STABILITY OF SECOND ORDER LINEAR RECURRENCE FUNCTIONAL EQUATIONS

Mr. Vipach Krisaravudh

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By	Mr. Vipach Krisaravudh
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Thesis Advisor	Associate Professor Paisan Nakmahachalasint, Ph.D.
Thesis Co-advisor	Assistant Professor Nataphan Kitisin, Ph.D.

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....Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

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....Thesis Advisor
(Associate Professor Paisan Nakmahachalasint, Ph.D.)

....Thesis Co-advisor
(Assistant Professor Nataphan Kitisin, Ph.D.)

....Examiner
(Sujin Khomrutai, Ph.D.)

....External Examiner
(Watcharapon Pimsert, Ph.D.)

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In this thesis, we prove the Hyers-Ulam stability of the linear recurrence functional equations, $f(x) = af(x-1) + bf(x-2)$, and prove the Hyers-Ulam-Aoki-Rassias stability of the Fibonacci functional equations, $f(x) = f(x-1)+f(x-2)$, for the class of functions $f : \mathbb{R} \to X$, where X is a real Banach space, a, b are complex numbers and $b \neq 0$.

Department : Mathematics and Computer Science	Student's Signature ....
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CONTENTS

CHAPTER I
INTRODUCTION

In this chapter, we will introduce functional equations and provide the overview of literature reviews.

1.1 Functional Equations

'What is a functional equation?', this question may arise when one heard of the word 'functional equations'. Unfortunately, there is no formal definition of what functional equation is, but it is widely accepted that functional equations concern with equations whose unknown are functions. J. Aczél ([1]) describe functional equations as follows.

functional equations are equations, both sides of which are terms constructed from a finite number of unknown functions (of a finite number of variables) and from a finite number of independent variables. This construction is effected by a finite number of known functions of one or several variables (including the four species) and by finitely many substitutions of terms which contain known and unknown functions into other known and unknown functions. The functional equations determine the unknown functions. We speak of functional equations or systems of functional equations, depending on whether we have one or several equations.

Functional equations grew rapidly in the last century. A number of books have been written on this subject. For instance, [1], [2] and [5] are notable.

In this thesis, we investigate an interesting property of functional equations, known as 'stability'. In chapter II, the basic knowledge and necessary backgrounds will be given. Afterward, in chapter III and IV, we will prove the stability of certain functional equations, which is the main result of our work.

1.2 Literature Review

In 1940, S. M. Ulam [9] proposed the following problem.

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\epsilon > 0$, does there exist a $\delta > 0$ such that for a function $h : G_1 \to G_2$ satisfying the inequality $d(h(xy), h(x)h(y)) < \delta$, for all $x, y \in G_1$, there exists a group homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$, for all $x \in G_1$?

If the answer to this question is affirmative, we say that the functional equation $f(xy) = f(x)f(y)$ is stable. The first answer to this question was given by D. H. Hyers [6] in 1941 as follows.

Theorem 1.1. (Hyers) *Let $f : E_1 \to E_2$ be a mapping between Banach spaces E_1, E_2 such that*

$$||f(x+y) - f(x) - f(y)|| \leq \epsilon \quad \textit{for all } x_1, x_2 \in E_1,$$

for some $\epsilon > 0$. Then there exists exactly one additive mapping $A : E_1 \to E_2$:

$$A(x+y) = A(x) + A(y) \quad \textit{for all } x, y \in E_1$$

such that

$$||A(x) - f(x)|| \leq \epsilon \quad \textit{for all } x \in E_1,$$

given by the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), x \in E_1.$$

Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then A is linear.

This result marks the starting point of the theory of Hyers-Ulam stability of functional equations.

Later, T. Aoki [3] and Th. M. Rassias [8] generalized the concept of the Hyers-Ulam stability which propeled many mathematicians to study this kind of stability for a number of important functional equations. Rassias' result is given in the following theorem.

Theorem 1.2. (Rassias) *Let $f : E_1 \to E_2$ be a mapping between Banach spaces E_1, E_2 such that*

$$||f(x+y) - f(x) - f(y)|| \leq Q(||x||^p + ||y||^p) \quad \text{for all } x_1, x_2 \in E_1,$$

for some constants $Q > 0$ and $0 \leq p < 1$. Then there exists a unique additive mapping $A : E_1 \to E_2$: such that

$$||A(x) - f(x)|| \leq \frac{2Q}{2-2^p}||x||^p \quad \text{for all } x \in E_1,$$

Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then A is linear.

In 2009, S. M. Jung [7] applied this idea of stability to the other functional equation as follows.
For $n \in \mathbb{N}$, let F_n be the n^{th} Fibonacci number. It is well known that

$$F_n = F_{n-1} + F_{n-2},$$

for all $n \geq 2$. Consequently, the functional equation

$$f(x) = f(x-1) + f(x-2)$$

is called the *Fibonacci functional equation.* Furthermore, a function $f : \mathbb{R} \to X$ will be called a *Fibonacci function* if it satisfies the Fibonacci functional equation, for all $x \in \mathbb{R}$, where X is a real vector space. Jung found the general solutions of the Fibonacci functional equation and proved its Hyer-Ulam stability for the certain class of functions $f : \mathbb{R} \to X$.

Theorem 1.3. (Jung) *Let $(X, || \cdot ||)$ be a real Banach space. If a function $f : \mathbb{R} \to X$ satisfies the inequality,*

$$||f(x) - f(x-1) - f(x-2)|| \leq \epsilon,$$

for all $x \in \mathbb{R}$ and for some $\epsilon > 0$, then there exists a Fibonaaci function $G : \mathbb{R} \to X$ such that

$$||f(x) - G(x)|| \leq (1 + \frac{2}{\sqrt{5}})\epsilon,$$

for all $x \in \mathbb{R}$.

We extend the definition of the Fibonacci functional equation in the following manner. For $a_1, a_2, \dots, a_k \in \mathbb{C}$ with $a_k \neq 0$, we call a functional equation

$$f(x) = a_1 f(x-1) + a_2 f(x-2) + \cdots + a_k f(x-k)$$

the *linear recurrence functional equation of order* k. A function $f : \mathbb{R} \to X$ is called a *recurrence function of order* k if it satisfies the recurrence functional equation of order k, for all $x \in \mathbb{R}$, where X is a complex vector space.

1.3 Proposed Work

We first give Jung's result of a stability of the Fibonacci functional equation in the sense of Hyers-Ulam-Aoki-Rassias. Then, we will prove the Hyers-Ulam stability of second order linear recurrence functional equations.

CHAPTER II
PRELIMINARIES

In combinatorics, recurrence relation is one of the most important topics. Our thesis concerns much about recurrence relations, especially linear homogeneous recurrence relations. In this chapter, we will review some basic knowledge of linear homogeneous recurrence relations.

One of the well-known sequences, the Fibonacci sequence $\{F_n\}$, can be constructed by the recurrence relation that every term is a sum of two predecessive terms, i.e.,

$$F_n = F_{n-1} + F_{n-2},$$

for all $n \geq 2$, with $F_0 = 0$ and $F_1 = 1$.

