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TRIANGULAR MEAN-VALUE FUNCTIONAL EQUATION

Mr. Worawit Tepsan

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By	Mr. Worawit Tepsan
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Thesis Advisor	Assistant Professor Nataphan Kitisin, Ph.D.
Thesis Co-advisor	Associate Professor Paisan Nakmahachalasint, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

....Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

....Chairman
(Associate Professor Patanee Udomkavanich, Ph.D.)

....Thesis Advisor
(Assistant Professor Nataphan Kitisin, Ph.D.)

....Thesis Co-advisor
(Associate Professor Paisan Nakmahachalasint, Ph.D.)

....Examiner
(Khamron Mekchay, Ph.D.)

....External Examiner
(Watcharapon Pimsert, Ph.D.)

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ในวิทยานิพนธ์ฉบับนี้ มีจุดประสงค์เพื่อหาผลเฉลยทั่วไป ของสมการเชิงฟังก์ชันค่าเฉลี่ยรูปสามเหลี่ยม ซึ่งมีข้อกำหนดว่า แต่ละรูปสามเหลี่ยมซึ่งเกิดจากการเลื่อนหรือขยายจากรูปสามเหลี่ยมที่ถูกกำหนดมารูปหนึ่ง ค่าของฟังก์ชันที่จุดเซนทรอยของรูปสามเหลี่ยมเท่ากับค่าเฉลี่ยเลขคณิตของค่าฟังก์ชันที่จุดยอด โดยมีรูปแบบสมการคือ

$$f(z+tz_1)+f(z+tz_2)+f(z+tz_3)=3f(z)$$

สำหรับทุก $z\in\mathbb{R}^2$ และ $t>0$ โดย z_1,z_2,z_3 เป็นจุดคงที่บนระนาบ

ภาควิชา คณิตศาสตร์ ลายมือชื่อนิสิต..

และวิทยาการคอมพิวเตอร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก.....................

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม.....................

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The main objective of this thesis is to determine the general solution of the triangular mean-value functional equation with the constrain that for each triangle obtained by translations and dilations of an arbitrary fixed triangle, the value of the function at its centroid is the arithmetic mean of its values at the vertices. In particular, we obtain a general solution to the equation

$$f(z+tz_1)+f(z+tz_2)+f(z+tz_3)=3f(z),$$

for all $z \in \mathbb{R}^2$ and $t>0$ where z_1, z_2, z_3 are fixed points in $\mathbb{R}^2$.

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Student's Signature ....
Advisor's Signature ....
Co-advisor's Signature ....

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CONTENTS

CHAPTER I
INTRODUCTION

1.1 Functional Equations

Functional equations are equations whose unknown variables are functions. The objective of study functional equation is to find all functions satisfying equations. The following example illustrates a functional equation problem and its solution.

Example 1.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 f(x) + f(-x) = x^3 - x$$

for all $x \in \mathbb{R}$.

Solution: Replace x by $-x$ in the equation, we have

$$x^2 f(-x) + f(x) = -x^3 + x$$

By adding both equations, we get

$$(x^2 + 1)(f(x) + f(-x)) = 0.$$

Since $x^2 + 1 > 0$, we have

$$f(-x) = -f(x).$$

Substituting this into the first equation and solving for $f(x)$, we get

$$f(x) = \begin{cases} x & \text{if } x \notin \{1, -1\} \\ c & \text{if } x = 1 \\ -c & \text{if } x = -1 \end{cases}$$

where c are constants. Conversely, the solution can be verified by substituting back into the first equation. $\square$

In addition, an unknown function in functional equations can be defined in any domain and range. We will give some example of functional equations whose unknown is defined on $\mathbb{R}^2$.

Example 1.2. Find all functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,x) + xyf(x,y) + f(y,y) = xy + 2$$

for all $x, y \in \mathbb{R}$.

<u>**Solution**</u>: Let $x = 0$ and $y = 0$ into the equation, we have

$$f(0,0) = 1.$$

Replacing $y = 0$ into the equation, we get

$$f(x,x) + f(0,0) = 2.$$

So,

$$f(x,x) = 1.$$

Substituting back into the first equation, we get

$$xyf(x,y) = xy.$$

So,

$$f(x,y) = 1,$$

if $x, y \neq 0$. So

$$f(x,y) = \begin{cases} 1 & \text{if } x, y \neq 0 \\ g_1(x) & \text{if } y = 0 \\ g_2(y) & \text{if } x = 0 \end{cases}$$

where g_1, g_2 are functions on $\mathbb{R}$ such that $g_1(0) = 1 = g_2(0)$. Conversely, the solution can be verified by substituting back into the first equation. $\square$

1.2 Literature Review

The functional equation has widely been studied. In this section, we review some researches which are related to our work. In 1968, J.Aczél, H.Haruki, M.A.Mckiernan, and G.N.Sakovič [1] published a paper which determined the general solution of a functional equation

$$f(x+\alpha, y+\beta)+f(x+\alpha, y-\beta)+f(x-\alpha, y+\beta)+f(x-\alpha, y-\beta)=4f(x,y).$$

This equation states that the value of f at the center of any rectangle with its sides parallel to the coordinate axes, equals to the arithmetic mean of its values at all vertices. The authors also showed that if $f \in C(\mathbb{R}^2)$ and the value of f at the center of any homothetic regular n-gons in $\mathbb{R}^2$ is the arithmetic mean of the values of vertices, then f is a harmonic polynomial of degree n. Later in 1982, S. Haruki [5] determined the general solution of triangular mean-value functional equation

$$f(x-t, y-\frac{t}{\sqrt{3}})+f(x+t, y-\frac{t}{\sqrt{3}})+f(x, y+\frac{2t}{\sqrt{3}})=3f(x,y).$$

Note that the triangles embedded in the above equation have the certain configurations that they are homothetic equilateral triangles with one side parallel to the x-axis. In 1995, J.A. Baker [2] studied a triangular mean-value functional equation of the form

$$f(z+e^{it})+f(z+e^{it}\omega)+f(z+e^{it}\overline{\omega})=3f(z),$$

where $z \in \mathbb{C}$ and $\omega = e^{2\pi i/3}$. This functional equation states that the value of f at the centroid of any triangle, obtained by rotations and translation of equilateral triangle, equals to the arithmetic mean of its values at all vertices. He found that a solution of this equation must be a harmonic polynomial provided that the function f is continuous. Recently, R. Kotnara [3] studied the functional equation which state that given $z_1, z_2 \in \mathbb{R}^2$,

$$f(z)+f(z+z_1t)+f(z+z_2t)=0$$

for all $z \in \mathbb{R}^2$ and $t \in \mathbb{R} \setminus \{0\}$. He found that a solution of this functional equation is zero function.

1.3 Purposed Problem

The objective of our studying is to solve the triangular mean-value functional equation. Through out this thesis, *the triangular mean-value functional equation of a triangle with vertices* $z_1, z_2, z_3 \in \mathbb{R}^2$ (it may be sometime called in short as *triangular mean-value functional equation of* z_1, z_2, z_3) is referred to a functional equation of the form

$$f(z+tz_1)+f(z+tz_2)+f(z+tz_3)=3f\big(z+\frac{t}{3}(z_1+z_2+z_3)\big), \qquad (1.1)$$

for all $z \in \mathbb{R}^2$ and $t > 0$.

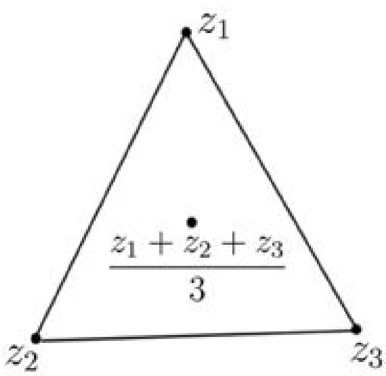

Figure 1.1

Geometrically, this equation says that given an arbitrary triangle with points z_1, z_2, z_3 as its vertices (as Figure 1.1.), for each triangles obtained by translations and dilations of this fixed triangle, the value of the function at its centroid is the arithmetic mean of the values at its vertices. For example, triangular mean-value functional equation of $(0, 2/\sqrt{3}), (1, -1/\sqrt{3}), (-1, -1/\sqrt{3})$ is of the form

$$f(x-t, y-\frac{t}{\sqrt{3}})+f(x+t, y-\frac{t}{\sqrt{3}})+f(x, y+\frac{2t}{\sqrt{3}})=3f(x,y),$$

for $x, y \in \mathbb{R}$ and $t > 0$. Note that the above equation is equivalent to the functional equation which Haruki had studied [5].

