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SERIES AND PRODUCT EXPANSIONS FOR ELEMENTS IN FUNCTION FIELDS AND CHARACTERIZATIONS OF RATIONAL ELEMENTS

Miss Narakorn Rompurk

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By	Miss Narakorn Rompurk
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Thesis Advisor	Associate Professor Ajchara Harnchoowong, Ph.D.
Thesis Co-advisor	Associate Professor Vichian Laohakosol, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctor's Degree

.... Dean of Faculty of Science

(Professor Piamsak Menasveta, Ph.D.)

THESIS COMMITTEE

...Yupaporn...Kemprasit...... Chairman

(Professor Yupaporn Kemprasit, Ph.D.)

...Ajchara Harnchoowong.......... Thesis Advisor

(Associate Professor Ajchara Harnchoowong, Ph.D.)

...Vichian Laohakosol......... Thesis Co-advisor

(Associate Professor Vichian Laohakosol, Ph.D.)

...P. Udomkavanich........................ Member

(Associate Professor Patanee Udomkavanich, Ph.D.)

.....Amorn....Wasanawichit...... Member

(Assistant Professor Amorn Wasanawichit, Ph.D.)

.....Pattira...Ruengsinsub........... Member

(Pattira Ruengsinsub, Ph.D.)

นรากร ร่มพฤกษ์ : การกระจายอนุกรมและผลคูณสำหรับสมาชิกในสนามฟังก์ชัน และ การให้ลักษณะของสมาชิกตรรกยะ (SERIES AND PRODUCT EXPANSIONS FOR ELEMENTS IN FUNCTION FIELDS AND CHARACTERIZATIONS OF RATIONAL ELEMENTS) อ.ที่ปรึกษา : รศ. ดร. อัจฉรา หาญชูวงศ์, อ.ที่ปรึกษาร่วม : รศ. ดร. วิเชียร เลาหโกศล 55 หน้า. ISBN 974-53-2852-9.

เป็นที่ทราบกันดีว่าจำนวนจริงบาก สามารถเขียนแทนด้วยอนุกรมอนันต์ หรือ ผลคูณอนันต์ ได้หลายรูปแบบ เช่น อนุกรมซิลเวสเตอร์ อนุกรมเองเกล อนุกรมรูรอท และ ผลคูณแคนทอร์ แต่ละรูปแบบเหล่านี้มีผลที่คล้ายกันในสนามของจำนวนพี-แอดิก เริ่มต้นในปี ค.ศ. 1987 A. Knopfmacher และ J. Knopfmacher ได้เสนอขั้นตอนวิธีทั่วไปสองแบบ แบบแรกสำหรับหาตัวแทนที่เป็นผลคูณ และ แบบที่สองสำหรับหาตัวแทนที่เป็นอนุกรม ของจำนวนพี-แอดิกโดยที่แต่ละจำนวนสามารถเขียนแทนได้ด้วยผลคูณหรืออนุกรมดังกล่าวได้เพียงอย่างละแบบเดียว และ ตัวอย่างการกระจายข้างต้นเป็นกรณีพิเศษที่ได้จากขั้นตอนวิธีนี้

วิทยานิพนธ์ฉบับนี้มีจุดมุ่งหมายเพื่อ นำขั้นตอนวิธีของ Knopfmachers ไปใช้ในกรณีของสนามฟังก์ชัน และเพื่อหาความเป็นไปได้ของการให้ลักษณะของสมาชิกตรรกยะผ่านการกระจายเหล่านี้ สนามฟังก์ชันที่พิจารณา คือ สนาม $\mathbb{F}_q((p(x)))$ และ $\mathbb{F}_q((1/x))$ ซึ่งเป็น ภาคบริบูรณ์ของสนามของฟังก์ชันตรรกยะเทียบกับ $p(x)$-แอดิกแวลูเอชัน และ แวลูเอชันอนันต์ ตามลำดับ โดยที่ $p(x)$ เป็นพหุนามลดทอนไม่ได้ เหนือ $\mathbb{F}_q$

โดยการดำเนินการที่คล้ายกับของ Knopfmachers ในส่วนแรกจะกล่าวถึงขั้นตอนวิธีของการกระจายอนุกรม และ การกระจายผลคูณสำหรับสมาชิกในสนาม $\mathbb{F}_q((p(x)))$ และ $\mathbb{F}_q((1/x))$ โดยมีการแสดงการพิสูจน์การลู่เข้า ความเป็นได้อย่างเดียว และ ระดับขั้นการประมาณ พร้อมทั้งให้ตัวอย่างที่ได้มาจากขั้นตอนวิธีเหล่านี้ ส่วนที่สองจะกล่าวถึงปัญหาในการให้ลักษณะของสมาชิกตรรกยะในสนามทั้งสอง ผ่านการกระจายอนุกรม และการกระจายผลคูณ ซึ่งพบว่าสำหรับการกระจายอนุกรมทุกแบบ ยกเว้นแบบรูรอท สมาชิกตรรกยะแต่ละตัวจะเขียนแทนได้ด้วยการกระจายอนุกรมจำกัด ส่วนการกระจายอนุกรมแบบรูรอท ซึ่งอาจเป็นกรณีที่ยากที่สุด พบว่าสมาชิกตรรกยะจะเขียนแทนได้ด้วยการกระจายอนุกรมจำกัด หรือ การกระจายอนุกรมที่เป็นคาบ การให้ลักษณะของสมาชิกตรรกยะในสนามฟังก์ชันสำหรับการกระจายอนุกรมมีผลที่ครบถ้วน สิ่งนี้ต่างกับกรณี พี-แอดิก ซึ่งในบางกรณียังคงเป็นปัญหาเปิด สำหรับการกระจายผลคูณมีเพียงการกระจายผลคูณแบบแคนทอร์เท่านั้นที่ให้ลักษณะของสมาชิกตรรกยะได้ครบถ้วน ทั้งนี้เพราะผลคูณดังกล่าวถูกสร้างมาจากการกระจายอนุกรมแบบที่ 4 ซึ่งมีการให้ลักษณะของสมาชิกตรรกยะที่สมบูรณ์แล้ว ส่วนการกระจายผลคูณแบบที่เหลือ ผลที่ได้คือเงื่อนไขที่เพียงพอสำหรับการเป็นสมาชิกตรรกยะ ซึ่งนับว่าดีกว่ากรณี พี-แอดิกอีกเช่นกัน เพราะการกระจายผลคูณสำหรับกรณี พี-แอดิกเท่าที่ทราบ มีเพียงเงื่อนไขที่เพียงพอสำหรับการเป็นจำนวนตรรกยะเท่านั้น

ภาควิชา ...คณิตศาสตร์... ลายมือชื่อนิสิต....................................

สาขาวิชา ...คณิตศาสตร์... ลายมือชื่ออาจารย์ที่ปรึกษา....................................

ปีการศึกษา ......2548...... ลายมือชื่ออาจารย์ที่ปรึกษาร่วม....................................

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KEY WORDS : VALUATIONS / COMPLETIONS / OPPENHEIM EXPANSIONS / SYLVESTER SERIES / ENGEL SERIES / LÜROTH SERIES /CANTOR PRODUCT

NARAKORN ROMPURK : SERIES AND PRODUCT EXPANSIONS FOR ELEMENTS IN FUNCTION FIELDS AND CHARACTERIZATIONS OF RATIONAL ELEMENTS. THESIS ADVISOR : ASSOC. PROF. AJCHARA HARNCHOOWONG, Ph.D. THESIS CO-ADVISOR : ASSOC. PROF. VICHIAN LAOHAKOSOL, Ph.D., 55 pp. ISBN 974-53-2852-9

It is well-known that a positive real number can be uniquely represented as an infinite series or as an infinite product in many different forms, such as, base representations, Sylvester series, Engel series, Lüroth series, and Cantor products. Most of these representations also have counterparts in the field of p-adic numbers. Starting from 1987, A. Knopfmacher and J. Knopfmacher introduced two major general algorithms, one for product and the other for series expansions, which give unique representations for p-adic numbers. Such algorithms embrace all the above-mentioned expansions as special cases.

The aims of this thesis are to carry over the Knopfmachers algorithms to the case of function fields and to investigate the possibility of characterizing rational elements via these expansions. By function fields, we refer to $\mathbb{F}_q((p(x)))$ and $\mathbb{F}_q((1/x))$, the completions of the field of rational functions $\mathbb{F}_q(x)$ with respect to the $p(x)$-adic valuation and the infinite valuation, respectively, where $p(x)$ is an irreducible polynomial over $\mathbb{F}_q$.

Following the processes similar to those of the Knopfmachers, in the first part, the algorithms for constructing series and product expansions for elements in the fields $\mathbb{F}_q((p(x)))$ and $\mathbb{F}_q((1/x))$ are described. Detailed proofs of their convergence, uniqueness and the degrees of approximation are proved together with examples derivable from these algorithms. The second part deals with the problem of characterizing rational elements in both fields through their series and product expansions. Using the concept of digit set as expounded in the works of the Knopfmachers, it is found that all, but one, series expansions represent rational elements if and only if they are finite. In the exceptional case, the Lüroth-type perhaps the hardest case, it is shown that rational elements correspond either to finite or periodic expansions. The characterization of rational elements is deemed complete for series expansions. This is in stark contrast to the p-adic case where this problem remains open in a few cases. Regarding product expansions, only one particular type, called Cantor product expansion, which is constructed from one of the series expansions, the Type 4 expansion, is completely characterized using the result from the series case. Sufficient conditions for rationality are established for the remaining cases. This again is more favorable than in the p-adic case where only some sufficient conditions are known for all such product expansions.

Department ...**Mathematics**...	Student's signature...Narakorn Rompurk
Field of study ...**Mathematics**...	Advisor's signature...Ajchara Harnchoowong
Academic year ....**2005**............	Co-advisor's signature...Vichian Laohakosol

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CONTENTS

CHAPTER I
INTRODUCTION

In 1987, A. Knopfmacher and J. Knopfmacher, [3], introduced a new algorithm that leads to a representation for any real number greater than 1 as an infinite product of rational numbers. A similar algorithm was later given in [2], [4], [5] and [6] leading to a unique infinite product representation for a given p-adic integer with leading coefficient 1. In another direction, in [7], [8], the Knopfmachers introduced an algorithm which leads to various unusual and unique series expansions of p-adic numbers as sums of rational numbers. Both the algorithms for product and series expansions can be extended to any discrete-valued non-archimedean fields, and brief discussions were given in [2], [4], [5] and [7]. A natural question posed by the Knopfmachers is whether there are any simple characterizations of rational elements via these expansions. For series expansions, in [8] one such characterization is given which asserts that a p-adic number is rational if and only if its Engel-type expansion, with coefficients belonging to an appropriate digit set, is finite. Recently, [9], another characterization, answering an open problem of the Knopfmachers, [7], for Lüroth-type expansions is given for both the p-adic number field and the function fields. As for product expansions, apart from many examples, no satisfactory characterization has been given so far.

In this work, we confine ourselves to the case of function fields. In the next chapter, we give definitions of valuation, completion and fractional parts of elements in the completions. We list some basic properties, without proofs, to be used throughout the entire thesis. In Chapter III, the Knopfmachers' algorithm for series expansions in this setting is given. In short, the algorithm for series expansion starts by writing each elements A in its canonical Laurent series expansion. Taking the first coefficient, a_0, to be the fractional part $\langle A \rangle$, and subtracting the first coefficient from the canonical representation, we get the tail part, whose valuation is less than 1. Inverting this tail part, we take the second coefficient, a_1, to be its fractional part $\left\langle \frac{1}{A-a_0} \right\rangle$. Inverting the second coefficient, subtracting it from the tail part and multiplying by the quotient of two parameters $\frac{r}{s}$, we obtain an expression of the form $\left(A - a_0 - \frac{1}{a_1}\right) \frac{r}{s}$, which we take as our next tail part and repeat the procedure. The coefficient a_n are referred to as "digits ". Apart from the description of the general algorithm, we give detailed proofs of their convergence, uniqueness, degrees of approximation and provide some interesting examples. We end this chapter with a characterization of rational elements, similar to the one proved by the Knopfmachers for p-adic numbers, [8]. This characterization states roughly that subject to a technical assumption on the algorithm parameters, a series expansion is finite if and only if it represents a rational element. This technical assumption, though applicable to many known examples in the p-adic case, is subject to a restriction on the p-adic digit set. As we shall see, the situation in the series fields is more favorable because such technical assumption applies to most well-known examples *without any restriction on the digit set*. This is due to the fact that there is no "carry" for multiplication in the series field. Chapter IV treats the product expansions for elements so normalized that the leading coefficient in its Laurent series is 1. There are three types of product expansions considered, each of which has different but similar algorithm. The algorithm of Type I products, is based on an identity connecting them with series expansions of Type 4. The algorithms for the remaining two product types are not related to series expansions. The ideas consist of cleverly inverting certain product, expanding the inverted expansion, and grouping terms to get a new expression of the same shape. The chapter ends with a complete characterization of Type I products and some sufficient conditions for rationality of the remaining types. As in the case of series, the results on rationality characterization via product expansions proved here for the two function fields are more satisfactory than in the p-adic case, where only some sufficient conditions are known.

CHAPTER II
PRELIMINARIES

In this chapter, we shall give some definitions, notations and results in the first section, mainly without proofs, to be used throughout the entire thesis. Details and proofs can be found in Bachman [1]. The second section deals with the completion $\mathbb{Q}_p$ with respect to the p-adic valuation of $\mathbb{Q}$ and the last section treats the completion $\mathbb{F}_q((p(x)))$ and $\mathbb{F}_q((1/x))$ with respect to the $p(x)$-adic valuation $|\cdot|_{p(x)}$ and the infinite valuation $|\cdot|_\infty$ of $\mathbb{F}_q(x)$, respectively.

2.1 Basic Definitions and Results

Definition 2.1. A valuation of a field K is a real-valued function $|\cdot| : K \to \mathbb{R}_0^+$ which satisfies the following conditions : for all $\alpha,\ \beta \in K$,

(i) $|\alpha| \geq 0$ and $|\alpha| = 0 \Leftrightarrow \alpha = 0$,

(ii) $|\alpha\beta| = |\alpha|\,|\beta|$ and

(iii) $|\alpha + \beta| \leq |\alpha| + |\beta|$.

There is always at least one valuation on K, namely, that given by setting $|a| = 1$ if $a \in K\setminus\{0\}$ and $|0| = 0$. This valuation is called the *trivial valuation* on K.

Definition 2.2. A valuation $|\cdot|$ on K is called *non-archimedean* if the condition (iii) in Definition 2.1 is replaced by a stronger condition, called the *strong triangle inequality*

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}.$$

Any other valuations on K is called *archimedean.*

Example 2.3. 1) The usual absolute value $|\cdot|$ is an archimedean valuation on $\mathbb{Q}$.

2) For the field of rational numbers $\mathbb{Q}$, p a prime number. Every $\alpha \in \mathbb{Q}\setminus\{0\}$ can be written in the form

$$\alpha = p^{\nu_p(\alpha)}\frac{a}{b}$$

where $\nu_p(\alpha) \in \mathbb{Z}$, $a \in \mathbb{Z}\setminus\{0\}$, $b \in \mathbb{N}$, such that $p \nmid ab$. Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_0^+$ by

$$|0|_p = 0, \ \ |\alpha|_p = p^{-\nu_p(\alpha)} \ \text{ if } \alpha \neq 0.$$

Then, $|\cdot|_p$ is a non-archimedean valuation on $\mathbb{Q}$ called the *p - adic valuation.*

3) For $\mathbb{F}_q(x)$, the field of rational functions over $\mathbb{F}_q$, the finite field of q elements. We introduce two types of valuations on $\mathbb{F}_q(x)$,namely the $p(x)$-adic valuation $|\cdot|_{p(x)}$ and the infinite valuation $|\cdot|_\infty$ as follows :

i) Let $p(x)$ be an irreducible polynomial of degree d over $\mathbb{F}_q$. Then any $\alpha \in \mathbb{F}_q(x)\setminus\{0\}$ can be written uniquely in the form

$$\alpha = p(x)^m\frac{r(x)}{s(x)}$$

where $m \in \mathbb{Z}$, $r(x)$, $s(x) \in \mathbb{F}_q[x]\setminus\{0\}$ such that $p(x) \nmid r(x)s(x)$. Define the $p(x)$-adic valuation $|\cdot|_{p(x)}$ by

$$|0|_{p(x)} = 0, \ \ |\alpha|_{p(x)} = q^{-md}.$$

ii) Define the infinite valuation $|\cdot|_\infty$ by

$$|0|_\infty = 0, \ \ \left|\frac{f(x)}{g(x)}\right|_\infty = q^{\deg f(x) - \deg g(x)},$$

where $f(x), g(x) \in \mathbb{F}_q[x]\setminus\{0\}$.

