

# CHAPTER IV

## APPLICATION OF QUANTUM BROWNIAN MOTION ON A VORTEX ESCAPING OUT OF A METASTABLE POTENTIAL

### 4.1 MAGNUS FORCE ON VORTEX AND HAMILTONIAN OF THE PROBLEM

"Magnus force" or sometimes called "lift force" has been discovered for a very long time ago within the domain of classical hydrodynamics. This force occurs and acts on a body when a body with a flow circulation around it moves through a fluid (Kutta-Joukowski theorem). For example, when rotating the cylinder clockwise in the fluid which initially has a uniform flow from left to right, the cylinder then feels the Magnus force acts in the upward direction (see fig. 4.1). This comes from the fact that the velocity at the top is greater than at the bottom of the cylinder which implies due to the Bernoulli's equation that the pressure at the top is lesser than at the bottom of the cylinder which again implies that the force acting at the top is lesser than at the bottom and the net force is in the upward direction because of the symmetry of the cylinder.

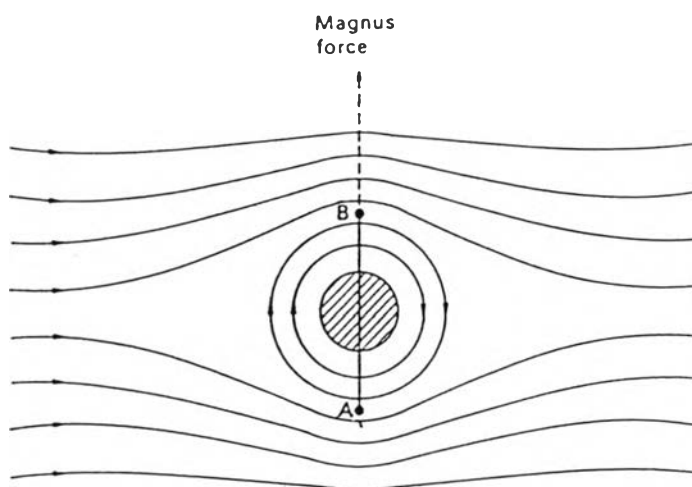


Fig. 4.1 : Magnus force on the rotating cylinder (see ref. [25]).

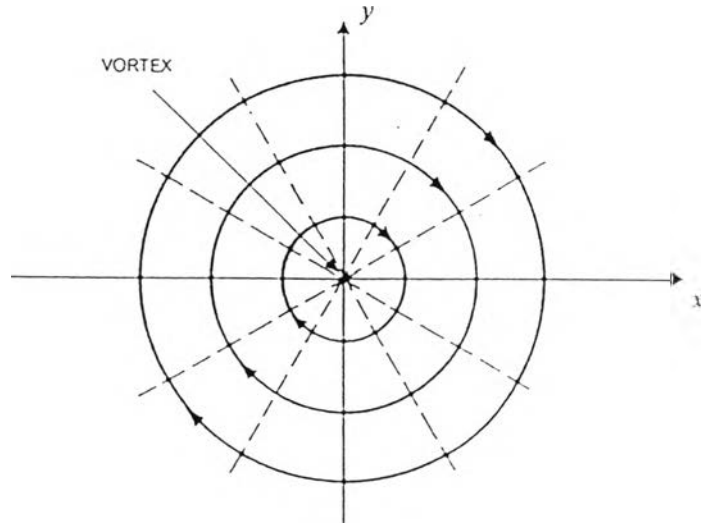


Fig. 4.2 : The vortex flow which its center called a "vortex" (see ref. [25]).

From classical hydrodynamics, this force can be computed exactly by various methods such as the method using theorem of Blasius [22]. The Magnus force per unit length of a cylinder can be written in the form

$$\vec{F}_M = \rho \vec{K} \times \vec{V} \quad (4.1)$$

where  $\rho$  is the fluid density,  $\vec{V}$  is the velocity of the cylinder relative to the (uniform flow) velocity of the fluid, and  $\vec{K}$  is the circulation vector of magnitude  $K$ ,

$$K = \oint_c \vec{v} \cdot d\vec{l} \quad (4.2)$$

,where  $\vec{v}$  is the velocity field of the fluid and  $c$  is any close path containing the cylinder, directed parallel to the axis that is perpendicular to the plane of flow with the right-hand rule.

"Vortex" in classical hydrodynamics is defined as a center of the vortex flow defining as the flow that each stream line has a symmetry about the center i.e., the fluid flows in a circular path with its speed depending only on the distance from the center (see fig. 4.2). Moreover, if the flow is irrotation (as assumed in this section), the speed of the fluid must be proportional to  $1/r$  where  $r$  is the distance from the vortex. In real nature, the vortex is not exactly a point but it forms the "core" of some radius. This core radius is

of the order of distance  $r$  at which the speed is of the order of the sound velocity. From fig. 4.2, one can think that the vortex flow is similar to the flow around the cylinder which has the radius close to zero and the vortex line (the line which has the vorticity  $(\vec{\nabla} \times \vec{v})$  as its tangent vector) itself is similar to this zero radius cylinder. From eq. (4.1), one can see that the Magnus force on the cylinder is independent of its radius. By these reasons, one may ask the question: can the vortex or vortex line feel the Magnus force of the form in eq. (4.1) if it moves through a fluid like in the case of cylinder in fig. 4.1? The answer is that it does provided that it can actually move through a fluid or, in other words, move with respect to the fluid. For the perfect fluid (incompressible, inviscid, and barotropic fluid) in a conservative force field such as gravity, the Helmholtz vortex equation [23] and Kelvin's circulation theorem [23] tell us that if the two dimensional flows which initially have the vortex at some points, then the vorticity at those points will remain the same as they flow in the course of time and they also move with the same velocity of the uniform flow (velocity far away from these vortices). In other words, the vortex lines in a perfect fluid move with the fluid or are "frozen" into the fluid if the fluid is in a conservative force field. Hence, in our case of irrotational vortex flow (we have only one vortex line which is the straight line perpendicular to the flow plane which passes through the center of the vortex flow), if it is in the conservative force fields, then the vortex line moves with the fluid which implies that the relative velocity between vortex and fluid must be equal to zero and so, from eq. (4.1), the Magnus force on a vortex does not appear. However, the non conservative forces usually arise in the real nature such as the friction so one can guess that the vortex in real nature should feel the Magnus force. The derivation of Magnus force on vortex, without using the analogy of vortex line with rotating cylinder as we have discussed above, has been shown excellently by Sonin [24]. His idea is that he supposes the vortex to move with constant velocity, which is not equal to the velocity of the uniform flow, by the influences of some external forces acting on the vortex line. In order to keep the vortex in a motion which has constant velocity, he found that the resultant external force per unit length of a vortex line must be in balance with the force which is in the same form of the Magnus force, eq. (4.1). From this derivation, one can conclude that the vortex, once moving with respect to the fluid, must feel the Magnus force of the form in eq. (4.1).

The vortex can also occur in both superfluids and superconductors. The main difference between these vortices and vortex in plain fluid is that the circulation in the case of superfluids or superconductors is, instead of being continuous as in plain fluid, quantized and taking the values of [25]

$$K = \oint_C \vec{v} \cdot d\vec{l} = \frac{nh}{m} \quad ; n = 0, 1, 2, \dots \quad (4.3)$$

where  $m$  is the atomic mass, and  $h$  is the Plank's constant. The first experiment to demonstrate that circulation is quantized in superfluid (He II) was that of Vinen (1961). There is a close similarity between vortices in superfluids and super conductors: in both cases they consist of particle currents circulation round a cylindrical core. On the other hand, there are important distinctions to be drawn, the most obvious one is that vortices in superconductors are coupled to the applied magnetic field, whereas a magnetic field has no influence on superfluid vortices. Note that the vortices in superfluid are very easily formed e.g., by rotating a cylindrical bucket containing He II, one vortex can be created by the rotation which only has the angular velocity of the order  $10^{-3} \text{ sec}^{-1}$  [25].

By analogy with the Magnus force in classical hydrodynamics, one may expect that a vortex in both superfluids and superconductors should also feel the Magnus force which is of the form in eq. (4.1) provided that the magnitude of circulation vector obeys the quantization rule (4.3). Although this above conclusion about the Magnus force on vortex in superfluids and superconductors is made by the analogy with the Magnus force in classical hydrodynamics, the Magnus force on vortex in superfluids and superconductors which is in the form as in eq. (4.1), has been derived based on quantum theory by many authors [26], [27]. Moreover, the experimental point of view of the Magnus force has been described in many papers such as in ref. [30]. In real nature such as in rotating He II, it contains a regular array of vortex lines all having the smallest possible circulation  $h/m$  (which implies that the total number of lines at that instant is at maximum) because the circulation must occur in such a way that it minimizes the free energy [25]. So, in the real nature situation, when inserting eq. (4.43), taken  $n = 1$ , into eq. (4.1), we obtain the Magnus force acting on a vortex line in superfluid in the form

$$\vec{F}_M = q_v \rho_s h d\vec{z} \times (\dot{\vec{r}} - \vec{v}_s) \quad (4.4)$$

where  $q_v = +1(-1)$  stands for the vorticity paralleling (antiparalleling) to the unit vector  $\hat{z}$  in the  $z$  direction,  $\rho_s$  is the superfluid atom number density,  $\dot{\vec{r}}$  is the vortex velocity in two dimensions ( $x-y$  plane).  $\vec{v}_s$  is the uniform superfluid velocity (velocity of superfluid far from vortex), and  $d$  is the thickness of the sample (it is equal to the length of a vortex line). Similarly, the Magnus force acting on a vortex line in superconductor is in the form

$$\vec{F}_M = \frac{q_v \rho_s h d}{2} \hat{z} \times (\dot{\vec{r}} - \vec{v}_s) \quad (4.5)$$

where  $\rho_s$  is the superfluid electron number density with the factor  $1/2$  counting for the Cooper pairing.

From eq. (4.4) or eq. (4.5), one can see that the Magnus force acting on a vortex is similar to the Lorentz force acting on an electron which is confined in the  $x-y$  plane under the magnetic field  $\vec{B}$  applying in  $z$  direction. This Lorentz force is in the form

$$\vec{F}_L = -e\dot{\vec{r}} \times \vec{B} = eB\hat{z} \times \dot{\vec{r}} \quad (4.6)$$

where  $e$  is the electron charge.

From this similarity, it is the convenience for someone to imagine the motion of a vortex under the Magnus force as the motion of an electron under the well-known Lorentz force. Note that the Magnus force on a vortex will occur if there exists the relative velocity  $\dot{\vec{r}} - \vec{v}_s$ , while the Lorentz force on electron will occur if there exists the magnetic field  $\vec{B}$ .

The vortex motion is an important behavior in many physical situation such as the flow of the supercurrent in superconductors. The motion of the vortices is the main mechanism for electrical resistance because if the vortices are free to move, then the supercurrent will flow with difficulty but, on the other hand, if the vortices are pinned or trapped, then the supercurrent will flow more easily and the resistivity is reduced. Hence, the behavior of pinned vortices is one of the interesting problem. In real situation, we can't completely trap the vortices i.e., the vortices can, by some means, leave the trap. In other words, the pinning potential can't be a stable potential but it is a "metastable potential". In many physical situations, the pinning of vortices will occur from the "disorder effect" but this effect can also produce the dissipation on vortices. In general, both pinning and dissipation effects frequently occur in real life situation.

Now, in our problem, we shall study the general problem of one vortex which is pinned in the metastable potential and treating everything in the sample except this vortex as the environment causing the dissipation on this vortex. As in section 3.4, we will model this environment like in the Caldeira-Leggett model. Since a vortex core is very small i.e., it is much smaller than the length scale in most case studies, a vortex can then be regarded as a point particle. By these above discussions and eq. (4.5), the Hamiltonian of the system of a vortex of finite mass  $M$  in superconductor plus its environment can be generally written as

$$H = \frac{1}{2M} \left| \bar{P} - q_v \bar{A}(\bar{r}) \right|^2 + V(\bar{r}) + \sum_a \left[ \frac{|\bar{P}_a|^2}{2m_a} + \frac{1}{2} m_a \omega_a^2 \left| \bar{q}_a - \frac{c_a \bar{r}}{m_a \omega_a^2} \right|^2 \right] \quad (4.7)$$