An important concept for solving linear homogeneous recurrence relations involves the *characteristic equation*. Loosely speaking, the characteristic equation is a polynomial equation which is used to find a general solution of the given linear homogeneous recurrence relation. To see this, we first give the following theorem ([4]) without proof:

Theorem 2.1. *Let q be a nonzero number. Then the sequence $\{h_n = q^n\}_{n=1}^{\infty}$ is a solution of the linear homogeneous recurrence relation*

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}, (a_k \neq 0, n \geq k) \tag{2.1}$$

with constant coefficients $a_1, a_2, \ldots, a_k$ if and only if q is a root of the polynomial equation

$$x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k = 0. \tag{2.2}$$

If the polynomial equation has k distinct roots $q_1, q_2, \ldots, q_k$, then

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n \tag{2.3}$$

is the general solution of (2.1) *in the following sense: No matter what initial values for* $h_0, h_1, \ldots, h_{k-1}$ *are given, there are constants* $c_1, c_2, \ldots, c_k$ *so that* (2.3) *is the unique sequence which satisfies both the recurrence relation* (2.1) *and the initial values.*

The polynomial equation (2.2) is called the *characteristic equation* of the recurrence relation (2.1) and its k roots (possibly complex) are the *charateristic roots*. If those k roots are pairwise distinct, then (2.3) is the general solution of (2.1).

In the case where the characteristic roots are repeated, we can handle it with the more general result as follows.

Theorem 2.2. *Let* $q_1, q_2, \ldots, q_t$ *be the distinct roots of the characteristic equation of the linear homogeneous recurrence relation with constant coefficients:*

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}, (a_k \neq 0, n \geq k) \tag{2.4}$$

If q_i *is a root of* (2.4) *with multiplicity* s_i, *the part of the general solution of this recurrence relation corresponding to* q_i *is*

$$H_n^{(i)} = (c_1 + c_2 n + \cdots + c_{s_i} n^{s_i - 1}) q_i^n.$$

The general solution of the recurrence relation (2.4) *is*

$$h_n = H_n^{(1)} + H_n^{(2)} + \cdots + H_n^{(t)}.$$

CHAPTER III

HYERS-ULAM-AOKI-RASSIAS STABILITY OF FIBONACCI FUNCTIONAL EQUATION

In this chapter, we will give a proof of a stability of the Fibonacci functional equation in the sense of Hyers-Ulam-Aoki-Rassias. Throughout this chapter, we denote the positive root and the negative root of the characteristic equation $x^2 - x - 1 = 0$ by α and β, respectively. To be precise, let

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2}.$$

Since we consider a function with the domain $\mathbb{R}$, we will use the absolute value as the norm to investigate the stability. We now prove the stability of the Fibonacci functional equation.

Theorem 3.1. *Let $(X, ||\cdot||)$ be a real Banach space and $0 < p < 1$. If a function $f : \mathbb{R} \to X$ satisfies the inequality,*

$$||f(x) - f(x-1) - f(x-2)|| \leq \epsilon |x|^p,$$

for all $x \in \mathbb{R}$ and for some $\epsilon > 0$, then there exists a Fibonaaci function $G : \mathbb{R} \to X$ such that

$$||f(x) - G(x)|| \leq \epsilon \cdot 2^p \left((\frac{5+2\sqrt{5}}{5})|x|^p + (\sqrt{5}-2) \right),$$

for all $x \in \mathbb{R}$.

Proof. Observe that

$$f(x) - (\alpha+\beta)f(x-1) + \alpha\beta f(x-2) = f(x) - f(x-1) - f(x-2). \tag{3.1}$$

Hence

$$\|f(x) - \alpha f(x-1) - \beta\Big(f(x-1) - \alpha f(x-2)\Big)\| \leq \epsilon |x|^p.$$

Fix a non-negative integer k. Replacing x by $x-k$ in the above inequality, we get

$$\|f(x-k)-\alpha f(x-k-1)-\beta\Big(f(x-k-1)-\alpha f(x-k-2)\Big)\| \leq \epsilon|x-k|^p.$$

So,

$$\|\beta^k\Big(f(x-k)-\alpha f(x-k-1)\Big)-\beta^{k+1}\Big(f(x-k-1)-\alpha f(x-k-2)\Big)\| \leq |\beta^k|(\epsilon|x-k|^p). \tag{3.2}$$

Note that $|x-k| \leq |x|+k \leq 2\max\{|x|,k\}$. Hence $|x-k|^p \leq 2^p\max\{|x|^p,k^p\} \leq 2^p(|x|^p+k^p)$. Then, (3.2) becomes

$$\|\beta^k\Big(f(x-k)-\alpha f(x-k-1)\Big)-\beta^{k+1}\Big(f(x-k-1)-\alpha f(x-k-2)\Big)\| \leq |\beta^k|\Big(\epsilon\cdot 2^p(|x|^p+k^p)\Big) \tag{3.3}$$

By a telescoping sum,

$$\begin{aligned} f(x)-&\alpha f(x-1)-\beta^n\Big(f(x-n)-\alpha f(x-n-1)\Big) \\ &=\sum_{k=0}^{n-1}\big(\beta^k\big(f(x-k)-\alpha f(x-k-1)\big)-\beta^{k+1}\big(f(x-k-1)-\alpha f(x-k-2)\big)\big). \end{aligned}$$

Using the triangle inequality and (3.3), we have

$$\begin{aligned} &||f(x)-\alpha f(x-1)-\beta^n\Big(f(x-n)-\alpha f(x-n-1)\Big)|| \\ &\leq \sum_{k=0}^{n-1}\|\beta^k\Big(f(x-k)-\alpha f(x-k-1)\Big)-\beta^{k+1}\Big(f(x-k-1)-\alpha f(x-k-2)\Big)\| \\ &\leq \sum_{k=0}^{n-1}|\beta^k|\Big(\epsilon\cdot 2^p(|x|^p+k^p)\Big)=\epsilon\cdot 2^p\left(|x|^p\sum_{k=0}^{n-1}|\beta^k|+\sum_{k=0}^{n-1}k^p|\beta^k|\right), \end{aligned} \tag{3.4}$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Since $p<1$ and recall that $|\beta|<1$, $\sum_{k=0}^{n-1}k^p|\beta^k| \leq \sum_{k=0}^{n-1}k|\beta^k| < \sum_{k=0}^{\infty}k|\beta^k|$. It can be verified that $\sum_{k=0}^{\infty}kx^k=\dfrac{x}{(1-x)^2}$ provided that$|x|<1$. Hence $\sum_{k=0}^{\infty}(k|\beta|^k)=\dfrac{|\beta|}{(1-|\beta|)^2}=\sqrt{5}-2$. By (3.4) and the above remarks, we get

$$||f(x)-\alpha f(x-1)-\beta^n\Big(f(x-n)-\alpha f(x-n-1)\Big)|| \leq \epsilon\cdot 2^p\left(|x|^p\sum_{k=0}^{n-1}|\beta^k|+(\sqrt{5}-2)\right) \tag{3.5}$$

Furthermore, for any $x \in \mathbb{R}$ and for any $n, m \in \mathbb{N}$ with $n \leq m$, (3.3) implies that

$$\|\beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big) - \beta^m\Big(f(x-m) - \alpha f(x-m-1)\Big)\| \leq \sum_{k=0}^{m-n-1} |\beta^{n+k}|\Big(\epsilon \cdot 2^p(|x|^p + (n+k)^p)\Big).$$