It is well-known that both $|\cdot|_{p(x)}$ and $|\cdot|_\infty$ are non-archimedean valuations on $\mathbb{F}_q(x)$ and the infinite valuation can be considered formally as the $(1/x)$-adic valuation for the field $\mathbb{F}_q(1/x)$ $(= \mathbb{F}_q(x))$.

Theorem 2.4. *Let $|\cdot|$ be a non-archimedean valuation on a field K and $\alpha, \beta \in K$ such that $|\alpha| \neq |\beta|$. Then*

$$|\alpha + \beta| = \max\{|\alpha|, |\beta|\}.$$

Let $|\cdot|$ be a valuation on a field K. We define a distance function $\rho : K \times K \to \mathbb{R}_0^+$ by

$$\rho(\alpha,\ \beta) = |\alpha - \beta|.$$

It follows immediately from the definition of valuation that ρ satisfies the following properties : for all $\alpha, \beta \in K$,

1) $\rho(\alpha, \beta) \geq 0$ and $\rho(\alpha, \beta) = 0$ if and only if $\alpha = \beta$,

2) $\rho(\alpha, \beta) = \rho(\beta, \alpha)$ and

3) $\rho(\alpha, \beta) \leq \rho(\alpha, \gamma) + \rho(\gamma, \beta)$ for $\gamma \in K$.

Hence K is a metric space with respect to the metric ρ defined above.

The statements concerning limits familiar in real and complex numbers hold in our more general situation since the valuation satisfies the same conditions that the usual absolute value does and which are needed in proving those statements.

If K is a complete field with respect to a valuation $|\cdot|$, we introduce the notion of convergence of an infinite series $\sum_{n=1}^{\infty} a_n$ and an infinite product $\prod_{n=1}^{\infty} a_n$, in the usual way. It follows immediately that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. It is useful to note that if $|\cdot|$ is non-archimedean, the converse statement holds, namely :

Theorem 2.5. *If K is a complete field with respect to a non-archimedean valuation $|\cdot|$, and if $\{a_n\}$ is a sequence of elements of K such that $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.*

Theorem 2.6. *Given a field K and a valuation $|\cdot|$ on K, there exists a field $\hat{K}$ (unique up to an isometric isomorphism), called the completion of K with respect to $|\cdot|$, such that $\hat{K}$ is a complete field with respect to a valuation extending $|\cdot|$, and K is dense in $\hat{K}$. Moreover, the field $\hat{K}$ has the same characteristic as K, and if the given valuation is non-archimedean, then so is the extended valuation on $\hat{K}$.*

Example 2.7. 1) The completion of $\mathbb{Q}$ with respect to the usual absolute value is the field of real numbers $\mathbb{R}$.

2) The completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\cdot|_p$ is denoted by $\mathbb{Q}_p$ and called the field of *p-adic numbers*.

3) The completion of the field $\mathbb{F}_q(x)$ of rational functions with respect to the $p(x)$-adic valuation $|\cdot|_{p(x)}$ and the infinite valuations $|\cdot|_\infty$ are denoted by $\mathbf{F}^p := \mathbb{F}_q\left((p(x))\right)$ and $\mathbf{F}^{1/x} := \mathbb{F}_q\left((1/x)\right)$, respectively, we shall refer to both fields as the *function fields.*

2.2 The completion $\mathbb{Q}_p$

Let $\mathbb{Q}$ be the field of rational numbers, p a prime numbers, and $\mathbb{Q}_p$ the field of p-adic numbers with the p-adic valuation $|\cdot|_p$. The extension of $|\cdot|_p$ to the completion $\mathbb{Q}_p$ is still denoted by $|\cdot|_p$ and it is non-archimedean.

For $A \in \mathbb{Q}_p$, we define the *order* $\nu(A)$ of A by $|A|_p = p^{-\nu(A)}$, with $\nu(0) = +\infty$. Then the order $\nu(A)$ has the properties

(1) $\nu(AB) = \nu(A) + \nu(B)$,

(2) $\nu(A/B) = \nu(A) - \nu(B)$ if $B \neq 0$ and

(3) $\nu(A+B) \geq \min\{\nu(A), \nu(B)\}$ with equality when $\nu(A) \neq \nu(B)$.

It is well known that every $A \in \mathbb{Q}_p$ has two unique up to the digit set series representations of the form

$$A = \sum_{n=\nu(A)}^{\infty} c_n p^n,\ c_n \in \{0, 1, 2, \ldots, p-1\}$$

and

$$A = \sum_{n=\nu(A)}^{\infty} b_n p^n,\ b_n \in \left\{0, \pm 1, \pm 2, \ldots, \pm\frac{p-1}{2}\right\}.$$

Each representation is unique and we call the finite series

$$\langle A\rangle = \sum_{n=\nu(A)}^{0} c_n p^n \text{ if } \nu(A) \leq 0, \text{ and 0 otherwise,}$$

and

$$\langle\langle A\rangle\rangle = \sum_{n=\nu(A)}^{0} b_n p^n \text{ if } \nu(A) \leq 0, \text{ and 0 otherwise}$$

the *fractional parts* of A. Let

$$S_p := \{\langle A\rangle\,; A \in \mathbb{Q}_p\} \subset \mathbb{Q}$$

and

$$S'_p := \{\langle\langle A\rangle\rangle\,;A \in \mathbb{Q}_p\} \subset \mathbb{Q}.$$

2.3 The completions $\mathbf{F}^p$ and $\mathbf{F}^{1/x}$

Let $\mathbf{F}^p := \mathbb{F}_q\left((p(x))\right)$ and $\mathbf{F}^{1/x} := \mathbb{F}_q\left((1/x)\right)$ be the function fields with respect to the $p(x)$-adic and the infinite valuations, respectively. The extension of the valuations to $\mathbf{F}^p$ and $\mathbf{F}^{1/x}$ are also denoted by $|\cdot|_{p(x)}$ and $|\cdot|_\infty$ and they are non-archimedean.

For $A \in \mathbf{F}^p$, respectively, $B \in \mathbf{F}^{1/x}$, we define the *orders* $\nu(A)$ of A, respectively, $\mu(B)$ of B by $|A|_{p(x)} = q^{-d\nu(A)}$, $|B|_\infty = q^{-\mu(B)}$, with $\nu(0) = +\infty = \mu(0)$. Then the order $\nu\,(A)$ and $\mu\,(B)$ have the properties : for $A, B \in \mathbf{F}^p$, respectively, $A, B \in \mathbf{F}^{1/x}$.

(1) $\nu\,(AB) = \nu\,(A) + \nu\,(B)$,

(2) $\nu\,(A/B) = \nu\,(A) - \nu\,(B)$ if $B \neq 0$,

(3) $\nu\,(A+B) \geq \min\{\nu\,(A)\,, \nu\,(B)\}$ with equality when $\nu\,(A) \neq \nu\,(B)$

and

(1) $\mu\,(AB) = \mu\,(A) + \mu\,(B)$,

(2) $\mu\,(A/B) = \mu\,(A) - \mu\,(B)$ if $B \neq 0$,

(3) $\mu\,(A+B) \geq \min\{\mu\,(A)\,, \mu\,(B)\}$ with equality when $\mu\,(A) \neq \mu\,(B)$.

It is well-known that each $A \in \mathbf{F}^p$, respectively, each $B \in \mathbf{F}^{1/x}$ are uniquely representable in the form

$$A = \sum_{n=\nu(A)}^{\infty} b_n(x)p(x)^n, \quad b_n(x) \in \mathbb{F}_q[x], \;\; \deg b_n(x) < d, \;\; b_{\nu(A)}(x) \neq 0,$$

respectively,

$$B = \sum_{n=\mu(B)}^{\infty} c_n \left(\frac{1}{x}\right)^n, \quad c_n \in \mathbb{F}_q, \;\; c_{\mu(B)} \neq 0.$$

The *fractional part* $\langle A\rangle_{p(x)}$ of A, respectively, the *fractional part* $\langle B\rangle_{1/x}$ of B, are defined as the finite series

$$\langle A\rangle_{p(x)} = \sum_{n=\nu(A)}^{0} b_n(x)p(x)^n \quad \text{if } \nu(A) \leq 0, \text{ and 0 otherwise,}$$

respectively,

$$\langle B\rangle_{1/x} = \sum_{n=\mu(B)}^{0} c_n \left(\frac{1}{x}\right)^n \quad \text{if } \mu(B) \leq 0, \text{ and 0 otherwise.}$$

Define the *digit sets* to be the two sets of all fractional parts by

$$S_{p(x)} := \left\{\langle A\rangle_{p(x)} ; A \in \mathbf{F}^p\right\} \subset \mathbb{F}_q(x)$$

respectively,

$$S_{1/x} := \left\{\langle B\rangle_{1/x} ; B \in \mathbf{F}^{1/x}\right\} \subset \mathbb{F}_q(x).$$

We say that $S_{p(x)}$, respectively $S_{1/x}$, is *multiplicatively closed* if $\langle A\rangle_{p(x)} \langle B\rangle_{p(x)} \in S_{p(x)}$, respectively, $\langle A\rangle_{1/x} \langle B\rangle_{1/x} \in S_{1/x}$ for all $A, B \in \mathbf{F}^p$, respectively, $A, B \in \mathbf{F}^{1/x}$. It is easy to see that S_x is multiplicatively closed and so is $S_{1/x}$. However if $\deg p(x) = d \geq 2$, then $S_{p(x)}$ is not multiplicatively closed. Let $f(x) = x^{d-1} + 1$. Then $f(x) \in S_{p(x)}$. Since

$$f(x)^2 = \left(x^{d-1} + 1\right)\left(x^{d-1} + 1\right) = x^{2d-2} + 2x^{d-1} + 1,$$

we see that $\deg f(x)^2 = 2d - 2 = d + d - 2 \geq d$. Thus there exist polynomials $q(x)$ and $r(x)$ over $\mathbb{F}_q$ such that

$$f(x)^2 = q(x)p(x) + r(x)$$

where $0 \leq \deg r(x) < d$. It follows that $\deg q(x) = d - 2 < d$ and so $f(x)^2 \notin S_{p(x)}$.

CHAPTER III
KNOPFMACHERS'S SERIES EXPANSIONS AND CHARACTERIZATIONS OF RATIONAL ELEMENTS

In this chapter, we give a general algorithm to construct series expansions for elements in function fields. Apart from the description of the general algorithm, we give detailed proofs of their convergence, uniqueness, degree of approximation and provide some interesting examples. In the last section, a characterization of rational elements, similar to the one proved by the Knopfmachers for p-adic numbers, [8] is derived.

3.1 Series Expansions

I. The general algorithm for elements in the field $\mathbf{F}^p$ proceeds exactly as in the Knopfmachers' algorithm for p-adic numbers. For $A \in \mathbf{F}^p$, let $a_0 := \langle A \rangle_{p(x)} \in S_{p(x)}$. Define $A_1 := A - a_0$. If $A_n \neq 0 \ \ (n \geq 1)$ is already defined, put

$$a_n = \left\langle \frac{1}{A_n} \right\rangle_{p(x)}, \quad A_{n+1} = \left(A_n - \frac{1}{a_n} \right) \frac{s_n}{r_n}$$

if $a_n \neq 0$ (in this case $\nu(a_n) \leq 0$), where r_n, s_n are non-zero rational functions which may depend on $a_1, ..., a_n$. Then

$$A = a_0 + A_1 = a_0 + \frac{1}{a_1} + \frac{r_1}{s_1} A_2 = \ldots = a_0 + \frac{1}{a_1} + \frac{r_1}{s_1}\frac{1}{a_2} + \cdots + \frac{r_1 \cdots r_{n-1}}{s_1 \cdots s_{n-1}} \frac{1}{a_n} + \frac{r_1 \cdots r_n}{s_1 \cdots s_n} A_{n+1}.$$

The process ends in a finite expansion if some $A_{n+1} = 0$. If some $a_n = 0$, then A_{n+1} is not defined. In order to deal with this, we note that

$$A_1 \neq 0 \Rightarrow \nu(A_1) \geq 1 \Rightarrow a_1 \neq 0, \ \nu(a_1) = \nu\left(\frac{1}{A_1}\right) \leq -1. \tag{3.1}$$

Next

$$A_n \neq 0 \Rightarrow \nu(a_n A_n - 1) \geq \nu(A_n) + 1. \tag{3.2}$$

Since $A_n \neq 0$, writing $1/A_n = a_n + \sum_{r \geq 1} c_r(x)' p(x)^r$, say, we get

$$\nu\left(a_n A_n - 1\right) = \nu\left(A_n \sum_{r \geq 1} c_r(x)' p(x)^r\right) \geq \nu(A_n) + 1.$$

Since $a_n = \langle 1/A_n \rangle_{p(x)}$ and $\nu\left(A_{n+1}\right) = \nu\left(a_n A_n - 1\right) - \nu\left(a_n\right) + \nu\left(s_n\right) - \nu\left(r_n\right)$, then

$$A_n \neq 0,\ a_n \neq 0 \Rightarrow \nu\left(a_n\right) = -\nu\left(A_n\right),\ \nu\left(A_{n+1}\right) \geq 1 - 2\nu\left(a_n\right) + \nu\left(s_n\right) - \nu\left(r_n\right). \tag{3.3}$$

Thus, if

$$\nu\left(s_n\right) - \nu\left(r_n\right) \geq 2\nu\left(a_n\right) - 1, \tag{3.4}$$

then $\nu\left(A_{n+1}\right) \geq 0$ and so $a_{n+1} \neq 0$ provided that $A_{n+1} \neq 0$. Therefore, (3.4) is a sufficient condition for the expansion to exist in $\mathbf{F}^p$.

II. The general algorithm for elements in the field $\mathbf{F}^{1/x}$ also proceeds as above. For $B \in \mathbf{F}^{1/x}$, let $b_0 := \langle B \rangle_{1/x} \in S_{1/x}$. Define $B_1 := B - b_0$. If $B_n \neq 0 \quad (n \geq 1)$ is already defined, put

$$b_n = \left\langle \frac{1}{B_n} \right\rangle_{1/x}, \quad B_{n+1} = \left(B_n - \frac{1}{b_n}\right)\frac{v_n}{u_n}$$

if $b_n \neq 0$, where u_n, v_n are non-zero rational functions which may depend on $b_1, \ldots, b_n$. Then

$$B = b_0 + B_1 = b_0 + \frac{1}{b_1} + \frac{u_1}{v_1}B_2 = \ldots = b_0 + \frac{1}{b_1} + \frac{u_1}{v_1}\frac{1}{b_2} + \cdots + \frac{u_1 \cdots u_{n-1}}{v_1 \cdots v_{n-1}}\frac{1}{b_n} + \frac{u_1 \cdots u_n}{v_1 \cdots v_n}B_{n+1}.$$

The process ends in a finite expansion if some $B_{n+1} = 0$. If some $b_n = 0$, then B_{n+1} is not defined. In order to deal with this, we note that

$$B_1 \neq 0 \Rightarrow \mu(B_1) \geq 1 \Rightarrow b_1 \neq 0,\ \mu(b_1) = \mu\left(\frac{1}{B_1}\right) \leq -1.$$

Next

$$B_n \neq 0 \Rightarrow \mu(b_n B_n - 1) \geq \mu(B_n) + 1.$$

Since $B_n \neq 0$, writing $1/B_n = b_n + \sum_{r \geq 1} c_r' \left(\frac{1}{x}\right)^r$, say, we get

$$\mu(b_n B_n - 1) = \mu\left(B_n \sum_{r \geq 1} c_r' \left(\frac{1}{x}\right)^r\right) \geq \mu(B_n) + 1.$$

Since $b_n = \langle 1/B_n\rangle_{1/x}$ and $\mu(B_{n+1}) = \mu(b_nB_n - 1) - \mu(b_n) + \mu(v_n) - \mu(u_n)$, then

$$B_n \neq 0,\ b_n \neq 0 \Rightarrow \mu(b_n) = -\mu(B_n), \mu(B_{n+1}) \geq 1 - 2\mu(b_n) + \mu(v_n) - \mu(u_n).$$

Thus, if

$$\mu(v_n) - \mu(u_n) \geq 2\mu(b_n) - 1, \tag{3.5}$$

then $\mu(B_{n+1}) \geq 0$ and so $b_{n+1} \neq 0$ provided that $B_{n+1} \neq 0$. Therefore, (3.5) is a sufficient condition for the expansion to exist in $\mathbf{F}^{1/x}$.