This Hamiltonian has been used to study the tunneling of a quantized vortex, with special attention paid to the effects of pinning and dissipation, by P.Ao and D.J. Thouless [19]. The meaning of each term in eq. (4.7) is as follows. The vector potential  $\bar{A}$ , which has the property that  $\bar{\nabla} \times \bar{A} = h \rho_s d\hat{z}/2$ , comes from the vortex velocity dependent part (VVDP) of the Magnus force eq. (4.5) while the potential due to the superfluid velocity dependent part (SFVDP) of the Magnus force is absorbed in the vortex potential  $V(\bar{r})$ . The last term, as in Caldeira – Leggett model, describes the environment consisting of the set of uncoupled harmonic oscillator linearly coupling with the vortex. By following P.Ao and D.J. Thouless [19], the gauge of vector potential will be chosen (it is easy to show that the (Euclidean) action is independent of the choice of gauge) so that the vector potential is in the form

$$\bar{A} = \frac{h}{2} \rho_s d(y, 0, 0) \quad (4.8)$$

and the vortex potential  $V(\bar{r})$  which allows an extensive analytical study is in the form

$$V(\bar{r}) = V(y) + \frac{1}{2} k_x x^2 \quad (4.9)$$

The potential  $V(y)$  consists of the contribution from the SFVDP Magnus force (this means that we “assume” the superfluid velocity  $\bar{v}_s$ , or supercurrent is along the  $x$  direction) and the pinning potential in  $y$  direction, which has a metastable point chosen at  $y = 0$ . The second term in eq. (4.9) is the pinning potential in  $x$  direction which is approximated by

the harmonic potential, and  $k_x$  should be determined experimentally. Although eq. (4.7) with the vector potential (4.8) is derived in the case of vortex in superconductor, the Hamiltonian for vortex in superfluid or for electron in magnetic field can be written in the same form because the Lorentz force and Magnus force in both superfluid and superconductor are in the same form. Hence, from eqs. (4.4)–(4.6) and (4.9), the Hamiltonian (4.7) can be generally written as

$$H = \frac{1}{2M} \left| \bar{p} - q_v \bar{A}(\bar{r}) \right|^2 + \frac{1}{2} k_x x^2 + V(y) + \sum_a \left[ \frac{|\bar{p}_a|^2}{2m_a} + \frac{1}{2} m_a \omega_a^2 \left| \bar{q}_a - \frac{c_a \bar{r}}{m_a \omega_a^2} \right|^2 \right] \quad (4.10)$$

with the vector potential in the form

$$\bar{A} = \frac{M\Omega}{q_v} (y, 0, 0) \quad (4.11)$$

where the frequency dimensional parameter  $\Omega = q_v h \rho_s d / 2M$  for a vortex in superconductor,  $\Omega = q_v h \rho_s d / M$  for a vortex in superfluid, and  $\Omega = eB/M$  for electron in magnetic field provided that  $q_v$  in both eqs. (4.10) and (4.11) are replaced by electron charge  $e$ . Note that in the case of electron in magnetic field, the metastable potential  $V(y)$  consists of the contribution from pinning potential in  $y$  direction only since it doesn't need its velocity relative to something in the sample (in fact, it must have relative velocity with respect to the source of magnetic field) in order to produce the Lorentz force on it as in the case of Magnus force in both superfluids and superconductors. (see eqs. (4.4) and (4.5)).

Now, let us summarize the problem that we want to deal with. Our problem is to study the vortex (electron) moving in the infinite two dimensional plane, and having, for superfluids or superconductors, the superfluid velocity  $\bar{v}_s$  or supercurrent along the  $x$  direction, which is influenced by the Magnus force (Lorentz force), the pinning potential, and the environment. In the Hamiltonian of the problem (4.10), the Magnus force (Lorentz force) is described by the vector potential  $\bar{A}$ , the environment is described by the last term in eq. (4.10), the pinning potential in  $x$  direction is described by the harmonic potential  $k_x x^2 / 2$ , and the pinning potential in  $y$  direction is contained in the metastable potential  $V(y)$  which is chosen so that the metastable point is at  $y = 0$ . Since the vortex (electron) potential  $V(y)$  is the metastable potential in  $y$  direction, the vortex (electron) must have some chances to escape out of this potential in  $y$  direction by both quantum

mechanical tunneling and thermal hopping. In our problems, we will specifically study on this behavior of vortex so the other sections in this chapter will be dealing with the study of the behavior of a vortex (electron) escaping out of a metastable potential.

#### 4.2 THE ESCAPE RATE FORMULA [16], [36]

The theory of rate coefficients has a long tradition in physics, chemistry and biology since the days of Arrhenius. H.A. Kramer's article of 1940 [28] represents a cornerstone in the quantitative analysis of thermally activated rate processes. An excellent review which covers extensive knowledge in this field is in ref. [37].

In this section, we assume that the system in question can be visualized as a particle of mass  $M$  describing by coordinate  $y$  moving in a metastable potential  $V(y)$  while coupled with the environment. The metastable potential  $V(y)$  has a single metastable minimum at a point which we choose at the origin of  $y$ . The bottom of the metastable potential is chosen to lie at zero i.e.,  $V(0) = 0$ . We assume that the potential  $V(y)$  is fairly smooth and has the general form depicted in fig. 4.3 i.e., a metastable quadratic-plus-cubic potential well. Since  $V(y)$  is a metastable potential of the form depicted in fig. 4.3, a particle confined to the metastable region will ultimately escape to the region of lower potential on the other side of the barrier.

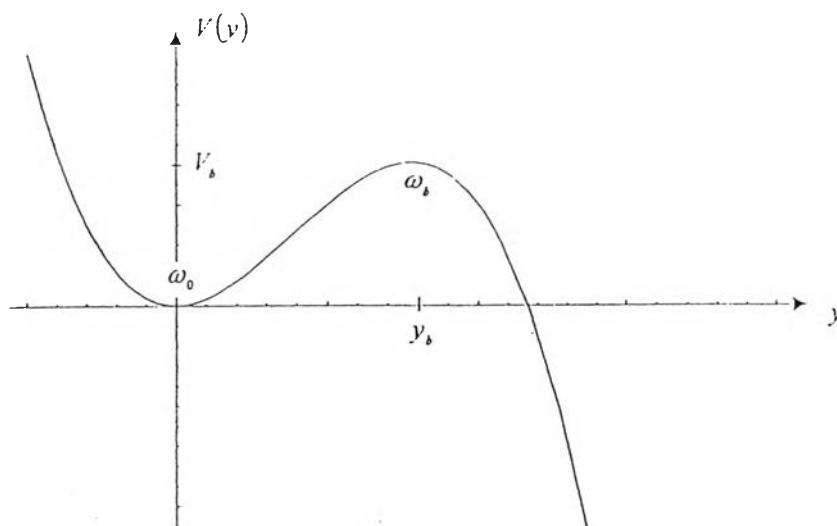


Fig. 4.3 : A metastable "quadratic-plus-cubic" potential well.



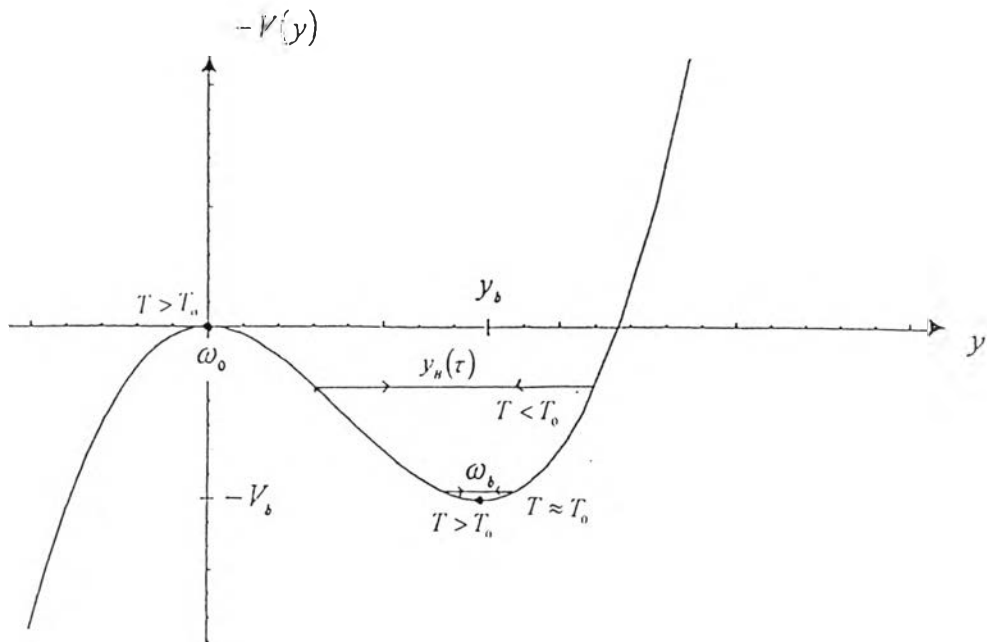


Fig. 4.4 : The inverted potential  $-V(y)$ .

Naturally, the concept of metastability only makes sense when the barrier is large enough that the decay time of the metastable state is very long comparing to all the other characteristic time scales of the system dynamics e.g., the correlation time of the noise and the time scales  $\omega_0^{-1}$  and  $\omega_b^{-1}$  related to the curvature of the potential at the metastable minimum and at the barrier top. This situation is called the "weak metastability". It requires that the barrier height  $V_b$  is by far the largest energy relevant to the problem.

$$V_b \gg k_B T, V_b \gg \hbar \omega_0 \quad (4.12)$$

Here,  $\omega_0$  is the frequency of small oscillation around the metastable minimum

$$\omega_0 = (V''(0)/M)^{1/2}, \quad (4.13)$$

where  $V''(0)$  is the curvature of  $V(y)$  at  $y = 0$ ,

and the barrier frequency

$$\omega_b = (-V''(y_b)/M)^{1/2}, \quad (4.14)$$

where  $V''(y_b)$  is the curvature of  $V(y)$  at  $y = y_b$ ,

characterizes the width of the parabolic top of the barrier hindering the decay process, see fig. 4.3. It represents the frequency of small oscillation around the minimum of the inverted potential  $-V(y)$ , see fig. 4.4.

From fig. 4.3, it is clear that the particle escapes out of the metastable potential by thermal hopping at high temperature while the quantum mechanical tunneling effect will be more incorporated in the escape process as the temperature is lower. The temperature

$T_0$ , where the change of dominating mechanism of the escape process from thermal activation to quantum tunneling is roughly to occur, is called the "crossover temperature" (it will be discussed in more details in sections 4.4 and 4.5). Various ranges of escape process are depicted in fig. 4.5

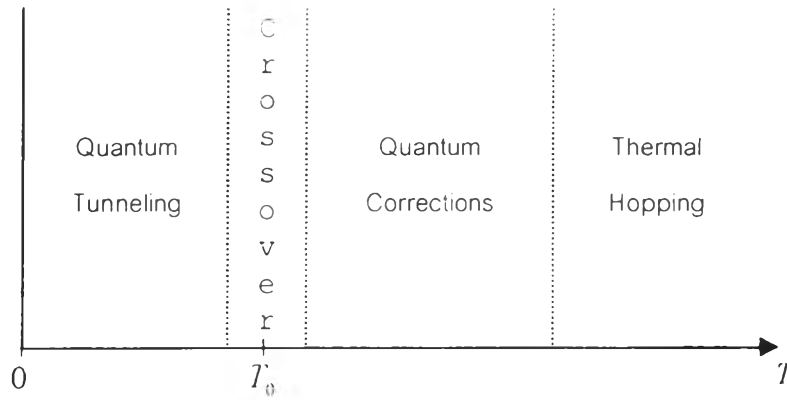


Fig. 4.5 : Dominant escape mechanism depicted schematically as a function of temperature.

Our study of the decay of a metastable state in thermal equilibrium with environment will be based on thermodynamic method pioneered by Langer [29] in 1967. His procedure of analytic continuation leads to an imaginary part of the free energy of the metastable state which then is interpreted in the same way as the imaginary part of a resonance energy in quantum field theory. His rate formula is formulated by analogy with the zero temperature formula. First consider the zero temperature formula. Let  $\psi(t)$  be the ground state wave function with complex ground state energy  $E_0 = \varepsilon + i\sigma$ . Then the decay probability  $k$  per unit time is determined from  $|\psi(t)|^2 = e^{2\sigma t/\hbar} |\psi(0)|^2 = e^{-kt} |\psi(0)|^2$  as

$$k = -\frac{2}{\hbar} \text{Im} E_0. \quad (4.15)$$

By analogy with this formula, the thermodynamic rate is given by

$$k = -\frac{2}{\hbar} \text{Im} F, \quad \text{for } T < T_0, \quad (4.16)$$

where  $F$  is the free energy. Note that the rate formula (4.16) can be used only for  $T < T_0$ . However, Affleck [31] has shown that the rate formula above  $T_0$  can be calculated by means of the modified formula

$$k = -\frac{2}{\hbar} \frac{\beta}{\beta_0} \text{Im} F, \quad \text{for } T > T_0, \quad (4.17)$$

where  $\beta_0 = 1/k_B T_0$  is the inverse crossover temperature.

An explicit calculation by Affleck [31] for an undamped system has demonstrated that Langer's method yields the same result as a Boltzmann average of energy dependent decay rates. Moreover, also the effects of frequency-dependent damping [32] as well as quantum corrections to thermally activated decay calculated by dynamical method [33] are reproduced quantitatively by the Langer's thermodynamic method. Since Langer's approach is a thermodynamic method, it cannot account for effect related to nonequilibrium occupation of states in the well. This means that Langer's method gives the correct result for the decay rate whenever nonequilibrium effects within the metastable well can be neglected. This is the case when the environmental coupling is strong enough to maintain thermal equilibrium within the metastable well. In case of very weak environmental coupling of friction (this is the case where Langer's method can't be used), the influence of the environment is not strong enough to maintain thermal equilibrium in the well. The escape over the barrier leads to a depletion of the Boltzmann distribution in an energy region of width  $k_B T$  just below the barrier top and the escape is limited by energy diffusion from the lower to the higher states in the well [34]. However, this weak coupling region characterizing by  $\gamma \leq \omega_s K_B T / I_s$  [16] is very small in particular for system with high barrier e.g., the system with the requirement of weak metastability. While in the absence of a fully dynamical justification of the approach which its range of validity is not exactly known [35], it is highly suggestive that Langer's method gives the correct result for the decay rate whenever nonequilibrium effects within the metastable well are neglected.