Since $|\beta| < 1$, the right hand side tends to zero as $m, n \to \infty$. We conclude that $\Big\{\beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big)\Big\}$ is a Cauchy sequence. Since X is complete, a function $G_1 : \mathbb{R} \to X$ given by

$$G_1(x) = \lim_{n\to\infty} \beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big),$$

is well-defined. Moreover, we obtain that

$$\begin{aligned}
&G_1(x-1) + G_1(x-2) \\
&\qquad = \beta^{-1} \lim_{n\to\infty} \beta^{n+1}\Big(f(x-(n+1)) - \alpha f(x-(n+1)-1)\Big) \\
&\qquad\quad + \beta^{-2} \lim_{n\to\infty} \beta^{n+2}\Big(f(x-(n+2)) - \alpha f(x-(n+2)-1)\Big) \\
&\qquad = (\beta^{-1} + \beta^{-2})G_1(x).
\end{aligned}$$

Since β is a root of $x^2 - x - 1 = 0$, we have $\beta^{-1} + \beta^{-2} = 1$. So $G_1(x-1) + G_1(x-2) = G_1(x)$ for all $x \in \mathbb{R}$. Hence G_1 is a Fibonacci function. If we let $n \to \infty$, then (3.5) yields

$$\|G_1(x) - (f(x) - \alpha f(x-1))\| \leq \epsilon \cdot 2^p \left((\frac{\sqrt{5}+3}{2})|x|^p + (\sqrt{5}-2) \right), \qquad (3.6)$$

for every $x \in \mathbb{R}$.

On the other hand, (3.1) can be rearranged to

$$\|f(x) - \beta f(x-1) - \alpha\Big(f(x-1) - \beta f(x-2)\Big)\| \leq \epsilon|x|^p.$$

By replacing x by $x+k$ in the above inequality, we get

$$\|f(x+k) - \beta f(x+k-1) - \alpha\Big(f(x+k-1) - \beta f(x+k-2)\Big)\| \leq \epsilon \cdot |x+k|^p.$$

Note that $|x+k| \leq |x| + k \leq 2\max\{|x|, k\}$. Hence $|x+k|^p \leq 2^p \max\{|x|^p, k^p\} \leq 2^p(|x|^p + k^p)$.

$$\|\alpha^{-k}\Big(f(x+k) - \alpha f(x+k-1)\Big) - \alpha^{-k+1}\Big(f(x+k-1) - \beta f(x+k-2)\Big)\| \leq \alpha^{-k}\Big(\epsilon \cdot 2^p(|x|^p + k^p)\Big). \qquad (3.7)$$

By a telescoping sum,

$$\begin{aligned}
f(x)-&\beta f(x-1)-\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big)\\
&=\sum_{k=1}^{n}\Big(\alpha^{-k}\big(f(x+k)-\beta f(x+k-1)\big)-\alpha^{-k+1}\big(f(x+k-1)-\beta f(x+k-2)\big)\Big).
\end{aligned}$$

Using the triangle inequality and (3.7), we have that

$$\begin{aligned}
&||f(x)-\beta f(x-1)-\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big)||\\
&\le\sum_{k=1}^{n}\|\alpha^{-k}\Big(f(x+k)-\beta f(x+k-1)\Big)-\alpha^{-k+1}\Big(f(x+k-1)-\beta f(x+k-2)\Big)\|\\
&\le\sum_{k=0}^{n-1}\alpha^{-k}\left(\epsilon\cdot 2^p(|x|^p+k^p)\right)=\epsilon\cdot 2^p\left(|x|^p\sum_{k=0}^{n-1}\alpha^{-k}+\sum_{k=0}^{n-1}k^p|\alpha^{-k}|\right), \qquad (3.8)
\end{aligned}$$

for $x\in\mathbb{R}$ and $n\in\mathbb{N}$. Since $p<1$ and recall that $\alpha^{-1}<1$, $\sum_{k=0}^{n-1}k^p\alpha^{-k}\le\sum_{k=0}^{n-1}k\alpha^{-k}<\sum_{k=0}^{\infty}k\alpha^{-k}=\dfrac{\sqrt{5}-2}{2}$. By (3.8) and the above facts, we get

$$||f(x)-\beta f(x-1)-\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big)||\le\epsilon\cdot 2^p(|x|^p\sum_{k=0}^{n-1}\alpha^{-k}+\frac{\sqrt{5}-2}{2}). \qquad (3.9)$$

Furthermore, for any $x\in\mathbb{R}$ and for any $n,m\in\mathbb{N}$ with $n\le m$, (3.7) implies that

$$\begin{aligned}
\|\alpha^{-n}\Big(f(x+n)-\alpha f(x+n-1)\Big)-\alpha^{-m}\Big(f(x+m)-\beta f(x+m-1)\Big)\|\\
\le\sum_{k=0}^{m-n-1}\alpha^{-(n+k)}\Big(\epsilon\cdot 2^p(|x|^p+(n+k)^p)\Big).
\end{aligned}$$

Since $\alpha^{-1}<1$, the right hand side tends to zero as $m,n\to\infty$. We conclude that $\Big\{\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big)\Big\}$ is a Cauchy sequence. Since X is complete, a function $G_2:\mathbb{R}\to X$ given by

$$G_2(x)=\lim_{n\to\infty}\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big),$$

is also well-defined. Moreover, we obtain that

$$\begin{aligned}
&G_2(x-1)+G_2(x-2)\\
&\qquad=\alpha^{-1}\lim_{n\to\infty}\alpha^{-(n-1)}\Big(f(x+(n-1))-\beta f(x+(n-1)-1)\Big)\\
&\qquad\quad+\alpha^{-2}\lim_{n\to\infty}\alpha^{-(n-2)}\Big(f(x+(n-2))-\beta f(x+(n-2)-1)\Big)\\
&\qquad=(\alpha^{-1}+\alpha^{-2})G_2(x).
\end{aligned}$$

Since α is a root of $x^2-x-1=0$, we have $\alpha^{-1}+\alpha^{-2}=1$. So $G_2(x-1)+G_2(x-2)=G_2(x)$ for all $x\in\mathbb{R}$. Hence G_2 is a Fibonacci function. If we let $n\to\infty$, then (3.9) becomes

$$\|G_2(x)-(f(x)-\beta f(x-1))\|\le\epsilon\cdot 2^p\Big((\frac{\sqrt{5}+1}{2})|x|^p+\frac{\sqrt{5}-2}{2}\Big),\qquad(3.10)$$

for every $x\in\mathbb{R}$.