In the next section, we show that every $A \in \mathbf{F}^p$ has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \frac{1}{a_{n+1}},$$

where $a_0 = \langle A\rangle_{p(x)}, a_n \in S_{p(x)}, r_n, s_n \in \mathbb{F}_q(x)\backslash\{0\}$ which may depend on $a_1, ..., a_n$, and every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent (relative to $|\cdot|_\infty$) expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{u_1 \cdots u_n}{v_1 \cdots v_n} \frac{1}{b_{n+1}},$$

where $b_0 = \langle B\rangle_{1/x}, b_n \in S_{1/x}, u_n, v_n \in \mathbb{F}_q(x)\backslash\{0\}$ which may depend on $b_1, ..., b_n$. The above two series are called the *Oppenheim-type* expansions. Such a representation is said to be *periodic* if there are indices m and l such that $a_n = a_{n+l}, r_n = r_{n+l}$ and $s_n = s_{n+l}$ for every $n \geq m$.

3.2 Convergence Uniqueness and Degree of Approximation of Expansions

For convergence, we need one more auxiliary result.

Lemma 3.1. *Any series*

$$\frac{1}{c_1} + \sum_{n\geq 1} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{c_{n+1}}$$

with $c_n \in S_{p(x)}$ *and*

$$\nu(c_1) \leq -1, \quad \nu(c_{n+1}) \leq 2\nu(c_n) - 1 + \nu(r_n) - \nu(s_n), \tag{3.6}$$

converges to an element $C \in \boldsymbol{F}^p$ such that $\nu(C) = -\nu(c_1)$. Furthermore $C \neq 0$ and $c_1 = \langle 1/C \rangle_{p(x)}$.

Proof. By equation (3.6) and $\nu(c_n) \leq 0$, we get $-\nu(c_{n+1}) \geq 1 - \nu(c_n) + \nu(s_n) - \nu(r_n)$ $(n \geq 1)$, and so

$$\nu\left(\frac{r_n}{s_n} \cdot \frac{1}{c_{n+1}}\right) \geq 1 + \nu\left(\frac{1}{c_n}\right) \qquad (n \geq 1). \tag{3.7}$$

It follows that

$$\nu\left(\frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{c_{n+1}}\right) \geq 1 + \nu\left(\frac{r_1 \cdots r_{n-1}}{s_1 \cdots s_{n-1}} \cdot \frac{1}{c_n}\right) \geq \ldots \geq n+1 \to \infty \quad (n \to \infty)$$

which implies

$$\frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{c_{n+1}} \to 0$$

relative to $|\cdot|_{p(x)}$. Since $\mathbf{F}^p$ is a complete metric space with respect to $|\cdot|_{p(x)}$, which is a non-archimedean valuation, the series converges to an element $C \in \mathbf{F}^p$. Then $C = \lim_{n\to\infty} S_n$, where

$$S_n = \frac{1}{c_1} + \sum_{i=1}^{n} \frac{r_1 \cdots r_i}{s_1 \cdots s_i} \cdot \frac{1}{c_{i+1}},$$

and equation (3.7) implies

$$\nu(S_n) = \min\left\{\nu\left(\frac{1}{c_1}\right),\ \nu\left(\frac{r_1}{s_1} \cdot \frac{1}{c_2}\right), \ldots,\ \nu\left(\frac{r_1 \cdots r_{n-1}}{s_1 \cdots s_{n-1}} \cdot \frac{1}{c_n}\right)\right\} = \nu\left(\frac{1}{c_1}\right).$$

By continuity of $|\cdot|_{p(x)}$, we obtain

$$\nu(C) = -\log_c |C|_{p(x)} = -\log_c \left|\lim_{n\to\infty} S_n\right|_{p(x)} = \lim_{n\to\infty} \nu(S_n) = \nu\left(\frac{1}{c_1}\right)$$

where $c = q^d$. Then $C \neq 0$. Now

$$C = \frac{1}{c_1} + \sum_{n\geq 1} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{c_{n+1}},$$

and so

$$c_1 = \frac{1}{C} + \frac{c_1}{C} \sum_{n\geq 1} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{c_{n+1}}.$$

Thus

$$\nu\left(c_1-\frac{1}{C}\right)=\nu(c_1)-\nu(C)+\nu\left(\sum_{n\geq 1}\frac{r_1\cdots r_n}{s_1\cdots s_n}\cdot\frac{1}{c_{n+1}}\right)$$
$$=2\nu(c_1)+\nu\left(\frac{r_1}{s_1}\cdot\frac{1}{c_2}\right)=2\nu(c_1)+\nu(r_1)-\nu(s_1)-\nu(c_2)\geq 1,$$

by (3.6), and so $c_1=\langle 1/C\rangle_{p(x)}$. □

Theorem 3.2. *Every $A\in\boldsymbol{F}^p$ has a finite or convergent (relative to $|\cdot|_{p(x)}$) expansion*

$$A=a_0+\frac{1}{a_1}+\sum_{n=1}^{\infty}\frac{r_1\cdots r_n}{s_1\cdots s_n}\frac{1}{a_{n+1}},\tag{3.8}$$

where $a_0=\langle A\rangle_{p(x)}, a_n\in S_{p(x)}$ and $r_n,s_n\in\mathbb{F}_q(x)\backslash\{0\}$ which may depend on $a_1,...,a_n$ and

$$\nu(a_1)\leq -1,\quad \nu(a_{n+1})\leq 2\nu(a_n)-1+\nu(r_n)-\nu(s_n)\tag{3.9}$$

provided that

$$\nu(s_n)-\nu(r_n)\geq 2\nu(a_n)-1\ \ \textit{for}\ \ n\geq 1.$$

The expansion is unique subject to these conditions on a_n, r_n and s_n.

Proof. Let $A\in\mathbf{F}^p$. Then using the above algorithm, we get

$$A=a_0+\frac{1}{a_1}+\frac{r_1}{s_1}\frac{1}{a_2}+\ldots+\frac{r_1\cdots r_{n-1}}{s_1\cdots s_{n-1}}\frac{1}{a_n}+\frac{r_1\cdots r_n}{s_1\cdots s_n}A_{n+1}\tag{3.10}$$

where $a_0=\langle A\rangle_{p(x)}, a_n\in S_{p(x)}, r_n,s_n\in\mathbb{F}_q(x)\backslash\{0\}$ which may depend on $a_1,\ldots,a_n$ and

$$\nu(a_1)\leq -1,\nu(a_{n+1})\leq 2\nu(a_n)-1+\nu(r_n)-\nu(s_n)\qquad(n\geq 1).$$

By equation (3.9), $-\nu(a_{n+1})\geq 1-\nu(a_n)+\nu(s_n)-\nu(r_n)$ $(n\geq 1)$, or

$$\nu\left(\frac{r_n}{s_n}\cdot\frac{1}{a_{n+1}}\right)\geq 1+\nu(\frac{1}{a_n})\qquad(n\geq 1).$$

Thus

$$\nu\left(\frac{r_1\cdots r_n}{s_1\cdots s_n}A_{n+1}\right)=\nu\left(\frac{r_1\cdots r_n}{s_1\cdots s_n}\cdot\frac{1}{a_{n+1}}\right)$$
$$\geq 1+\nu\left(\frac{r_1\cdots r_{n-1}}{s_1\cdots s_{n-1}}\cdot\frac{1}{a_n}\right)\geq\ldots\geq n+1\to\infty\qquad(n\to\infty)$$

which implies

$$\frac{r_1 \cdots r_n}{s_1 \cdots s_n} A_{n+1} \to 0$$

relative to $|\cdot|_{p(x)}$, and shows that the right hand sum in (3.8) converges as asserted.

Next, we prove uniqueness. Suppose that A has two expansions

$$A \;=\; a_0 + \frac{1}{a_1} + \sum_{n\geq 1} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{a_{n+1}} = a_0' + \frac{1}{a_1'} + \sum_{n\geq 1} \frac{r_1' \cdots r_n'}{s_1' \cdots s_n'} \cdot \frac{1}{a_{n+1}'},$$

where $r_i = r_i(a_1, \ldots, a_i),\ s_i = s_i(a_1, \ldots, a_i),\ r_i' = r_i(a_1', \ldots, a_i')$ and $s_i' = s_i(a_1', \ldots, a_i')$ subject to the above conditions for a_i and a_i', respectively. Since $a_0 = \langle A \rangle_{p(x)} = a_0'$, we have

$$A_1 = A_1' \;:= \frac{1}{a_1} + \sum_{n\geq 1} \frac{r_1 \cdots r_n}{s_1 \cdots s_n} \cdot \frac{1}{a_{n+1}} = \frac{1}{a_1'} + \sum_{n\geq 1} \frac{r_1' \cdots r_n'}{s_1' \cdots s_n'} \cdot \frac{1}{a_{n+1}'}.$$

If $A_1' \neq 0$, then Lemma 3.1 implies that $a_1 = \langle 1/A_1' \rangle_{p(x)} = a_1'$. It follows that

$$r_1 = r_1(a_1) = r_1(a_1') = r_1',\ s_1 = s_1(a_1) = s_1(a_1') = s_1'$$

and so

$$A_2' \;:= \frac{1}{a_2} + \sum_{n\geq 2} \frac{r_2 \cdots r_n}{s_2 \cdots s_n} \cdot \frac{1}{a_{n+1}} = \frac{1}{a_2'} + \sum_{n\geq 2} \frac{r_2' \cdots r_n'}{s_2' \cdots s_n'} \cdot \frac{1}{a_{n+1}'}.$$

If $A_2' \neq 0$, by Lemma 3.1, we obtain $a_2 = \langle 1/A_2' \rangle_{p(x)} = a_2'$. Thus

$$r_2 = r_2(a_1,\ a_2) = r_2(a_1',\ a_2') = r_2',\ s_2 = s_2(a_1,\ a_2) = s_2(a_1',\ a_2') = s_2'.$$

In the same way, we have $a_n = a_n',\ r_n = r_n',\ s_n = s_n'$ for all $n \geq 1$. Therefore the expansion is unique. $\square$

In order to derive a degree of approximation, we introduce the notation

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1} + \sum_{i=2}^{n} \frac{r_1 \cdots r_{i-1}}{s_1 \cdots s_{i-1}} \frac{1}{a_i}, \quad \text{and } q_n = a_1 a_2 \cdots a_n \quad (n \geq 1). \tag{3.11}$$

Theorem 3.3. *For $n \geq 1$, the rational approximation p_n/q_n to $A \in \boldsymbol{F}^p$ satisfies*

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}.$$

Proof. We proved by induction on n. First, for $n = 1$,

$$\nu\left(A - \frac{p_1}{q_1}\right) = \nu\left(\frac{r_1}{s_1}A_2\right) = \nu(r_1) - \nu(s_1) - \nu(a_2) \geq 1 - 2\nu(a_1) \geq 2 - \nu(a_1).$$

So

$$\left|A - \frac{p_1}{q_1}\right|_{p(x)} = q^{-d\nu\left(A - \frac{p_1}{q_1}\right)} \leq q^{-d(2-\nu(a_1))} = \frac{1}{q^{2d}|q_1|_{p(x)}}.$$

For $n \geq 2$, assume that $|A - p_{n-1}/q_{n-1}|_{p(x)} \leq 1/\left(q^{nd}|q_{n-1}|_{p(x)}\right)$. Then

$$\begin{aligned}
\left|A - \frac{p_n}{q_n}\right|_{p(x)} &= \left|\frac{r_1\cdots r_n}{s_1\cdots s_n}A_{n+1}\right|_{p(x)} = \left|\frac{r_1\cdots r_n}{s_1\cdots s_n}\left(A_n - \frac{1}{a_n}\right)\frac{s_n}{r_n}\right|_{p(x)} \\
&= \left|\frac{r_1\cdots r_{n-1}}{s_1\cdots s_{n-1}}\left(a_n - \frac{1}{A_n}\right)\frac{A_n}{a_n}\right|_{p(x)} \\
&= q^{-d\nu(a_n - \frac{1}{A_n})}\frac{1}{|a_n|_{p(x)}}\left|\frac{r_1\cdots r_{n-1}}{s_1\cdots s_{n-1}}A_n\right|_{p(x)} \\
&\leq \frac{1}{q^d|a_n|_{p(x)}}\left|A - \frac{p_{n-1}}{q_{n-1}}\right|_{p(x)}, \qquad \text{since } \nu\left(a_n - \frac{1}{A_n}\right) \geq 1 \\
&\leq \frac{1}{q^{(n+1)d}|q_n|_{p(x)}},
\end{aligned}$$

by induction hypothesis. □

For the field $\mathbf{F}^{1/x}$, since the infinite valuation can be formally considered as $(1/x)$-adic valuation, the results of Lemma 3.1, Theorem 3.2 and Theorem 3.3 translate into :

Lemma 3.4. *Any series*

$$\frac{1}{b_1} + \sum_{n\geq 1}\frac{u_1\cdots u_n}{v_1\cdots v_n}\cdot\frac{1}{b_{n+1}}$$

with $b_n \in S_{1/x}$ *and* $\mu(b_1) \leq -1, \mu(b_{n+1}) \leq 2\mu(b_n) - 1 + \mu(u_n) - \mu(v_n)$ *converges to an element* $B' \in \boldsymbol{F}^{1/x}$ *such that* $\mu(B') = -\mu(b_1)$. *Furthermore* $B' \neq 0$ *and* $b_1 = \langle 1/B'\rangle_{1/x}$.

Theorem 3.5. *Every* $B \in \boldsymbol{F}^{1/x}$ *has a finite or convergent (relative to* $|\cdot|_\infty$*) expansion*

$$B = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty}\frac{u_1\cdots u_n}{v_1\cdots v_n}\frac{1}{b_{n+1}},$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ *and* $u_n, v_n \in \mathbb{F}_q(x)\backslash\{0\}$ *which may depend on* $b_1, \ldots, b_n$ *and*

$$\mu(b_1) \leq -1, \quad \mu(b_{n+1}) \leq 2\mu(b_n) - 1 + \mu(u_n) - \mu(v_n) \tag{3.12}$$

provided that

$$\mu(v_n) - \mu(u_n) \geq 2\mu(b_n) - 1 \qquad (n \geq 1).$$

The expansion is unique subject to these conditions on b_n, u_n *and* v_n.

For the degree of approximation, we have

Theorem 3.6. *For* $n \geq 1$, *the rational approximation*

$$\frac{p_n}{q_n} = b_0 + \frac{1}{b_1} + \sum_{i=2}^{n} \frac{u_1 \cdots u_{i-1}}{v_1 \cdots v_{i-1}} \frac{1}{b_i},$$

with $q_n = b_1 b_2 \cdots b_n$ *to*

$$B = b_0 + \frac{1}{b_1} + \sum_{i=1}^{\infty} \frac{u_1 \cdots u_i}{v_1 \cdots v_i} \frac{1}{b_{i+1}} \in \boldsymbol{F}^{1/x}$$

satisfies

$$\left| B - \frac{p_n}{q_n} \right|_\infty \leq \frac{1}{q^{n+1} \left| q_n \right|_\infty}.$$

3.3 Examples

We turn now to specific examples.

Type $1_{p(x)}$ **(Sylvester-type)**. Taking $r_n = s_n = 1$ in Theorem 3.2, we deduce that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$ and $\nu(a_n) \leq 1 - 2^n$, $\nu(a_{n+1}) \leq 2\nu(a_n) - 1 \quad (n \geq 1)$. Its degree of approximation is

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}, \quad \nu\left(A - \frac{p_n}{q_n} \right) = -\nu(a_{n+1}) \geq 2^{n+1} - 1.$$

Proof. Let $A \in \mathbf{F}^p$. Since $r_n = s_n = 1$ and $\nu(a_n) \le 0$ for $n \ge 1$, $0 = \nu(s_n) - \nu(r_n) \ge 2\nu(a_n) - 1 \quad (n \ge 1)$. Thus, by Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n},$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_{n+1}) \le 2\nu(a_n) - 1 \quad (n \ge 1)$. Next, we will show $\nu(a_n) \le 1 - 2^n$ for $n \ge 1$ by induction on n. First, if $n = 1$, then $\nu(a_1) \le -1 = 1 - 2^1$ by (3.1). For $n \ge 1$, assume $\nu(a_n) \le 1 - 2^n$. Then

$$\nu(a_{n+1}) \le 2\nu(a_n) - 1 \le 2(1 - 2^n) - 1 = 1 - 2^{n+1}.$$

If we define p_n/q_n and q_n as above, then by Theorem 3.3, we have

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \le \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}.$$

Therefore $\nu(A - p_n/q_n) = \nu(A_{n+1}) = -\nu(a_{n+1}) \ge 2^{n+1} - 1$. $\square$

Type $2_{p(x)}$ (Engel-type). Taking $r_n = 1,\ s_n = a_n$ in Theorem 3.2, we deduce that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$ and $\nu(a_n) \le -n,\ \nu(a_{n+1}) \le \nu(a_n) - 1 \quad (n \ge 1)$. Its degree of approximation is

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \le \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}},\ \ \nu\left(A - \frac{p_n}{q_n} \right) = -\nu(q_{n+1}) \ge \frac{(n+1)(n+2)}{2}.$$

Proof. Let $A \in \mathbf{F}^p$. Since $r_n = 1, s_n = a_n$ and $\nu(a_n) \le 0$ for $n \ge 1$, $\nu(s_n) - \nu(r_n) = \nu(a_n) \ge 2\nu(a_n) - 1 \quad (n \ge 1)$. Thus, by Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n},$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_{n+1}) \le \nu(a_n) - 1 \quad (n \ge 1)$. Next, we will show $\nu(a_n) \le -n$ for $n \ge 1$ by induction on n. If $n = 1$, then $\nu(a_1) \le -1$ by (3.1).