Now, we have the escape rate formulae (4.16) and (4.17). From these equations, it is clear that our problem now hinges on finding the imaginary part of free energy. In section 4.4, we will find the imaginary part of free energy of a vortex and then, by using these escape rate formulae, find the escape rate formula for a vortex escaping out of the metastable potential  $V(y)$  in  $y$  direction.

#### 4.3 EFFECTIVE ONE-DIMENSIONAL REDUCED PARTITION FUNCTION

From Hamiltonian of the problem (4.10), one can view a vortex as the Brownian particle moving under the influence of its environment. From section 3.6, we have known that the reduced description on Brownian particle can be found by tracing out all of the

environmental degrees of freedom. Notice that our problem of vortex is in two dimensions but we are interested in  $y$  direction only, since the metastable potential depends on coordinate  $y$  only which means that the escaping behavior can occur only in  $y$  direction. By these reasons, one should treat only  $y$  coordinate of a vortex as a coordinate of the Brownian particle (system) and leave out other degrees of freedom, which are  $x$  degrees of freedom of vortex itself and all of the environmental degrees of freedom, as the constitution of the environment. So, in our problem, it is clear that the elimination of environmental coordinates is more complicated than in section 3.6 since the  $x$  degree of freedom couples linearly with the  $y$  degree of freedom via its velocity instead of coordinate like the coupling between  $y$  and  $q_n$  degree of freedom.

Now, by using the same method as in section 3.6, we will find, the effective one-dimensional reduced partition function. From eqs. (4.10) and (4.11), the corresponding Euclidean action is in the form

$$S^E = \int_0^U d\tau \left\{ \frac{1}{2} M |\dot{\bar{r}}|^2 + iM\Omega \dot{x}y + V(y) + \frac{1}{2} k_x x^2 + \sum_\alpha \left[ \frac{1}{2} m_\alpha |\dot{\bar{q}}_\alpha|^2 + \frac{1}{2} m_\alpha \omega_\alpha^2 \left| \bar{q}_\alpha - \frac{c_\alpha \bar{r}}{m_\alpha \omega_\alpha^2} \right|^2 \right] \right\} \quad (4.18)$$

where  $\bar{r} = (x, y, 0)$  and  $\bar{q}_n = (q_n^x, q_n^y, 0)$ . As in eq. (3.72), after integrating the (normalized) density matrix of the universe,  $K(y'', x', \bar{q}'; y', x', \bar{q}')$ , with respect to  $\bar{q}'$  and  $x'$ , we have from eq. (4.18) the effective one-dimensional reduced density matrix, similar to eq. (3.73), in the form

$$\rho(y', y'') = Z_d^{-1} \int_{y(0)=y'}^{y(U)=y''} Dy(\tau) \exp(-S_S^E[y]/\hbar) F^E[y] \tilde{F}^E[y]; \quad Z_d = \frac{Z}{Z_R^2 Z_{X,R}} \quad (4.19)$$

where

$$\tilde{F}^E[y] = Z_{X,R}^{-1} \oint Dx(\tau) \exp(-S_{X,I}^E[x, y]/\hbar) F^E[x] \quad (4.20)$$

with

$$S_{X,I}^E[y, x] = S_X^E[x] + S_I^E[y, x]$$

$$F^E[x] = Z_R^{-1} \oint D^N q^x(\tau) \exp(-S_{R,I}^E[x, \bar{q}^x]/\hbar); \quad D^N q^x(\tau) = Dq_1^x(\tau) \dots Dq_N^x(\tau), \\ \bar{q}^x = (q_1^x, \dots, q_N^x) \quad (4.21)$$

with

$$S_{R,1}^E[x, \bar{q}^x] = S_R^E[\bar{q}^x] + S_1^E[x, \bar{q}^x]$$

$$F^E[y] = Z_R^{-1} \oint D^N q^v(\tau) \exp(-S_{R,1}^E[y, \bar{q}^v] / \hbar); D^N q^v(\tau) = Dq_1^v(\tau) \dots Dq_N^v(\tau),$$

$$\bar{q}^v = (q_1^v, \dots, q_N^v)$$

(4.22)

with

$$S_{X,1}^E[y, \bar{q}^v] = S_R^E[\bar{q}^v] + S_1^E[y, \bar{q}^v]$$

and

$$\left. \begin{aligned} S_S^E[y] &= \int_0^U \left( \frac{1}{2} M \dot{y}^2 + V(y) \right) d\tau \\ S_X^E[x] &= \int_0^U \left( \frac{1}{2} M \dot{x}^2 + \frac{1}{2} k_c x^2 \right) d\tau \\ S_1^E[y, x] &= iM\Omega \int_0^U \dot{x}y d\tau \\ S_R^E[\bar{q}^x] &= \int_0^U \sum_{\alpha=1}^N \frac{m_\alpha}{2} \left( \dot{q}_\alpha^{x^2} + \omega_\alpha^2 q_\alpha^{x^2} \right) d\tau \\ S_R^E[\bar{q}^v] &= \int_0^U \sum_{\alpha=1}^N \frac{m_\alpha}{2} \left( \dot{q}_\alpha^{v^2} + \omega_\alpha^2 q_\alpha^{v^2} \right) d\tau \\ S_1^E[x, \bar{q}^x] &= \int_0^U \sum_{\alpha=1}^N \left( -c_\alpha q_\alpha^x + \frac{1}{2} \frac{c_\alpha^2 x^2}{m_\alpha \omega_\alpha^2} \right) d\tau \\ S_1^E[y, \bar{q}^v] &= \int_0^U \sum_{\alpha=1}^N \left( -c_\alpha q_\alpha^v + \frac{1}{2} \frac{c_\alpha^2 y^2}{m_\alpha \omega_\alpha^2} \right) d\tau \end{aligned} \right\} \quad (4.23)$$

Here  $Z$  is the partition function of the universe,  $Z_R$  is expressed by eq. (3.76) representing the partition function normalizes  $F^E[x]$  or  $F^E[y]$ , so that  $F^E[x] = 1$  or  $F^E[y] = 1$  when the coupling is switched off i.e.,  $S_1^E[x, \bar{q}^x] = 0$  or  $S_1^E[y, \bar{q}^v] = 0$ ,  $Z_{X,R}$  is the partition function (which will be determined later) normalizes  $\tilde{F}^E[y]$ , so that  $\tilde{F}^E[y] = 1$  when the coupling is switched off i.e.,  $S_1^E[y, x]$  and  $Z_d$ , which is related with other partition functions by  $Z_d = Z/Z_R^2 Z_{X,R}$  is the effective one-dimensional reduced partition function normalizes the reduced density operator  $\hat{\rho}(\rho(y', y'') = \langle y' | \hat{\rho} | y'' \rangle)$ , so that  $\text{Tr} \hat{\rho} = 1$ . Note that in eq. (4.19) we have two influence functionals,  $F^E[y]$  which has

already been calculated (see eq. (3.81)) describing the pure influence from harmonic bath and  $\tilde{F}^E[y]$  describing the influence from both harmonic bath and  $x$  degree of freedom of a vortex (see eq. (4.20)). The influence functional  $\tilde{F}^E[y]$  appears in eq. (4.19) because of the fact that we treat  $x$  degree of freedom of a vortex as one of the degree of freedom of the environment. Notice that the influence functional  $\tilde{F}^E[y]$  can be computed exactly since everything in the path integration is Gaussian. Inserting eq. (3.81) into eq. (4.20) and using the definition of  $S_{x,1}^E[y, x]$  in eq. (4.20) together with eq. (4.23), we obtain

$$\oint Dx(\tau) \exp(-S_{x,1}^E[y, x]/\hbar) \tilde{F}^E[x] = \oint Dx(\tau) \exp\left\{-\frac{1}{\hbar} \int_0^U d\tau \left(\frac{1}{2} M \dot{x}^2 + \frac{1}{2} k_x x^2 + iM\Omega \dot{x}y\right) + \frac{1}{2} \int_0^U \int_0^U d\tau d\tau' K(|\tau - \tau'|) [x(\tau) - x(\tau')]^2\right\} \quad (4.24)$$

This path integration can be evaluated by various methods but the most convenient one is, perhaps, the Fourier series method. This method gives both prefactor and exponential term simultaneously. Since the path integration here runs over all periodic path with period  $U = \beta\hbar$  taken by the coordinate  $x(\tau)$ , the coordinate  $x(\tau)$  can then be represented by Fourier series,

$$x(\tau) = \sum_{n=-\infty}^{\infty} x_n e^{i\nu_n \tau} ; \nu_n = 2n\pi/U \quad (4.25)$$

and the functional measure can be defined by

$$\oint Dx(\tau) = \int \frac{dx_0}{\sqrt{2\pi\hbar U/M}} \prod_{n=1}^{\infty} \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d(\text{Re}x_n) d(\text{Im}x_n)}{(\pi\hbar MU\nu_n^2)} \right] \quad (4.26)$$

Note that this functional measure is applicable to general action [5]. From eqs. (4.25) and (4.26), it is clear that eq. (4.24) can be easily evaluated if one knows the Fourier series representation of  $K(\tau)$ . So, first considering eq. (3.82). Notice that

$$\frac{\cosh(\omega[|\tau| - U/2])}{\sinh(\omega U/2)} = \frac{2}{T} \sum_{n=-\infty}^{\infty} \frac{\omega}{\nu_n^2 + \omega^2} e^{i\nu_n \tau} = \frac{2}{\omega T} \sum_{n=-\infty}^{\infty} e^{i\nu_n \tau} - \frac{2}{\omega T} \sum_{n=-\infty}^{\infty} \frac{\nu_n^2 e^{i\nu_n \tau}}{\omega^2 + \nu_n^2}.$$

When inserting the above equation into eq. (3.82),  $K(|\tau|)$  can then be written in the form

$$K(|\tau|) = \mu : \delta(\tau) : -k(\tau) \quad (4.27)$$

where  $\mu = (1/\pi) \int_0^\infty J(\omega)/\omega d\omega$ ,  $\delta(\tau) := (1/U) \sum_{n=-\infty}^{\infty} e^{iv_n \tau} = \sum_{n=-\infty}^{\infty} \delta(\tau - nU)$  is a  $\delta$ -function periodically repeated at time  $\tau = \pm nU$ , and

$$k(\tau) = \frac{1}{2U} \sum_{n=-\infty}^{\infty} \xi_n e^{iv_n \tau}; \quad \xi_n = \frac{2}{\pi} \int_0^\infty \frac{J(\omega) v_n^2}{\omega(\omega^2 + v_n^2)} d\omega \quad (4.28)$$

From eq. (4.27), the nonlocal term (the fourth term) in the exponent of the integrand in eq. (4.24) can be written in the form

$$\begin{aligned} \frac{1}{2} \int_0^U \int_0^U d\tau d\tau' K(|\tau - \tau'|) [x(\tau) - x(\tau')]^2 &= -\frac{1}{2} \int_0^U \int_0^U k(\tau - \tau') [x(\tau) - x(\tau')]^2 d\tau d\tau' \\ &= \int_0^U \int_0^U k(\tau - \tau') x(\tau) x(\tau') d\tau d\tau' \end{aligned} \quad (4.29)$$

To arrive at this result, we have used the fact that, from eq. (4.28),  $\int_0^U k(\tau - \tau') d\tau = \int_0^U k(\tau - \tau') d\tau = 0$ . In our problem, we want only the reduced partition function which can be obtained if one knows the diagonal part of the reduced density matrix i.e.,  $\rho(y', y')$ . By this reason,  $y(\tau)$  can be represented by Fourier series,

$$y(\tau) = \sum_{n=-\infty}^{\infty} y_n e^{iv_n \tau}; \quad v_n = 2n\pi/U \quad (4.30)$$

When inserting eq. (4.29) into eq. (4.24) and using eqs. (4.25), (4.28), and (4.30), we obtain, by using the orthogonality relation  $\int_0^U e^{iv_n \tau} e^{-iv_m \tau} d\tau = U \delta_{n,m}$ , the exponent of the integrand of eq. (4.24) in the form

$$-\frac{U}{2\hbar} \sum_{n=-\infty}^{\infty} \gamma_n |x_n|^2 - \frac{UM\Omega}{\hbar} \sum_{n=-\infty}^{\infty} v_n y_n x_n; \quad \gamma_n = Mv_n^2 + k_x + \xi_n \quad (4.31)$$

Substituting (4.31) for the exponent of the integrand of eq. (4.24) and using eq. (4.26), one get (see Appendix B)

$$\begin{aligned} \oint Dx(\tau) \exp(-S_N^E[y, x]/\hbar) F^E[x] &= \left[ \frac{1}{U\omega_x} \prod_{n=1}^{\infty} \left( \frac{Mv_n^2}{Mv_n^2 + M\omega_x^2 + \xi_n} \right) \right] \times \\ &\exp\left( -\frac{UM^2\Omega^2}{\hbar} \sum_{n=1}^{\infty} \frac{v_n^2}{\gamma_n} |y_n|^2 \right) \end{aligned} \quad (4.32)$$

where  $\omega_x^2 = k_x/M$

From eqs. (4.32) and (4.20), it is clear that  $\widetilde{F}^E[y]$  is in the form

$$\widetilde{F}^E[y] = \exp\left(-\frac{UM^2\Omega^2}{\hbar} \sum_{n=1}^{\infty} \frac{\nu_n^2}{\gamma_n} |y_n|^2\right) \quad (4.33)$$

while

$$Z_{X,R} = \frac{1}{U\omega_x} \prod_{n=1}^{\infty} \left(\frac{M\nu_n^2}{M\nu_n^2 + M\omega_x^2 + \xi_n}\right) \quad (4.34)$$