We now set

$$G(x)=\frac{\beta}{\beta-\alpha}G_1(x)-\frac{\alpha}{\beta-\alpha}G_2(x)$$

Since both G_1 and G_2 are Fibonacci functions, it is straight forward to show that G is also a Fibonacci function. Furthermore,

$$\begin{aligned}
&||f(x)-G(x)||\\
&\quad=||f(x)-\Big(\frac{\beta}{\beta-\alpha}G_1(x)-\frac{\alpha}{\beta-\alpha}G_2(x)\Big)||\\
&\quad=\frac{1}{\alpha-\beta}||(\beta-\alpha)f(x)-\Big(\beta G_1(x)-\alpha G_2(x)\Big)||\\
&\quad\le\frac{1}{\alpha-\beta}(||\beta f(x)-\alpha\beta f(x-1)-\beta G_1(x)||+||\alpha G_2(x)-\alpha f(x)+\alpha\beta f(x-1)||)\\
&\quad=\frac{1}{\sqrt{5}}(|\beta|\cdot\|G_1(x)-(f(x)-\alpha f(x-1))\|+\alpha\cdot\|G_2(x)-(f(x)-\beta f(x-1))\|).
\end{aligned}$$

Applying the inequalities (3.6) and (3.10), we have

$$||f(x)-G(x)||\le\epsilon\cdot 2^p\left((\frac{5+2\sqrt{5}}{5})|x|^p+(\sqrt{5}-2)\right),$$

for all $x\in\mathbb{R}$. This completes the proof. □

CHAPTER IV
SECOND ORDER LINEAR RECURRENCE FUNCTIONAL EQUATIONS

4.1 General Solution

After we have already presented a linear recurence functional equation, it is natural to ask for an existence of linear recurrence function (or a solution of the linear recurrence functional equation).

Assume that α is a root of the characteristic equation $x^2 - ax - b = 0$. We easily see that a function f, which is defined by

$$f(x) = \begin{cases} \alpha^x, & \text{if } x \in \mathbb{Z}; \\ 0, & \text{if } x \notin \mathbb{Z}, \end{cases}$$

satisfies the linear recurrence functional equation $f(x) = af(x-1) + bf(x-2)$. (Note that it is clear that $\alpha \neq 0$ since $b \neq 0$.) This example gives us a hint how to find a general solution of the linear recurrence functional equation.

Assume that f is a solution of the linear recurrence functional equation. Note that for every point $x, y \in \mathbb{R}$ with $|x-y| < 1$, there is no correlation between $f(x)$ and $f(y)$. We may say informally that the value of f at x and y are independent.

As we can see in the above example, once we carefully assign the value of points having integral-valued distance, we can leave the other points vanished without breaking the recurrence condition.

Moreover, note that if we assign the value to only two points having distance 1, e.g. $f(-0.5)$ and $f(0.5)$, then the other values, says $f(0.5+k)$ for all $k \in \mathbb{Z}$, can be obtained immediately by the condition $f(x) = af(x-1) + bf(x-2)$.

From this observation, we can think of the interval $[-1, 1)$ as the 'basis' for extending other values outside the interval $[-1, 1)$ recursively. By using the notation $[x]$ for the greatest integer which is not greater than x for any $x \in \mathbb{R}$, we have the following theorem.

Theorem 4.1. *Let $(X, ||\cdot||)$ be a complex Banach space, $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and β be roots of the characteristic equation $x^2 - ax - b = 0$.*

(i) For $\alpha \neq \beta$: A function $f : \mathbb{R} \to X$ is a second order linear recurrence function of the form $f(x) = af(x-1) + bf(x-2)$ if and only if there exists a function $g : [-1, 1) \to X$ such that

$$f(x) = \frac{\alpha^{[x]+1} - \beta^{[x]+1}}{\alpha - \beta} g(x - [x]) - \alpha\beta \frac{\alpha^{[x]} - \beta^{[x]}}{\alpha - \beta} g(x - [x] - 1), \qquad (4.1)$$

for all $x \in \mathbb{R}$.

(ii) For $\alpha = \beta$: A function $f : \mathbb{R} \to X$ is a second order linear recurrence function of the form $f(x) = af(x-1) + bf(x-2)$ if and only if there exists a function $g : [-1, 1) \to X$ such that

$$f(x) = ([x] + 1)\alpha^{[x]} g(x - [x]) - [x]\alpha^{[x]+1} g(x - [x] - 1), \qquad (4.2)$$

for all $x \in \mathbb{R}$.

Proof.

(i) For $\alpha \neq \beta$, observe that

$$f(x) - (\alpha + \beta)f(x-1) + \alpha\beta f(x-2) = f(x) - af(x-1) - bf(x-2) = 0.$$

Thus,

$$f(x) - \alpha f(x-1) = \beta\big(f(x-1) + \alpha f(x-2)\big)$$

By mathematical induction, we get

$$f(x) - \alpha f(x-1) = \beta^n\big(f(x-n) + \alpha f(x-n-1)\big), \qquad (4.3)$$

for all $x \in \mathbb{R}$ and arbitrary non-negative integer n. Substitue x by $x + n$ in (4.3) and divide both sides by β^n, we have that

$$f(x) - \alpha f(x-1) = \beta^{-n}\big(f(x+n) + \alpha f(x+n-1)\big).$$

This implies that (4.3) is true for all $x \in \mathbb{R}$ and for all $n \in \mathbb{Z}$. Similarly,

$$f(x) - \beta f(x-1) = \alpha^n \big(f(x-n) + \beta f(x-n-1) \big), \tag{4.4}$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{Z}$.

We multiply (4.3) by β and (4.4) by α, respectively. After that, we subtract the first resulting equation from the second equation, we obtain

$$f(x) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} f(x-n) - \alpha\beta \frac{\alpha^n - \beta^n}{\alpha - \beta} f(x-n-1),$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{Z}$. Taking $n = [x]$ in the above equation, we have

$$f(x) = \frac{\alpha^{[x]+1} - \beta^{[x]+1}}{\alpha - \beta} f(x-[x]) - \alpha\beta \frac{\alpha^{[x]} - \beta^{[x]}}{\alpha - \beta} f(x-[x]-1).$$

Since $0 \leq x - [x] < 1$ and $-1 \leq x - [x] - 1 < 0$, we define $g = f|_{[-1,1)}$ and the necessary condition is done.

Conversely, to prove the sufficient condition, assume that f is a function of the form (4.1), where $g : [-1,1) \to X$ is an arbitrary function. We will show that f satisfies $f(x) = af(x-1) + bf(x-2)$ by a direct computation. Recall that

$$f(x) = \frac{\alpha^{[x]+1} - \beta^{[x]+1}}{\alpha - \beta} g(x-[x]) - \alpha\beta \frac{\alpha^{[x]} - \beta^{[x]}}{\alpha - \beta} g(x-[x]-1) \tag{4.5}$$

We substitute x by $x-1$ and $x-2$ in (4.5), respectively. Using the fact that $[x+k] = [x] + k$ for all $x \in \mathbb{R}$ and for all $k \in \mathbb{Z}$, we have

$$f(x-1) = \frac{\alpha^{[x]} - \beta^{[x]}}{\alpha - \beta} g(x-[x]) - \alpha\beta \frac{\alpha^{[x]-1} - \beta^{[x]-1}}{\alpha - \beta} g(x-[x]-1), \tag{4.6}$$

and

$$f(x-2) = \frac{\alpha^{[x]-1} - \beta^{[x]-1}}{\alpha - \beta} g(x-[x]) - \alpha\beta \frac{\alpha^{[x]-2} - \beta^{[x]-2}}{\alpha - \beta} g(x-[x]-1). \tag{4.7}$$