For $n \geq 1$, assume $\nu(a_n) \leq -n$. Then $\nu(a_{n+1}) \leq \nu(a_n) - 1 \leq -n - 1 = -(n+1)$. If we define p_n/q_n and q_n as above, then by Theorem 3.3, we have

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}$$

and

$$\begin{aligned} \nu\left(A - \frac{p_n}{q_n}\right) &= -\left\{\nu(a_1) + \cdots + \nu(a_{n+1})\right\} = -\nu(q_{n+1}) \\ &\geq 1 + 2 + \cdots + (n+1) = \frac{(n+1)(n+2)}{2}. \end{aligned}$$

□

Type $3_{p(x)}$ (Lüroth-type). Taking $r_n = 1,\ s_n = a_n(a_n - 1)$ in Theorem 3.2, we deduce that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n},$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$ and $\nu(a_n) \leq -1 \quad (n \geq 1)$. Its degree of approximation is

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}, \quad \nu\left(A - \frac{p_n}{q_n}\right) = -\nu(q_{n+1}) - \nu(q_n) \geq 2n + 1.$$

Proof. Let $A \in \mathbf{F}^p$ and $r_n = 1, s_n = a_n(a_n - 1) \quad (n \geq 1)$. First, we will prove that $\nu(a_n) \leq -1$ for $n \geq 1$. We have seen that $\nu(a_1) \leq -1$. For $n \geq 1$, assume $\nu(a_n) \leq -1$ which implies $\nu(a_n - 1) = \nu(a_n)$. It follows that

$$\nu(a_{n+1}) \leq 2\nu\left(a_n\right) - 1 - \nu\left(a_n\right) - \nu\left(a_n - 1\right) = -1$$

by (3.9). Then $\nu(s_n) - \nu(r_n) = 2\nu(a_n) \geq 2\nu(a_n) - 1 \quad (n \geq 1)$. Thus, by Theorem 3.2, we deduce that A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=2}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_{n-1}(a_{n-1}-1)a_n},$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_n) \leq -1 \ \ (n \geq 1)$. If we define p_n/q_n and q_n as above, then by Theorem 3.3, we have

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}$$

and

$$\begin{aligned} \nu\left(A - \frac{p_n}{q_n}\right) &= -\left\{\nu(a_1) + \cdots + \nu(a_{n+1})\right\} - \left\{\nu(a_1) + \cdots + \nu(a_n)\right\} \\ &= -\nu(q_{n+1}) - \nu(q_n) \geq 2n + 1. \end{aligned}$$

□

Type $4_{p(x)}$. Taking $r_n = a_n + 1, \ s_n = a_n$ in Theorem 3.2, we deduce that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=2}^{\infty} \left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_{n-1}}\right) \frac{1}{a_n}$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$ and $\nu(a_n) \leq 1 - 2^n, \ \nu(a_{n+1}) \leq 2\nu(a_n) - 1 \ \ (n \geq 1)$. Its degree of approximation is

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}, \quad \nu\left(A - \frac{p_n}{q_n}\right) = \nu\left(A_{n+1}\right) = -\nu(a_{n+1}) \geq 2^{n+1} - 1.$$

Proof. Let $A \in \mathbf{F}^p$ and $r_n = a_n + 1, s_n = a_n \ \ (n \geq 1)$. First, we will show $\nu(a_n) \leq 1 - 2^n$ for $n \geq 1$ by induction on n. If $n = 1$, then $\nu(a_1) \leq -1 = 1 - 2^1$ by (3.1). For $n \geq 1$, assume $\nu(a_n) \leq 1 - 2^n$. Then $\nu(a_n + 1) = \nu(a_n)$ and so

$$\nu(a_{n+1}) \leq 2\nu(a_n) - 1 \leq 2(1 - 2^n) - 1 = 1 - 2^{n+1}.$$

It follows that $\nu(s_n) - \nu(r_n) = \nu(a_n) - \nu(a_n + 1) = \nu(a_n) - \nu(a_n) = 0 \geq 2\nu(a_n) - 1 \quad (n \geq 1)$, since $\nu(a_n) \leq 0$. Thus, by Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=2}^{\infty} \left(1 + \frac{1}{a_1}\right) \cdots \left(1 + \frac{1}{a_{n-1}}\right) \frac{1}{a_n},$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_{n+1}) \leq 2\nu(a_n) - 1 \quad (n \geq 1)$. If we define p_n/q_n and q_n as above, then by Theorem 3.3, we have

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} |q_n|_{p(x)}}.$$

Therefore

$$\nu\left(A - \frac{p_n}{q_n}\right) = \nu\left(\frac{a_1+1}{a_1} \cdots \frac{a_n+1}{a_n} A_{n+1}\right) = \nu(A_{n+1}) = -\nu(a_{n+1}) \geq 2^{n+1} - 1.$$

□

Type $5_{p(x)}$. Taking $r_n = a_n^k$ $(k \in \mathbb{N} \cup \{0\})$, $s_n = 1$ in Theorem 3.2, we deduce that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{(a_1 \cdots a_n)^k}{a_{n+1}}$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$ and $\nu(a_{n+1}) \leq (k+2)\nu(a_n) - 1$, $\nu(a_n) \leq \frac{-(k+2)^n+1}{k+1}$ $(n \geq 1)$. Its degree of approximation is

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} |q_n|_{p(x)}}.$$

Proof. Let $A \in \mathbf{F}^p$. Since $r_n = a_n^k, s_n = 1$ and $\nu(a_n) \leq 0$ for $n \geq 1$, $\nu\left(a_n^k\right) = k\nu(a_n) \leq 1 - 2\nu(a_n)$. Thus $\nu(s_n) - \nu(r_n) = -\nu\left(a_n^k\right) \geq 2\nu(a_n) - 1 \quad (n \geq 1)$. Hence, by Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{(a_1 \cdots a_n)^k}{a_{n+1}}$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_{n+1}) \leq 2\nu(a_n) - 1 + \nu\left(a_n^k\right) = (k+2)\nu(a_n) - 1 \quad (n \geq 1)$. Next, we will show $\nu(a_n) \leq (-(k+2)^n + 1)/(k+1)$ for $n \geq 1$ by induction on n. First, if $n = 1$, then $\nu(a_1) \leq -1 = (-(k+2)+1)/(k+1)$ by (3.1). For $n \geq 1$, assume $\nu(a_n) \leq (-(k+2)^n + 1)/(k+1)$. Then

$$\nu(a_{n+1}) \leq (k+2)\nu(a_n) - 1 \leq (k+2)\frac{-(k+2)^n+1}{k+1} - 1 = \frac{-(k+2)^{n+1}+1}{k+1}.$$

If we define p_n/q_n and q_n as above, then by Theorem 3.3,

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\,|q_n|_{p(x)}}.$$

□

Note that the case where $k = 0$ of the Type $5_{p(x)}$ coincides with the Sylvester-type (Type $1_{p(x)}$) representation.

Type $6_{p(x)}$ **(radix-type 1)**. Taking $s_n = 1$ and $r_n = r$, where $r \in \mathbb{F}_q(x)$, $\nu(r) \leq 1$, in Theorem 3.2, we deduce that very $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{r^{n-1}}{a_n}$$

where $\nu(a_1) \leq -1$, $\nu(a_{n+1}) \leq 2\nu(a_n)$, and $\nu(a_n) \leq -2^{n-1}$ $(n \geq 1)$. Its degree of approximation is

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\,|q_n|_{p(x)}}.$$

Proof. Let $A \in \mathbf{F}^p$ and $s_n = 1, r_n = r \in \mathbb{F}_q(x)$ such that $\nu(r) \leq 1$. Since $\nu(a_n) \leq 0$ for $n \geq 1$, $\nu(s_n) - \nu(r_n) = -\nu(r) \geq -1 \geq 2\nu(a_n) - 1$. By Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{r^{n-1}}{a_n}$$

where $\nu(a_1) \leq -1$, $\nu(a_{n+1}) \leq 2\nu(a_n) - 1 + \nu(r) \leq 2\nu(a_n)$ $(n \geq 1)$. Next, we will show that $\nu(a_n) \leq -2^{n-1}$ for $n \geq 1$. If $n = 1$, then $\nu(a_1) \leq -1 = -2^{1-1}$, by (3.1). For $n \geq 1$, assume that $\nu(a_n) \leq -2^{n-1}$. Then $\nu(a_{n+1}) \leq 2\nu(a_n) \leq -2^n$. If we define p_n/q_n and q_n as above, then by Theorem 3.3,

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\,|q_n|_{p(x)}}.$$

□

Type $7_{p(x)}$ **(radix-type 2)**. Taking $r_n = 1,\ s_n = s \in \mathbb{F}_q(x),\ \nu(s) \geq -2$ in Theorem 3.2, we have every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{s^{n-1}a_n},$$

where $\nu\left(a_n\right) \leq -1$, and $\nu\left(a_{n+1}\right) \leq 2\nu(a_n) - 1 - \nu(s) \quad (n \geq 1)$. Its degree of approximation is

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\left|q_n\right|_{p(x)}}.$$

Proof. Let $A \in \mathbf{F}^p$ and $r_n = 1, s_n = s \in \mathbb{F}_q(x) \quad (n \geq 1)$, where $\nu(s) \geq -2$. First, we will prove that $\nu(a_n) \leq -1$ for $n \geq 1$. We have seen that $\nu(a_1) \leq -1$. For $n \geq 1$, assume $\nu(a_n) \leq -1$. Then

$$\nu(a_{n+1}) \leq 2\nu\left(a_n\right) - 1 - \nu\left(s\right) \leq -1.$$

Thus we have $\nu(s_n) - \nu(r_n) = \nu(s) \geq -2 > 2\nu(a_n) - 1 \quad (n \geq 1)$. By Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{s^{n-1}a_n},$$

where $\nu\left(a_n\right) \leq -1, \quad \nu(a_{n+1}) \leq 2\nu(a_n) - 1 - \nu(s) \quad (n \geq 1)$. If we define p_n/q_n and q_n as above, then by Theorem 3.3,

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\left|q_n\right|_{p(x)}}.$$

□

Type $8_{p(x)}$. Taking $s_n = a_n,\ r_1 = 1,\ r_n = a_{n-1} \quad (n \geq 2)$ in Theorem 3.2, we deduce that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}},$$

where $\nu\left(a_2\right) \leq \nu\left(a_1\right) - 1$, and $\nu\left(a_{n+1}\right) \leq \nu\left(a_n\right) + \nu\left(a_{n-1}\right) - 1 \quad (n \geq 2)$. Its degree of approximation is

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\left|q_n\right|_{p(x)}}.$$

Proof. Let $A \in \mathbf{F}^p$. Since $s_n = a_n,\ r_1 = 1,\ r_n = a_{n-1}\quad (n \geq 2)$, $\nu(s_1) - \nu(r_1) = \nu(a_1) \geq 2\nu(a_1) - 1$ and for $n \geq 2, \nu(s_n) - \nu(r_n) = \nu(a_n) - \nu(a_{n-1}) \geq \nu(a_n) \geq 2\nu(a_n) - 1$. This choice leads to a unique finite or a convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}},$$

where $\nu(a_2) \leq \nu(a_1) - 1$, and $\nu(a_{n+1}) \leq \nu(a_n) + \nu(a_{n-1}) - 1\quad (n \geq 2)$. If we define p_n/q_n and q_n as above, then by Theorem 3.3,

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} |q_n|_{p(x)}}.$$

$\square$

Type $9_{p(x)}$. In Theorem 3.2, taking $r_n = 1,\ s_n = a_n g_n$, with $g_n \in \mathbb{F}_q(x),\ \nu(g_n) \geq \nu(a_n) - 1\quad (n \geq 1)$, we have that every $A \in \mathbf{F}^p$ has a unique finite or convergent series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{(a_1 g_1) \cdots (a_n g_n)} \frac{1}{a_{n+1}}$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$ and $\nu(a_1) \leq -1,\ \nu(a_{n+1}) \leq -1 + \nu(a_n) - \nu(g_n) \leq 0\quad (n \geq 1)$. Its degree of approximation is

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} |q_n|_{p(x)}},\ \nu\left(A - \frac{p_n}{q_n}\right) \geq n + 2.$$

Proof. Let $A \in \mathbf{F}^p$ and $r_n = 1,\ s_n = a_n g_n\quad (n \geq 1)$ where $g_n \in \mathbb{F}_q(x),\ \nu(g_n) \geq \nu(a_n) - 1\quad (n \geq 1)$. Then $\nu(s_n) - \nu(r_n) = \nu(a_n g_n) \geq 2\nu(a_n) - 1$. By Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{(a_1 g_1) \cdots (a_n g_n)} \frac{1}{a_{n+1}}.$$

In this case we have by equation (3.9) that $\nu(a_{n+1}) \leq 2\nu(a_n) - 1 - \nu(a_n) - \nu(g_n) = -1 + \nu(a_n) - \nu(g_n)$. If we define p_n/q_n and q_n as above, then by Theorem 3.3,

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} |q_n|_{p(x)}},$$

and

$$\begin{aligned}
\nu\left(A-\frac{p_n}{q_n}\right) &= \nu\left(\frac{r_1\cdots r_n}{s_1\cdots s_n}A_{n+1}\right)\\
&= \nu\left(A_{n+1}\right)-\nu\left(s_1\cdots s_n\right)\\
&\geq \nu\left(A_n\right)+1+\nu\left(g_n\right)-\nu\left(s_1\cdots s_n\right)\\
&\geq \nu\left(A_{n-1}\right)+2+\nu\left(g_{n-1}\right)+\nu\left(g_n\right)-\nu\left(s_1\cdots s_n\right)\\
&\vdots\\
&\geq \nu\left(A_1\right)+n+\nu\left(g_1\right)+\cdots+\nu\left(g_n\right)-\nu\left(s_1\cdots s_n\right)\\
&= \nu\left(A_1\right)+n-\left\{\nu\left(a_1\right)+\cdots+\nu\left(a_n\right)\right\}, \quad \text{since } s_n=a_ng_n\\
&\geq \nu\left(A_1\right)+n+1, \quad \text{by } (3.1)\\
&\geq n+2.
\end{aligned}$$

□

The choices $g_n = 1$, respectively, $g_n = a_n - 1 \ \ (n \geq 1)$ coincide with the Engel-type (Type $2_{p(x)}$), respectively, the Lüroth-type (Type $3_{p(x)}$) representations.

Type $10_{p(x)}$. Taking $r_n = 1,\ \nu\left(s_n\right) = \nu\left(a_n\right) \quad (n \geq 1)$ in Theorem 3.2 leads to a unique finite or convergent series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty}\frac{1}{s_1\cdots s_n}\frac{1}{a_{n+1}},$$

where $a_0 = \langle A\rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_n) \leq -n,\ \nu(a_{n+1}) \leq \nu(a_n) - 1$. Its degree of approximation is

$$\left|A-\frac{p_n}{q_n}\right|_{p(x)} \leq \frac{1}{q^{(n+1)d}\left|q_n\right|_{p(x)}},\ \nu\left(A-\frac{p_n}{q_n}\right) \geq \frac{(n+1)(n+2)}{2}.$$

Proof. Let $A \in \mathbf{F}^p$ and $r_n = 1, s_n \in \mathbb{F}_q(x)$ such that $\nu\left(s_n\right) = \nu\left(a_n\right) \quad (n \geq 1)$. Then $\nu(s_n) - \nu(r_n) = \nu\left(a_n\right) \geq 2\nu(a_n) - 1 \ \ (n \geq 1)$, since $\nu(a_n) \leq 0$. Thus, by Theorem 3.2, A has a unique finite or convergent (relative to $|\cdot|_{p(x)}$) series expansion of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty}\frac{1}{s_1\cdots s_n}\frac{1}{a_{n+1}},$$

where $a_0 = \langle A \rangle_{p(x)}, a_n \in S_{p(x)}$, and $\nu(a_{n+1}) \leq \nu(a_n) - 1 \quad (n \geq 1)$. Next, we will show $\nu(a_n) \leq -n$ for $n \geq 1$ by induction on n. If $n = 1$, then $\nu(a_1) \leq -1$ by (3.1). For $n \geq 1$, assume $\nu(a_n) \leq -n$. Then $\nu(a_{n+1}) \leq \nu(a_n) - 1 \leq -n - 1 = -(n+1)$. If we define p_n/q_n and q_n as above, then we have

$$\left| A - \frac{p_n}{q_n} \right|_{p(x)} \leq \frac{1}{q^{(n+1)d} \left| q_n \right|_{p(x)}}$$

and

$$\begin{aligned} \nu\left(A - \frac{p_n}{q_n}\right) &= \nu\left(\frac{r_1 \cdots r_n}{s_1 \cdots s_n} A_{n+1}\right) \\ &= \nu\left(A_{n+1}\right) - \nu\left(s_1 \cdots s_n\right) \\ &= -\nu\left(a_{n+1}\right) - \{\nu(a_1) + \cdots + \nu(a_n)\} \\ &\geq 1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}. \end{aligned}$$

□

Next are standard examples of expansions for elements in $\mathbf{F}^{1/x}$.

