}-\frac{513\omega^8t^{10}}{10!}+\frac{2049\omega^{10}t^{12}}{12!}-\ldots\ldots\right\}\frac{bx^2}{m}$$

$$=\left\{\frac{-3\omega^2t^2}{2!}+\frac{9\omega^4t^4}{4!}-\frac{33\omega^6t^6}{6!}+\frac{129\omega^8t^8}{8!}-\frac{513\omega^{10}t^{10}}{10!}+\frac{2049\omega^{12}t^{12}}{12!}-.......\right\}\frac{bx^2}{m\omega^2}$$

$$=\left(\cos(\omega t)+\frac{\cos(2\omega t)}{2}-\frac{3}{2}\right)\frac{bx^2}{m\omega^2}, \tag{4.3.38}$$

$$\left\{\frac{-6bt^3}{3!m^2}+\frac{60abt^5}{5!m^3}-\frac{504a^2bt^7}{7!m^4}+\frac{4080a^3bt^9}{9!m^5}-\frac{32736a^4bt^{11}}{11!m^6}+\frac{262080a^5bt^{13}}{13!m^7}-.......\right\}xp$$

$$=\left\{\frac{-3\omega^3t^3}{3!}+\frac{15\omega^5t^5}{5!}-\frac{63\omega^7t^7}{7!}+\frac{255\omega^9t^9}{9!}-\frac{1023\omega^{11}t^{11}}{11!}+\frac{4095\omega^{13}t^{13}}{13!}-.......\right\}\frac{2b}{m^2\omega^3}xp$$

$$=\left(\frac{1}{2}\sin(2\omega t)-\sin(\omega t)\right)\frac{2b}{m^2\omega^2}xp, \tag{4.3.39}$$

$$\left\{\frac{t}{m}-\frac{2at^3}{3!m^2}+\frac{4a^2t^5}{5!m^3}-\frac{8a^3t^7}{7!m^4}+\frac{16a^4t^9}{9!m^5}-\frac{32a^5t^{11}}{11!m^6}+\frac{64a^6t^{13}}{13!m^7}-\frac{128a^7t^{15}}{15!m^8}+.......\right\}p$$

$$=\left\{t\omega-\frac{\omega^3t^3}{3!}+\frac{\omega^5t^5}{5!}-\frac{\omega^7t^7}{7!}+\frac{\omega^9t^9}{9!}-\frac{\omega^{11}t^{11}}{11!}+\frac{\omega^{13}t^{13}}{13!}-\frac{\omega^{15}t^{15}}{15!}+.......\right\}\frac{p}{m\omega}$$

$$=\frac{\sin(\omega t)}{m\omega}p, \qquad (4.3.40)$$

$$\left\{\frac{3t^3}{3!m^2}-\frac{30at^5}{5!m^3}+\frac{252a^2t^7}{7!m^4}-\frac{2040a^3t^9}{9!m^5}+\frac{16368a^4t^{11}}{11!m^6}-\frac{131040a^5t^{13}}{13!m^7}+.......\right\}bi\hbar U$$

$$=\left\{\frac{3\omega^3t^3}{3!}-\frac{15\omega^5t^5}{5!}+\frac{63\omega^7t^7}{7!}-\frac{255\omega^9t^9}{9!}+\frac{1023\omega^{11}t^{11}}{11!}-\frac{4095\omega^{13}t^{13}}{13!}+.......\right\}\frac{bi\hbar}{m^2\omega^3}U$$

$$=\left(\sin(\omega t)-\frac{1}{2}\sin(2\omega t)\right)\frac{bi\hbar}{m^2\omega^3}U, \qquad (4.3.41)$$

which we have used $a=m\omega^2/2$. Therefore, when $\hat{x}(-t)$ has operated on the eigenfunction $K(x,\xi,t)$, then the result of addition of the eq. (4.3.37), eq.(4.3.38) eq. (4.3.39), eq. (4.3.40) and eq. (4.3.41) operated on $K(x,\xi,t)$, has been written as

$$\hat{x}(-t)K=\left(x\cos(\omega t)+\frac{bx^2}{2m\omega^2}\cos(\omega t)+\frac{bx^2}{2m\omega^2}\cos(2\omega t)-\frac{3bx^2}{2m\omega^2}-\frac{bx}{m^2\omega^3}\sin(2\omega t)p\right)K$$

$$+\left(\frac{2bx}{m^2\omega^3}\sin(\omega t)p-\frac{\sin(\omega t)}{m\omega}p-\frac{bi\hbar\ \sin(\omega t)}{m^2\omega^3}U+\frac{bi\hbar\ \sin(2\omega t)}{2m^2\omega^3}U\right)K=\xi K.$$