Recall that $a = \alpha + \beta$ and $b = -\alpha\beta$. So,

$$
\begin{aligned}
&af(x{-}1) + bf(x-2) \\
&\quad = (\alpha+\beta)\Big(\frac{\alpha^{[x]} - \beta^{[x]}}{\alpha-\beta} g(x-[x]) - \alpha\beta \frac{\alpha^{[x]-1} - \beta^{[x]-1}}{\alpha-\beta} g(x-[x]-1)\Big) \\
&\qquad - (\alpha\beta)\Big(\frac{\alpha^{[x]-1} - \beta^{[x]-1}}{\alpha-\beta} g(x-[x]) - \alpha\beta \frac{\alpha^{[x]-2} - \beta^{[x]-2}}{\alpha-\beta} g(x-[x]-1)\Big) \\
&\quad = \big(\frac{(\alpha+\beta)(\alpha^{[x]} - \beta^{[x]}) - (\alpha\beta)(\alpha^{[x]-1} - \beta^{[x]-1})}{\alpha-\beta}\big) g(x-[x]) \\
&\qquad - \alpha\beta\big(\frac{(\alpha+\beta)(\alpha^{[x]-1} - \beta^{[x]-1}) - (\alpha\beta)(\alpha^{[x]-2} - \beta^{[x]-2})}{\alpha-\beta}\big) g(x-[x]-1) \\
&\quad = \frac{\alpha^{[x]+1} - \beta^{[x]+1}}{\alpha-\beta} g(x-[x]) - \alpha\beta \frac{\alpha^{[x]} - \beta^{[x]}}{\alpha-\beta} g(x-[x]-1) \\
&\quad = f(x).
\end{aligned}
$$

This completes the sufficient condition of the first case.

(ii) For $\alpha = \beta$, we claim that

$$f(x) = (n+1)\alpha^n f(x-n) - n\alpha^{n+1} f(x-n-1), \tag{4.8}$$

for all $x \in \mathbb{R}$ and for all $n \in \mathbb{Z}$. We use mathematical induction as follows. Since $\alpha = \beta$, by the relation between roots and coefficients, we have

$$f(x) = 2\alpha f(x-1) - \alpha^2 f(x-2). \tag{4.9}$$

Assume $f(x) = n\alpha^{n-1} f(x-n+1) - (n-1)\alpha^n f(x-n)$ for $n > 0$. Then

$$
\begin{aligned}
f(x) &= n\alpha^{n-1} f(x-n+1) - (n-1)\alpha^n f(x-n) \\
&= n\alpha^{n-1}\big(2\alpha f(x-n) + \alpha^2 f(x-n-1)\big) - (n-1)\alpha^n f(x-n) \\
&= (n+1)\alpha^n f(x-n) - n\alpha^{n+1} f(x-n-1).
\end{aligned}
$$

On the other hand, substitute $x = x+2$ in (4.9) and dividing both sides by α^2, we derive an equation

$$f(x) = 2\alpha^{-1} f(x+1) - \alpha^{-2} f(x+2). \tag{4.10}$$

Assume $f(x) = (n-1)\alpha^{-n+2} f(x+n-2) + (n-2)\alpha^{-n+1} f(x+n-1)$ for $n > 0$. Then

$$\begin{aligned} f(x) &= (n-1)\alpha^{-n+2} f(x+n-2) + (n-2)\alpha^{-n+1} f(x+n-1) \\ &= (n-1)\alpha^{-n+2}(2\alpha^{-1} f(x+n-1) - \alpha^{-2} f(x+n)) + (n-2)\alpha^{-n+1} f(x+n-1) \\ &= n\alpha^{-n+1} f(x+n-1) - (n-1)\alpha^{-n} f(x+n). \end{aligned}$$

That is,

$$f(x) = (-n+1)\alpha^{-n} f(x+n) + n\alpha^{-n+1} f(x+n-1).$$

This implies that (4.8) is true for all $x \in \mathbb{R}$ and for all $n \in \mathbb{Z}$ as desired. Taking $n = [x]$ in the above equation, we have

$$f(x) = ([x]+1)\alpha^{[x]} f(x-[x]) - [x]\alpha^{[x]+1} f(x-[x]-1).$$

Since $0 \leq x - [x] < 1$ and $-1 \leq x - [x] - 1 < 0$, we define $g = f|_{[-1,1)}$ and the necessary condition is done.

Conversely, to prove the sufficient condition, assume that f is a function of the form (4.2), where $g : [-1,1) \to X$ is an arbitrary function. We again show that f satisfies $f(x) = af(x-1) + bf(x-2)$ by a direct computation. Recall that

$$f(x) = ([x]+1)\alpha^{[x]} g(x-[x]) - [x]\alpha^{[x]+1} g(x-[x]-1). \tag{4.11}$$

Substitute x by $x-1$ and $x-2$ in 4.11, respectively, we get

$$f(x-1) = [x]\alpha^{[x]-1} g(x-[x]) - ([x]-1)\alpha^{[x]} g(x-[x]-1) \tag{4.12}$$

and

$$f(x-2) = ([x]-1)\alpha^{[x]-2} g(x-[x]) - ([x]-2)\alpha^{[x]-1} g(x-[x]-1). \tag{4.13}$$

Therefore,

$$
\begin{aligned}
af(x-1) &+ bf(x-2) \\
&= 2\alpha\Big([x]\alpha^{[x]-1}g(x-[x]) - ([x]-1)\alpha^{[x]}g(x-[x]-1)\Big) \\
&\quad - \alpha^2\Big(([x]-1)\alpha^{[x]-2}g(x-[x]) - ([x]-2)\alpha^{[x]-1}g(x-[x]-1)\Big) \\
&= \big(2[x]\alpha^{[x]} - ([x]-1)\alpha^{[x]}\big)g(x-[x]) \\
&\quad - \big(2([x]-1)\alpha^{[x]+1} - ([x]-2)\alpha^{[x]+1}\big)g(x-[x]-1) \\
&= ([x]+1)\alpha^{[x]}g(x-[x]) - [x]\alpha^{[x]+1}g(x-[x]-1) \\
&= f(x).
\end{aligned}
$$

This final assertion completes the entire proof. □

In short, we can assign a value to all point in the interval $[-1, 1)$ independently. After that, we extend the domain $[-1, 1)$ to the entire real line by the recurrence condition relying on the roots of the characteristic equation.

4.2 Stability

In this section, we will prove the Hyers-Ulam stability of second order linear recurrence functional equations. We give an overview of the entire section here.

(i) If the characteristic equation has no root which lies on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$, then the second order linear recurrence functional equation is stable.

(ii) Otherwise, the second order linear recurrence functional equation has no stability.

We begin with the following two lemmas.