Type 1_∞ **(Sylvester-type)**. Taking $u_n = v_n = 1$ in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \sum_{n=1}^{\infty} \frac{1}{b_n},$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ and $\mu(b_n) \leq 1 - 2^n, \ \mu(b_{n+1}) \leq 2\mu(b_n) - 1 \quad (n \geq 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \leq \frac{1}{q^{n+1} \left| q_n \right|_\infty}, \ \mu\left(B - \frac{p_n}{q_n}\right) = -\mu(b_{n+1}) \geq 2^{n+1} - 1.$$

Type 2_∞ **(Engel-type)**. Taking $u_n = 1, \ v_n = b_n$ in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \cdots b_n},$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ and $\mu(b_n) \le -n,\ \mu(b_{n+1}) \le \mu(b_n) - 1\ \ (n \ge 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty},\quad \mu\left(B - \frac{p_n}{q_n} \right) = -\mu(q_{n+1}) \ge \frac{(n+1)(n+2)}{2}.$$

Type 3_∞ **(Lüroth-type)**. Taking $u_n = 1,\ v_n = b_n(b_n - 1)$ in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=2}^{\infty} \frac{1}{b_1(b_1 - 1) \cdots b_{n-1}(b_{n-1} - 1) b_n},$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ and $\mu(b_n) \le -1\ \ (n \ge 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty},\quad \mu\left(B - \frac{p_n}{q_n} \right) = -\mu(q_{n+1}) - \mu(q_n) \ge 2n + 1.$$

Type 4_∞. Taking $u_n = b_n + 1,\ v_n = b_n$ in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=2}^{\infty} \left(1 + \frac{1}{b_1} \right) \cdots \left(1 + \frac{1}{b_{n-1}} \right) \frac{1}{b_n}$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ and $\mu(b_n) \le 1 - 2^n,\ \mu(b_{n+1}) \le 2\mu(b_n) - 1\ \ \ (n \ge 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty},\quad \mu\left(B - \frac{p_n}{q_n} \right) = \mu\left(B_{n+1} \right) = -\mu(b_{n+1}) \ge 2^{n+1} - 1.$$

Type 5_∞. Taking $u_n = b_n^k\ (k \in \mathbb{N} \cup \{0\}),\ v_n = 1$ in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{\left(b_1 \cdots b_n \right)^k}{b_{n+1}}$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ and $\mu(b_{n+1}) \le (k+2)\mu\left(b_n \right) - 1, \mu\left(b_n \right) \le \left(-\left(k+2 \right)^n + 1 \right) / \left(k+1 \right)$ for $(n \ge 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty}.$$

Note that the case where $k = 0$ of the Type 5_∞ coincides with the Sylvester-type (Type 1_∞) representation.

Type 6_∞ **(radix-type 1)**. Taking $v_n = 1$ and $u_n = u$, where $u \in \mathbb{F}_q(x),\ \mu(u) \le 1$, in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \sum_{n=1}^{\infty} \frac{u^{n-1}}{b_n}$$

where $\mu(b_1) \le -1,\ \ \mu(b_{n+1}) \le 2\mu(b_n)$, and $\mu(b_n) \le -2^{n-1}\ \ (n \ge 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty}.$$

Type 7_∞ **(radix-type 2)**. Taking $u_n = 1,\ v_n = v \in \mathbb{F}_q(x),\ \mu(v) \ge -2$ in Theorem 3.5, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \sum_{n=1}^{\infty} \frac{1}{v^{n-1} b_n},$$

where $\mu(b_n) \le -1$, and $\mu(b_{n+1}) \le 2\mu(b_n) - 1 - \mu(v)\ \ (n \ge 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty}.$$

Type 8_∞. Taking $v_n = b_n,\ u_1 = 1,\ u_n = b_{n-1}\ \ (n \ge 2)$ leads to a unique finite or convergent series expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{a_n a_{n+1}}\ .$$

where $\mu(b_2) \le \mu(b_1) - 1$, and $\mu(b_{n+1}) \le \mu(b_n) + \mu(b_{n-1}) - 1\ \ (n \ge 2)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \le \frac{1}{q^{n+1} \left| q_n \right|_\infty}.$$

Type 9_∞. In Theorem 3.5, taking $u_n = 1,\ v_n = b_n g_n$, with $g_n \in \mathbb{F}_q(x),\ \mu(g_n) \geq \mu(b_n) - 1\ \ (n \geq 1)$, we deduce that every $B \in \mathbf{F}^{1/x}$ has a unique finite or convergent series expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{(b_1 g_1) \cdots (b_n g_n)} \frac{1}{b_{n+1}}$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$ and $\mu(b_1) \leq -1,\ \mu(b_{n+1}) \leq -1 + \mu(b_n) - \mu(g_n)\ \ (n \geq 1)$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \leq \frac{1}{q^{n+1} |q_n|_\infty},\ \mu\left(B - \frac{p_n}{q_n}\right) \geq n + 2.$$

The choices $g_n = 1$, respectively, $g_n = b_n - 1\ \ (n \geq 1)$ coincide with the Engel-type (Type 2_∞), respectively, the Lüroth-type (Type 3_∞) representations.

Type 10_∞. Taking $u_n = 1,\ \mu(v_n) = \mu(b_n)\ \ (n \geq 1)$ in Theorem 3.5 leads to a unique finite or convergent series expansion of the form

$$B = b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{v_1 \cdots v_n} \frac{1}{b_{n+1}},$$

where $b_0 = \langle B \rangle_{1/x}, b_n \in S_{1/x}$, and $\mu(b_n) \leq -n,\ \mu(b_{n+1}) \leq \mu(b_n) - 1$. Its degree of approximation is

$$\left| B - \frac{p_n}{q_n} \right|_\infty \leq \frac{1}{q^{n+1} |q_n|_\infty},\ \mu\left(B - \frac{p_n}{q_n}\right) \geq \frac{(n+1)(n+2)}{2}.$$

3.4 Characterizations of Rational Elements

We first state some characterizations of rational elements via Oppenheim expansion and Engel-type expansion for p-adic numbers, [8], based on the sets of fractional parts S'_p and S_p, respectively,

Theorem 3.7. [8] *If* $a_n r_n / s_n \in S'_p\ \ (n \geq 1)$, *then the Oppenheim expansion of* $A \in \mathbb{Q}_p$ *with* $a_n \in S'_p$ *is finite if and only if* $A \in \mathbb{Q}$.

Theorem 3.8. [8] *The Engel-type expansion of $A \in \mathbb{Q}_p$ with $a_n \in S_p$ is finite if and only if $A \in \mathbb{Q}$.*

We now prove that the above two theorems hold in the case of function fields $\mathbf{F}^p$ and $\mathbf{F}^{1/x}$. Moreover, we give some other characterizations of rational elements via other types of series expansions, given in Section 3.3.

We start with an easy estimate.

Lemma 3.9. *If $a \in S_{p(x)}$, then*

$$|a|_\infty \leq \frac{1}{q}\,|p(x)|_\infty\,. \tag{3.13}$$

Proof. Writing

$$a = b_{-m}(x)p(x)^{-m} + b_{-m+1}(x)p(x)^{-m+1} + \cdots + b_{-1}(x)p(x)^{-1} + b_0(x) \in S_{p(x)},$$

where $m \in \mathbb{N} \cup \{0\}$ and $b_{-m}(x)\ (\neq 0), \ldots, b_0(x)$ are polynomials over $\mathbb{F}_q$ of degree less than d, we have

$$|a|_\infty \leq \max\left\{\left|b_{-m}(x)p(x)^{-m}\right|_\infty, \ldots, \left|b_{-1}(x)p(x)^{-1}\right|_\infty, |b_0(x)|_\infty\right\} \leq \frac{1}{q}\,|p(x)|_\infty\,.$$

$\square$

Theorem 3.10. *If $a_n r_n/s_n \in S_{p(x)}$ $(n \geq 1)$, then the Oppenheim series of $A \in \boldsymbol{F}^p$ is finite if and only if $A \in \mathbb{F}_q(x)$.*

Proof. The fact that a finite expansion yields a rational element is trivial. Conversely, assume that $A \in \mathbb{F}_q(x)$. Thus, each A_n is also rational, which allows us to write it uniquely under the form

$$A_n = p(x)^{\nu(A_n)}\frac{\alpha_n(x)}{\beta_n(x)},$$

where $\alpha_n(x),\ \beta_n(x)\ (\neq 0) \in \mathbb{F}_q[x]$ with $\gcd(\alpha_n(x), \beta_n(x)) = 1,\ p(x) \nmid \alpha_n(x)\beta_n(x)$. Since a_n and $a_n r_n/s_n$ are elements of $S_{p(x)}$, write

$$a_n = b_n(x)p(x)^{\nu(a_n)}, \quad \frac{a_n r_n}{s_n} = c_n(x)p(x)^{\nu(a_n)+\nu(r_n)-\nu(s_n)},$$

where $b_n(x), c_n(x) \in \mathbb{F}_q[x]$ satisfy, by (3.13),

$$|b_n(x)|_\infty \le \frac{1}{q}\,|p(x)|_\infty^{-\nu(a_n)+1}, \quad |c_n(x)|_\infty \le \frac{1}{q}\,|p(x)|_\infty^{\nu(s_n)-\nu(r_n)-\nu(a_n)+1}. \tag{3.14}$$

Rewriting the recurrence

$$A_{n+1} = \left(A_n - \frac{1}{a_n}\right)\frac{s_n}{r_n} = (a_nA_n - 1)\,\frac{s_n}{a_nr_n},$$

we obtain

$$\beta_n(x)c_n(x)\alpha_{n+1}(x)p(x)^{-\nu(a_{n+1})+\nu(a_n)+\nu(r_n)-\nu(s_n)} = \beta_{n+1}(x)\left\{b_n(x)\alpha_n(x) - \beta_n(x)\right\}. \tag{3.15}$$

Note from (3.9) that

$$-\nu\left(a_{n+1}\right) + \nu\left(a_n\right) + \nu\left(r_n\right) - \nu\left(s_n\right) \ge 1 - \nu\left(a_n\right) > 0.$$

Since $\gcd\left(\alpha_{n+1}(x)p(x)^{-\nu(a_{n+1})+\nu(a_n)+\nu(r_n)-\nu(s_n)},\ \beta_{n+1}(x)\right) = 1$, it follows from (3.15) that $\beta_{n+1}(x) \mid \beta_n(x)c_n(x)$, i.e.,

$$|\beta_{n+1}(x)|_\infty \le |\beta_n(x)c_n(x)|_\infty \qquad (n \ge 1) \tag{3.16}$$

and so (3.15) yields

$$|\alpha_{n+1}(x)|_\infty \le |p(x)|_\infty^{\nu(a_n)-1} \max\left\{|b_n(x)|_\infty\,|\alpha_n(x)|_\infty, |\beta_n(x)|_\infty\right\}.$$

Using (3.16) repeatedly and (3.14), we deduce

$$\begin{aligned}|\beta_n(x)|_\infty &\le |c_{n-1}(x)|_\infty\,|\beta_{n-1}(x)|_\infty \le \ldots \le |c_{n-1}(x)|_\infty\,|c_{n-2}(x)|_\infty \cdots |c_1(x)|_\infty\,|\beta_1(x)|_\infty \\ &\le \frac{1}{q^{n-1}}\,|\beta_1(x)|_\infty\,|p(x)|_\infty^{n-1+\sum_{i=1}^{n-1}(\nu(s_i)-\nu(r_i))-\nu(a_1\cdots a_{n-1})}.\end{aligned}$$

From the degree of approximation in Theorem 3.3, we have

$$n-\nu\left(q_{n-1}\right) \le \nu\left(A - \frac{p_{n-1}}{q_{n-1}}\right) = \nu\left(\frac{r_1\cdots r_{n-1}}{s_1\cdots s_{n-1}}\cdot\frac{1}{a_n}\right) = \sum_{i=1}^{n-1}\left(\nu\left(r_i\right) - \nu\left(s_i\right)\right) - \nu\left(a_n\right)$$

and so

$$n - 1 + \sum_{i=1}^{n-1}\left(\nu\left(s_i\right) - \nu\left(r_i\right)\right) - \nu\left(q_{n-1}\right) \le -\nu\left(a_n\right) - 1.$$

Consequently, $|\beta_n(x)|_\infty \le (1/q^{n-1})\,|\beta_1(x)|_\infty\,|p(x)|_\infty^{-\nu(a_n)-1}$ which implies

$$|\alpha_{n+1}(x)|_\infty \le \max\left\{\frac{|\alpha_n(x)|_\infty}{q}, \frac{|\beta_1(x)|_\infty}{|p(x)|_\infty^2\, q^{n-1}}\right\}.$$

This shows that $|\alpha_{n+1}(x)|_\infty \le (1/q)\,|\alpha_n(x)|_\infty$, for all large n. Thus, from some m onwards, $\alpha_m(x) = 0$, and so $A_m = 0$, i.e., the expansion is finite. □

In contrast to the case of p-adic numbers in [8], Theorem 3.10 applies to most well-known expansions in the function field case as we now record.

Corollary 3.11. *Let $A \in \boldsymbol{F}^p$. Then the Sylvester-type (Type $1_{p(x)}$), or the Engel-type (Type $2_{p(x)}$), or the Type $4_{p(x)}$, or the Type $8_{p(x)}$ expansion of A is finite if and only if $A \in \mathbb{F}_q(x)$.*

Furthermore, the same conclusion holds for

1. *the Type $5_{p(x)}$ expansion provided $d = 1$;*
2. *the radix-type 1 (Type $6_{p(x)}$) expansion provided $d = 1$ and $r_n = r \in S_{p(x)}$; and*
3. *the radix-type 2 (Type $7_{p(x)}$) expansion provided $s_n = s = cp(x)^{\nu(s)}$ with $c \in \mathbb{F}_q \setminus \{0\}, \nu(s) \ge 0$.*

Lemma 3.12. *If the corresponding Oppenheim-type expansion of $A \in \boldsymbol{F}^p$ is ultimately periodic, then $A \in \mathbb{F}_q(x)$.*

Proof. Let $A \in \mathbf{F}^p$ and assume that its Oppenheim-type expansion is ultimately periodic. Then there exist positive integers m and l such that $a_n = a_{n+l}, r_n = r_{n+l}$ and $s_n = s_{n+l}$ for all $n \ge m$. For $n \ge 1$, define

$$b_n = \frac{r_1 \cdots r_{n-1}}{s_1 \cdots s_{n-1}} \text{ and } b = \frac{b_{m+l}}{b_m}.$$

Then $b^k = \frac{r_m \cdots r_{m+kl-1}}{s_m \cdots s_{m+kl-1}}$ for all $k \ge 1$. By periodicity (with $1 = b_1, a_0 = p_0/q_0$ if

$m = 1$), we have

$$A = \frac{p_{m-1}}{q_{m-1}} + b_m \left(\frac{1}{a_m} + \frac{r_m}{s_m} \cdot \frac{1}{a_{m+1}} + \cdots + \frac{r_m \cdots r_{m+l-2}}{s_m \cdots s_{m+l-2}} \cdot \frac{1}{a_{m+l-1}} \right)$$
$$+ b_m b \left(\frac{1}{a_m} + \frac{r_m}{s_m} \cdot \frac{1}{a_{m+1}} + \cdots + \frac{r_m \cdots r_{m+l-2}}{s_m \cdots s_{m+l-2}} \cdot \frac{1}{a_{m+l-1}} \right)$$
$$+ b_m b^2 \left(\frac{1}{a_m} + \frac{r_m}{s_m} \cdot \frac{1}{a_{m+1}} + \cdots + \frac{r_m \cdots r_{m+l-2}}{s_m \cdots s_{m+l-2}} \cdot \frac{1}{a_{m+l-1}} \right) + \cdots$$
$$= \frac{p_{m-1}}{q_{m-1}} + c b_m \left(1 + b + b^2 + \cdots \right),$$

where

$$c = \frac{1}{a_m} + \frac{r_m}{s_m} \cdot \frac{1}{a_{m+1}} + \cdots + \frac{r_m \cdots r_{m+l-2}}{s_m \cdots s_{m+l-2}} \cdot \frac{1}{a_{m+l-1}} \in \mathbb{F}_q(x).$$

Since $1 + b + b^2 + \cdots = 1/(1-b)$, then $A = p_{m-1}/q_{m-1} + c b_m 1/(1-b) \in \mathbb{F}_q(x)$. $\square$

The only remaining type, Type 3 (Lüroth-type) expansion, perhaps the hardest in this setting, is a special case of the following theorem :

Theorem 3.13. *Let $r_n = 1, s_n = a_n f(a_n)$, for some fixed $f(x) \in \mathbb{F}_q[x]$. If $|a_n|_\infty \leq 1$ for n sufficiently large, then the corresponding Oppenheim-type expansion of $A \in \boldsymbol{F}^p$ is finite or ultimately periodic if and only if $A \in \mathbb{F}_q(x)$.*

Proof. The fact that finite Oppenheim expansions yield rational elements is trivial, while that periodic expansions also give rational elements are already checked as in Lemma 3.12.