CHAPTER II
PRELIMINARIES

In this chapter, we provide background knowledge which plays important role in this thesis. A function $A : \mathbb{R} \to \mathbb{R}$ is said to be additive provided $A(x + y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}$. If $n \in \mathbb{N}$ and $A_n : \mathbb{R}^n \to \mathbb{R}$, we say that A_n is $n-$additive (multi-additive) provided it is additive in each variable. A_n is symmetric provided $A_n(x_1, \ldots, x_n) = A_n(y_1, \ldots, y_n)$ whenever $(y_1, \ldots, y_n)$ is a permutation of $(x_1, \ldots, x_n)$. Throughout this paper, we use the alphabet A_0 or A^0 to denote a constant and A or A^1 to denote an additive function. For any $m, n \in \mathbb{N} \cup \{0\}$, we use the following notations for specific meanings:

- If $z = (x, y) \in \mathbb{R}^2$, we denote $\mathrm{Re}(z) = x, \mathrm{Im}(z) = y$ and $||z|| = \sqrt{x^2 + y^2}$.
- $S(z_1, z_2, z_3)$ is the set of all solutions of a triangular mean-value functional equation of $z_1, z_2, z_3 \in \mathbb{R}^2$.
- $A_m(x_1, x_2, \ldots, x_m) : \mathbb{R}^m \to \mathbb{R}$ is a symmetric $m-$additive function and $A^m(x) : \mathbb{R} \to \mathbb{R}$ be the diagonalization of A_m, i.e., $A^m(x) = A_m(x, \ldots, x)$. We say that $A^m(x)$ is a diagonalization of order m.
- $A_{m,n}(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n) : \mathbb{R}^{m+n} \to \mathbb{R}$ is a $(m+n)$-additive function which is symmetric in its first m entries while the last n entries are fixed and vice versa.
- $A^{0,0}(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a constant function.
- $A^{m,0}(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a diagonalization of a symmetric m-additive function while y is fixed, and is a constant while x is fixed. For convenience, we use $A^{m,0}(x)$ to denote $A^{m,0}(x, y)$. We define $A^{0,n}(x, y)$ and $A^{0,n}(y)$ similarly.

- $A^{m,n}(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$A^{m,n}(x,y) = A_{m,n}(\underbrace{x, x, \ldots, x}_{m \text{ entries}}; \underbrace{y, y, \ldots, y}_{n \text{ entries}}).$$

In this study, the difference operator is extensively used. We define the difference operator of order $n \in \mathbb{N}$ for a function $f : \mathbb{R} \to \mathbb{R}$ as follow;

$$\underset{t}{\Delta}^n f(x) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\big(x + (n-i)t\big).$$

The following theorem shows the relationship between difference operator and the multi-additive function.

Theorem 2.1. *[4] If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$\underset{t}{\Delta}^{M+1} f(x) = 0$$

for all $x, t \in \mathbb{R}$ then

$$f(x) = \sum_{n=0}^{M} A^n(x).$$

The difference operator can be extended to the function on $\mathbb{R}^2$. We define the difference operator of order $n \in \mathbb{N}$ for a function $f : \mathbb{R}^2 \to \mathbb{R}$ with respect to x as follow;

$$\underset{x,t}{\Delta}^n f(x,y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\big(x + (n-i)t, y\big).$$

Similarly, the difference operator of order $n \in \mathbb{N}$ with respect to y is defined as;

$$\underset{y,t}{\Delta}^n f(x,y) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\big(x, y + (n-i)t\big),$$

where $(x,y) \in \mathbb{R}^2$ and $t \in \mathbb{R}$.

The following theorem shows the relationship between difference operator of a function on $\mathbb{R}^2$ and the multi-additive function.

Theorem 2.2. *[1] If a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies both equations*

$$\underset{x,t}{\Delta}^{M+1} f(x,y) = 0 \quad \text{and} \quad \Delta_{y,t}^{N+1} f(x,y) = 0$$

for all $x, y, t \in \mathbb{R}$ *then*

$$f(x,y) = \sum_{n=0}^{M} \sum_{m=0}^{N} A^{n,m}(x,y).$$

S. Haruki[5] determined solution of the triangular mean-value functional equation of the certain equilateral triangle. His result will be applied in the next chapter. The Haruki's theorem can be rewritten in our notations as follow:

Theorem 2.3. *[5] The general solution of the functional equation*

$$f(x-t, y-\frac{t}{\sqrt{3}}) + f(x+t, y-\frac{t}{\sqrt{3}}) + f(x, y+\frac{2t}{\sqrt{3}}) = 3f(x,y),$$

for all $x, y, t \in \mathbb{R}$, *can be written in the form of*

$$f(x,y) = A^0 + A^{1,0}(x) + A^{2,0}(x) + A^{3,0}(x) + A^{0,1}(y) + A^{1,1}(x,y) + A^{0,2}(y) + A^{1,2}(x,y),$$

where $A^{0,2}(\sqrt{3}t) = -3A^{2,0}(t)$ *and* $A^{1,2}(x, \sqrt{3}t) = -9A_{3,0}(x,t,t)$, *for* $x, y, t \in \mathbb{R}$.

CHAPTER III

Equilateral Triangular Mean-Value Functional Equation

According to Haruki's paper [5], Haruki determined solution of the triangular mean-value functional equation of the certain equilateral triangle. In this chapter, we extend Harukis's work by solving a triangular mean-value functional equation of any equilateral triangle.

First, we will establish fundamental knowledge which is important for solving our problem. Recall that $S(z_1, z_2, z_3)$ is the set of all solutions of a triangular mean-value functional equation of $z_1, z_2, z_3 \in \mathbb{R}^2$, i.e.,

$$\begin{aligned} S(z_1, z_2, z_3) = \{f : \mathbb{R}^2 \to \mathbb{R} | f(z + tz_1) + f(z + tz_2) + f(z + tz_3) \\ = 3f\big(z + \frac{t}{3}(z_1 + z_2 + z_3)\big), \text{ for all } z \in \mathbb{R}^2 \text{ and } t > 0\}. \end{aligned}$$

The following lemma shows that the set of solutions of triangular mean-value functional equation of two triangles, where one triangle is obtained from dialation and translation from the other, are equal.

Lemma 3.1. *Let* $z_0, z_1, z_2, z_3 \in \mathbb{R}^2$ *and* $\alpha > 0$*. Then*

$$S(z_1, z_2, z_3) = S(\alpha z_1 + z_0, \alpha z_2 + z_0, \alpha z_3 + z_0).$$

Proof. Let $f \in S(z_1, z_2, z_3)$. So f satisfies the functional equation

$$f(z + tz_1) + f(z + tz_2) + f(z + tz_3) = 3f\big(z + \frac{t}{3}(z_1 + z_2 + z_3)\big),$$

for all $z \in \mathbb{R}^2$ and $t > 0$. By substituting z as $z + tz_0$ and t as αt in the above equation, the equation becomes

$$\begin{aligned} f(z + t(\alpha z_1 + z_0)) + f(z + t(\alpha z_2 + z_0)) + f(z + t(\alpha z_3 + z_0)) \\ = 3f\big(z + t\frac{(\alpha z_1 + z_0) + (\alpha z_2 + z_0) + (\alpha z_3 + z_0)}{3}\big) \end{aligned}$$

for all $z \in \mathbb{R}^2$ and $t > 0$. Hence, $f \in S\{\alpha z_1+z_0, \alpha z_2+z_0, \alpha z_3+z_0\}$. Conversely, we can similarly prove that $S(\alpha z_1+z_0, \alpha z_2+z_0, \alpha z_3+z_0) \subset S(z_1, z_2, z_3)$ by substituting z as $z - (tz_0)/\alpha$ and t as t/α. $\square$

Let $z_1, z_2, z_3 \in \mathbb{R}^2$ be noncollinear points. Define $\omega_i = z_i - (z_1 + z_2 + z_3)/3$ for $i = 1, 2, 3$. This is translation and dialation the triangle with vertices z_1, z_2, z_3 to the triangle whose centroid is the origin point (as Figure 3.1).