Notice that  $\widetilde{F}^E[y]$  in eq. (4.33) is expressed in term of the Fourier coefficient  $y_n$ . To express it as a functional of  $y(\tau)$ , first considering the exponent of eq. (4.33) (reminded that  $y(\tau)$  is represented by Fourier series (4.30)),

$$\begin{aligned} -\frac{UM^2\Omega^2}{\hbar} \sum_{n=1}^{\infty} \frac{\nu_n^2}{\gamma_n} |y_n|^2 &= \frac{UM^2\Omega^2}{2\hbar} \sum_{n=-\infty}^{\infty} \frac{\nu_n^2}{\gamma_n} y_n y_{-n} \\ &= -\frac{M^2\Omega^2}{2\hbar} \sum_{n=-\infty}^{\infty} \frac{\nu_n^2}{\gamma_n} \int_0^U \int_0^U y(\tau) y(\tau') e^{i\nu_n(\tau-\tau')} d\tau d\tau' \end{aligned} \quad (4.35)$$

Note that to arrive this result, we have used the relations  $y_{-n} = y_n^*$ ,  $\nu_{-n} = -\nu_n$ , and  $\gamma_{-n} = \gamma_n$ . By using the relations  $y(\tau)y(\tau') = (y^2(\tau) + y^2(\tau') - [y(\tau) - y(\tau')]^2)/2$ ,  $\nu_0 = 0$ , and  $\int_0^U e^{i\nu_n\tau} e^{-i\nu_n\tau'} d\tau = U\delta_{n,0}$  and recalling the definition of  $\gamma_n$  in eq. (4.31), eq. (4.35)

becomes

$$-\frac{UM^2\Omega^2}{\hbar} \sum_{n=1}^{\infty} \frac{\nu_n^2}{\gamma_n} |y_n|^2 = -\frac{1}{2\hbar} \int_0^U \int_0^U d\tau d\tau' [y(\tau) - y(\tau')]^2 \left(-\frac{M^2\Omega^2}{U} \sum_{n=-\infty}^{\infty} \frac{M\nu_n^2 e^{i\nu_n(\tau-\tau')}}{M\nu_n^2 + M\omega_x^2 + \xi_n}\right) \quad (4.36)$$

To change this formula to another form (the form used in ref. [19]), first noticing that

$$0 = \int_0^U \int_0^U \delta(\tau - \tau') [y(\tau) - y(\tau')]^2 d\tau d\tau'$$

Recalling the definition of  $\delta(\tau)$  in eq. (4.27) and multiplying both sides of this equation by  $-M\Omega^2/4\hbar$ , the result is

$$0 = -\frac{1}{2\hbar} \int_0^U \int_0^U d\tau d\tau' [y(\tau) - y(\tau')]^2 \left(\frac{M\Omega^2}{2U} \sum_{n=-\infty}^{\infty} e^{i\nu_n(\tau-\tau')}\right) \quad (4.37)$$

Adding eq. (4.36) with eq. (4.37), one obtains

$$-\frac{UM^2\Omega^2}{\hbar} \sum_{n=1}^{\infty} \frac{\nu_n^2}{\gamma_n} |y_n|^2 = -\frac{1}{2\hbar} \int_0^U \int_0^U g(\tau - \tau') [y(\tau) - y(\tau')]^2 d\tau d\tau' \quad (4.38)$$

where

$$g(\tau) = \frac{M^2\Omega^2}{2U} \sum_{n=-\infty}^{\infty} \left(\frac{M\omega_x^2 + \xi_n}{M\nu_n^2 + M\omega_x^2 + \xi_n}\right) e^{i\nu_n\tau} \quad (4.39)$$



Inserting eq. (4.39) into eq. (4.33), we obtain

$$\widetilde{F}^E[y] = \exp\left(-\frac{1}{2\hbar} \int_0^U \int_0^U g(\tau - \tau') [y(\tau) - y(\tau')]^2 d\tau d\tau'\right) \quad (4.40)$$

Inserting eqs. (4.40) and (3.81) into eq. (4.19) and using eq. (4.23), the diagonal part of the effective one-dimensional reduced density matrix (4.19) can be written in the form

$$\rho(y', y) = Z_d^{-1} \int_{y(0)=y'}^{y(U)=y'} Dy(\tau) \exp(-S_{\text{eff}}^E[y]/\hbar) \quad (4.41)$$

where the effective action  $S_{\text{eff}}^E[y]$  is given by the expression

$$S_{\text{eff}}^E[y] = \int_0^U \left( \frac{i}{2} M \dot{y}^2 + V(y) \right) d\tau + \frac{1}{2} \int_0^U \int_0^U [K(|\tau - \tau'|) + g(\tau - \tau')] [y(\tau) - y(\tau')]^2 d\tau d\tau' \quad (4.42)$$

From eq. (4.41), it is clear that the effective one-dimensional reduced partition function can be written as

$$Z_d = \oint Dy(\tau) \exp(-S_{\text{eff}}^E[y]/\hbar) \quad (4.43)$$

From ref. [19],  $g(\tau)$  defined by eq. (4.39) is called the "anomalous damping kernel" and  $K(\tau)$ , which is called the damping kernel in chapter III, will be now called the "normal damping kernel" in order to make it different from the anomalous damping kernel. Now, we already have reduced the original two-dimensional system to the effective one-dimensional system described by coordinate  $y$ , and the thermodynamics reduced description is now contained in the reduced partition function (4.43). The dissipative effect from harmonic bath is effectively contained in both normal and anomalous damping kernel while the dissipative effect from  $x$  degree of freedom of a vortex is effectively contained in anomalous damping kernel only.

#### 4.4 THE ESCAPE RATE FORMULA OF A VORTEX

In this section we will study the behavior of a vortex escaping out of the metastable potential  $V(y)$  of the form depicted in fig. 4.3. The quantity which will be used to study this behavior is the escape rate  $k$  which has been discussed in section 4.2. The escape rate formula is divided into two regions,  $T < T_0$  and  $T > T_0$ . From eqs. (4.16) and (4.17), it is clear that these formulae can be obtained if one knows the free energy. Since the

escape process occurs in  $y$  direction only, the free energy which we have to calculate must be related, by the relation in eq. (2.3), with the effective one-dimensional reduced partition function (4.43). Now, our problem is hinging on evaluating the functional integral in eq. (4.43). As mentioned in section 4.2, weak metastability requires that the barrier height  $V_b$  obeys eq. (4.12). Hence, the functional integral (4.43) can be evaluated in the "semiclassical approximation" since the dominant contributions come from the region in function space around the stationary points of action. From eqs. (4.42), when substituting eq. (4.27) for  $K(\tau - \tau')$  in eq. (4.42) and using the variational principle i.e.,  $\delta S_{\text{eff}}^E[y_c] = 0$ , one can see that the classical path  $y_c(\tau)$  obeys the equation of motion

$$-M\ddot{y}_c + V'(y_c) + 2 \int_0^U [g(\tau - \tau') - k(\tau - \tau')] [y_c(\tau) - y_c(\tau')] d\tau' = 0 \quad (4.44)$$

with the periodic boundary condition  $y_c(0) = y_c(U)$ .

In the absence of dissipation (from both harmonic bath and  $x$  degree of freedom) i.e.,  $g(\tau) = k(\tau) = 0$ , eq. (4.44) describes the motion in the inverted potential  $-V(y)$ . It is clear from the inspection of fig. 4.4 that there are three solutions for  $y_c(\tau)$  where the periodic boundary condition is concerned. First, the trivial periodic solution  $y_c(\tau) = y_b$ , where the particle sits just in the minimum of the inverted potential, second, another trivial periodic solution  $y_c(\tau) = 0$  where the particle sits at the maximum of the inverted potential, third, the periodic solution  $y_c(\tau) = y_B(\tau)$  bouncing back and forth in the potential well of the inverted potential. For this reason, this trajectory is frequently called the "bounce". However, the bounce trajectory  $y_B(\tau)$  can only occur when the temperature is low enough, so that the period  $U = \beta\hbar$  is long enough to admit a bounce solution  $y_B(\tau)$ . In the absence of dissipation, the smallest period that admits a bounce solution must be equal to the period of small oscillation about  $y_b$  in the inverted potential i.e.,  $U = 2\pi/\omega_b$ . We denote the corresponding temperature to this period by  $T_0$  so  $T_0 = \hbar\omega_b/2\pi k_B$ . In the presence of dissipation, it is clear from eq. (4.44) that the trivial solutions  $y_c(\tau) = 0$  and  $y_c(\tau) = y_b$  are not effected by dissipation since the nonlocal term (the third term of eq. (4.44) is equal to zero while the bounce solution is affected by dissipation described by the nonlocal term.

It is now important that the action  $S_B = S_{\text{eff}}^E[y_B]$  of the bounce is smaller than the action  $\beta\hbar V_b$  of the trivial path  $y_c(\tau) = y_b$ . Hence below  $T_0$ , the functional integral (4.43) is dominated by the bounce and from eqs. (4.45) and (4.61) one can conclude that  $T_0$  here is the temperature where the change of dominating mechanism of the escape process from thermal activation to quantum tunneling is roughly to occurs. This means that (as mentioned in section 4.2)  $T_0$  here, which is the temperature where the bounce trajectory begins to be the dominant path of the functional integral (4.43), is the crossover temperature. In the presence of dissipation, one may guess that the smallest period that admits a bounce solution must be greater than  $2\pi/\omega_b$  because the particle dissipates its energy while bouncing back and forth in the well. So, in the presence of dissipation, one can think as if there is no dissipation but  $\omega_b$ , instead, decrease which implies that the crossover temperature should be decreased in the presence of dissipation. We will discuss about the crossover temperature in more details in section 4.5. For  $T < T_0$ , the escape rate can be written as [16], [36]

$$K = f_{\text{qm}} \exp(-S_B/\hbar) \quad (4.45)$$

where  $f_{\text{qm}}$  is the quantum-mechanical prefactor

Note that  $S_B$  will be calculated if one knows the bounce trajectory  $y_B(\tau)$ . From eq. (4.44), one can see that  $y_B(\tau)$  is the solution of the nonlinear integro-differential equation with periodic boundary condition which is very difficult to solve. I think that, now, no one can completely determine the action  $S_B$ . By this reason, we will deal with the escape rate in the region where  $T > T_0$  (the quantum correction region, see fig. 4.5) which the exact expression can be found. Although we are interested in the case of  $T > T_0$  where the escape process is dominated by thermally activation, the tunneling behavior of a vortex can be roughly known from the quantum correction factor (it will be discussed later). Now, we will calculate the escape rate in the temperature region  $T > T_0$ . As I have discussed above, since  $y_c(\tau)$  for  $T > T_0$  consists of two constant paths  $y_c(\tau) = 0$  and  $y_c(\tau) = y_b$ , the functional integral (4.43) can be written as

$$Z_d \approx Z_d^{(0)} + Z_d^{(b)} = Z_d^{(0)} \left( 1 + \frac{Z_d^{(b)}}{Z_d^{(0)}} \right) \quad (4.46)$$

where

$$Z_d^{(0)} = \oint Dy(\tau) \exp(-S_{\text{eff}}^{E(0)}[y]/\hbar) \quad (4.47)$$

and

$$Z_d^{(b)} = \oint Dy(\tau) \exp(-S_{\text{eff}}^{E(b)}[y]/\hbar) \quad (4.48)$$

where  $S_{\text{eff}}^{E(0)}[y]$  and  $S_{\text{eff}}^{E(b)}[y]$  are the semiclassical effective action about  $y = 0$  and  $y = y_b$  respectively.

In order to evaluate the path integrations (4.47) and (4.48), it is again convenient to use the Fourier series method. By this reason, we should first find the effective action  $S_{\text{eff}}^{E(0)}[y]$  and  $S_{\text{eff}}^{E(b)}[y]$  in terms of, the Fourier coefficient  $y_n$ . When substituting eq. (4.27) for  $K(\tau - \tau')$  in eq. (4.42) and using eqs. (4.28), (4.30) and (4.39), after using the orthogonality relation  $\int_0^U e^{in\tau} e^{-im\tau} d\tau = U\delta_{n,m}$ , we obtain

$$S_{\text{eff}}^E(y) = \frac{MU}{2} \sum_{n=-\infty}^{\infty} \left[ v_n^2 + \frac{\xi_n}{M} + \frac{\Omega^2 M v_n^2}{M v_n^2 + M \omega_x^2 + \xi_n} \right] |y_n|^2 + \int_0^U V(y) d\tau \quad (4.49)$$

From eqs. (4.28) and (3.67), one can see that  $\xi_n$  must be related with the Laplace transform of the retarded friction (see eqs. (3.45) and (3.60)) by

$$\xi_n = M |v_n| \hat{\gamma}(|v_n|) \quad (4.50)$$

Substituting eq. (4.50) for  $\xi_n$  in eq. (4.49), we get

$$S_{\text{eff}}^E(y) = \frac{MU}{2} \sum_{n=-\infty}^{\infty} \lambda_n |y_n|^2 + \int_0^U V(y) d\tau \quad ; \lambda_n = v_n^2 + |v_n| \hat{\gamma}_M(|v_n|) \quad (4.51)$$

where

$$\hat{\gamma}_M(x) = \hat{\gamma}(x) + \frac{\Omega^2 x}{x^2 + \omega_x^2 + x \hat{\gamma}(x)} \quad (4.52)$$

Note that  $\hat{\gamma}_M(x)$ , where the subscript  $M$  on  $\hat{\gamma}_M$  denotes the abbreviated name of Magnus force, reduces to  $\hat{\gamma}(x)$  when the Magnus force is absent i.e.,  $\hat{\gamma}_M \rightarrow \hat{\gamma}$  as  $\Omega \rightarrow 0$ . From eq. (4.51), develop  $V(y)$  in a Taylor series around  $y = 0$  and  $y = y_b$  and keep only the first and second terms of the series, we then get the semiclassical action in the form (see Appendix C)

$$S_{\text{eff}}^{E(0)}[y] = \frac{MU}{2} \lambda_0^{(0)} y_0^2 + MU \sum_{n=1}^{\infty} \lambda_n^{(0)} |y_n|^2 \quad ; \lambda_n^{(0)} = v_n^2 + \omega_0^2 + v_n \hat{\gamma}_M(v_n) \quad (4.53)$$

and

$$S_{\text{eff}}^{E(b)}[y] = V_b U + \frac{MU}{2} \lambda_0^{(b)} y_0^2 + MU \sum_{n=1}^{\infty} \lambda_n^{(b)} |y_n|^2; \lambda_n^{(b)} = v_n^2 - \omega_b^2 + v_n \hat{\gamma}_M(v_n) \quad (4.54)$$

Now, we will divide the situations into two cases,  $\omega_v \neq 0$  and  $\omega_v = 0$ .