It remains to show that if the Oppenheim expansion of $A \in \mathbb{F}_q(x)$ is infinite, then it is ultimately periodic. Since A is rational, so is each A_n $(n \geq 1)$, which allows us to write it uniquely under the form

$$A_n = p(x)^{\nu(A_n)} \frac{\alpha_n(x)}{\beta_n(x)},$$

where $\alpha_n(x), \beta_n(x) \in \mathbb{F}_q[x]$ with $\gcd(\alpha_n(x), \beta_n(x)) = 1, p(x) \nmid \alpha_n(x)\beta_n(x)$. For $n \geq 1$, since $a_n \in S_{p(x)} \setminus \{0\}$, we can write uniquely

$$a_n = \frac{c_{\nu_n}(x)}{p(x)^{\nu_n}} + \cdots + \frac{c_{D_n}(x)}{p(x)^{D_n}} = \frac{c_{\nu_n}(x) + \cdots + c_{D_n}(x) p(x)^{\nu_n - D_n}}{p(x)^{\nu_n}} =: \frac{b_n(x)}{p(x)^{\nu_n}}$$

where $c_{\nu_n}(x)(\neq 0), \ldots, c_{D_n}(x)(\neq 0) \in \mathbb{F}_q[x]$; $\nu_n := \nu(A_n) = -\nu(a_n) \geq D_n \geq 0$ and

$$b_n(x) = c_{\nu_n}(x) + \cdots + c_{D_n}(x) p(x)^{\nu_n - D_n} \in \mathbb{F}_q[x] \setminus \{0\}.$$

For $0 \leq k \leq \nu_n - D_n - 1$, if $-d(D_n + k + 1) + \deg c_{D_n+k+1}(x) \geq -d(D_n + k) + \deg c_{D_n+k}(x)$, then $-d + \deg c_{D_n+k+1}(x) \geq \deg c_{D_n+k}(x)$, because $\deg c_{D_n+k+1}(x)$, $\deg c_{D_n+k}(x) < d$. Thus $-d(D_n + k + 1) + \deg c_{D_n+k+1}(x) < -d(D_n + k)$ $+ \deg c_{D_n+k}(x)$. It follows that

$$\begin{aligned}
|a_n|_\infty &= \left| \frac{c_{\nu_n}(x)}{p(x)^{\nu_n}} + \cdots + \frac{c_{D_n+1}(x)}{p(x)^{D_n+1}} + \frac{c_{D_n}(x)}{p(x)^{D_n}} \right|_\infty \\
&= \max \left\{ \left| \frac{c_{\nu_n}(x)}{p(x)^{\nu_n}} \right|_\infty, \ldots, \left| \frac{c_{D_n+1}(x)}{p(x)^{D_n+1}} \right|_\infty, \left| \frac{c_{D_n}(x)}{p(x)^{D_n}} \right|_\infty \right\} \\
&= \max \left\{ q^{-d\nu_n + \deg c_{\nu_n}(x)}, \ldots, q^{-d(D_n+1) + \deg c_{D_n+1}(x)}, q^{-dD_n + \deg c_{D_n}(x)} \right\} \\
&= q^{-dD_n + \deg c_{D_n}(x)}. \qquad (3.17)
\end{aligned}$$

Similarly,

$$|b_n(x)|_\infty = q^{\deg c_{D_n}(x) + d\nu_n - dD_n} \leq q^{d-1+d\nu_n - dD_n}. \qquad (3.18)$$

Let $s_n = a_n f(a_n) := a_n (f_m a_n^m + \cdots + f_1 a_n + f_0) \in a_n \mathbb{F}_q[a_n]$. Rewriting

$$A_{n+1} = \left(A_n - \frac{1}{a_n} \right) s_n, \qquad (3.19)$$

we get

$$\begin{aligned}
p(x)^{\nu_{n+1}+m\nu_n} \alpha_{n+1}(x) \beta_n(x) &= \beta_{n+1}(x) \{\alpha_n(x) b_n(x) - \beta_n(x)\} \cdot \\
&\quad \left\{ f_m b_n(x)^m + f_{m-1} b_n(x)^{m-1} p(x)^{\nu_n} + \cdots + f_1 b_n(x) p(x)^{(m-1)\nu_n} + f_0 p(x)^{m\nu_n} \right\}. \qquad (3.20)
\end{aligned}$$

Since $p(x) \nmid \beta_{n+1}(x)$, $\gcd(\alpha_{n+1}(x), \beta_{n+1}(x)) = 1$, equation (3.20) implies that $\beta_{n+1}(x) \mid \beta_n(x)$, which yields $0 \leq \deg \beta_{n+1}(x) \leq \deg \beta_n(x)$ $(n \geq 1)$. Thus there exist a positive integer N_0 and a non-negative integer M such that $\deg \beta_n(x) = M$ for all $n \geq N_0$, i.e.,

$$\{\deg \beta_n(x)\} \quad \text{is a finite set} \qquad (3.21)$$

and

$$\beta_n(x) = c_n \beta_{N_0}(x), \quad c_n \in \mathbb{F}_q \setminus \{0\} \quad (n \geq N_0). \qquad (3.22)$$

Since $\nu_n \geq 1, p(x) \nmid \beta_{n+1}(x)$, and $p(x) \nmid b_n(x)$, equation (3.20) also implies that for $n \geq N_0$

$$p(x)^{\nu_{n+1}+m\nu_n} \mid \{\alpha_n(x) b_n(x) - c_n \beta_{N_0}(x)\}$$

and so

$$d\nu_{n+1} + md\nu_n \leq \max\{\deg \alpha_n(x) + \deg b_n(x),\ M\}. \tag{3.23}$$

If there exists an infinite subsequence $(n_k) \subset \mathbb{N}$, with all $n_k \geq N_0$, such that

$$\deg \alpha_{n_k}(x) + \deg b_{n_k}(x) \leq M,$$

then the two sequences of non-negative integers $(\deg \alpha_{n_k}(x) + \deg b_{n_k}(x))$ and $(\deg \alpha_{n_k}(x))$ are bounded, and from (3.23) so is the sequence (ν_{n_k}). We deduce from the unique representation $A_{n_k} = p(x)^{\nu_{n_k}} (\alpha_{n_k}(x)/\beta_{n_k}(x))$ that $\{A_{n_k}\}$ is a finite set. Hence, there exist indices $K < L$ such that all $A_{n_K} = A_{n_L}$ and so $a_{n_K} = \langle 1/A_{n_K} \rangle = \langle 1/A_{n_L} \rangle = a_{n_L}$ implying $s_{n_K} = s_{n_L}$. Invoking upon (3.19), we see that $A_{n_K+1} = A_{n_L+1}$ and so the expansion is ultimately periodic.

Henceforth, assume that there exists a positive integer $N_1 \geq N_0$ such that

$$\deg \alpha_n(x) + \deg b_n(x) > M \quad (n \geq N_1). \tag{3.24}$$

Going back to equation (3.20) and using equation (3.22), for $n \geq N_1$, we get

$$\begin{aligned} p(x)^{\nu_{n+1}+m\nu_n} \alpha_{n+1}(x) c_n &= c_{n+1} \{\alpha_n(x) b_n(x) - c_n \beta_{N_0}(x)\} \times \\ &\times \left\{ f_m b_n(x)^m + f_{m-1} b_n(x)^{m-1} p(x)^{\nu_n} + \cdots + f_1 b_n(x) p(x)^{(m-1)\nu_n} + f_0 p(x)^{m\nu_n} \right\}. \end{aligned}$$

Taking infinite valuation, we get

$$\begin{aligned} q^{d\nu_{n+1}+md\nu_n+\ \deg \alpha_{n+1}(x)} \leq\ & \max\left\{ |\alpha_n(x)|_\infty |b_n(x)|_\infty,\ q^M \right\} \times \\ & \times \max\left\{ |b_n(x)|_\infty^m, |b_n(x)|_\infty^{m-1} q^{d\nu_n}, \ldots, |b_n(x)|_\infty q^{(m-1)d\nu_n}, q^{md\nu_n} \right\} \\ = |\alpha_n(x)|_\infty |b_n(x)|_\infty & \max\left\{ |b_n(x)|_\infty^m, |b_n(x)|_\infty^{m-1} q^{d\nu_n}, \ldots, |b_n(x)|_\infty q^{(m-1)d\nu_n}, q^{md\nu_n} \right\}, \end{aligned} \tag{3.25}$$

using (3.24). From the given hypothesis, we may assume, without loss of generality, that $|a_n|_\infty \leq 1$ when $n \geq N_1$. We distinguish two possible cases.

Case 1: $|a_n|_\infty < 1$. In this case, $D_n > 0$ yielding $d\nu_n - dD_n + d - 1 < d\nu_n$ and $|b_n(x)|_\infty < q^{d\nu_n}$ by (3.18). Relation (3.25) becomes

$$q^{d\nu_{n+1}+md\nu_n+\ \deg\ \alpha_{n+1}(x)} < q^{\deg\ \alpha_n(x)+d\nu_n+md\nu_n},$$

i.e., we have the strict inequality

$$d\nu_{n+1} + \deg \alpha_{n+1}(x) < d\nu_n + \deg \alpha_n(x).$$

Case 2: $|a_n|_\infty = 1$. In this case, $D_n = 0$ and $c_{D_n}(x) = c \in \mathbb{F}_q \setminus \{0\}$, which yields $|b_n(x)|_\infty = q^{d\nu_n}$ by (3.18). Relation (3.25) thus implies

$$\begin{aligned} q^{d\nu_{n+1}+md\nu_n+\ \deg \alpha_{n+1}(x)} &\le q^{\deg \alpha_n(x)+d\nu_n} \max\{q^{md\nu_n}, q^{(m-1)d\nu_n+d\nu_n}, \ldots, q^{d\nu_n+(m-1)d\nu_n}, q^{md\nu_n}\} \\ &= q^{\deg \alpha_n(x)+d\nu_n+md\nu_n}, \end{aligned}$$

i.e., we have the non-strict inequality

$$d\nu_{n+1} + \deg \alpha_{n+1}(x) \le d\nu_n + \deg \alpha_n(x).$$

From both cases, we see that $(d\nu_n + \deg \alpha_n(x))_{n \ge N_1}$ is a non-increasing sequence of positive integers. It follows that both $\{\nu_n\}_{n\ge N_1}$ and $\{\deg \alpha_n(x)\}_{n \ge N_1}$ are bounded sets of non-negative integers. Consequently, there are infinitely many suffixes $n \ge N_1$ such that all ν_n are equal and all $\deg \alpha_n(x)$ are the same. Taking into account (3.22), we deduce that there are suffixes $M > L > N_1$ such that $A_M = A_L$. Invoking upon (3.19), we conclude that the expansion is ultimately periodic. □

The condition $|a_n|_\infty \le 1$ in Theorem 3.13 can be removed by taking $p(x) = x$, and we get

Corollary 3.14. *For $r_n = 1, s_n = a_n f(a_n)$, for some fixed $f(x) \in \mathbb{F}_q[x]$, the Oppenheim-type expansion of $A \in \mathbb{F}_q((x))$ is finite or ultimately periodic if and only if $A \in \mathbb{F}_q(x)$.*

Specializing $f(x) = x-1$ in Theorem 3.13, we get a characterization of rational elements by their Lüroth expansions.

Corollary 3.15. *If $|a_n|_\infty \le 1$, then the corresponding Lüroth-type expansion of $A \in \boldsymbol{F}^p$ is finite or ultimately periodic if and only if $A \in \mathbb{F}_q(x)$.*

Corollary 3.16. *The Lüroth-type expansion of $A \in \mathbb{F}_q((x))$ is finite or ultimately periodic if and only if $A \in \mathbb{F}_q(x)$.*

It is also possible to remove the restriction $d = 1$ in the case of Type $6_{p(x)}$ (radix-type 1) expansion at the expense of limiting the range of values of $r_n = r$.

Theorem 3.17. *Let $s_n = 1$ and $r_n = r = R(x)p(x)^{\nu(r)}$, where $R(x) \in \mathbb{F}_q[x]$ and $\nu(r) \le (-\deg R(x)/d)$. Then the radix-type 1 (Type $6_{p(x)}$) expansion of $A \in \boldsymbol{F}^p$ is finite if and only if $A \in \mathbb{F}_q(x)$.*

Proof. It is clear that a finite expansion yields a rational element. Conversely, assume that $A \in \mathbb{F}_q(x)$. Thus, each $A_n \quad (n \ge 1)$ is also rational and so we can write uniquely

$$A_n = p(x)^{\nu(A_n)} \frac{\alpha_n(x)}{\beta_n(x)},$$

where $\alpha_n(x),\ \beta_n(x)\ (\neq 0) \in \mathbb{F}_q[x]$ with $\gcd(\alpha_n(x), \beta_n(x)) = 1,\ p(x) \nmid \alpha_n(x)\beta_n(x)$. Since $a_n \in S_{p(x)}$, we can write uniquely $a_n = b_n(x)p(x)^{\nu(a_n)}$, where $b_n(x) \in \mathbb{F}_q[x]$ satisfy

$$|b_n(x)|_\infty \le \frac{1}{q} |p(x)|_\infty^{-\nu(a_n)+1}, \tag{3.26}$$

by (3.13). Equation (3.4) and (3.9) yield

$$\nu(r) \le 1 - 2\nu(a_n) \tag{3.27}$$

and

$$\nu(a_{n+1}) \le 2\nu(a_n) + \nu(r) - 1. \tag{3.28}$$

Rewriting the recurrence

$$A_{n+1} = \left(A_n - \frac{1}{a_n}\right)\frac{1}{r} = \frac{a_nA_n - 1}{a_n r},$$

we get

$$\beta_n(x)\alpha_{n+1}(x)b_n(x)R(x)p(x)^{-\nu(a_{n+1})+\nu(a_n)+\nu(r)} = \beta_{n+1}(x)\left\{b_n(x)\alpha_n(x) - \beta_n(x)\right\}. \tag{3.29}$$

By equation (3.28), we have $-\nu(a_{n+1}) + \nu(a_n) + \nu(r) \ge 1 - \nu(a_n) > 0$. Then $\beta_{n+1}(x) | \beta_n(x)b_n(x)R(x)$ and so $|\beta_{n+1}(x)|_\infty \le |\beta_n(x)b_n(x)R(x)|_\infty$. It follows that

$$\begin{aligned} |\beta_n(x)|_\infty &\le |\beta_1(x)|_\infty |b_{n-1}(x)|_\infty \cdots |b_1(x)|_\infty |R(x)|_\infty^{n-1} \\ &\le |\beta_1(x)|_\infty \left(\frac{|R(x)|_\infty}{q}\right)^{n-1} |p(x)|_\infty^{n-1-\nu(q_{n-1})}, \end{aligned}$$

by equation (3.26). Taking the infinite valuation through equation (3.29), we get

$$\begin{aligned}
|\alpha_{n+1}(x)|_\infty &\le |p(x)|_\infty^{\nu(a_{n+1})-\nu(a_n)-\nu(r)} \max\{|b_n(x)|_\infty |\alpha_n(x)|_\infty, |\beta_n(x)|_\infty\} \\
&\le \max\left\{|p(x)|_\infty^{\nu(a_n)-1}\frac{1}{q}|p(x)|_\infty^{-\nu(a_n)+1}|\alpha_n(x)|_\infty, |p(x)|_\infty^{\nu(a_{n+1})-\nu(a_n)-\nu(r)}|\beta_n(x)|_\infty\right\} \\
&\le \max\left\{\frac{1}{q}|\alpha_n(x)|_\infty, |\beta_1(x)|_\infty\left(\frac{|R(x)|_\infty}{q}\right)^{n-1}|p(x)|_\infty^{\nu(a_{n+1})-\nu(a_n)-\nu(r)-\nu(q_{n-1})+n-1}\right\}.
\end{aligned}$$