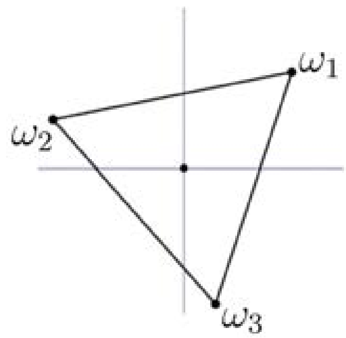

Figure 3.1

By the property of ω_i, we have

$$\omega_1 + \omega_2 + \omega_3 = 0$$

and the triangular mean-value functional equation of $\omega_1, \omega_2, \omega_3$ is, therefore, of the form

$$f(z + t\omega_1) + f(z + t\omega_2) + f(z + t\omega_3) = 3f(z). \tag{3.1}$$

According to Lemma 3.1, we have $S(z_1, z_2, z_3) = S(\omega_1, \omega_2, \omega_3)$. Hence, to solve triangular mean-value functional equation of z_1, z_2, z_3 is sufficient to solve the functional equation (3.1) instead. Thus, we will concentrate on solving only triangular mean-value functional equation of a triangle which sum of its vertices is zero (its centroid is the origin).

Next, we consider the relationship between solutions of triangular mean-value functional equation of two related triangles. Suppose that there exists a linear bijection $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps three vertices of a triangle to three vertices of another triangle (as Figure 3.2), we find that each solution of triangular mean-value functional equations can be formed by composition of one solution with the linear bijection T. This fact is stated precisely as in the following lemma.

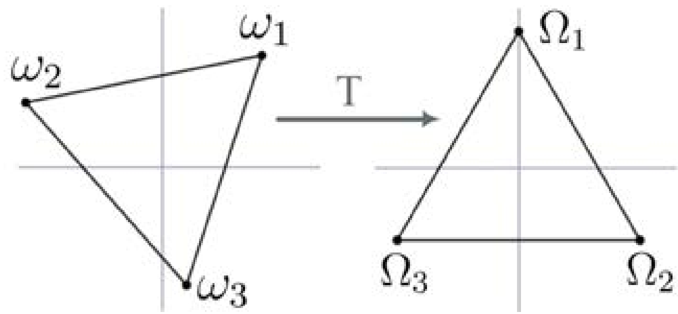

Figure 3.2

Lemma 3.2. *Let $\omega_i, \Omega_i \in \mathbb{R}^2$ for $i = 1, 2, 3$ be such that $\sum_{i=1}^{3} \omega_i = \sum_{i=1}^{3} \Omega_i = 0$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear bijection with $T(\omega_i) = \Omega_i$ for $i = 1, 2, 3$, then*

$$S(\omega_1, \omega_2, \omega_3) = \{g \circ T | g \in S(\Omega_1, \Omega_2, \Omega_3)\}.$$

Proof. We will show that $S(\omega_1, \omega_2, \omega_3) = \{g \circ T | g \in S(\Omega_1, \Omega_2, \Omega_3)\}$.
($\supset$) Let $g \in S\{\Omega_1, \Omega_2, \Omega_3\}$. So,

$$g(z + t\Omega_1) + g(z + t\Omega_2) + g(z + t\Omega_3) = 3g(z).$$

Since $z + t\Omega_i = T(T^{-1}(z)) + T(t\omega_i) = T(T^{-1}(z) + t\omega_i)$ for $i = 1, 2, 3$, we have that

$$g(T(T^{-1}(z) + t\omega_1)) + g(T(T^{-1}(z) + t\omega_2)) + g(T(T^{-1}(z) + t\omega_3)) = 3g(T(T^{-1}(z))).$$

Hence,

$$g \circ T(T^{-1}(z) + t\omega_1) + g \circ T(T^{-1}(z) + t\omega_2) + g \circ T(T^{-1}(z) + t\omega_3) = 3g \circ T(T^{-1}(z)).$$

Since T is a linear bijection, $T^{-1}(z)$ can be chosen arbitrary. So, we obtain that $g \circ T \in S\{\omega_1, \omega_2, \omega_3\}$.

($\subset$) Let $f \in S\{\omega_1, \omega_2, \omega_3\}$. Since $T(\omega_i) = \Omega_i$, we get that $T^{-1}(\Omega_i) = \omega_i$. So $f \circ T^{-1} \in S\{\Omega_1, \Omega_2, \Omega_3\}$. Since $f = (f \circ T^{-1}) \circ T$, we have that $f \in \{g \circ T | g \in S(\Omega_1, \Omega_2, \Omega_3)\}$. $\square$

Next, we will show that solutions of triangular mean-value functional equation of $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^2$ and $-\omega_1, -\omega_2, -\omega_3 \in \mathbb{R}^2$ are the same.

Proposition 3.3. *Let $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^2$ be noncollinear and satisfied $\omega_1+\omega_2+\omega_3 = 0$. Then*

$$S(\omega_1, \omega_2, \omega_3) = S(-\omega_1, -\omega_2, -\omega_3).$$

Proof. It is easy to see that $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(z) = -z$ is a linear bijection which is satisfied $T(\omega_i) = -\omega_i$ for $i = 1, 2, 3$. By Lemma 3.2, there exists $g \in S(-\omega_1, -\omega_2, -\omega_3)$ such that $f(z) = g \circ T(z) = g(-z)$. Since $f \in S(\omega_1, \omega_2, \omega_3)$, we have

$$3f(z) = f(z + t\omega_1) + f(z + t\omega_2) + f(z + t\omega_3).$$

Because $f(z) = g(-z)$, we obtain that $f(z+t\omega_1) = g(-z-t\omega_i)$ for $i = 1, 2, 3$. Then

$$3f(z) = g(-z - t\omega_1) + g(-z - t\omega_2) + g(-z - t\omega_3).$$

Since $g \in S(-\omega_1, -\omega_2, -\omega_3)$, $g(z) = (g(z - t\omega_1) + g(z - t\omega_2) + g(z - t\omega_3))/3$ for all $z \in \mathbb{R}^2$ and $t > 0$. Then

$$\begin{aligned} 3f(z) =& \frac{1}{3}(g(-z - t\omega_1 - t\omega_1) + g(-z - t\omega_1 - t\omega_2) + g(-z - t\omega_1 - t\omega_3) \\ &+ g(-z - t\omega_2 - t\omega_1) + g(-z - t\omega_2 - t\omega_2) + g(-z - t\omega_2 - t\omega_3) \\ &+ g(-z - t\omega_3 - t\omega_1) + g(-z - t\omega_3 - t\omega_2) + g(-z - t\omega_3 - t\omega_3)). \end{aligned}$$

Since $\omega_1 + \omega_2 + \omega_3 = 0$, the above equation becomes

$$\begin{aligned} 3f(z) =& \frac{1}{3}(g(-z - 2t\omega_1) + g(-z + t\omega_3) + g(-z + t\omega_2) \\ &+ g(-z + t\omega_3) + g(-z - 2t\omega_2) + g(-z + t\omega_1) \\ &+ g(-z + t\omega_2) + g(-z + t\omega_1) + g(-z - 2t\omega_3)) \\ =& \frac{1}{3}(g(-z - 2t\omega_1) + g(-z - 2t\omega_2) + g(-z - 2t\omega_3) \\ &+ 2(g(-z + t\omega_1) + g(-z + t\omega_2) + g(-z + t\omega_3))). \end{aligned}$$

Because $f(z) = g(-z)$ for all $z \in \mathbb{R}^2$ and $g \in S(-\omega_1, -\omega_2, -\omega_3)$, we have form the previous equation that

$$\begin{aligned} 3f(z) &= \frac{1}{3}(3g(-z) + 2(g(-z + t\omega_1) + g(-z + t\omega_2) + g(-z + t\omega_3))) \\ &= \frac{1}{3}(3f(z) + 2(f(z - t\omega_1) + f(z - t\omega_2) + f(z - t\omega_3))). \end{aligned}$$

Consequently, we have

$$3f(z) = f(z - t\omega_1) + f(z - t\omega_2) + f(z - t\omega_3).$$

Hence, $S(\omega_1, \omega_2, \omega_3) \subset S(-\omega_1, -\omega_2, -\omega_3)$. On the other hand, we can similarly prove that $S(-\omega_1, -\omega_2, -\omega_3) \subset S(\omega_1, \omega_2, \omega_3)$ by considering ω_i as $-\omega_i$. Therefore, $S(\omega_1, \omega_2, \omega_3) = S(-\omega_1, -\omega_2, -\omega_3)$. □

As a result of Proposition 3.3, we immediately have the following corollary.