(4.3.42)

Considering the eq. (4.3.42) in configuration space we obtain

$$\left(x\ \cos(\omega t)+\frac{bx^2\cos(\omega t)}{m\omega^2}+\frac{bx^2\cos(2\omega t)}{2m\omega^2}-\frac{3bx^2}{2m\omega^2}-\frac{bi\hbar\ \sin(\omega t)}{m^2\omega^3}+\frac{bi\hbar\ \sin(2\omega t)}{2m^2\omega^3}-\xi\right)K+$$

$$\left(\frac{bi\hbar x\ \sin(2\omega t)}{m^2\omega^3}\frac{-2bi\hbar x\ \sin(\omega t)}{m^2\omega^3}+\frac{i\hbar\ \sin(\omega t)}{m\omega}\right)\frac{\partial K}{\partial x}=0$$

(4.3.43a)

Form eq. (4.3.43a), we can write in the integrated form

$$\ln k+f(t)=\frac{im\omega}{\hbar\ \sin\ \omega t}\int\left(\frac{1}{\left(1-\frac{4bx}{m\omega^2}\sin^2\left(\frac{\omega t}{2}\right)\right)}\left(x\ \cos\ \omega t-\xi+\frac{bx^2}{m\omega^2}\ \cos(\omega t)\right.\right.$$

$$\left.\left.+\frac{bx^2}{2m\omega^2}\cos(2\omega t)-\frac{i\hbar b}{m^2\omega^3}\sin\ \omega t+\frac{i\hbar b}{2m^2\omega^3}\sin(2\omega t)-\frac{3bx^2}{2m\omega^2}\right)\right)dx.$$

(4.3.43b)

Then, we expand $\left(1-\frac{4bx}{m\omega^2}\sin^2\frac{\omega t}{2}\right)^{-1}$ in Taylor's series when $bx^3 << ax^2$, we rewrite eq. (4.3.43b) that

$$\ln k + f(t) = \frac{im\omega}{\hbar\ \sin\ \omega t}\int\left\{\left(1+\frac{4bx}{m\omega^2}\sin^2\frac{\omega t}{2}\right)\left(x\ \cos\ \omega t - \xi + \frac{bx^2}{m\omega^2}\ \cos\ \omega t\right.\right.$$

$$\left.\left. + \frac{bx^2}{2m\omega^2}\cos 2\omega t - \frac{i\hbar b}{m^2\omega^3}\sin\ \omega t + \frac{i\hbar b}{2m^2\omega^3}\sin 2\omega t - \frac{3bx^3}{2m\omega^2}\right)\right\}dx. \quad (4.3.43c)$$

We can use computer to calculate eq. (4.3.43c) (see Appendix B).

And finally, after normalization G 4, we get propagator

$$K(x,\xi,t) = \sqrt{\frac{m\omega}{2\pi i\hbar\ \sin(\omega t)}}\left(\cos\left(\frac{\omega t}{2}\right)\right)^{-\frac{4b}{m\omega^2}(2x+\xi)} \exp\left[\frac{im\omega}{2\hbar\ \sin(\omega t)}\left\{\left(x^2+\xi^2\right)(\cos(\omega t) - 2x\xi)\right\}\right.$$

$$+b\left\{\frac{x}{m\omega^2}\left(3-4\ \sin^2\left(\frac{\omega t}{2}\right)\right) - \frac{3i\xi}{m\omega^2} + \frac{ix^2}{\hbar}(2x+3\xi)t\right.$$

$$\left.\left. -\frac{ix^3}{2\hbar\omega}\left(\sin(\omega t) + \tan\left(\frac{\omega t}{2}\right)\cos(\omega t) + 7\tan\left(\frac{\omega t}{2}\right)\right) - \frac{ix\xi}{\hbar\omega}(5x+2\xi)\tan\left(\frac{\omega t}{2}\right)\right\}\right]$$

(4.3.44)