From the degree of approximation in Theorem 3.3, we have

$$n-\nu(q_{n-1}) \le (n-1)\nu(r)-\nu(a_n).$$

Then

$$\nu(a_{n+1})-\nu(a_n)-\nu(r)-\nu(q_{n-1})+n-1 \le \nu(a_{n+1})-2\nu(a_n)+(n-2)\nu(r)-1.$$

Thus

$$|\alpha_{n+1}(x)|_\infty \le \max\left\{\frac{1}{q}|\alpha_n(x)|_\infty, |\beta_1(x)|_\infty q^{(n-1)(\deg R(x)-1)+d(-1+(n-2)\nu(r)-2\nu(a_n)+\nu(a_{n+1}))}\right\}.$$

Since $\nu(r) \le (-\deg R(x)/d)$, we have

$$\begin{aligned}
&(n-1)(\deg R(x)-1)+d(-1+(n-2)\nu(r)-2\nu(a_n)+\nu(a_{n+1})) \\
&= n(\deg R(x)+d\nu(r)-1)+1-\deg R(x)+d(-2\nu(r)-2\nu(a_n)+\nu(a_{n+1})-1) \\
&\le -n+1-\deg R(x)-d(\nu(r)+2),
\end{aligned}$$

by (3.28). Thus

$$|\alpha_{n+1}(x)|_\infty \le \max\left\{\frac{1}{q}|\alpha_n(x)|_\infty, \frac{|\beta_1(x)|_\infty}{q^{\deg R(x)+d(\nu(r)+2)-1}}\cdot\frac{1}{q^n}\right\}.$$

If $\alpha_n(x) \ne 0$ for all $n \ge 1$, then for n sufficiently large, we have

$$\frac{|\beta_1(x)|_\infty}{q^{\deg R(x)+d(\nu(r)+2)-1}}\cdot\frac{1}{q^n} < \frac{1}{q} \le \frac{1}{q}|\alpha_n(x)|_\infty,$$

showing that $|\alpha_{n+1}(x)|_\infty < |\alpha_n(x)|_\infty$, or equivalently, $\deg\alpha_{n+1}(x) < \deg\alpha_n(x)$. Thus,

$\{\deg\alpha_n(x)\}$ is eventually a strictly decreasing sequence of non-negative integers implying that from some m onwards, $\alpha_m(x)=0$, and so $A_m=0$ which renders the expansion finite.

□

For the field $\mathbf{F}^{1/x}$, similar procedures as for $\mathbf{F}^p$ lead to analogous results as follows :

Lemma 3.18. *If $b \in S_{1/x}$, then*

$$b = c\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{\mu(b)}, \ c\left(\frac{1}{x}\right) \in \mathbb{F}_q\left[\frac{1}{x}\right] \ \text{ and } \left|c\left(\frac{1}{x}\right)\right|_x \leq q^{-\mu(b)}. \tag{3.30}$$

Proof. For $b \in S_{1/x}$, we can write

$$\begin{aligned} b &= c_{\mu(b)}\left(\frac{1}{x}\right)^{\mu(b)} + c_{\mu(b)+1}\left(\frac{1}{x}\right)^{\mu(b)+1} + \cdots + c_0 \\ &= \left(\frac{1}{x}\right)^{\mu(b)}\left\{c_{\mu(b)} + c_{\mu(b)+1}\left(\frac{1}{x}\right) + \cdots + c_0\left(\frac{1}{x}\right)^{-\mu(b)}\right\} \\ &= c\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{\mu(b)} \end{aligned}$$

where $c\,(1/x) := c_{\mu(b)} + c_{\mu(b)+1}\,(1/x) + \cdots + c_0\,(1/x)^{-\mu(b)} \in \mathbb{F}_q\,[1/x]$ and

$$b = c_{\mu(b)}x^{-\mu(b)} + c_{\mu(b)+1}x^{-\mu(b)-1} + \cdots + c_{-1}x + c_0.$$

Then

$$|b|_x = \max\left\{\left|c_{\mu(b)}x^{-\mu(b)}\right|_x, \left|c_{\mu(b)+1}x^{-\mu(b)-1}\right|_x, \ldots, |c_{-1}x|_x, |c_0|_x\right\} \leq q^0 = 1.$$

Thus we have $|c\,(1/x)|_x \leq q^{-\mu(b)}$ as required. □

Theorem 3.19. *If $b_n u_n/v_n \in S_{1/x}$, then the corresponding Oppenheim-type expansion of $A \in \boldsymbol{F}^{1/x}$ is finite if and only if $A \in \mathbb{F}_q(x)$.*

Proof. Assume that $b_n u_n/v_n \in S_{1/x}$ for $n \geq 1$. The fact that finite Oppenheim expansions yield rational elements is trivial.

Conversely, assume that $B \in \mathbb{F}_q\,(1/x)$. Thus, each B_n $(n \geq 1)$ is also rational, which allows us to write it uniquely in the form

$$B_n = \left(\frac{1}{x}\right)^{-\mu(b_n)} \frac{\alpha_n\left(\frac{1}{x}\right)}{\beta_n\left(\frac{1}{x}\right)},$$

where $\alpha_n(1/x), \beta_n(1/x) \in \mathbb{F}_q[1/x]$ with $\gcd(\alpha_n(1/x), \beta_n(1/x)) = 1, 1/x \nmid \alpha_n(1/x)\beta_n(1/x)$. For $n \geq 1$, since $b_n, b_n u_n/v_n \in S_{1/x}$,

$$b_n = c_n\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{\mu(b_n)} \text{ and } \frac{b_n u_n}{v_n} = d_n\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{\mu(b_n)+\mu(u_n)-\mu(v_n)},$$

where $c_n(1/x), d_n(1/x) \in \mathbb{F}_q[1/x]$ such that

$$\left|c_n\left(\frac{1}{x}\right)\right|_x \leq q^{-\mu(b_n)} \text{ and } \left|d_n\left(\frac{1}{x}\right)\right|_x \leq q^{\mu(v_n)-\mu(u_n)-\mu(b_n)}.$$

Now we have

$$B_{n+1} = \left(B_n - \frac{1}{b_n}\right)\frac{v_n}{u_n} = (b_n B_n - 1)\frac{v_n}{b_n u_n}. \tag{3.31}$$

Substituting all this into (3.31) leads to

$$\begin{aligned}\beta_n\left(\frac{1}{x}\right) d_n\left(\frac{1}{x}\right)\alpha_{n+1}\left(\frac{1}{x}\right)\left(\frac{1}{x}\right)^{-\mu(b_{n+1})+\mu(b_n)+\mu(u_n)-\mu(v_n)}&\\ = \beta_{n+1}\left(\frac{1}{x}\right)\left\{c_n\left(\frac{1}{x}\right)\alpha_n\left(\frac{1}{x}\right) - \beta_n\left(\frac{1}{x}\right)\right\}&\end{aligned}$$

considered as polynomial in $1/x$, where $-\mu(b_{n+1}) + \mu(b_n) + \mu(u_n) - \mu(v_n) \geq 1 - \mu(b_n) > 0$ by equation (3.12). Since $\gcd\Big(\alpha_{n+1}(1/x)(1/x)^{-\mu(b_{n+1})+\mu(b_n)+\mu(u_n)-\mu(v_n)}, \beta_{n+1}(1/x)\Big) = 1, \beta_{n+1}(1/x) \mid \beta_n(1/x)\, d_n(1/x)$ and so

$$\left|\beta_{n+1}\left(\frac{1}{x}\right)\right|_x \leq \left|\beta_n\left(\frac{1}{x}\right)\right|_x \left|d_n\left(\frac{1}{x}\right)\right|_x.$$

Thus

$$\begin{aligned}\left|\alpha_{n+1}\left(\frac{1}{x}\right)\right|_x &= \left|\frac{1}{x}\right|_x^{\mu(b_{n+1})-\mu(b_n)-\mu(u_n)+\mu(v_n)} \frac{\left|\beta_{n+1}\left(\frac{1}{x}\right)\right|_x}{\left|c_n\left(\frac{1}{x}\right)\beta_n\left(\frac{1}{x}\right)\right|_x}\left|c_n\left(\frac{1}{x}\right)\alpha_n\left(\frac{1}{x}\right) - \beta_n\left(\frac{1}{x}\right)\right|_x\\
&\leq \left|\frac{1}{x}\right|_x^{\mu(b_n)-1} \max\left\{\left|c_n\left(\frac{1}{x}\right)\right|_x\left|\alpha_n\left(\frac{1}{x}\right)\right|_x, \left|\beta_n\left(\frac{1}{x}\right)\right|_x\right\}\\
&\leq q^{\mu(b_n)-1}\max\left\{q^{-\mu(b_n)}\left|\alpha_n\left(\frac{1}{x}\right)\right|_x, \left|\beta_n\left(\frac{1}{x}\right)\right|_x\right\}\\
&= \max\left\{\frac{1}{q}\left|\alpha_n\left(\frac{1}{x}\right)\right|_x, q^{\mu(b_n)-1}\left|\beta_n\left(\frac{1}{x}\right)\right|_x\right\}.\end{aligned}$$

Now since $|\beta_n(1/x)|_x \leq |d_{n-1}(1/x)|_x |\beta_{n-1}(1/x)|_x$ for all $n \geq 2$, it follows that

$$\begin{aligned}
\left|\beta_n\left(\frac{1}{x}\right)\right|_x &\leq \left|d_{n-1}\left(\frac{1}{x}\right)\right|_x \left|\beta_{n-1}\left(\frac{1}{x}\right)\right|_x \\
&\leq \left|d_{n-1}\left(\frac{1}{x}\right)\right|_x \left|d_{n-2}\left(\frac{1}{x}\right)\right|_x \left|\beta_{n-2}\left(\frac{1}{x}\right)\right|_x \\
&\vdots \\
&\leq \left|d_{n-1}\left(\frac{1}{x}\right)\right|_x \cdots \left|d_1\left(\frac{1}{x}\right)\right|_x \left|\beta_1\left(\frac{1}{x}\right)\right|_x \\
&\leq \left|\beta_1\left(\frac{1}{x}\right)\right|_x q^{\sum_{i-1}^{n-1}(\mu(v_i)-\mu(u_i))-\mu(b_1 b_2\cdots b_{n-1})},
\end{aligned}$$

because $|d_i(1/x)|_x \leq q^{\mu(v_i)-\mu(u_i)-\mu(b_i)}$ for all $i \geq 1$. By Theorem 3.6, we get

$$q^{-\mu\left(B-\frac{p_{n-1}}{q_{n-1}}\right)} = \left|B - \frac{p_{n-1}}{q_{n-1}}\right|_\infty \leq \frac{1}{q^n |q_{n-1}|_\infty} = q^{\mu(q_{n-1})-n}$$

and so

$$n-\mu(q_{n-1}) \leq \mu\left(B - \frac{p_{n-1}}{q_{n-1}}\right) = \mu\left(\frac{u_1\cdots u_{n-1}}{v_1\cdots v_{n-1}}\cdot\frac{1}{b_n}\right) = \sum_{i=1}^{n-1}(\mu(u_i)-\mu(v_i)) - \mu(b_n).$$

Thus

$$\begin{aligned}
\sum_{i=1}^{n-1}(\mu(v_i)-\mu(u_i)) - \mu(q_{n-1}) &= -\sum_{i=1}^{n-1}(\mu(u_i)-\mu(v_i)) - \mu(q_{n-1}) \\
&\leq -n + \mu(q_{n-1}) - \mu(b_n) - \mu(q_{n-1}) \\
&= -\mu(b_n) - n.
\end{aligned}$$

It follows that $|\beta_n(1/x)|_x \leq |\beta_1(1/x)|_x\, q^{-\mu(b_n)-n}$ and so

$$\begin{aligned}
\left|\alpha_{n+1}\left(\frac{1}{x}\right)\right|_x &\leq \max\left\{\frac{1}{q}\left|\alpha_n\left(\frac{1}{x}\right)\right|_x, q^{\mu(b_n)-1}\left|\beta_1\left(\frac{1}{x}\right)\right|_x q^{-\mu(b_n)-n}\right\} \\
&= \max\left\{\frac{1}{q}\left|\alpha_n\left(\frac{1}{x}\right)\right|_x, \frac{\left|\beta_1\left(\frac{1}{x}\right)\right|_x}{q^{n+1}}.\right\}
\end{aligned}$$

If $\alpha_n(1/x) \neq 0$ for all $n \geq 1$, then for n sufficiently large,

$$\frac{\left|\beta_1\left(\frac{1}{x}\right)\right|_x}{q^{n+1}} < \frac{1}{q} \leq \frac{1}{q} q^{\deg \alpha_n\left(\frac{1}{x}\right)} = \frac{1}{q}\left|\alpha_n\left(\frac{1}{x}\right)\right|_x.$$

This implies

$$q^{\deg \alpha_{n+1}\left(\frac{1}{x}\right)} = \left|\alpha_{n+1}\left(\frac{1}{x}\right)\right|_x < \left|\alpha_n\left(\frac{1}{x}\right)\right|_x = q^{\deg \alpha_n\left(\frac{1}{x}\right)}.$$

Thus $\{\deg \alpha_n(1/x)\}$ is eventually a strictly decreasing sequence of non-negative integer, a contradiction. Therefore $\alpha_m(1/x) = 0$ for some $m \geq 1$. Hence $A_m = 0$ and the algorithm terminates.

□

In the case of the function field $\mathbf{F}^{1/x}$, Theorem 3.19 applies to most well-known expansions.

Corollary 3.20. *Let $B \in \boldsymbol{F}^{1/x}$. Then the Sylvester-type (Type 1_∞), or the Engel-type (Type 2_∞), or the Type 4_∞, or the Type 5_∞, or the Type 8_∞ expansion of B is finite if and only if $B \in \mathbb{F}_q(x)$.*

The same conclusion holds for

1. *the radix-type 1 (Type 6_∞) expansion provided $u_n = u \in S_{1/x}$;*
2. *the radix-type 2 (Type 7_∞) expansion provided $v_n := v = e(1/x)^{\mu(v)}$ with $e \in \mathbb{F}_q \setminus \{0\}, \mu(v) \geq 0$.*

Since the infinite valuation can be considered as the $(1/x)$-adic valuation for the field $\mathbb{F}_q(1/x)$, Corollary 3.16 holds for the case of the function field $\mathbf{F}^{1/x}$ and we show

Theorem 3.21. *In $\boldsymbol{F}^{1/x}$, the Lüroth-type expansion of A is finite or ultimately periodic if and only if $A \in \mathbb{F}_q(x)$.*

CHAPTER IV
PRODUCT EXPANSIONS AND SOME CHARACTERIZATIONS OF RATIONAL ELEMENTS

In this chapter, we introduce three main types of product expansions for elements in certain specific subsets of the function fields. We give detailed proofs of their convergence, uniqueness and the degree of approximation for each type. Characterizations of rational elements are established in the first type. In the remaining types, sufficient conditions of rationality are given. First, we introduce two subsets of $\mathbf{F}^p$ and $\mathbf{F}^{1/x}$ for first two types. Let

$$P := \left\{ 1 + \sum_{i=1}^{\infty} c_i(x)p(x)^i;\ c_i(x) \in \mathbb{F}_q[x],\ \ \deg c_i(x) < d \right\} \ \subset \ \mathbf{F}^p$$

and

$$Q := \left\{ 1 + \sum_{i=1}^{\infty} c_i x^{-i};\ c_i \in \mathbb{F}_q \right\} \ \subset \ \mathbf{F}^{1/x}.$$

4.1 Products of Type $\mathrm{I}_{p(x)}$ and Type I_∞ and Characterizations of Rational Elements

The first type of product is derived from the Type $4_{p(x)}$ series expansion as seen in Theorem 4.2. For convergence of product of Type $\mathrm{I}_{p(x)}$, we need one more auxiliary result.

Lemma 4.1. *Any product* $\prod_{i\geq 1}(1 + 1/b_i)$ *with*

$$b_i \in S_{p(x)}, \nu(b_i) \leq 1 - 2^i, \nu(b_{i+1}) \leq 2\nu(b_i) - 1, \tag{4.1}$$

converges (relative to $|\cdot|_{p(x)}$*) to an element* $B' \in P$ *such that* $B' \neq 1$ *and* $b_1 = \left\langle 1/\left(B' - 1\right)\right\rangle_{p(x)}$.