Corollary 3.4. *Let $z_1, z_2, z_3 \in \mathbb{R}^2$. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies* (1.1) *i.e.*

$$f(z + tz_1) + f(z + tz_2) + f(z + tz_3) = 3f\big(z + \frac{t}{3}(z_1 + z_2 + z_3)\big),$$

for all $z \in \mathbb{R}^2$ and $t > 0$. Then f satisfies (1.1) *for all $z \in \mathbb{R}^2$ and $t \in \mathbb{R}$.*

Originally, triangular mean value functional equation (1.1) is only for $t > 0$. The above corollary, however, implies that the equation hold for all $t \in \mathbb{R}$. This fact implies that solutions of triangular mean-value functional equation of $(0, 2/\sqrt{3}), (1, -1/\sqrt{3}), (-1, -1/\sqrt{3})$ which is of the form

$$f(x - t, y - \frac{t}{\sqrt{3}}) + f(x + t, y - \frac{t}{\sqrt{3}}) + f(x, y + \frac{2t}{\sqrt{3}}) = 3f(x, y),$$

for all $x, y \in \mathbb{R}$ and $t > 0$ is also Haruki's solution as in Theorem 2.3. Applying this fact and Lemma 3.2, we can find solutions of triangular mean-value functional equation of any equilateral triangular by composition of Haruki's solution with the linear bijection which maps its three vertices to $(0, 2/\sqrt{3}), (1, -1/\sqrt{3}), (-1, -1/\sqrt{3})$, respectively.

In the following theorem, we reformulate Theorem 2.3 by incorporating the conditions on $A^{0,2}$ and $A^{1,2}$ into the solution.

Theorem 3.5. *The solution in Theorem 2.3 is equivalent to*

$$\begin{aligned} f(x, y) = A_0 + B(x) + \widetilde{B}(y) + C(x + \sqrt{3}y, 3x - \sqrt{3}y) + \widetilde{C}(y, x) \\ + D(x, x + \sqrt{3}y, x - \sqrt{3}y), \end{aligned}$$

where A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function.

Proof. (⇒) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the Haruki's solution as in Theorem 2.3;

$$f(x,y) = A^0 + A^{1,0}(x) + A^{2,0}(x) + A^{3,0}(x) + A^{0,1}(y) + A^{1,1}(x,y) + A^{0,2}(y) + A^{1,2}(x,y),$$

where $A^{0,2}(\sqrt{3}t) = -3A^{2,0}(t)$ and $A^{1,2}(x, \sqrt{3}t) = -9A_{3,0}(x,t,t)$, for all $x, y, t \in \mathbb{R}$. By choosing

$$B(x) = A^{1,0}(x), \quad \widetilde{B}(y) = A^{0,1}(y), \quad D(x,y,z) = A_{3,0}(x,y,z).$$

$$C(x,y) = A_{2,0}(x,y)/3 \text{ and } \widetilde{C}(y,x) = A^{1,1}(x,y) + A_{2,0}(x, \sqrt{3}y)/3 - A_{2,0}(\sqrt{3}y, x),$$

we have that A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function and

$$\begin{aligned} f(x,y) &= A_0 + B(x) + \widetilde{B}(y) + C(x + \sqrt{3}y, 3x - \sqrt{3}y) + \widetilde{C}(\sqrt{3}y, x) \\ &\quad + D(x, x + \sqrt{3}y, x - \sqrt{3}y). \end{aligned}$$

(⇐) Let

$$\begin{aligned} f(x,y) &= A_0 + B(x) + \widetilde{B}(y) + C(x + \sqrt{3}y, 3x - \sqrt{3}y) + \widetilde{C}(y, x) \\ &\quad + D(x, x + \sqrt{3}y, x - \sqrt{3}y), \end{aligned}$$

where A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function. By properties of m-additive function, we have

$$\begin{aligned} f(x,y) &= A_0 + B(x) + \widetilde{B}(y) + C(x, 3x) - C(x, \sqrt{3}y) + C(\sqrt{3}y, 3x) \\ &\quad - C(\sqrt{3}y, \sqrt{3}y) + \widetilde{C}(y, x) + D(x, x, x) - D(x, \sqrt{3}y, \sqrt{3}y). \end{aligned}$$

By choosing

$$A^{1,0}(x) = B(x), \quad A^{2,0}(x) = C(x, 3x), \quad A^{3,0}(x) = D(x, x, x),$$

$$A^{0,1}(y) = \widetilde{B}(y), \quad A^{1,1}(x,y) = -C(x, \sqrt{3}y) + C(\sqrt{3}y, 3x) + \widetilde{C}(y, x),$$

$$A^{0,2}(y) = -C(\sqrt{3}y, \sqrt{3}y) \quad \text{and} \quad A^{1,2}(x,y) = -D(x, \sqrt{3}y, \sqrt{3}y),$$

we obtain that

$$A^{0,2}(\sqrt{3}t) = -C(3t,3t) = -3C(t,3t) = -3A^{2,0}(t),$$

$$A^{1,2}(x,\sqrt{3}t) = -D(x,3t,3t) = -9D(x,t,t) = -9A_{3,0}(x,t,t)$$

and

$$f(x,y) = A^0 + A^{1,0}(x) + A^{2,0}(x) + A^{3,0}(x) + A^{0,1}(y) + A^{1,1}(x,y) + A^{0,2}(y) + A^{1,2}(x,y),$$

for all $x, y, t \in \mathbb{R}$. □

Now, we will determine the solution of triangular mean-value functional equation of arbitrary equilateral triangle. First, we find a linear bijection formula which maps three vertices of any equilateral triangle to $(0, 2/\sqrt{3}), (1, -1/\sqrt{3})$, $(-1, -1/\sqrt{3})$. We know that any equilateral triangle whose centroid is the original point can be obtained by rotation or dilation of the triangle whose vertices are at $(0, 2/\sqrt{3}), (1, -1/\sqrt{3}), (-1, -1/\sqrt{3})$. Let $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^2$ be three vertices of a triangle whose centroid is the original point (as Figure 3.3).

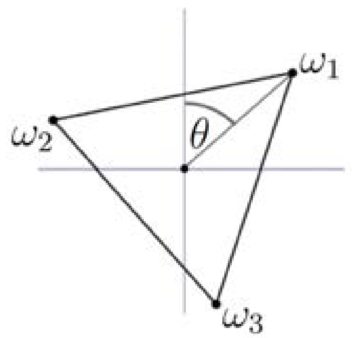

Figure 3.3

Without loss of generality, we can assume that vertices of equilateral triangle are

$$\omega_1 = r(2/\sqrt{3}\sin\theta, 2/\sqrt{3}\cos\theta),$$
$$\omega_2 = r(\cos\theta - 1/\sqrt{3}\sin\theta, -1/\sqrt{3}\cos\theta - \sin\theta), \text{ and}$$
$$\omega_3 = r(-\cos\theta - 1/\sqrt{3}\sin\theta, -1/\sqrt{3}\cos\theta + \sin\theta),$$

for some $\theta \in \mathbb{R}$ and $r > 0$. On the other hand, define

$$T(x,y) = \frac{2}{\sqrt{3}r}(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Then T is a linear bijection and

$$T(\omega_1) = (0, 2/\sqrt{3}),$$
$$T(\omega_2) = (1, -1/\sqrt{3}), \text{ and}$$
$$T(\omega_3) = (-1, -1/\sqrt{3}).$$

By applying Lemma 3.2 with the above linear bijection T, we are able to solve the triangular mean-value functional equation as follow:

Theorem 3.6. *Let z_1, z_2, z_3 be vertices of an equilateral triangle. Without loss of generality, assume that $\omega_1 = z_1 - (z_1 + z_2 + z_3)/3$ with $\mathrm{Re}(\omega_1) > 0$. Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies (1.1),*

$$f(z + tz_1) + f(z + tz_2) + f(z + tz_3) = 3f\big(z + \frac{t}{3}(z_1 + z_2 + z_3)\big),$$

for all $z \in \mathbb{R}$ and $t > 0$. Then f is of the form

$$\begin{aligned} f(x, y) = A_0 &+ B(x) + \widetilde{B}(y) + C(\beta x + \alpha y, \alpha x - \beta y) \\ &+ \widetilde{C}((\alpha + \sqrt{3}\beta)x + (\sqrt{3}\alpha - \beta)y, (3\alpha - \sqrt{3}\beta)x + (-\sqrt{3}\alpha - 3\beta)y) \\ &+ D(\alpha x - \beta y, (\alpha + \sqrt{3}\beta)x + (\sqrt{3}\alpha - \beta)y, (\alpha - \sqrt{3}\beta)x + (-\sqrt{3}\alpha - \beta)y) \end{aligned}$$

where $\theta = \arccos(\mathrm{Im}(\omega_1)/||\omega_1||)$, $\alpha = \cos\theta$, $\beta = \sin\theta$, A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function.

Proof. Let $\omega_i = z_i - (z_1 + z_2 + z_3)$ for $i = 1, 2, 3$. By Lemma 3.1, we have that the set of solution of (1.1) can be solved from the functional equation (3.1),

$$f(z + t\omega_1) + f(z + t\omega_2) + f(z + t\omega_3) = 3f(z),$$

for all $z \in \mathbb{R}^2$ and $t > 0$. Therefore, we will determine solution of (3.1) instead of (1.1). Given a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies (3.1). Since $\omega_1 + \omega_2 + \omega_3 = 0$, $\mathrm{Re}(\omega_1 + \omega_2 + \omega_3) = \mathrm{Re}(\omega_1) + \mathrm{Re}(\omega_2) + \mathrm{Re}(\omega_3) = 0$. Because $\omega_1, \omega_2, \omega_3$ are vertices of a nondegerated equilateral triangle, they are noncollinear. Therefore, $\mathrm{Re}(\omega_i) > 0$

for some $i = 1, 2, 3$. Without loss of generality, we assume that $\mathrm{Re}(\omega_1) > 0$. Let $r = ||\omega_1||$ and $\theta = \arccos(\mathrm{Im}(\omega_1)/r)$. Define

$$T(x, y) = \frac{2}{\sqrt{3}r}(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta).$$

Then T is linear bijection which

$$\{T(\omega_1), T(\omega_2), T(\omega_3)\} = \{(0, 2/\sqrt{3}), (1, -1/\sqrt{3}), (-1, -1/\sqrt{3})\}.$$

By corollary 3.4, we have the solution of triangular mean-value functional equation of $(0, 2/\sqrt{3}), (1, -1/\sqrt{3}), (-1, -1/\sqrt{3})$ is Haruki's solution. Assume that g is a Haruki's solution. Because of Theorem 3.5, g can be written in the form of

$$\begin{aligned} g(x, y) = {} & A'_0 + B'(x) + \widetilde{B'}(y) + C'(x + \sqrt{3}y, 3x - \sqrt{3}y) + \widetilde{C'}(y, x) \\ & + D'(x, x + \sqrt{3}y, x - \sqrt{3}y), \end{aligned}$$

where A'_0 is a constant, $B', \widetilde{B'} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C', \widetilde{C'} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D' : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function. Applying Lemma 3.2 with the above linear bijection T, we have

$$f(x, y) = g \circ T(x, y).$$

Making simple manipulations using additive properties of the functions, we get

$$\begin{aligned} f(x, y) = {} & A_0 + B(x) + \widetilde{B}(y) + C(\beta x + \alpha y, \alpha x - \beta y) \\ & + \widetilde{C}((\alpha + \sqrt{3}\beta)x + (\sqrt{3}\alpha - \beta)y, (3\alpha - \sqrt{3}\beta)x + (-\sqrt{3}\alpha - 3\beta)y) \\ & + D(\alpha x - \beta y, (\alpha + \sqrt{3}\beta)x + (\sqrt{3}\alpha - \beta)y, (\alpha - \sqrt{3}\beta)x + (-\sqrt{3}\alpha - \beta)y) \end{aligned}$$

where $\alpha = \cos\theta$, $\beta = \sin\theta$, A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function. Conversely, the solution can be verified by substituting back into (1.1). □

CHAPTER IV

FUNDAMENTAL CASE OF TRIANGULAR MEAN-VALUE FUNCTIONAL EQUATION

In this section, we prove the fundamental case of (1.1) when the three fixed points z_1, z_2, z_3 are $(0,1), (1,0)$, and $(-1,-1)$ respectively (as Figure 3.1).

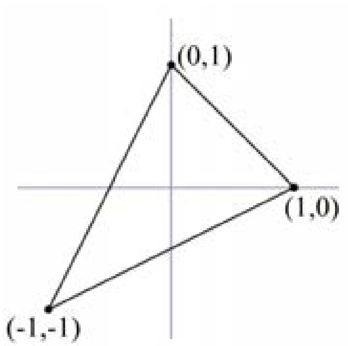

Figure 4.1

Note that in this case, the directions of the median lines of the triangle are parallel to the $x-$axis and the $y-$axis. In addition, its centroid is at the origin. So, (1.1) is of the form

$$f(x+t,y) + f(x,y+t) + f(x-t,y-t) = 3f(x,y), \qquad (4.1)$$

for all $x, y \in \mathbb{R}$ and $t > 0$. First, we prove the following proposition and lemmas which are needed in the proof of Theorem 4.4.

Proposition 4.1. *Let $z_1, z_2, z_3 \in \mathbb{R}^2$ and $\omega_i = z_i - (z_1 + z_2 + z_3)/3$. Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies (1.1). Then*

$$f(z + 3t\omega_i) - 3f(z + 2t\omega_i) + 3f(z + t\omega_i) - f(z) = 0,$$

for all $z \in \mathbb{R}^2$, $t \in \mathbb{R}$ and $i = 1, 2, 3$.

Proof. By Corollary 3.4, we have f satisfies (1.1) for all $z \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Since (1.1) is equivalent to (3.1), we get that $\sum_{i=1}^{3} f(z+\omega_i t) = 3f(z)$ for all $z \in \mathbb{R}^2$ and $t \in \mathbb{R}$. Define

$$F(z,t) = 3f(z) - \big(f(z+t\omega_1) + f(z+t\omega_2) + f(z+t\omega_3)\big).$$

From (3.1), we have $F(z,t) = 0$ for all $z \in \mathbb{R}^2, t \in \mathbb{R}$. For $i = 1,2,3$. We obtain that

$$\begin{aligned}
0 &= F(z+t\omega_i, -t) - F(z+2t\omega_i, t) \\
&= 3f(z+t\omega_i) - \big(f(z+t\omega_i - t\omega_1) + f(z+t\omega_i - t\omega_2) + f(z+t\omega_i - t\omega_3)\big) \\
&- \big(3f(z+2t\omega_i) - (f(z+2t\omega_i + t\omega_1) + f(z+2t\omega_i + t\omega_2) + f(z+2t\omega_i + t\omega_3))\big) \\
&= 3f(z+t\omega_i) - f(z+t(\omega_i-\omega_1)) - f(z+t(\omega_i-\omega_2)) - f(z+t(\omega_i-\omega_3)) \\
&- 3f(z+2t\omega_i) + f(z+t(2\omega_i+\omega_1)) + f(z+t(2\omega_i+\omega_2)) + f(z+t(2\omega_i+\omega_3))
\end{aligned}$$