Case 1 ( $\omega_v \neq 0$ ):

This is the case where we have the pinning potential in  $x$  direction. In this case, it is clear from eq. (4.52) that  $v_0 \hat{\gamma}_M(v_0) = 0$  which implies that  $\lambda_0^{(b)} = \omega_0^2$  and  $\lambda_0^{(b)} = -\omega_b^2$ . When inserting eq. (4.53) into eq. (4.47) and using the functional measure (4.26) (here the variable  $x_n$  must be replaced by  $y_n$ ), we obtain, after some integration procedures similar to eq. (B.3) in Appendix B when  $\Omega = 0$  and  $v_n$  is replaced by  $\lambda_n^{(b)}$ , the reduced partition function  $Z_d^{(b)}$  in the form

$$Z_d^{(a)} = \frac{1}{\omega_0 U} \prod_{n=1}^{\infty} \frac{v_n^2}{\lambda_n^{(a)}} \quad (4.55)$$

Similarly, when inserting eq. (4.54) into eq. (4.48),  $Z_d^{(b)}$  can be written as

$$Z_d^{(b)} = \sqrt{\frac{M}{2\pi l/\hbar}} \left( \prod_{n=1}^{\infty} \frac{v_n^2}{\lambda_n^{(b)}} \right) \left( \int_{-\infty}^{\infty} e^{\left( \frac{M \omega_b^2}{2\hbar} y_0^2 \right)} dy_0 \right) e^{-\beta V_b} \quad (4.56)$$

From eq. (4.56), it is surprising that the divergent integral over  $y_0$  appears. It appears since  $\lambda_0^{(b)} = -\omega_b^2$  is "negative". The divergence is related to the fact that the action for the trajectory  $y_c(\tau) = y_b$  is a saddle point in function space with the unstable direction along  $y_0$ . In fact, this should not be a surprise since we are trying to evaluate the free energy of an unstable system. Langer [29] was the first who explained that the functional integral can still be defined by distorting the integration contour of the variable  $y_0$  into the upper half of the complex plane along the direction of steepest descent. This leads to a positive imaginary part of the quantity  $Z_d^{(b)}$  in the form

$$\text{Im} Z_d^{(b)} = \frac{1}{2\omega_b U} \left( \prod_{n=1}^{\infty} \frac{v_n^2}{\lambda_n^{(b)}} \right) e^{-\beta V_b} \quad (4.57)$$

From eq. (4.12), it is clear that  $e^{-\beta V_b}$  is an exponentially small factor which is contained in  $Z_d^{(b)}$ . Hence, eq. (4.46) can be written, by a negligible error, in the form

$$Z_d = Z_d^{(0)} e^{z_d^{(b)}/z_d^{(0)}} \quad (4.58)$$

From eq. (2.3), the free energy must be related with the reduced partition function  $Z_d$  by  $F = -(1/\beta) \ln Z_d$ . When inserting  $Z_d$  from eq. (4.58) into this expression, we obtain

$$F = -\frac{1}{\beta} \ln Z_d^{(0)} - \frac{Z_d^{(b)}}{Z_d^{(0)}}$$

Since  $Z_d^{(0)}$  is real, see eq. (4.55), the imaginary part of the free energy expressed above can be written as

$$\text{Im } F = -\left(\beta Z_d^{(0)}\right)^{-1} \text{Im } Z_d^{(b)} \quad (4.59)$$

Inserting eqs. (4.57) and (4.55) into eq. (4.59), we get

$$\text{Im } F = -\frac{\omega_0}{2\beta\omega_b} \left( \prod_{n=1}^{\infty} \frac{\lambda_n^{(0)}}{\lambda_n^{(b)}} \right) e^{-\beta E_b} \quad (4.60)$$

Now, insert eq. (4.60) into eq. (4.17), we then get the escape rate formula of a vortex in the form

$$k = \frac{\omega_0}{2\pi} \rho c_{\text{qm}} e^{-\beta E_b} \quad (4.61)$$

where the factor

$$\rho = \frac{\omega_R}{\omega_b}; \omega_R = \frac{2\pi}{\beta_0 \hbar} = \frac{2\pi k_B T_0}{\hbar} \quad (4.62)$$

describes the effect of recrossings of the barrier top in the moderate to large damping region and the quantum-mechanical enhancement factor or quantum correction factor

$$c_{\text{qm}} = \prod_{n=1}^{\infty} \frac{\lambda_n^{(0)}}{\lambda_n^{(b)}} = \prod_{n=1}^{\infty} \frac{v_n^2 + \omega_0^2 + v_n \hat{\gamma}_M(v_n)}{v_n^2 - \omega_b^2 + v_n \hat{\gamma}_M(v_n)} \geq 1 \quad (4.63)$$

describes the quantum effect on escape process. The leading quantum corrections have a rather simple origin. The escape is enhanced in two different ways by quantum fluctuations. First, they increase the average energy in the well. Second, when a particle is thermally excited almost to the barrier top, they allow for tunneling through the remaining small barrier. Both effects lead to an effective reduction of the barrier. Note that in the classical limit ( $T \gg T_0$ ), where the factor  $c_{\text{qm}}$  approaches unity the escape rate formula (4.61) reduces to the classical escape rate formula. The dissipation effect on the rate is contained in the frequency  $\omega_R$  which is related with the crossover temperature  $T_0$  by  $\omega_R = (2\pi k_B / \hbar) T_0$  (it will be discussed in more detail in section 4.5).

Case 2 ( $\omega_x = 0$ ):

This is the case where we have no pinning potential in  $x$  direction. In this case, it is clear from eqs. (4.52), (4.50) and (4.28) that

$$\nu_0 \hat{\gamma}_{bM}(\nu_0) = \Omega^2 \sqrt{1 + (2M\pi) \int_0^{\nu_0} J(\omega) \omega^3 d\omega} \neq 0$$

which imply, by eqs. (4.53) and (4.54), that

$$\lambda_0^{(0)} = \omega_0^2 + \Omega^2 \sqrt{1 + (2M\pi) \int_0^{\infty} J(\omega) \omega^3 d\omega} \equiv \omega_{bM}^2$$

and

$$\lambda_0^{(b)} = -\omega_b^2 + \Omega^2 \sqrt{1 + (2M\pi) \int_0^{\nu_0} J(\omega) \omega^3 d\omega}$$

Now, in this case, we will "assume" that  $\lambda_0^{(b)} = -\omega_{bM}^2 < 0$  (the case where  $\lambda_0^{(b)} > 0$  will be discussed in section 4.7) where

$$\omega_{bM}^2 = \omega_b^2 - \frac{\Omega^2}{1 + \frac{2}{M\pi} \int_0^{\nu_0} \frac{J(\omega)}{\omega^3} d\omega} > 0 \quad (4.64)$$

Here the subscript  $M$  on  $\omega_{bM}$  and  $\omega_{\alpha M}$  denotes the abbreviated name of Magnus force. Based on this assumption, following the same procedure in case1, we finally get the escape rate formula of a vortex, denoted by  $\tilde{K}$  for this case, in the form

$$\tilde{K} = \frac{\omega_0}{2\pi} \tilde{\rho} c_{\text{qm}} e^{-\beta \tilde{b}} \quad (4.65)$$

where  $\tilde{\rho}$  is defined here, by analogy with  $\rho$  in the rate expression (4.61) in case 1 where  $\omega_x \neq 0$ , by

$$\tilde{\rho} = \frac{\omega_{\alpha M} \omega_R}{\omega_0 \omega_{bM}} \quad (4.66)$$

Now, we have two escape rate formulae of a vortex corresponding to two cases,  $K$  for  $\omega_x \neq 0$  and  $\tilde{K}$  for  $\omega_x = 0$ . In both cases, the quantum effect on escape process is contained in quantum correction factor  $c_{\text{qm}}$ . On the other hand, for  $\omega_x \neq 0$ , the factor  $\rho$  contains the effects implicitly from both environment (characterized by  $J(\omega)$  or  $\hat{\gamma}$ ) and  $x$  degree of freedom of a vortex (characterized by Magnus force described by  $\Omega$ ) via  $\omega_R$  (See eqs. (4.52) and (4.70)) while, for  $\omega_x = 0$ , both effects are contained implicitly in  $\tilde{\rho}$  via  $\omega_R$  and are also contained explicitly in  $\tilde{\rho}$  via  $\omega_{\alpha M}$  and  $\omega_{bM}$ .

#### 4.5 CROSSOVER TEMPERATURE AND ITS IMPORTANT THEOREMS

From fig. 4.4, it is clear that, slightly below  $T_0$ , the bounce trajectory is just represented by a harmonic oscillator about  $y_b$  with the frequency  $\omega_R = 2\pi/U_0 = 2\pi \beta_0 \hbar = 2\pi k_B T_0 / \hbar$ . Hence the bounce  $y_B(\tau)$  can be written as

$$y_B(\tau) = y_b + \varepsilon \cos(\omega_R \tau) \quad (4.67)$$

where  $\varepsilon$  is an infinitesimal amplitude i.e.,  $O(\varepsilon^2) = 0$ . From eqs. (4.28) and (4.39), it is easy to show that

$$g(\tau) - k(\tau) = \frac{M\Omega^2}{2U} \sum_{n=-\infty}^{\infty} F_n e^{i\nu_n \tau}; \quad F_n = \frac{M\omega_x^2 + \xi_n}{M\nu_n^2 + M\omega_x^2 + \xi_n} = \frac{\xi_n}{M\Omega^2} \quad (4.68)$$

substituting  $y_B(\tau)$  from eq. (4.67) for  $y_c(\tau)$  in eq. (4.44) and using eqs. (4.50) and (4.68) (when  $U$  is replaced by  $U_0$ ), one can linearize the equation of motion (4.44), since  $O(\varepsilon^2) = 0$ , and get (see Appendix D)

$$\left\{ M\omega_R^2 + V''(y_b) + M\Omega^2 - M\Omega^2 \left[ \frac{\omega_x^2 + \omega_R \hat{\gamma}(\omega_R)}{\omega_R^2 + \omega_x^2 + \omega_R \hat{\gamma}(\omega_R)} - \frac{\omega_R \hat{\gamma}(\omega_R)}{\Omega^2} \right] \right\} \varepsilon \cos(\omega_R \tau) = 0 \quad (4.69)$$

When equating the quantity in  $\{ \}$  with zero and substituting  $-M\omega_b^2$  for  $V''(y_b)$ , we get

$$\omega_R^2 + \omega_R \hat{\gamma}_M(\omega_R) = \omega_b^2 \quad (4.70)$$

where  $\hat{\gamma}_M$  has already been defined in eq. (4.52)

Since  $\omega_R$  relates with crossover temperature  $T_0$  by  $\omega_R = 2\pi k_B T_0 / \hbar$ ,  $T_0$  can be obtained via  $\omega_R$  which is the solution of eq. (4.70). In general, the solution of eq. (4.70) should be more than one. But in the physical situation, the crossover temperature, which we confine it to have only positive value i.e.,  $T_0 \geq 0$  (since the absolute temperature must be greater than zero), should have only one or equivalently the positive solution of eq. (4.70) i.e.,  $\omega_R$  where  $\omega_R \geq 0$  should be unique. This comes from the fact that the crossover temperature is the temperature where the change of dominating mechanism of the escape process from thermal activation to quantum tunneling is roughly to occur, and once this change has occurred it should not reoccur again. Another interesting question is that "is it possible that the crossover temperature does not exist, or equivalently is it



possible that eq. (4.70) has no solution?" To prove about the uniqueness of the solution of eq. (4.70) and to answer this question, we should consider the following theorems. However, before considering these theorems, I had to construct two lemmas as follows:

Lemma 1. Let  $f : \mathbb{R}_0^+ (= [0, \infty)) \rightarrow \mathbb{R}$  be the continuous increasing function. Let  $a \in \mathbb{R}$  be any constant. Then  $\exists!$ (there exists a unique)  $x_0 \in \mathbb{R}_0^+$  such that  $x_0^2 + f(x_0) = a$  if and only if  $\lim_{x \rightarrow 0} f(x) \leq a$ .