Lemma 4.2. *Let $(X, ||\cdot||)$ be a complex Banach space, $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and β be roots of the characteristic equation $x^2 - ax - b = 0$ and assume that $|\beta| < 1$. Given $\epsilon > 0$, if a function $f : \mathbb{R} \to X$ satisfies the inequality,*

$$||f(x) - af(x-1) - bf(x-2)|| \leq \epsilon,$$

for all $x \in \mathbb{R}$, then there exists a recurrence function of order two $G_1 : \mathbb{R} \to X$ such that

$$\|G_1(x) - (f(x) - \alpha f(x-1))\| \le (\frac{|\beta|}{1-|\beta|})\epsilon$$

Proof. Let α, β be characteristic roots of $x^2 - ax - b = 0$ and assume that $|\beta| < 1$. Recall that

$$f(x) - (\alpha+\beta)f(x-1) + \alpha\beta f(x-2) = f(x) - af(x-1) - bf(x-2). \quad (4.14)$$

Hence

$$\|f(x) - \alpha f(x-1) - \beta\Big(f(x-1) - \alpha f(x-2)\Big)\| < \epsilon.$$

Fix a non-negative integer k. Replacing x by $x-k$ in the above inequality, we get

$$\|f(x-k) - \alpha f(x-k-1) - \beta\Big(f(x-k-1) - \alpha f(x-k-2)\Big)\| \le \epsilon.$$

So,

$$\|\beta^k\Big(f(x-k) - \alpha f(x-k-1)\Big) - \beta^{k+1}\Big(f(x-k-1) - \alpha f(x-k-2)\Big)\| \le |\beta^k|\epsilon \quad (4.15)$$

By a telescoping sum,

$$\begin{aligned} f(x) - &\alpha f(x-1) - \beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big) \\ &= \sum_{k=0}^{n-1} \big(\beta^k\big(f(x-k) - \alpha f(x-k-1)\big) - \beta^{k+1}\big(f(x-k-1) - \alpha f(x-k-2)\big)\big). \end{aligned}$$

Using the triangle inequality and (4.15), we have

$$\begin{aligned} &||f(x) - \alpha f(x-1) - \beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big)|| \\ &\le \sum_{k=0}^{n-1} \|\beta^k\Big(f(x-k) - \alpha f(x-k-1)\Big) - \beta^{k+1}\Big(f(x-k-1) - \alpha f(x-k-2)\Big)\| \\ &\le \sum_{k=0}^{n-1} |\beta^k|\epsilon, \qquad (4.16) \end{aligned}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Furthermore, for any $x \in \mathbb{R}$ and for any $n, m \in \mathbb{N}$ with $n \le m$, (4.15) implies that

$$\|\beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big) - \beta^m\Big(f(x-m) - \alpha f(x-m-1)\Big)\| \le \sum_{k=n}^{m-1} |\beta^k|\epsilon.$$

Since $|\beta| < 1$, the right hand side tends to zero as $m, n \to \infty$. We conclude that $\Big\{\beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big)\Big\}$ is a Cauchy sequence. Since X is complete, a function $G_1 : \mathbb{R} \to X$ given by

$$G_1(x) = \lim_{n\to\infty} \beta^n\Big(f(x-n) - \alpha f(x-n-1)\Big),$$

is well-defined. Moreover, we obtain that

$$\begin{aligned}
&aG_1(x-1) + bG_1(x-2) \\
&\qquad = a\beta^{-1}\lim_{n\to\infty} \beta^{n+1}[f(x-(n+1)) - \alpha f(x-(n+1)-1)] \\
&\qquad\quad + b\beta^{-2}\lim_{n\to\infty} \beta^{n+2}[f(x-(n+2)) - \alpha f(x-(n+2)-1)] \\
&\qquad = (a\beta^{-1} + b\beta^{-2})G_1(x).
\end{aligned}$$

Since β is a root of $x^2 - ax - b = 0$, $a\beta^{-1} + b\beta^{-2} = 1$. So $aG_1(x-1) + bG_1(x-2) = G_1(x)$ for all $x \in \mathbb{R}$. Hence G_1 is a recurrence function of order two. If we let $n \to \infty$, then (4.16) yields

$$\|G_1(x) - (f(x) - \alpha f(x-1))\| \leq (\frac{|\beta|}{1-|\beta|})\epsilon,$$

for every $x \in \mathbb{R}$. $\square$

Lemma 4.3. *Let $(X, ||\cdot||)$ be a complex Banach space, $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and β be roots of the characteristic equation $x^2 - ax - b = 0$ and assume that $|\alpha| > 1$. Given $\epsilon > 0$, if a function $f : \mathbb{R} \to X$ satisfies the inequality,*

$$||f(x) - af(x-1) - bf(x-2)|| \leq \epsilon,$$

for all $x \in \mathbb{R}$ and for some $\epsilon > 0$, then there exists a recurrence function of order two $G_2 : \mathbb{R} \to X$ such that

$$\|G_2(x) - (f(x) - \beta f(x-1))\| \leq (\frac{|\alpha|^{-1}}{1-|\alpha|^{-1}})\epsilon$$

Proof. Let α, β be characteristic roots of $x^2 - ax - b = 0$ and assume that $|\alpha| > 1$. The equation (4.14) in the previous lemma can be rearranged to

$$\|f(x) - \beta f(x-1) - \alpha(f(x-1) - \beta f(x-2))\| \leq \epsilon.$$

By replacing x by $x+k$ in the above inequality, we get

$$\|f(x+k)-\beta f(x+k-1)-\alpha\Big(f(x+k-1)-\beta f(x+k-2)\Big)\| \le \epsilon.$$

So,

$$\|\alpha^{-k}\Big(f(x+k)-\alpha f(x+k-1)\Big)-\alpha^{-k+1}\Big(f(x+k-1)-\beta f(x+k-2)\Big)\| \le |\alpha^{-k}|\epsilon \tag{4.17}$$

By a telescoping sum,

$$\begin{aligned} f(x)-&\beta f(x-1)-\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big) \\ &= \sum_{k=1}^{n}\Big(\alpha^{-k}\big(f(x+k)-\beta f(x+k-1)\big)-\alpha^{-k+1}\big(f(x+k-1)-\beta f(x+k-2)\big)\Big). \end{aligned}$$

Using the triangle inequality and (4.17), we have that

$$\begin{aligned} ||f(x)-&\beta f(x-1)-\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big)|| \\ &\le \sum_{k=1}^{n}\|\alpha^{-k}\Big(f(x+k)-\beta f(x+k-1)\Big)-\alpha^{-k+1}\Big(f(x+k-1)-\beta f(x+k-2)\Big)\| \\ &\le \sum_{k=1}^{n}|\alpha^{-k}|\epsilon, \end{aligned} \tag{4.18}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Furthermore, for any $x \in \mathbb{R}$ and for any $n, m \in \mathbb{N}$ with $n \le m$, (4.17) implies that

$$\|\alpha^{-n}\Big(f(x+n)-\alpha f(x+n-1)\Big)-\alpha^{-m}\Big(f(x+m)-\beta f(x+m-1)\Big)\| \le \sum_{k=n}^{m-1}|\alpha^{-k}|\epsilon.$$

Since $|\alpha^{-1}| < 1$, the right hand side tends to zero as $m, n \to \infty$. We conclude that $\{\alpha^{-n}\big(f(x+n)-\beta f(x+n-1)\big)\}$ is a Cauchy sequence. Since X is complete, a function $G_2 : \mathbb{R} \to X$ given by

$$G_2(x) = \lim_{n\to\infty}\alpha^{-n}\Big(f(x+n)-\beta f(x+n-1)\Big),$$

is also well-defined. Moreover, we obtain that

$$\begin{aligned} aG_2(x-1)&+bG_2(x-2) \\ &= a\alpha^{-1}(\lim_{n\to\infty}\alpha^{-(n-1)}[f(x+(n-1))-\beta f(x+(n-1)-1)]) \\ &\quad + b\alpha^{-2}(\lim_{n\to\infty}\alpha^{-(n-2)}[f(x+(n-2))-\beta f(x+(n-2)-1)]) \\ &= (a\alpha^{-1}+b\alpha^{-2})G_2(x). \end{aligned}$$