Let $j,k \in \{1,2,3\}$, $i \neq j$, $i \neq k$ and $j \neq k$. We must have that one of $\omega_i - \omega_1, \omega_i - \omega_1, \omega_i - \omega_1$ is zero and one of $2\omega_i + \omega_1, 2\omega_i + \omega_2, 2\omega_i + \omega_3$ is $3\omega_i$. Since $\omega_1 + \omega_2 + \omega_3 = 0$, we get that $2\omega_i + \omega_j = \omega_i - \omega_k$. Therefore, two of $\omega_i - \omega_1, \omega_i - \omega_1, \omega_i - \omega_1$ are the same as two of $2\omega_i + \omega_1, 2\omega_i + \omega_2, 2\omega_i + \omega_3$. By eliminating the equal terms from the above equation, we have

$$0 = f(z+3t\omega_i) - 3f(z+2t\omega_i) + 3f(z+t\omega_i) - f(z).$$

□

Lemma 4.2. *Given $A^{0,2}(x,y)$, $A^{2,0}(x,y)$ and $A^{1,1}(x,y)$ satisfying*

$$2A^{2,0}(t) + 2A^{0,2}(t) + A^{1,1}(t,t) = 0, \tag{4.2}$$

for all $t \in \mathbb{R}$. There exist 2-additive functions C and $\widetilde{C}$ such that

$$A^{2,0}(x,y) + A^{1,1}(x,y) + A^{0,2}(x,y) = C(x, 2y-x) + \widetilde{C}(2x-y, y).$$

Proof. Since $A^{2,0}$ is a quadratic function, there exists 2-additive function C such that $A^{2,0}(x) = -C(x,x)$. Let

$$\widetilde{C}(x,y) = -C(x,y) + \frac{1}{2}A_{1,1}(x;y).$$

Then $\widetilde{C}(x,y)$ is also 2-additive function and

$$A_{1,1}(x;y) = 2C(x,y) + 2\widetilde{C}(x,y).$$

From (4.2), we have

$$-2C(t,t) + 2A^{0,2}(t) + 2C(t,t) + 2\widetilde{C}(t,t) = 0, \text{ for } t \in \mathbb{R}.$$

Therefore, $A^{0,2}(t) = -\widetilde{C}(t,t)$ for all $t \in \mathbb{R}$. So, we get

$$\begin{aligned} A^{1,1}(x,y) + A^{2,0}(x) + A^{0,2}(y) &= 2C(x,y) + 2\widetilde{C}(x,y) - C(x,x) - \widetilde{C}(y,y) \\ &= C(x, 2y - x) + \widetilde{C}(2x - y, y). \end{aligned}$$

□

Lemma 4.3. *Given $A^{1,2}(x,y)$ and $A^{2,1}(x,y)$ satisfying*

$$A^{1,2}(x,t) + A_{2,1}(x,t;t) = 0 \quad \textit{and} \quad A^{2,1}(t,x) + A_{1,2}(t;x,t) = 0 \tag{4.3}$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. There exists a symmetric 3-additive function D such that

$$A^{2,1}(x,y) + A^{1,2}(x,y) = D(x - y, x, y).$$

Proof. Substituting $t = y + z$ into (4.3), we get

$$2A_{1,2}(x;y,z) + A_{2,1}(x,y;z) + A_{2,1}(x,z;y) = 0, \text{ and} \tag{4.4}$$

$$2A_{2,1}(y,z;x) + A_{1,2}(y;z,x) + A_{1,2}(z;y,x) = 0. \tag{4.5}$$

Since (4.5) holds for arbitrary $x, y, z \in \mathbb{R}$, we have

$$\begin{aligned} &2A_{2,1}(x,z;y) + A_{1,2}(x;z,y) + A_{1,2}(z;x,y) = 0, \text{ and} \\ &2A_{2,1}(y,x;z) + A_{1,2}(y;x,z) + A_{1,2}(x;y,z) = 0. \end{aligned} \tag{4.6}$$

Using (4.4) and (4.6), we get

$$2A_{1,2}(x;y,z) = A_{1,2}(y;x,z) + A_{1,2}(z;x,y). \tag{4.7}$$

The cyclic permutations of (x, y, z) and (4.7) give

$$\begin{aligned} 2A_{1,2}(y; x, z) =& A_{1,2}(x; y, z) + A_{1,2}(z; x, y), \text{ and} \\ 2A_{1,2}(z; x, y) =& A_{1,2}(y; x, z) + A_{1,2}(x; y, z). \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we conclude that

$$A_{1,2}(x; y, z) = A_{1,2}(y; z, x) = A_{1,2}(z; y, x). \tag{4.9}$$

Hence $A_{1,2}$ is a symmetric 3-additive function and so is $A_{2,1}$. Moreover $A^{1,2}(x, y) = -A_{2,1}(x, y; y)$. Therefore,

$$\begin{aligned} A^{1,2}(x, y) + A^{2,1}(x, y) &= -A_{2,1}(x, y; y) + A_{2,1}(x, x; y) = -A_{2,1}(x - y, x; y) \\ &= D(x - y, x, y), \end{aligned}$$

where D is a symmetric 3-additive function. □

Now, we are ready to establish the general solutions of the mean-value functional equation (4.1).

Theorem 4.4. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies* (4.1),

$$f(x + t, y) + f(x, y + t) + f(x - t, y - t) = 3f(x, y),$$

for all $x, y \in \mathbb{R}$ and $t > 0$ if and only if

$$f(x, y) = A_0 + B(x) + \widetilde{B}(y) + C(x, 2y - x) + \widetilde{C}(2x - y, y) + D(x - y, x, y),$$

where A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function.

Proof. Since (4.1) is the triangular mean-value functional equation which is obtained from the three fixed points $(1, 0), (0, 1)$ and $(-1, -1)$, by Proposition 4.1, we have

$$\begin{aligned} &f(x + 3t, y) - 3f(x + 2t, y) + 3f(x + t, y) - f(x, y) = 0, \text{ and} \\ &f(x, y + 3t) - 3f(x, y + 2t) + 3f(x, y + t) - f(x, y) = 0 \end{aligned}$$

for all x, y and $t \in \mathbb{R}$. This means that $\underset{x,t}{\Delta^3} f(x,y) = 0$ and $\underset{y,t}{\Delta^3} f(x,y) = 0$ for all x, y and $t \in \mathbb{R}$. Consequently, Theorem 2.2 implies that

$$f(x,y) = \sum_{n=0}^{2}\sum_{m=0}^{2} A^{n,m}(x,y). \tag{4.10}$$

Substituting (4.10) into (4.1), we have the following equation:

$$\begin{aligned}
&\big(2A^{2,2}(t,y) + 2A^{2,2}(x,t) + 2(A_{2,2}(t,t;y,t) + A_{2,2}(x,t;t,t)) \\
&+ 4A_{2,2}(x,t;y,t) + A^{2,2}(t,t)\big) \\
&+ \big(2A^{2,1}(t,y) + A^{2,1}(t,t) + 2A_{2,1}(x,t;t)\big) \\
&+ \big(2A^{1,2}(x,t) + A^{1,2}(t,t) + 2A_{1,2}(t;y,t)\big) \\
&+ \big(A^{1,1}(t,t) + 2A^{2,0}(t) + 2A^{0,2}(t)\big) = 0.
\end{aligned} \tag{4.11}$$

We then substitute x, y and t by rx, ry and rt respectively, where r is rational. Note that after the substitution, we have the equation which is a polynomial in variable r. Since this is true for all r, by considering the coefficient of r^4, r^3 and r^2 we must have

$$\begin{aligned}
&2A^{2,2}(t,y) + 2A^{2,2}(x,t) + 2(A_{2,2}(t,t;y,t) \\
&+A_{2,2}(x,t;t,t)) + 4A_{2,2}(x,t;y,t) + A^{2,2}(t,t) = 0,
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\big(2A^{2,1}(t,y) + A^{2,1}(t,t) + 2A_{2,1}(x,t;t)\big) \\
&+\big(2A^{1,2}(x,t) + A^{1,2}(t,t) + 2A_{1,2}(t;y,t)\big) = 0,
\end{aligned} \tag{4.13}$$

and

$$A^{1,1}(t,t) + 2A^{2,0}(t) + 2A^{0,2}(t) = 0, \tag{4.14}$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. From (4.12), we again substitute t by rt, where r is rational, and let $y = 0$, and by considering the coefficient of r^2, we obtain that $A^{2,2}(x,t) = 0$ for $x \in \mathbb{R}$ and $t \in \mathbb{R}$. So

$$A^{2,2}(x,y) = 0$$

for all $(x,y) \in \mathbb{R}^2$. Similarly, from (4.13), we have,

$$\begin{aligned}
A^{2,1}(t,x) + A_{1,2}(t;x,t) &= 0, \text{ and} \\
A^{1,2}(x,t) + A_{2,1}(x,t;t) &= 0
\end{aligned} \tag{4.15}$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. By Lemma 4.2 and (4.14), we get that

$$A^{0,2}(x,y) + A^{1,1}(x,y) + A^{2,0}(x,y) = C(x, 2y - x) + \widetilde{C}(2x - y, y),$$

for some 2-additive functions C and $\widetilde{C}$. By Lemma 4.3 and (4.15), we have that

$$A^{2,1}(x,y) + A^{1,2}(x,y) = D(x - y, x, y),$$

for some symmetric 3-additive function $D : \mathbb{R}^3 \to \mathbb{R}$. Hence

$$f(x,y) = A^0 + A^{1,0}(x) + A^{0,1}(y) + C(x, 2y - x) + \widetilde{C}(2x - y, y) + D(x - y, x, y).$$