Proof.

Let  $g(x) = a - x^2$ . Then  $x_0$  must obey the relation  $g(x_0) = f(x_0)$ . It is clear that  $g(x)$  is a continuous decreasing function on  $\mathbb{R}_0^+$  which has the maximum value equal to  $a$  at  $x = 0$ . Suppose that  $\lim_{x \rightarrow 0} f(x) > a$ . Since  $f(x)$  is a continuous increasing function on  $\mathbb{R}_0^+$ ,  $g(x) \neq f(x) \forall x \in \mathbb{R}_0^+$  (see fig. 4.6). This statement equivalent to  $\exists x_0 \in \mathbb{R}_0^+$  such that  $g(x_0) = f(x_0) \Rightarrow \lim_{x \rightarrow 0} f(x) \leq a$ . Claim that  $\exists x_0 \in \mathbb{R}_0^+$  such that  $g(x_0) = f(x_0) \Rightarrow x_0$  is unique. Let  $x_1$  and  $x_2$  obey the equation  $g(x) = f(x)$ . Then  $a - x_1^2 = f(x_1)$  and  $a - x_2^2 = f(x_2)$  which imply that  $x_1^2 - x_2^2 = f(x_2) - f(x_1)$ . It is clear that if  $x_1 \neq x_2$ , then we always get the contradiction since  $f(x)$  is a continuous increasing function on  $\mathbb{R}_0^+$ . So, we have the claim. Hence,  $\exists! x_0 \in \mathbb{R}_0^+$  such that  $g(x_0) = f(x_0) \Rightarrow \lim_{x \rightarrow 0} f(x) \leq a$ .

Now, suppose that  $\lim_{x \rightarrow 0} f(x) \leq a$ . Since  $f(x)$  is a continuous increasing function on  $\mathbb{R}_0^+$ ,  $\lim_{x \rightarrow 0} f(x) \leq a \Rightarrow \exists x_0 \in \mathbb{R}_0^+$  such that  $g(x_0) = f(x_0) \Rightarrow \exists! x_0 \in \mathbb{R}_0^+$  such that  $g(x_0) = f(x_0)$  (see fig 4.6).

□

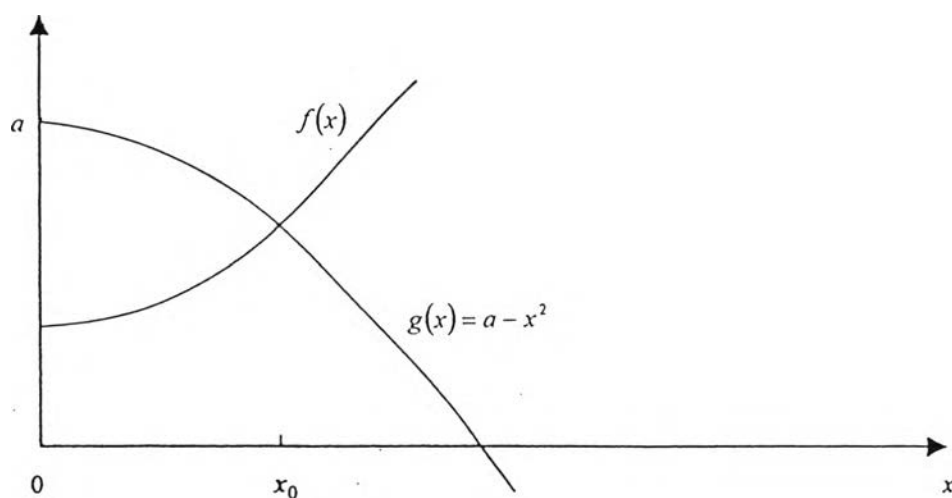


Fig. 4.6 : Intersection between the increasing and decreasing function.

Lemma 2.  $x\hat{\gamma}_M(x)$  is a continuous increasing function on  $\mathbb{R}_0^+$ .

Proof.

From eqs. (4.28) and (4.50), it is clear that  $x\hat{\gamma}(x)$  can be written as

$$x\hat{\gamma}(x) = \frac{2x^2}{M\pi} \int_0^\infty \frac{J(\omega)}{\omega(\omega^2 + x^2)} d\omega \quad (4.71)$$

In the physical point of view, any integrations over  $\omega$  concerning the spectral function  $J(\omega)$  are always the convergent integrals which imply that  $x\hat{\gamma}(x)$  in eq. (4.71) is a continuous function which imply again, from eq. (4.52), that  $x\hat{\gamma}_M(x)$  is a continuous function. Next, from eq. (4.71), it is easy to show that

$$\frac{d}{dx}(x\hat{\gamma}(x)) = \frac{4x}{M\pi} \int_0^\infty \frac{\omega J(\omega)}{(\omega^2 + x^2)^2} d\omega \geq 0 \quad \forall x \in \mathbb{R}_0^+ \quad (4.72)$$

and

$$\frac{d}{dx} \left( \frac{\Omega^2 x^2}{x^2 + \omega_x^2 + x\hat{\gamma}(x)} \right) = \frac{\Omega^2 \left[ 2x\omega_x^2 + \frac{4x^5}{M\pi} \int_0^\infty \frac{J(\omega)}{\omega(\omega^2 + x^2)^2} d\omega \right]}{\left[ x^2 + \omega_x^2 + \frac{2x^2}{M\pi} \int_0^\infty \frac{J(\omega)}{\omega(\omega^2 + x^2)^2} d\omega \right]^2} \geq 0 \quad \forall x \in \mathbb{R}_0^+ \quad (4.73)$$

From eqs. (4.72), (4.73), and (4.52), one can conclude that

$$\frac{d}{dx}(x\hat{\gamma}_M(x)) \geq 0 \quad \forall x \in \mathbb{R}_0^+ \quad (\text{equality hold at } x = 0 \text{ only}) \quad (4.74)$$

□

Now let us consider the following theorems.

Theorem 1 (Uniqueness). If  $\omega_x \neq 0$ , then the unique crossover temperature  $T_0$  always exists.

Proof.

From eq. (4.70),  $T_0$  can be obtained via  $\omega_R$  where  $\omega_R$  is the positive root of the equation  $x^2 + x\hat{\gamma}_M(x) = \omega_b^2$ . From eq. (4.52), it is clear that  $\lim_{x \rightarrow 0} x\hat{\gamma}_M(x) = 0 < \omega_b^2$ . Now, from lemmas 1 and 2, we have the theorem.

□

Theorem 2 (Non-existence). If  $\omega_x = 0$ , then the criterion for the non-existence of crossover temperature  $T_0$  is given by  $\Omega^2 / \left[ 1 + (2/M\pi) \int_0^\infty J(\omega)/\omega^3 d\omega \right] > \omega_b^2$ . If this criterion is violated, then the unique crossover temperature  $T_0$  still exists.

Proof.

From eq. (4.52), it is clear that  $\lim_{x \rightarrow 0} x \hat{\gamma}_M(x) = \Omega^2 \left/ \left[ 1 + (2/M\pi) \int_0^{\omega} J(\omega) / \omega^3 d\omega \right] \right.$ .

Now, from lemmas 1 and 2,

$$\Omega^2 \left/ \left[ 1 + (2/M\pi) \int_0^{\omega} J(\omega) / \omega^3 d\omega \right] > \omega_b^2 \Rightarrow x^2 + x \hat{\gamma}_M(x) \neq \omega_b^2 \quad \forall x \in \mathbb{R}_0^+$$

This means that there exists no  $\omega_R$  or equivalently  $T_0$  within this criterion. On the other

hand, if this criterion is violated i.e.,  $\Omega^2 \left/ \left[ 1 + (2/M\pi) \int_0^{\omega} J(\omega) / \omega^3 d\omega \right] \leq \omega_b^2$  then, from

lemmas 1 and 2,  $\exists! x_0 \in \mathbb{R}_0^+$  such that  $x_0^2 + x_0 \hat{\gamma}_M(x_0) = \omega_b^2$  which means that there exists the unique  $\omega_R$  or equivalently  $T_0$ .

□

Now, we can conclude in this section that the crossover temperature can be obtained via  $\omega_R$  which is the only one positive root of eq. (4.70). Moreover, from theorems 1 and 2, we found that the pinning potential in  $x$  direction characterizing by  $\omega_x$  is a very important quantity because if  $\omega_x \neq 0$ , then the crossover temperature always exists but if  $\omega_x = 0$ , then there exists no crossover temperature for the strong enough Magnus force characterized by  $\Omega$  (see the non-existence criterion in theorem 2). In other words, if we have the pinning potential in  $x$  direction, then the change of dominating mechanism of the escape process from thermal activation to quantum tunneling will occur at some temperature, which is called crossover temperature. On the other hand, if we don't have that pinning potential in  $x$  direction, this change will not occur because of the strong enough-Magnus force. From theorem 2, it is worth to note that the dissipation characterized by spectral function  $J(\omega)$  tends to suppress the Magnus force within the sense of the non-existence criterion.

#### 4.6 DISSIPATION AND MAGNUS FORCE EFFECTS ON THE CROSSOVER TEMPERATURE

Since the crossover temperature is the temperature where the change of dominating mechanism of the escape process from thermal activation to quantum

tunneling is roughly to occur, one of the interesting questions is that how this transition depends on dissipation and the Magnus force strength provided that the crossover temperature exist. At first sight, one may guess that if the dissipation or Magnus force strength increases, then the smallest period that admit a bounce solution will increase which implies that the crossover temperature will decrease. This guess, however, is not totally right. To investigate about the dependence of dissipation and the Magnus force on crossover temperature or equivalently  $\omega_R$ , we should first make a reasonable assumption that the spectral function can be written in the form  $J(\omega) = \eta j(\omega)$  where  $\eta$  is the only one parameter characterizing the dissipation strength and  $j(\omega)$  is any function depends on specific damping model. There are many examples of spectral function, as discussed in the end of section 3.6, corresponding to this form such as eq. (3.89) for the Ohmic damping case, eq. (3.90) for the Drude-regularized damping case, and eq. (3.91) for more general damping case. Now, based on this assumption, eq. (4.70) can be written as

$$\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R) + \frac{\Omega^2 \omega_R^2}{\omega_R^2 + \omega_x^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)} = \omega_b^2 ; \hat{\gamma}_\eta(\omega_R) = \frac{2\omega_R}{M\pi} \int_0^{\omega_R} \frac{j(\omega) d\omega}{\omega(\omega^2 + \omega_R^2)}. \quad (4.75)$$

From eq. (4.75), we can define  $\omega_R$  as an implicit function of  $\eta$ . When implicitly differentiating both side of eq. (4.75) with respect to  $\eta$  (denoted by the symbol “ ’ ”), we get

$$\left(\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)\right)' + \Omega^2 \left[ \frac{2(\omega_R^2 + \omega_x^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)) \omega_R \omega_R' - \omega_R^2 (\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R))'}{(\omega_R^2 + \omega_x^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R))^2} \right] = 0 \quad (4.76)$$

It is easy to show that

$$\left(\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)\right)' = \omega_R' \left[ 2\omega_R + \eta \frac{d}{d\omega_R} (\omega_R \hat{\gamma}_\eta(\omega_R)) \right] + \omega_R \hat{\gamma}_\eta(\omega_R) \quad (4.77)$$

Inserting eq. (4.77) into eq. (4.76) and using the fact that, from eq. (4.75),  $\Omega^2 \omega_R^2 / (\omega_R^2 + \omega_x^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)) = \omega_b^2 - \omega_R^2 - \eta \omega_R \hat{\gamma}_\eta(\omega_R)$ , we obtain

$$\begin{aligned} & \omega_R' \left\{ \left[ 2\omega_R + \eta \frac{d}{d\omega_R} (\omega_R \hat{\gamma}_\eta(\omega_R)) \right] \left[ 2(\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)) + (\omega_x^2 - \omega_b^2) \right] + 2\omega_R \Omega^2 \right\} \\ & = -\omega_R \hat{\gamma}_\eta(\omega_R) \left[ 2(\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R)) + (\omega_x^2 - \omega_b^2) \right] \end{aligned} \quad (4.78)$$

From eq. (4.78) it is clear that if  $\omega_x \geq \omega_b$ , then  $\omega'_R < 0 \forall \eta \in \mathbf{R}_0^+$  since  $\omega_R > 0, \hat{\gamma}_\eta(\omega_R) > 0$  (see eq. (4.75)), and  $d(\omega_R \hat{\gamma}_\eta(\omega_R))/d\omega_R > 0$  (See eq. (4.71)) for all  $\eta \geq 0$ . In other words,  $\omega_R$  is a decreasing function of  $\eta$  on  $\mathbf{R}_0^+$  if  $\omega_x \geq \omega_b$ . On the other hand, if  $\omega_x < \omega_b$ , then  $\omega_R$  has the relative extremum at  $\eta = \eta_0$  defined by the relation  $\omega'_R(\eta_0) = 0$ . It is clear from eq. (4.78) that the points  $\eta_0$  can be found by solving the transcendental equation

$$\omega_b^2 - \omega_x^2 = 2(\omega_R^2 + \eta \omega_R \hat{\gamma}_\eta(\omega_R))$$

or equivalently (by using eq. (4.75))

$$\omega_b^2 + \omega_x^2 = \frac{2\Omega^2 \omega_R^2}{\omega_R^2 + \omega_x^2 + \eta_0 \omega_R \hat{\gamma}_{\eta_0}(\omega_R)} \quad (4.79)$$

where  $\omega_R = \omega_R(\eta_0)$  can be obtained via eq. (4.75) when  $\eta$  is replaced by  $\eta_0$ .