Since α is a root of $x^2-ax-b=0$, $a\alpha^{-1}+b\alpha^{-2}=1$. So $aG_2(x-1)+bG_2(x-2)=G_2(x)$ for all $x\in\mathbb{R}$. Hence G_2 is a recurrence function of order two. If we let $n\to\infty$, then (4.18) becomes

$$\|G_2(x)-(f(x)-\beta f(x-1))\|\leq(\frac{|\alpha|^{-1}}{1-|\alpha|^{-1}})\epsilon,$$

for every $x\in\mathbb{R}$. □

For a convenient reason, we construct a real-valued function $\lambda:\mathbb{C}\backslash\{z\in\mathbb{C}\mid|z|\neq1\}\to\mathbb{R}$ by

$$\lambda(\gamma,\epsilon)=\begin{cases}\Big(\dfrac{|\gamma|}{1-|\gamma|}\Big)\epsilon, & \text{if } |\gamma|<1;\\ \Big(\dfrac{|\gamma|^{-1}}{1-|\gamma|^{-1}}\Big)\epsilon, & \text{if } |\gamma|>1.\end{cases}$$

It is valuable to note that for any fixed γ with $|\gamma|\neq1$, $\lambda(\gamma,\epsilon)$ is arbitrarily small depending on ϵ, i.e., $\lambda(\gamma,\epsilon)\to0$ as $\epsilon\to0$. Taking this function into account, we merge Lemma 4.2 and Lemma 4.3 into a single lemma.

Lemma 4.4. *Let $(X,||\cdot||)$ be a complex Banach space, $a,b\in\mathbb{C}$ and $b\neq0$. Let α and β be roots of the characteristic equation $x^2-ax-b=0$ and assume that $|\beta|\neq1$. Given $\epsilon>0$, if a function $f:\mathbb{R}\to X$ satisfies the inequality,*

$$||f(x)-af(x-1)-bf(x-2)||\leq\epsilon,$$

for all $x\in\mathbb{R}$, then there exists a recurrence function of order two $G':\mathbb{R}\to X$ such that

$$\|G'(x)-(f(x)-\alpha f(x-1))\|\leq\lambda(\beta,\epsilon).$$

Next, we introduce the *Kronecker delta* symbol δ_{xy}, which is defined by

$$\delta_{xy}=\begin{cases}1, & \text{if } x=y;\\ 0, & \text{if } x\neq y.\end{cases}$$

For the rest of the thesis, we use the function λ and the Kronecker delta symbol to represent our theorems elegantly and generalizable. Now, Lemma 4.4 will be applied to prove our main theorem as follows.

Theorem 4.5. [Stability of the second order linear recurrence functional equations]

Let $(X, ||\cdot||)$ be a complex Banach space, $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and β be distinct roots of the characteristic equation $x^2 - ax - b = 0$. Assume that $|\alpha| \neq 1$ and $|\beta| \neq 1$. Given $\epsilon > 0$, if a function $f : \mathbb{R} \to X$ satisfies the inequality,

$$||f(x) - af(x-1) - bf(x-2)|| \leq \epsilon,$$

for all $x \in \mathbb{R}$, then there exists a recurrence function of order two $G : \mathbb{R} \to X$ such that

$$||f(x) - G(x)|| \leq \left(\frac{|\beta|\lambda(\beta,\epsilon) + |\alpha|\lambda(\alpha,\epsilon)}{|\beta - \alpha|}\right)$$

for all $x \in \mathbb{R}$.

Proof. Let α, β be distinct characteristic roots of $x^2 - ax - b = 0$ such that $|\alpha| \neq 1$ and $|\beta| \neq 1$. Since $|\beta| \neq 1$, by using Lemma 4.4, there exist a recurrence function of order two $G_1 : \mathbb{R} \to X$ such that

$$\|G_1(x) - (f(x) - \alpha f(x-1))\| \leq \lambda(\beta, \epsilon). \tag{4.19}$$

Since $|\alpha| \neq 1$, by a similar argument, there exist a recurrence function of order two $G_2 : \mathbb{R} \to X$ and $\delta_2 > 0$ such that

$$\|G_2(x) - (f(x) - \beta f(x-1))\| \leq \lambda(\alpha, \epsilon). \tag{4.20}$$

The assumption $\alpha \neq \beta$ allows us to define

$$G(x) = \frac{\beta}{\beta - \alpha} G_1(x) - \frac{\alpha}{\beta - \alpha} G_2(x)$$

Since $G_1(x) = aG_1(x-1) + bG_1(x-2)$ and $G_2(x) = aG_2(x-1) + bG_2(x-2)$, it is straight forward to show that G is a recurrence function of order two, more explicitly, $G(x) = aG(x-1) + bG(x-2)$. By (4.19), (4.20) and the triangle inequality, we have

$$
\begin{aligned}
&||f(x) - G(x)|| \\
&\quad = ||f(x) - \Big(\frac{\beta}{\beta-\alpha}G_1(x) - \frac{\alpha}{\beta-\alpha}G_2(x)\Big)|| \\
&\quad = \frac{1}{|\beta-\alpha|}||(\beta-\alpha)f(x) - [\beta G_1(x) - \alpha G_2(x)]|| \\
&\quad \le \frac{1}{|\beta-\alpha|}(||\beta f(x) - \alpha\beta f(x-1) - \beta G_1(x)|| + ||\alpha G_2(x) - \alpha f(x) + \alpha\beta f(x-1)||) \\
&\quad \le \frac{|\beta|\lambda(\beta,\epsilon) + |\alpha|\lambda(\alpha,\epsilon)}{|\beta-\alpha|}.
\end{aligned}
$$

This completes the proof. □

The stability the recurrence functional equation $f(x) = af(x-1) + bf(x-2)$ fails in the case that (at least) one of its characteristic roots lies on the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ as we will show later.

To show that the stability fails, we need a counter-example. Let $\epsilon > 0$, $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and ω be roots of the characteristic equation $x^2 - ax - b = 0$ and assume that $|\omega| = 1$. We define a function $f : \mathbb{R} \to \mathbb{C}$ by

$$
f(x) = \begin{cases} \gamma\omega^x x^{\delta_{\alpha\omega}+1}, & \text{if } x \in \mathbb{Z}; \\ 0, & \text{if } x \notin \mathbb{Z}. \end{cases} \tag{4.21}
$$

where $\gamma = \min\{\frac{\epsilon}{2|\omega|}, \frac{\epsilon}{2|\alpha|}\} = \min\{\frac{\epsilon}{2}, \frac{\epsilon}{2|\alpha|}\}$. We call it a *pseudo second order linear recurrence function*. We will use this notation for the rest of our work, and we will show later that this function is a counter-example we desire. Moreover, the following remark is very useful. For any $x \in \mathbb{Z}$,

$$
\begin{aligned}
&|f(x) - af(x-1) - bf(x-2)| \\
&\qquad = \gamma|\omega^x x^{\delta_{\alpha\omega}+1} - a\omega^{x-1}(x-1)^{\delta_{\alpha\omega}+1} - b\omega^{x-2}(x-2)^{\delta_{\alpha\omega}+1}| \\
&\qquad = \gamma|(\omega^x - a\omega^{x-1} - b\omega^{x-2})x^{\delta_{\alpha\omega}+1} - \\
&\qquad\quad \Big(a\omega^{x-1}((x-1)^{\delta_{\alpha\omega}+1} - x^{\delta_{\alpha\omega}+1}) + b\omega^{x-2}((x-2)^{\delta_{\alpha\omega}+1} - x^{\delta_{\alpha\omega}+1})\Big)|.
\end{aligned}
$$