Conversely, the solution can be verified by substituting back into (4.1). $\square$

CHAPTER V

TRIANGULAR MEAN-VALUE FUNCTIONAL EQUATION

In the previous section, we have established the general solution for the fundamental case of the triangular mean-value functional equation which will play an important role for the main theorem of this section. Given three noncollinear points $z_1, z_2, z_3 \in \mathbb{R}^2$. So, these points generate an actual triangle which its vertices are at z_1, z_2, z_3 and its centroid is at $(z_1 + z_2 + z_3)/3$ (as Figure 5.1).

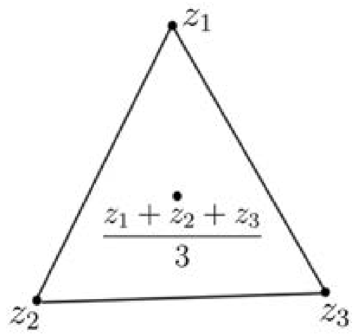

Figure 5.1

Let $\omega_i = z_i - (z_1 + z_2 + z_3)/3$. This mean that the triangle is translated and dialated to the triangle which its vertices are at $\omega_1, \omega_2, \omega_3$ and its centoid is at the origin point (as Figure 5.2).

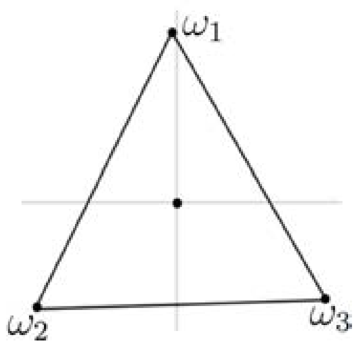

Figure 5.2

Assume that $\omega_1 = (\alpha_1, \beta_1)$ and $\omega_2 = (\alpha_2, \beta_2)$. Since z_1, z_2, z_3 are noncollinear, $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ (ω_1, ω_2 are linearly independent). Given $T : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $T(\omega_1) = (1, 0)$ and $T(\omega_2) = (0, 1)$. By simple calculation, we have

$$T(x, y) = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}(x\beta_2 - y\alpha_2, y\alpha_1 - x\beta_1).$$

Then T is linear bijection. Applying Lemma 3.2 with the linear bijection T, we can solve triangular mean-value functional equation of z_1, z_2, z_3. The following theorem shows the general solution of triangular mean-value functional equation.

Theorem 5.1. *Let $z_1 = (x_1, y_1), z_2 = (x_2, y_2), z_3 = (x_3, y_3) \in \mathbb{R}^2$ be noncollinear points. A function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies* (1.1) *if and only if*

$$\begin{aligned} f(z) = A_0 + B(x) + \widetilde{B}(y) + D(\beta_1 x - \alpha_1 y, \beta_2 x - \alpha_2 y, \beta_3 x - \alpha_3 y) \\ + C(\beta_2 x - \alpha_2 y, \delta_1 x - \gamma_1 y) + \widetilde{C}(\delta_2 x - \gamma_2 y, \beta_1 x - \alpha_1 y), \end{aligned}$$

where $\alpha_i = x_i - (x_1 + x_2 + x_3)/3$,$\beta_i = y_i - (y_1 + y_2 + y_3)/3$, $\gamma_i = x_3 - x_i$, $\delta_i = y_3 - y_i$ for all $i = 1, 2, 3$, A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function.

Proof. Given a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies (1.1). Let $\omega_i = z_i - (z_1 + z_2 + z_3)/3$, for $i = 1, 2, 3$. WLOG, assume that $\omega_1 = (\alpha_1, \beta_1), \omega_2 = (\alpha_2, \beta_2)$ and $\omega_2 = (\alpha_2, \beta_3)$. By Lemma 3.1, f also satisfies (3.1) i.e.

$$f(z + t\omega_1) + f(z + t\omega_2) + f(z + t\omega_3) = 3f(z).$$

Define

$$T(x, y) = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}(x\beta_2 - y\alpha_2, y\alpha_1 - x\beta_1).$$

Then T is a linear bijection and $T(\omega_1) = (1, 0)$, $T(\omega_2) = (0, 1)$ and $T(\omega_3) = (-1, -1)$. Let g be a solution of the triangular mean-value functional equation of $(1, 0), (0, 1), (-1, -1)$. By Theorem 4.4, we have

$$g(x, y) = A_0 + B(x) + \widetilde{B}(y) + C(x, 2y - x) + \widetilde{C}(2x - y, y) + D(x - y, x, y),$$

where A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function. Applying Lemma 3.2 with the linear bijection T, we have

$$f(x, y) = g \circ T(x, y) = g(\frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}(x\beta_2 - y\alpha_2, y\alpha_1 - x\beta_1)).$$

By making simple manipulations using additive properties of the functions and changing of variable, we have

$$\begin{aligned} f(x, y) = A_0 + B(x) + \widetilde{B}(y) + D(\beta_1 x - \alpha_1 y, \beta_2 x - \alpha_2 y, \beta_3 x - \alpha_3 y) \\ + C(\beta_2 x - \alpha_2 y, \delta_1 x - \gamma_1 y) + \widetilde{C}(\delta_2 x - \gamma_2 y, \beta_1 x - \alpha_1 y), \end{aligned}$$

$\alpha_i = x_i - (x_1 + x_2 + x_3)/3$,$\beta_i = y_i - (y_1 + y_2 + y_3)/3$, $\gamma_i = x_3 - x_i$, $\delta_i = y_3 - y_i$ for all $i = 1, 2, 3$, A_0 is a constant, $B, \widetilde{B} : \mathbb{R} \to \mathbb{R}$ are additive functions, $C, \widetilde{C} : \mathbb{R}^2 \to \mathbb{R}$ are 2-additive functions, and $D : \mathbb{R}^3 \to \mathbb{R}$ is a symmetric 3-additive function. Conversely, the solution can be verified by substituting back into (1.1). □

CHAPTER VI
DEGERNERATED TRIANGULAR MEAN-VALUE FUNCTIONAL EQUATION

For the sake of completion, we will treat triangular mean-value functional equation of a degenerated triangle (three vertices of triangle are collinear). In this case, we will show that the general solution varies independently among the lines parallel to the line generated by z_1, z_2, z_3.