Although those relative extremum of  $\omega_R$  appear when  $\omega_x < \omega_b$ ,  $\omega_R$  is always a decreasing function of  $\eta$  for large  $\eta$  limit. This comes from the fact that, for  $\eta \gg 0$  eq. (4.75) is approximately reduced to  $\eta \omega_R \hat{\gamma}_\eta(\omega_R) = \omega_b^2$  which implies that  $\omega'_R = -\omega_R \hat{\gamma}_\eta(\omega_R) / \eta d(\omega_R \hat{\gamma}_\eta(\omega_R)) / d\omega_R < 0 \forall \eta$ . By using eq. (4.72), the definition of  $\hat{\gamma}_\eta$  in eq. (4.75), and the fact that  $\omega_R \rightarrow 0^+$  as  $\eta \rightarrow \infty$  (see eq. (4.75)), this inequality tells us that  $\omega'_R \rightarrow 0^-$  as  $\eta \rightarrow \infty$ . Notice that when  $\omega_x < \omega_b$ ,  $\omega_x$  can be either zero or non-zero.

If it is not equal to zero, then nothing we have to worry about but if  $\omega_x = 0$ , then the dissipation strength  $\eta$  may be restricted by the criterion for the non-existence of crossover

temperature i.e.,  $\Omega^2 / \left[ 1 + \eta (2/M\pi) \int_0^\infty j(\omega) / \omega^3 d\omega \right] > \omega_b^2$ . From this criterion, it is clear that

the restriction on  $\eta$  appears when  $\Omega \geq \omega_b$  only, because the criterion is violated for all  $\eta$  if  $\Omega < \omega_b$ . If  $\Omega \geq \omega_b$ , then  $\eta$  must be restricted in the range  $[\eta_c, \infty)$ , where  $\eta_c \geq 0$

( $\eta_c = 0$  when  $\Omega = \omega_b$  only) obeys the equation  $\Omega^2 / \left[ 1 + \eta_c (2/M\pi) \int_0^\infty j(\omega) / \omega^3 d\omega \right] = \omega_b^2$ ,

since, for  $\eta < \eta_c$ , the criterion is fulfilled which implies that the crossover temperature is not exist. Moreover, from eq. (4.75), it is clear that  $\omega_R \rightarrow 0^+$  as  $\eta \rightarrow \eta_c^+$  and from eq.

(4.79) (with  $\omega_x = 0$ ), one can prove that  $\eta_0 > \eta_c$  (see Appendix E).

From all of the above results, since the crossover temperature  $T_0$  is proportional to  $\omega_R$ , the graph between  $T_0(\eta)/T_0(\eta=0)$ , in the case where  $\omega_x \neq 0$  and  $\omega_x = 0$  with the condition  $\Omega < \omega_b$ , and dissipation strength  $\eta$  can be generally depicted in fig. 4.7 and

the graph between  $T_0(\eta)$  (it can not be normalized since  $T_0(\eta = \eta_c) = 0$ ) and dissipation strength  $\eta$  can be generally depicted in fig. 4.8 as follows:

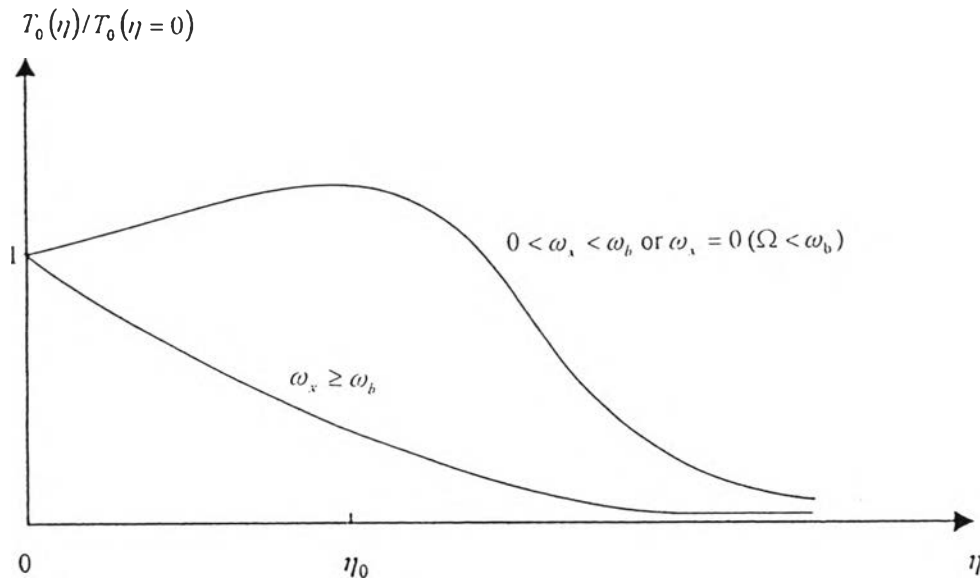


Fig. 4.7 : The normalized crossover temperature  $T_0(\eta)/T_0(\eta = 0)$  is shown for general damping as a function of dissipation strength  $\eta$  in the case where  $\omega_x \neq 0$  and  $\omega_x = 0$  with the condition  $\Omega < \omega_b$ .

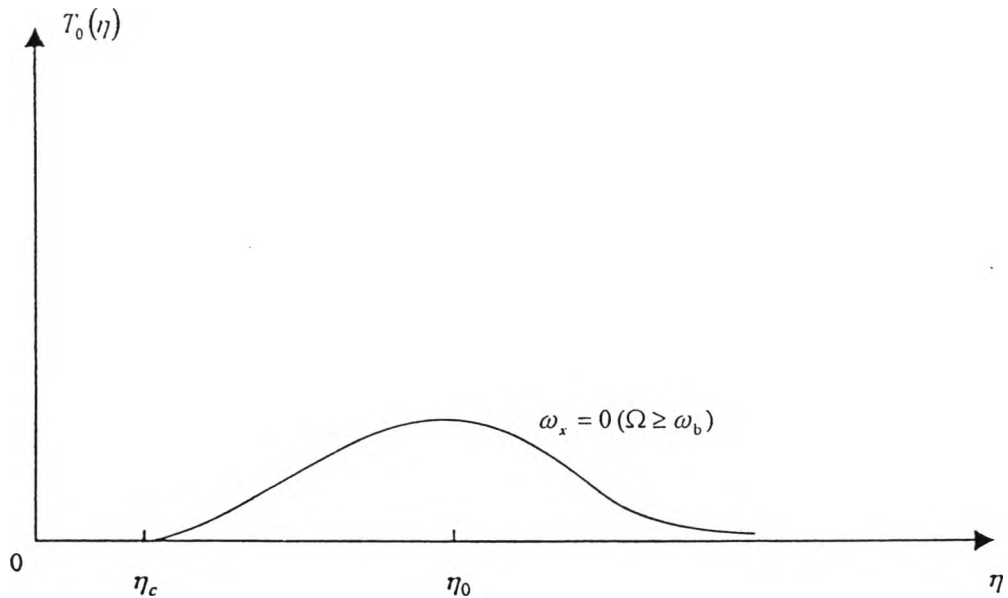


Fig. 4.8 : The crossover temperature  $T_0(\eta)$  is shown for general damping as a function of dissipation strength  $\eta$  in the case where  $\omega_x = 0$  with the condition  $\Omega \geq \omega_b$ .

However, we do not prove here whether there exists the points  $\eta_0$  obtained from eq. (4.79) or not and if they exist, we don't know that they are the points corresponding to

the maximum or minimum of  $\omega_R$ , and also do not prove whether these points are only one or not. Moreover, we can not prove by the use of eq. (4.78) that what the value of  $\omega'_R$  when  $\eta \rightarrow 0$  or  $\eta \rightarrow \eta_c^-$  is. By this reason, the graph in fig. 4.8 and the graph in fig. 4.7 corresponding to the case where  $0 < \omega_x < \omega_b$  or  $\omega_x = 0$  with the condition  $\Omega < \omega_b$  is not completely used. However, they are completely used if one can prove that there exists a unique  $\eta_c$ , which is the positive root of eq. (4.79).

Similar to the technique used for the variable  $\eta$ , if we define  $\omega_R$  as an implicit function of  $\Omega$  (it characterizes the Magnus force strength) via eq. (4.75), we also get

$$\omega'_R \left\{ \left[ 2\omega_R + \frac{d}{d\omega_R} (\omega_R \hat{\gamma}(\omega_R)) \right] \left[ 2(\omega_R^2 + \omega_R \hat{\gamma}(\omega_R)) + (\omega_x^2 - \omega_b^2) \right] + 2\omega_R \Omega^2 \right\} = -\frac{2\Omega \omega_R^2}{M} \quad (4.80)$$

where  $\omega'_R$  here denotes the differentiation of  $\omega_R$  with respect to  $\Omega$ .

From eq. (4.80), by the same reasons in previous case, it is clear that if  $\omega_x \geq \omega_b$ , then  $\omega'_R < 0 \forall \Omega \in \mathbb{R}_0^+$  which imply that  $\omega_R$  is a decreasing function of  $\Omega$  on  $\mathbb{R}_0^+$  but, for  $\omega_x < \omega_b$ , it is not sure whether  $\omega'_R < 0$  or not. However, we can still prove that  $\omega'_R$  is always less than or equal to zero by considering eq. (4.80) as follows: suppose that there exists set  $S \subseteq \mathbb{R}_0^+$  such that  $\omega'_R > 0 \forall \Omega \in S$ . Since, from eq. (4.75), we know that  $\omega_R^2 + \omega_R \hat{\gamma}(\omega_R) = \omega_b^2 - \Omega^2 \omega_R^2 / (\omega_R^2 + \omega_x^2 + \omega_R \hat{\gamma}(\omega_R))$  and  $\Omega \omega_R \rightarrow \omega_x \omega_b$  as  $\Omega \rightarrow \infty$  which imply by the use of eq. (4.80) that  $\omega'_R \rightarrow 0^-$  as  $\Omega \rightarrow \infty$  or  $\Omega \rightarrow 0^+$ , there exists at least  $\Omega_0^{(1)}, \Omega_0^{(2)} \in (0, \infty)$  such that  $\omega'_R(\Omega_0^{(1)}) = \omega'_R(\Omega_0^{(2)}) = 0$  but, from eq. (4.80),  $\omega'_R = 0$  at  $\Omega = 0$  only. This leads to the contradiction and hence  $\omega'_R \leq 0 \forall \Omega \in \mathbb{R}_0^+$  (the equality hold at  $\Omega = 0$  only). Moreover, from eq. (4.75), it is clear that  $\omega_R \rightarrow 0^+$  as  $\Omega \rightarrow \infty$ . As in the previous case of variable  $\eta$ , if  $\omega_x = 0$ , then the Magnus force strength  $\Omega$  must be restricted in the range  $[0, \Omega_c]$  by the criterion for the non-existence of crossover

temperature where  $\Omega_c$  obeys the equation  $\Omega_c^2 \sqrt{1 + (2/M\pi) \int_0^{\Omega_c} J(\omega) \omega^3 d\omega} = \omega_b^2$ .

Moreover, from eq. (4.75), it is clear that  $\omega_R \rightarrow 0^+$  as  $\Omega \rightarrow \Omega_c^-$ . By the same proof mentioned in the previous case of variable  $\eta$ , since  $\omega'_R \rightarrow 0^-$  as  $\Omega \rightarrow 0^+$  and  $\omega_R \rightarrow 0^+$  as  $\Omega \rightarrow \Omega_c^-$ , we can conclude that  $\omega'_R < 0 \forall \Omega \in (0, \Omega_c)$  and  $\omega'_R = 0$  at  $\Omega = 0$ .

Unfortunately, we can not prove by the use of eq. (4.80) that what is the value of  $\omega'_R$  when

$\Omega \rightarrow \Omega_c^-$  since the right hand side and  $\{ \}$  in the left hand side of eq. (4.80) are both equal to zero.

From the above discussion, since the crossover temperature  $T_0$  is proportional to  $\omega_R$ , the graph between  $T_0(\Omega)/T_0(\Omega=0)$  and the Magnus force strength  $\Omega$  can be generally depicted in figs. 4.9 and 4.10 as follows:

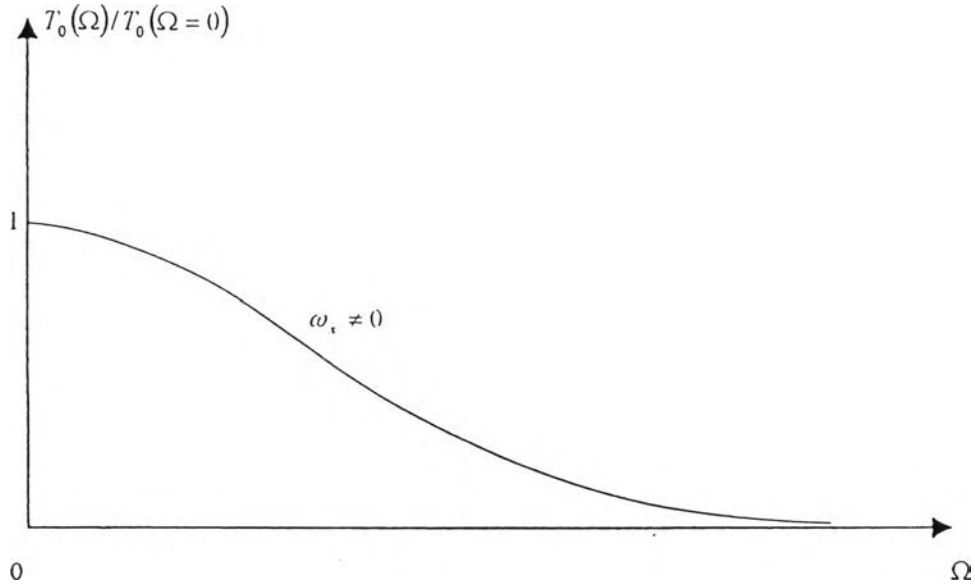


Fig. 4.9 : The normalized crossover temperature  $T_0(\Omega)/T_0(\Omega=0)$  is shown for general damping as a function of Magnus force strength  $\Omega$  in the case where  $\omega_x \neq 0$ .