Using the fact that $\omega^2 - a\omega - b = 0$, i.e., $\omega^x - a\omega^{x-1} - b\omega^{x-2} = 0$, we get

$$
|f(x)-af(x-1)-bf(x-2)| = \gamma|a\omega^{x-1}((x-1)^{\delta_{\alpha\omega}+1}-x^{\delta_{\alpha\omega}+1})+b\omega^{x-2}((x-2)^{\delta_{\alpha\omega}+1}-x^{\delta_{\alpha\omega}+1})|. \tag{4.22}
$$

The last two results of our work can be described in this way. The pseudo second order linear recurrence function f satisfies $|f(x)-af(x-1)-bf(x-2)| \leq \epsilon$ for all $x \in \mathbb{R}$. Suppose that there is a recurrence function G which approximates f, i.e., the quantity $|f(x) - G(x)|$ is bounded. Our technique is to show that for sufficient large natural number k, $\mid f(k) - G(k) \mid$ is always greater than the absolute value of a non-constant polynomial $p(k)$, i.e., it is unbounded. Hence, a contradiction.

Lemma 4.6. *Let $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and ω be roots of the characteristic equation $x^2 - ax - b = 0$ and assume that $|\omega| = 1$. The pseudo second order linear recurrence function defined by* (4.21) *satisfies the inequality*

$$|f(x) - af(x-1) - bf(x-2)| \leq \epsilon, \tag{4.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that the pseudo recurrence function is not vanished only at integers. Thus, (4.23) is true for all real number $x \notin \mathbb{Z}$. In this proof, the variable x will be taken as an integer.

We consider two cases as follows.

Case 1: $\alpha = \omega$.

In this case, it is straightforward that (4.22) becomes $|f(x)-af(x-1)-bf(x-2)| = \gamma|a\omega^{x-1} + 2b\omega^{x-2}|$. Since $|\omega| = 1$, $\gamma|a\omega^{x-1} + 2b\omega^{x-2}| = \gamma|a\omega + 2b| \cdot |\omega^{x-2}| = \gamma|a\omega + 2b|$. The relation between the roots and coefficients, says $a = \alpha + \omega$ and $b = -\alpha\omega$, implies that $|f(x) - af(x-1) - bf(x-2)| = \gamma|a\omega + 2b| = \gamma|(\alpha + \omega)\omega - 2\alpha\omega| = \gamma|\omega^2 - \alpha\omega| = \gamma|\omega - \alpha|$. Since $\gamma = \min\{\frac{\epsilon}{2}, \frac{\epsilon}{2|\alpha|}\}$, by the triangle inequality, $\gamma|\omega - \alpha| \leq \gamma|\omega| + \gamma|\alpha| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This asserts (4.23).

Case 2: $\alpha = \omega$.

Since ω is a root of the characteristic polynomial, $\omega^2 = a\omega + b$. Thus, $a\omega + 2b = \omega^2 + b$. Replacing $n = 2$ into (4.22) yields

$$\begin{aligned} |f(x) - af(x-1) - bf(x-2)| &= \gamma|a\omega(-2x+1) + b(-4x+4)| \cdot |\omega^{x-2}| \\ &= \gamma| - 2(a\omega + 2b)x + ((a\omega + 2b) + 2b)| \\ &\leq \gamma(|2(\omega^2 + b)x| + |(\omega^2 + b) + 2b|). \end{aligned} \tag{4.24}$$

Since ω has a multiplicity 2 , $2\omega = \alpha$ and $\omega^2 = -b$. So, $a\omega + 2b = (a\omega + b) + b = \omega^2 + b = 0$. Taking this fact into (4.24). We have

$$|f(x) - af(x-1) - bf(x-2)| \leq \gamma|2b| = 2\gamma|\omega^2| = 2\gamma \leq \epsilon,$$

as desired. □

Theorem 4.7. *Let $a, b \in \mathbb{C}$ and $b \neq 0$. Let α and ω be roots of the characteristic equation $x^2 - ax - b = 0$ and assume that $|\omega| = 1$. Then, the pseudo second order linear recurrence function f defined by (4.21) satisfies the inequality*

$$|f(x) - af(x-1) - bf(x-2)| \leq \epsilon, \tag{4.25}$$

for all $x \in \mathbb{R}$. But for any real number η, there is no recurrence function $G : \mathbb{R} \to \mathbb{C}$ such that $G(x) = aG(x-1) + bG(x-2)$ with

$$|f(x) - G(x)| \leq \eta, \tag{4.26}$$

for all $x \in \mathbb{R}$.

Proof. Lemma 4.6 confirms that (4.25) holds for every $x \in \mathbb{R}$, i.e.,

$$|f(x) - af(x-1) - bf(x-2)| \leq \epsilon,$$

for all $x \in \mathbb{R}$. Suppose on the contrary that such G exists. Let $k \in \mathbb{N}$ be arbitrary. Since α and ω are roots of the characteristic equation $x^2 - ax - b = 0$, by solving the linear homogeneous recurrence relation, there exist $c_1, c_2 \in \mathbb{C}$ such that $G(k) = c_1 k^{\delta_{\alpha\omega}} \alpha^k + c_2 \omega^k$. So,

$$\mid f(k) - G(k) \mid = \mid \gamma\omega^k k^{\delta_{\alpha\omega}+1} - (c_1 k^{\delta_{\alpha\omega}} \alpha^k + c_2\omega^k) \mid \tag{4.27}$$

If $\alpha \neq \omega$, then (4.27) gives

$$\begin{aligned}
\mid f(k) - G(k) \mid &= \mid \gamma\omega^k k - (c_1\alpha^k + c_2\omega^k) \mid \\
&= \mid \omega^k(\gamma k - c_2) - c_1\alpha^k \mid \\
&\geq \mid |\omega^k(\gamma k - c_2)| - |c_1\alpha^k| \mid \\
&= \mid |\gamma k - c_2| - |c_1\alpha^k| \mid .
\end{aligned}$$

Whether $|\alpha| = 1$ or $|\alpha| \neq 1$, the last term tends to infinity as $k \to \infty$. This contradicts (4.26).

If $\alpha = \omega$, then (4.27) gives

$$\begin{aligned} | f(k) - G(k) | &=| \gamma\omega^k k^2 - (c_1 k\omega^k + c_2\omega^k) | \\ &=| \gamma k^2 - (c_1 k + c_2) | . \end{aligned}$$

The last term tends to infinity as $k \to \infty$. This contradicts (4.26). $\square$

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VITA

Name	:	Mr. Vipach Krisaravudh
Date of Birth	:	16 July, 1988.
Place of Birth	:	Bangkok, Thailand
Education	:	B.Sc.(Mathematics), (Second Class Honors), Chulalongkorn University, 2010.
Scholarship	:	The H.M. King Bhumibol Adulyadej's 72nd Birthday Anniversary Scholarship