Now, we will solve the triangular mean-value functional equation of a degenerated triangle. Assume that $z_1, z_2, z_3 \in \mathbb{R}^2$ are collinear points and a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying (1.1). If $z_1 = z_2 = z_3$, then it is obvious that any function is a solution. In the case where z_1, z_2, z_3 are not all equal, let $\omega_i = z_i - (z_1+z_2+z_3)/3$ for all $i = 1, 2, 3$. Without loss of generality, we assume $||\omega_1|| \geq ||\omega_2|| \geq ||\omega_3||$. Since z_1, z_2, z_3 are not all equal and $\omega_1 + \omega_2 + \omega_3 = 0$, we get $||\omega_1|| > 0$. By letting $\alpha = ||\omega_2||/||\omega_1||$, we have $\omega_2 = -\alpha\omega_1$ and $\omega_3 = (\alpha - 1)\omega_1$. So, the triangular mean-value functional equation in this case is of the form

$$f(z + \omega_1 t) + f(z - \alpha\omega_1 t) + f(z + (\alpha - 1)\omega_1 t) = 3f(z) \tag{6.1}$$

for all $z \in \mathbb{R}^2$ and $t > 0$. By Proposition 4.1, we have

$$f(z + 3\omega_1 t) - 3f(z + 2\omega_1 t) + 3f(z + \omega_1 t) - f(z) = 0. \tag{6.2}$$

for all $z \in \mathbb{R}^2$ and $t \in \mathbb{R}$. First, we consider the case where $\mathrm{Re}(\omega_1) = 0$. Without loss of generality, we assume that $\omega_1 = (0, 1)$. Therefore, (6.1) in this case is of the form

$$f(x, y + t) + f(x, y - \alpha t) + f(x, y + (\alpha - 1)t)) = 3f(x, y) \tag{6.3}$$

for all x, y and $t > 0$. Fixed $x \in \mathbb{R}$, define $f_x(y) = f(x, y)$. Then (6.3) becomes

$$f_x(y + t) + f_x(y - \alpha t) + f_x(y + (\alpha - 1)t)) = 3f_x(y). \tag{6.4}$$

for all x, y and $t > 0$. By (6.2), we also have

$$f_x(y+3t) - 3f_x(y+2t) + 3f_x(y+t) - f_x(y) = 0$$

for all $y, t \in \mathbb{R}$. This implies that

$$\Delta_t^3 f_x(y) = 0 \tag{6.5}$$

for all $y, t \in \mathbb{R}$

We observe that the equation reduces to functional equation whose domain is $\mathbb{R}$. The following lemma shows the solution of any f_x.

Lemma 6.1. *Let $\alpha \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies*

$$f(x+t) + f(x - \alpha t) + f(x + (\alpha - 1)t) = 3f(x)$$

for all $x \in \mathbb{R}$ and $t > 0$. Then

$$f(x) = A^0 + A(x) + A^2(x)$$

where $A^2((1-\alpha)x) + A_2(\alpha x, x) = 0$ for all $x \in \mathbb{R}$. In particular, if α is rational number, $f(x) = A^0 + A(x)$.

Proof. By (6.5), we have

$$\Delta_t^3 f(x) = 0.$$

Because of Theorem 2.1, we get

$$f(x) = A^0 + A(x) + A^2(x).$$

Substituting back to the equation, we have a condition

$$A^2((1-\alpha)x) + A_2(\alpha x, x) = 0$$

for all $x \in \mathbb{R}$. If α is rational number, $1 - \alpha$ is rational number. By the property of multi-additive function, we get

$$\begin{aligned} A^2((1-\alpha)x) + A_2(\alpha x, x) &= (1-\alpha)^2 A^2(x) + \alpha A^2(x, x) \\ &= (\alpha^2 - \alpha + 1)A^2(x) \end{aligned}$$

From the condition of A^2 and $(\alpha^2 - \alpha + 1) > 0$, we have $A^2(x) = 0$ for all $x \in \mathbb{R}$. In conclusion,

$$f(x) = A^0 + A(x) + A^2(x)$$

where $A^2((1-\alpha)x) + A(\alpha x, x) = 0$ for all $x \in \mathbb{R}$. If α is rational number, $f(x) = A^0 + A(x)$. □

Therefore, Lemma 6.1 implies that

$$f_x(y) = A_x^0 + A_x^1(y) + B_x(y, y)$$

where A_x^0 is a constant, A_x^1 is additive function and B_x is symmetric 2−additive function equipped with the condition $B_x((1-\alpha)y, (1-\alpha)y) + B_x(\alpha y, y) = 0$ for all $y \in \mathbb{R}$. We can see that the general solution of this functional equation varies independently among the lines parallel to x-axis.

Now, we will determine the general solution of triangular mean-value functional equation of a degenerated triangle which the line generated by its vertices is not parallel to y-axis ($\mathrm{Re}(\omega_1) \neq 0$). We can rewrite (6.1) of the form

$$\begin{aligned} f(x + \mathrm{Re}(\omega_1)t, y + \mathrm{Im}(\omega_1)t) + f(x - \alpha\mathrm{Re}(\omega_1)t, y - \alpha\mathrm{Im}(\omega_1)t) \\ + f(x + (\alpha - 1)\mathrm{Re}(\omega_1)t, y + (\alpha - 1)\mathrm{Im}(\omega_1)t) = 3f(x, y). \end{aligned}$$

Let $m = \mathrm{Im}(\omega_1)/\mathrm{Re}(\omega_1)$. Substituting t as $t/\mathrm{Re}(\omega_1)$ in the above equation, we have

$$f(x+t, y+mt) + f(x-\alpha t, y-\alpha mt) + f(x+(\alpha-1)t, y+(\alpha-1)mt) = 3f(x, y).$$

Define

$$L_{y_0} = \{(x, y) | y = y_0 + mx\}.$$

Therefore, L_{y_0} is the line which parallel to the line generated by z_1, z_2, z_3 and pass $(0, y_0)$. We will show that the general solution varies independently among L_{y_0}. Define

$$f_{L_{y_0}}(x) = f(x, y_0 + mx).$$

Let $(x, y) \in \mathbb{R}^2$. Given $y_0 = y - mx$, we have

$$f(x+t, y_0 + m(x+t)) + f(x - \alpha t, y_0 + m(x - \alpha t)) \\ + f(x + (\alpha - 1)t, y_0 + m(x + (\alpha - 1))t) = 3f(x, y_0 + mx).$$

Hence,

$$f_{L_{y_0}}(x+t) + f_{L_{y_0}}(x - \alpha t)) + f_{L_{y_0}}(x + (\alpha - 1)t) = 3f_{L_{y_0}}(x).$$

Therefore, Lemma 6.1 implies that

$$f_{L_{y_0}}(x) = A^0_{y_0} + A^1_{y_0}(x) + B_{y_0}(x, x)$$

where $A^0_{y_0}$ is a constant, $A^1_{y_0}$ is additive function and B_{y_0} is symmetric 2$-$additive function equipped with the condition $B_{y_0}((1-\alpha)x, (1-\alpha)x) + B_{y_0}(\alpha x, x) = 0$ for all $x \in \mathbb{R}$. We can see that the general solution of this functional equation varies independently among the lines parallel to the line generated by z_1, z_2, z_3. In conclusion, we have the following theorem.

Theorem 6.2. *Let $z_1, z_2, z_3 \in \mathbb{R}^2$ be collinear points and not all equal. Choose $\omega_i = z_j - (z_1 + z_2 + z_3)/3$ for $i, j = 1, 2, 3$ such that $||\omega_1|| \geq ||\omega_2|| \geq ||\omega_3||$. A function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies* (1.1). *Let $\alpha = ||\omega_2||/||\omega_1||$ and $m = \text{Im}(\omega_1)/\text{Re}(\omega_1)$. If $\text{Re}(\omega_1) = 0$, then*

$$f(x, y) = A^0_x + A^1_x(y) + B_x(y, y)$$

where A^0_x is a constant, A^1_x is additive function and B_x is symmetric 2$-$additive function equipped with the condition $B_x((1-\alpha)y, (1-\alpha)y) + B_x(\alpha y, y) = 0$ for all $y \in \mathbb{R}$. If $\text{Re}(\omega_1) \neq 0$, then

$$f(x, y) = A^0_{y-mx} + A^1_{y-mx}(x) + B_{y-mx}(x, x)$$

where A^0_{y-mx} is a constant, A^1_{y-mx} is additive function and B_{y-mx} is symmetric 2$-$additive function equipped with the condition $B_{y-mx}((1-\alpha)x, (1-\alpha)x) + B_{y-mx}(\alpha x, x) = 0$ for all $x \in \mathbb{R}$.

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VITA

Name	Mr. Worawit Tepsan
Date of Birth	21 December 1987
Place of Birth	Lampang, Thailand
Education	B.Sc. (Mathematics)(First Class Honours), Chiang Mai University, 2010
Scholarship	Development and Promotion of Science and Technology Talents Project (DPST)