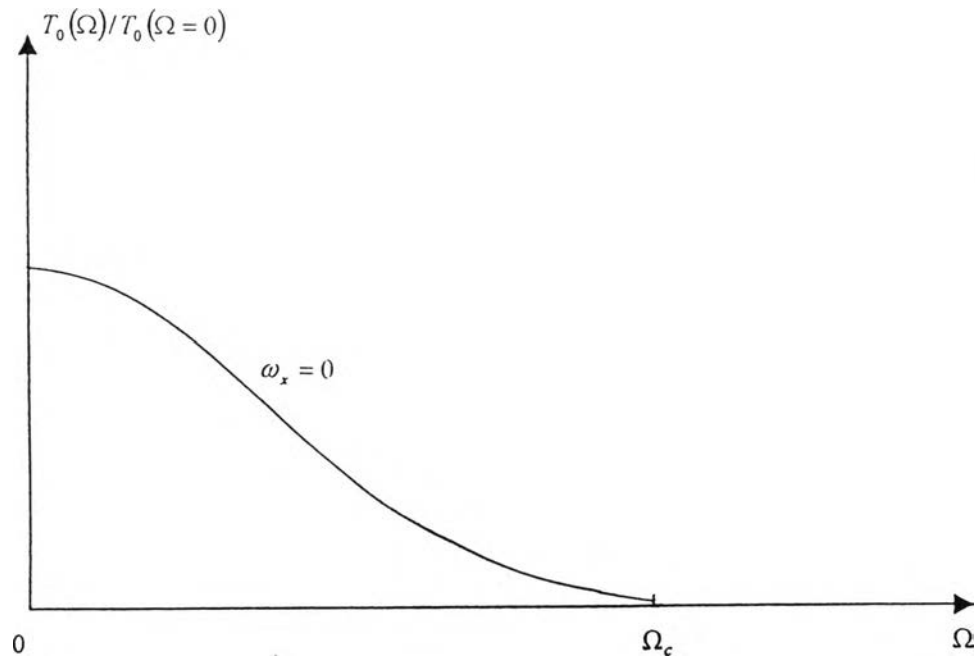


Fig. 4.10 : The normalized crossover temperature  $T_0(\Omega)/T_0(\Omega=0)$  is shown for general damping as a function of Magnus force strength  $\Omega$  in the case where  $\omega_x = 0$ .



From figs. (4.7), (4.8), (4.9), and (4.10), we can conclude in this section that the parameter  $\omega_x$  characterizing the pinning potential in  $x$  direction may affect the monotonically decreasing behavior of  $T_0$  on dissipation strength when  $\omega_x < \omega_b$  only while it can not affect this behavior of  $T_0$  on dissipation strength when  $\omega_x \geq \omega_b$  and Magnus force strength for all value of  $\omega_x$ . When  $\omega_x = 0$ , these behaviors still unchanged but, due to the criterion for the non-existence of  $T_0$ , the dissipation strength is restricted by the lower bound  $\eta_c \geq 0$  if  $\Omega \geq \omega_b$  while the Magnus fore strength is always restricted by the upper bound  $\Omega_c > 0$ . Moreover, when looking at the whole trend of these behaviors, one can conclude that if the dissipation or Magnus force strength is stronger, then the region of the temperature where the decay is dominated by quantum tunneling is thinner which imply that the temperature region where the decay is dominated by thermally activation is broader. Note that if the dissipation or Magnus force strength go to infinity, then the decay is totally dominated by thermal activation.

#### 4.7 LOCALIZATION OF A VORTEX

From previous section, it has been known that, for  $\omega_x \neq 0$ , the quantum tunneling always occurs since the crossover temperature always exists. This means that a vortex is not localized in the well for  $T < T_0$  i.e., it escapes out of metastable well by quantum tunneling mechanism describing by the escape rate formula (4.45). Moreover, for  $\omega_x \neq 0$  and  $T > T_0$ , a vortex also escapes out of a metastable well dominated by thermally activated decay mechanism described by the escape rate formula (4.61). Hence in the case where  $\omega_x \neq 0$ , a vortex is not localized in the well for the whole range of temperature. On the other hand, for  $\omega_x = 0$ , if the Magnus force characterized by  $\Omega$  is strong enough so that it fulfills the criterion in theorem 2, then the crossover temperature does not exist which implies that a vortex can not escape out of the well by quantum tunneling mechanism. One can guess at first sight that a vortex can escape out of the well by thermal activation mechanism for the whole temperate range. This guess is false! It comes from the fact that if the criterion in theorem 2 is fulfilled, then the assumption in eq. (4.64) is wrong which implies that there is no divergent integral in the reduced partition function  $Z_d^{(b)}$  (see eq. (4.56), because  $\lambda_0^{(b)}$  is now positive, and then make  $Z_d^{(b)}$  finite and real. The finite value of reduced partition function  $Z_d^{(b)}$  implies that the reduced partition

function  $Z_d = Z_d^{(a)} + Z_d^{(b)}$  (see eq. (4.46)) is also finite since  $Z_d^{(a)}$  is finite (See eq. (4.55)). The finite value of reduced partition function  $Z_d$  indicates that a vortex must be localized in the well. Since this conclusion of the localization of a vortex has no restriction on temperature, this conclusion must be true for the whole range of temperature. However, for  $\omega_x = 0$ , if the Magnus force is not strong enough so that the non-existence criterion in theorem 2 is violated, then a vortex is not localized in the well, as the assumption (4.64) is fulfilled so  $\lambda_{v_0}^{(b)}$  is negative, i.e., it can escape out of the well by both quantum tunneling and thermal hopping but, in this case where  $\omega_x = 0$ , we have to use the escape rate formula (4.65), instead of (4.61), for  $T > T_0$ .

Now, let us summarize this section. If  $\omega_x \neq 0$ , then a vortex is not always localized in the well at any temperature. On the other hand, if  $\omega_x = 0$  then the localization criterion at any temperature, which is the same as the criterion for the non-existence of crossover temperature, is in the form

$$\frac{\Omega^2}{1 + \frac{2}{M\pi} \int_0^\infty \frac{J(\omega)}{\omega^3} d\omega} > \omega_b^2 \quad (4.81)$$

Recall the definition of  $\Omega$  from eq. (4.11). In the cas of a vortex in superconductor, this criterion can be written in the form

$$\frac{(h\rho_s d/2)^2}{M + \frac{2}{\pi} \int_0^\infty \frac{J(\omega)}{\omega^3} d\omega} > |V''(y_b)| \quad (4.82)$$

It is worthwhile to point out that according to eq. (4.82) a large Magnus force inhibits vortex escaping while a large vortex mass favors vortex escaping. Note that when the dissipation is absent i.e.,  $J(\omega) = 0$ , the formula (4.82) reduces to the localization criterion in the case of no pinning and dissipation at zero temperature given by P.Ao and D.J.Thouless [19].

#### 4.8 EFFECTIVE MASS OF A VORTEX AND ITS INTERPRETATION

From eq. (4.81), the localization criterion in the absence of dissipation can be written as

$$\frac{(M\Omega)^2}{M} > |V''(y_b)| \quad (4.83)$$

where  $M\Omega$  is the mass independent parameter e.g.,  $M\Omega = h\rho_s d/2$  for a vortex in superconduction (see eq. (4.11)). Comparing eq. (4.83) with eq. (4.81), in the presence of dissipation, we can define the "effective mass" of a vortex  $M^*$  in the sense of localization criterion by

$$M^* = M + \frac{2}{\pi} \int_0^\infty \frac{J(\omega)}{\omega^3} d\omega \quad (4.84)$$

so that the localization criterion (4.81) can be written as

$$\frac{(M\Omega)^2}{M^*} > |V''(y_b)| \quad (4.85)$$

Comparing eq. (4.85) with eq. (4.83), the effective mass of a vortex can be described as follows: a vortex in contact with the environment which is localized in the pinning potential for sufficiently strong Magnus force behaves as if it is effectively not in contact with the environment any more but its mass must be effectively changed to the new bigger one called effective mass. This effective mass is equal to the original mass plus the extra mass originated from the environment since it depends on the spectral function. In other words, when a vortex in contact with the environment decides to escape out of the pinning potential, it behaves as if it is effectively free from the environment but its original mass must be changed to the effective one. Note that this extra mass (second term in the right hand side of eq. (4.84) is proportional to the effective mass of a vortex given by J.H.Han, P.Ao, and X.M.Zhu[18].

It is worth to note that since our definition of the effective mass of a vortex does not come directly from the dynamical approach (it is defined via the localization criterion), we don't know whether the coordinate of an effectively undamped vortex (a vortex which is effectively free from the environment) of mass  $M^*$  is identical with the coordinate  $\bar{r}$  of the original damped vortex or not. However, some conclusions can be made by first introducing new coordinates  $\bar{q}'_\alpha$  and new masses  $\mu_\alpha$  for the bath oscillator [39],

$$\left. \begin{aligned} \bar{q}'_\alpha &= \frac{m_\alpha \omega_\alpha^2 \bar{q}_\alpha}{c_\alpha} \\ \mu_\alpha &= \frac{c_\alpha^2}{m_\alpha \omega_\alpha^4} \end{aligned} \right\} \quad (4.86)$$

From eq. (4.86), the Hamiltonian (4.7) can be written as

$$H = \frac{1}{2M} |\bar{P} - q_v \bar{A}(\bar{r})|^2 + V(r) + \frac{1}{2} \sum_{\alpha=1}^N \mu_{\alpha} \left[ \dot{\bar{q}}_{\alpha}'^2 + \omega_{\alpha}^2 |\bar{q}_{\alpha}' - \bar{r}|^2 \right] \quad (4.87)$$

We can see that the model Hamiltonian (4.7), in fact, describes a vortex of mass  $M$  with many masses  $\mu_{\alpha}$  affected with springs to its coordinate  $\bar{r}$ . With the substitution (4.86) the spectral function of the environmental coupling  $J(\omega)$ , which has been defined in eq. (3.62), take the form [16]

$$J(\omega) = \frac{\pi}{2} \sum_{\alpha=1}^N \mu_{\alpha} \omega_{\alpha}^3 \delta(\omega - \omega_{\alpha}). \quad (4.88)$$

From eq. (4.88), it is clear that the total mass attached with springs to vortex's coordinate  $\bar{r}$  can be written as

$$\sum_{\alpha=1}^N \mu_{\alpha} = \frac{2}{\pi} \int_0^{\infty} \frac{J(\omega)}{\omega^3} d\omega. \quad (4.89)$$

From this point of view, from eqs. (4.87) and (4.89), our effective mass  $M^*$  of vortex (4.84) is, in fact, equal to the total mass of the system which is composed of a vortex of mass  $M$  with many masses  $\mu_{\alpha}$  attached with springs to its coordinate. Since, our problem is in the infinite two dimensional plane and the whole system (vortex plus environment) is in the thermal equilibrium, the environment will move randomly in such a way that the average of the center of mass coordinate of the system described by Hamiltonian (4.87) coincide with the average of the coordinate of a vortex at all time. Hence, the center of mass itself can be viewed as a vortex since it feels the Magnus force. By this reason, the whole system can be viewed as one undamped vortex, which has its mass equals to the total mass of the system, described by the center of mass coordinate. Since this vortex is an undamped vortex, the localization criterion which we have to use for this undamped vortex is the localization criterion (4.83). This localization criterion is now identical with the criterion (4.85) since the mass of this undamped vortex is equal to the mass of the whole system which is equal to our effective mass of vortex. By the above discussion, one can conclude that the damped vortex of mass  $M$  can be effectively viewed as an undamped vortex of mass  $M^*$  within the sense of the localization criterion, and the coordinate of the undamped vortex is identical to the coordinate of an original damped vortex which is also identical to the center of mass coordinate of the system described by Hamiltonian (4.87)

(this Hamiltonian actually describes our original problem since the physical behavior is independent of the choice of coordinate one used). However, if the system is confined in the "finite plane", then this conclusion will be true only for the case of sufficiently weak environmental coupling so that  $\sum_{\alpha=1}^N \mu_{\alpha} \ll M$ . This come from the fact that when  $\sum_{\alpha=1}^N \mu_{\alpha} \ll M$ , the center of mass coordinates of the system will approximately coincide with the original coordinates of a damped vortex, even when a vortex is located near the edges of the plane.