Proof. We need consider the case where the product is infinite. Let $B'_n = \prod_{i=1}^{n}(1+1/b_i)$. Then $B'_{n+k} - B'_n = B'_n\left(\prod_{i=n+1}^{n+k}(1+1/b_i) - 1\right)$ $(k \geq 1)$. Since $\nu(b_i) \leq -1$, $\nu(1+1/b_i) = \min\{\nu(1), \nu(1/b_i)\} = 0$, we get $\nu(B'_n) = 0$. From

$$\begin{aligned}\prod_{i=n+1}^{n+k}\left(1+\frac{1}{b_i}\right) - 1 &= \left(1+\frac{1}{b_{n+1}}\right)\left(1+\frac{1}{b_{n+2}}\right)\cdots\left(1+\frac{1}{b_{n+k}}\right) - 1,\\ &= \frac{1}{b_{n+1}} + \frac{1}{b_{n+2}} + \cdots + \frac{1}{b_{n+k}} + \cdots + \frac{1}{b_ib_j} + \cdots + \frac{1}{b_ib_jb_l}\\ &\quad + \cdots + \frac{1}{b_{n+1}b_{n+2}\cdots b_{n+k}},\end{aligned}$$

we have

$$\begin{aligned}&\nu\left(\prod_{i=n+1}^{n+k}\left(1+\frac{1}{b_i}\right) - 1\right)\\ &= \min\left\{\nu\left(\frac{1}{b_{n+1}}\right), \ldots, \nu\left(\frac{1}{b_{n+1}b_{n+2}}\right), \ldots, \nu\left(\frac{1}{b_{n+1}b_{n+2}\cdots b_{n+k}}\right)\right\}\\ &= \nu\left(\frac{1}{b_{n+1}}\right),\end{aligned}$$

because $\nu(1/b_i) \geq 1$ and $\nu(1/b_{i+1}) \geq 2\nu(1/b_i) + 1 > \nu(1/b_i)$. Consequently,

$$\nu\left(B'_{n+k} - B'_n\right) = \nu\left(\frac{1}{b_{n+1}}\right) \geq 2^{n+1} - 1 \to \infty \quad (n \to \infty),$$

i.e. (B'_n) is a Cauchy sequence (relative to $|\cdot|_{p(x)}$). Since $\mathbf{F}^p$ is a complete metric space, (B'_n) converges (relative to $|\cdot|_{p(x)}$) to some element $B' \in \mathbf{F}^p$. Since $B' - 1 = \lim_{n\to\infty}(B'_n - 1)$, and, as above, $\nu(B'_n - 1) = \nu(1/b_1)$, we get $\nu(B' - 1) = \nu(1/b_1) \geq 1$, i.e., $1 \neq B' \in P$ and $\langle B' \rangle_{p(x)} = 1$. Write now $B' = (1 + 1/b_1)B''$, where $B'' = \prod_{n=2}^{\infty}(1+1/b_i)$. Then $b_1 - 1/(B'-1) = (1+b_1)(B''-1)/(B'-1)$. Similarly, $\nu(B''-1) = \nu(1/b_2)$. Moreover,

$$\begin{aligned}\nu\left(b_1 - \frac{1}{B'-1}\right) &= \nu(1+b_1) + \nu(B''-1) - \nu(B'-1)\\ &= \nu(b_1) + \nu\left(\frac{1}{b_2}\right) - \nu\left(\frac{1}{b_1}\right) \geq 1,\end{aligned}$$

by (4.1). Hence $b_1 = \langle 1/(B'-1) \rangle_{p(x)}$. $\square$

Theorem 4.2. (Product of Type $\mathbf{I}_{p(x)}$) *Every $A \in P \setminus \{1\}$ has a finite or convergent (relative to $|\cdot|_{p(x)}$) product, termed Cantor product, expansion of the form*

$$A = \prod_{i=1}^{\infty} \left(1 + \frac{1}{a_i}\right),$$

where the "digits" $a_i \in S_{p(x)}$ satisfy $\nu(a_i) \le 1 - 2^i$, $\nu(a_{i+1}) \le 2\nu(a_i) - 1$. Subject to these conditions, the representation is unique.

In addition, if $\prod_{i=1}^{n}(1 + 1/a_i) = p_n/q_n$, with $q_n = a_1 a_2 \cdots a_n$, then

$$\left|A - \frac{p_n}{q_n}\right|_{p(x)} \le \frac{1}{q^{d(n+1)} |q_n|_{p(x)}} \le q^{-d\left(2^{n+1}-1\right)}.$$

Proof. In Theorem 3.2, let $A \in P \setminus \{1\}$ with $r_n = a_n + 1$ and $s_n = a_n$ $(n \ge 1)$. Then $a_0 = \langle A \rangle_{p(x)} = 1$ and

$$A = 1 + \frac{1}{a_1} + \left(1 + \frac{1}{a_1}\right)\frac{1}{a_2} + \ldots + \left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_{n-1}}\right)\frac{1}{a_n} + \left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_n}\right)A_{n+1}.$$

For $n \ge 1$, it follows easily by induction that

$$\prod_{i=1}^{n}\left(1 + \frac{1}{a_i}\right) = 1 + \frac{1}{a_1} + \left(1 + \frac{1}{a_1}\right)\frac{1}{a_2} + \cdots + \left(1 + \frac{1}{a_1}\right)\cdots\left(1 + \frac{1}{a_{n-1}}\right)\frac{1}{a_n} = \frac{p_n}{q_n},$$

where $q_n = a_1 \cdots a_n$ and so $A = (1 + A_{n+1})\prod_{i=1}^{n}(1 + 1/a_i)$. Also by induction, we have

$$\nu(A_n) = -\nu(a_n) \ge 2^n - 1 \quad (n \ge 1). \tag{4.2}$$

Thus, $A_{n+1} \to 0$ $(n \to \infty)$, which enables us to write

$$A = \prod_{i=1}^{\infty}\left(1 + \frac{1}{a_i}\right) \in \mathbf{F}^p.$$

Next, we show that

$$\nu(A_n) \ge n - \nu(q_{n-1}) \quad (n \ge 2) \tag{4.3}$$

by induction on n. First, $\nu(A_2) = -\nu(a_2) \ge 1 - 2\nu(a_1) \ge 2 - \nu(a_1)$ by (3.1). For $n \ge 2$, assume $\nu(A_n) \ge n - \nu(q_{n-1})$. Then $\nu(A_{n+1}) = -\nu(a_{n+1}) \ge 1 - 2\nu(a_n) = 1 - \nu(a_n) - \nu(a_n) \ge 1 - \nu(a_n) + n - \nu(q_{n-1}) = n + 1 - \nu(q_n)$. From (4.2), we get

$$\nu(q_n) = \nu(a_1) + \nu(a_2) + \cdots + \nu(a_n) \le (1 - 2^1) + (1 - 2^2) + \cdots + (1 - 2^n) = n - (2 + 2^2 + \ldots + 2^n),$$

and so by (4.3),

$$\nu\left(A-\frac{p_n}{q_n}\right)=\nu\left(A_{n+1}\prod_{i=1}^{n}\left(1+\frac{1}{a_i}\right)\right)=\nu\left(A_{n+1}\right)$$
$$\geq n+1-\nu(q_n)\geq 1+2+2^2+\cdots+2^n=2^{n+1}-1,$$

which yields the asserted approximation inequality.

To prove uniqueness, suppose that

$$A=\prod_{i=1}^{\infty}\left(1+\frac{1}{a_i}\right)=\prod_{i=1}^{\infty}\left(1+\frac{1}{a_i'}\right),$$

with both products being of the stated form. From Lemma 4.1, $a_1=\langle 1/\left(A-1\right)\rangle_{p(x)}=a_1'$. Cancelling out the first factor, we get

$$\prod_{i=2}^{\infty}\left(1+\frac{1}{a_i}\right)=\prod_{i=2}^{\infty}\left(1+\frac{1}{a_i'}\right),$$

Proceeding in the same manner, we successively obtain $a_i=a_i'$ for all $i\geq 1$. $\square$

Indeed from Theorems 3.2, 4.2 and the fact that

$$\prod_{i=1}^{n}\left(1+\frac{1}{a_i}\right)=a_0+\frac{1}{a_1}+\sum_{i=2}^{n}\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_{i-1}}\right)\frac{1}{a_i},$$

we obtain at once

Corollary 4.3. *Each $A\in P\setminus\{1\}$ has a series representation of the form*

$$A=1+\frac{1}{a_1}+\sum_{i=2}^{\infty}\left(1+\frac{1}{a_1}\right)\cdots\left(1+\frac{1}{a_{i-1}}\right)\frac{1}{a_i},$$

where $a_i\in S_{p(x)}$ satisfy $\nu\left(a_i\right)\leq 1-2^i,\ \nu\left(a_{i+1}\right)\leq 2\nu\left(a_i\right)-1$,if and only if it has a Cantor product representation of the form

$$A=\prod_{i=1}^{\infty}\left(1+\frac{1}{a_i}\right).$$

Using this last corollary and the characterization of Type $4_{p(x)}$ series expansion in Corollary 3.11, we get

Corollary 4.4. *The Cantor product of Type $I_{p(x)}$ of each $A \in P \backslash \{1\}$ is finite if and only if $A \in \mathbb{F}_q(x)$.*

For the field $\mathbf{F}^{1/x}$, similar proofs as in Lemma 4.1, Theorem 4.2, Corollary 4.3 and Corollary 4.4 yield

Lemma 4.5. *Any product $\prod_{i \geq 1} (1 + 1/b_i)$ with*

$$b_i \in S_{1/x}, \mu(b_i) \leq 1 - 2^i, \mu(b_{i+1}) \leq 2\mu(b_i) - 1,$$

converges (relative to $|\cdot|_\infty$) to an element $B' \in Q$ such that $B' \neq 1$ and $b_1 = \langle 1/\left(B' - 1\right) \rangle_{1/x}$.

Theorem 4.6. (Product of Type $\mathbf{I}_\infty$) *Every $B \in Q \setminus \{1\}$ has a finite or convergent (relative to $|\cdot|_\infty$) product, termed Cantor product, representation of the form*

$$B = \prod_{i=1}^{\infty} \left(1 + \frac{1}{b_i}\right),$$

where the "digits" $b_i \in S_{1/x}$ satisfy $\mu\left(b_i\right) \leq 1 - 2^i, \quad \mu\left(b_{i+1}\right) \leq 2\mu\left(b_i\right) - 1$. Subject to these conditions, the representation is unique.

In addition, if $\prod_{i=1}^{n} (1 + 1/b_i) = p_n/q_n$, with $q_n = b_1 b_2 \cdots b_n$, then

$$\left| B - \frac{p_n}{q_n} \right|_\infty \leq \frac{1}{q^{n+1} \left|q_n\right|_\infty} \leq q^{-\left(2^{n+1}-1\right)}.$$

Corollary 4.7. *Each $B \in Q \setminus \{1\}$ has a series representation of the form*

$$B = 1 + \frac{1}{b_1} + \sum_{i=2}^{\infty} \left(1 + \frac{1}{b_1}\right) \cdots \left(1 + \frac{1}{b_{i-1}}\right) \frac{1}{b_i},$$

where $b_i \in S_{1/x}$ satisfy $\mu\left(b_i\right) \leq 1 - 2^i, \nu\left(b_{i+1}\right) \leq 2\nu\left(b_i\right) - 1$, if and only if it has a Cantor product representation of the form

$$B = \prod_{i=1}^{\infty} \left(1 + \frac{1}{b_i}\right).$$

Corollary 4.8. *The Cantor product of Type I_∞ of $B \in Q \setminus \{1\}$ is finite if and only if $B \in \mathbb{F}_q\left(1/x\right)$.*

We continue with the second type product.

4.2 Products of Type $\mathbf{II}_{p(x)}$ and Type $\mathbf{II}_\infty$

We start with $\mathbf{F}^p$.

Lemma 4.9. *Any product* $\prod_{i=1}^{\infty}\left(1+c_i(x)p(x)^{d_i}\right)$ *with* $c_i(x)\in\mathbb{F}_q[x]\backslash\{0\}$ *such that* $\deg c_i(x)<d, d_i\in\mathbb{N}$ *and* $d_{i+1}>d_i$*, converges (relative to* $|\cdot|_{p(x)}$*) to an element* $B'\neq 1$ *in* P *with* $B'=1+c_1(x)p(x)^{d_1}B'',\ B''\in P$*.*

Proof. For $n\geq 1$, let $Q_n=\prod_{i=1}^{n}\left(1+c_i(x)p(x)^{d_i}\right)$. Then, for $k\geq 1$,

$$Q_{n+k}-Q_n=Q_n\left\{\prod_{i=n+1}^{n+k}\left(1+c_i(x)p(x)^{d_i}\right)-1\right\}=Q_nR_{n,k},$$

where $R_{n,k}=\prod_{i=n+1}^{n+k}\left(1+c_i(x)p(x)^{d_i}\right)-1$. Since $d_{i+1}>d_i$ for $i\geq 1$, we get

$$R_{n,k}=c_{n+1}(x)p(x)^{d_{n+1}}R'_{n,k},\ Q_n=1+c_1(x)p(x)^{d_1}S_n \ \text{ for } \ R'_{n,k},S_n\in P.$$

It is easy to see that $\nu(Q_n)=0$. Thus

$$\begin{aligned}\nu\left(Q_{n+k}-Q_n\right)&=\nu(R_{n,k})=\nu(c_{n+1}(x)p(x)^{d_{n+1}}R'_{n,k})\\&=d_{n+1}\geq 1+d_n\geq\ldots\geq n+1.\end{aligned}$$

Hence, for any $k\geq 1$, $\nu\left(Q_{n+k}-Q_n\right)\to\infty\ (n\to\infty)$, which implies that (Q_n) is a Cauchy sequence. Since $\mathbf{F}^p$ is a complete metric space, the sequence (Q_n) converges (relative to $|\cdot|_{p(x)}$) to an element B' in $\mathbf{F}^p$. It follows that $\nu(B'-Q_n)>d_1$ for n sufficiently large. Therefore $B'=1+c_1(x)p(x)^{d_1}B''$ for $B''\in P$, because $Q_n=1+c_1(x)p(x)^{d_1}S_n$. □

Theorem 4.10. (Product of Type $\mathbf{II}_{p(x)}$) *Every* $A\in P\setminus\{1\}$ *has a unique product representation of the form*

$$A=\prod_{i=1}^{\infty}\left(1+b_i(x)p(x)^{e_i}\right),$$

where $b_i(x)\in\mathbb{F}_q[x]\backslash\{0\}$, $\deg b_i(x)<d$, $e_i\in\mathbb{N}$ *and* $e_{i+1}>e_i$*. Further,*

$$\nu\left(A-P_n\right)\geq e_{n+1}>n,$$

where $P_n=\prod_{i=1}^{n}\left(1+b_i(x)p(x)^{e_i}\right)$*.*

Proof. Let $\hat{A}_1 := A \in P \backslash \{1\}$. Then we can write

$\hat{A}_1 = 1 + b_1(x)p(x)^{e_1}A'_1,$ with $e_1 \in \mathbb{N},\ b_1(x) \in \mathbb{F}_q[x]\backslash\{0\},\ \deg b_1(x) < d,\ A'_1 \in P.$

For $n \geq 1$, if $\hat{A}_n = 1 + b_n(x)p(x)^{e_n}A'_n$ has already been defined with $e_n \in \mathbb{N},\ A'_n \in P \setminus \{1\}$, $b_n(x) \in \mathbb{F}_q[x]\backslash\{0\}$ and $\deg b_n(x) < d$, then define

$$\begin{aligned}\hat{A}_{n+1} &= (1 + b_n(x)p(x)^{e_n})^{-1}\,\hat{A}_n \\ &= \left(1 - b_n(x)p(x)^{e_n} + b_n(x)^2p(x)^{2e_n} - \cdots\right)(1 + b_n(x)p(x)^{e_n} + \cdots) \\ &= 1 + b_{n+1}(x)p(x)^{e_{n+1}}A'_{n+1},\end{aligned}$$

where $A'_{n+1} \in P$ (if $A'_{n+1} \neq 0$), $e_{n+1} > e_n$, $b_{n+1}(x) \in \mathbb{F}_q[x]\backslash\{0\}$ and $\deg b_{n+1}(x) < d$. If $A'_n = 1$, then $\hat{A}_{n+1} = 1$ and the algorithm terminates yielding

$$A = \hat{A}_1 = (1 + b_1(x)p(x)^{e_1})\,\hat{A}_2 = \ldots = \hat{A}_{n+1}\prod_{i=1}^{n}(1 + b_i(x)p(x)^{e_i}).$$

Otherwise, $\hat{A}_{n+1} \neq 1$ and so $\nu\left(\hat{A}_{n+1} - 1\right) = e_{n+1} \geq 1 + e_n \geq \ldots \geq n + e_1 \geq n+1$, implying $\lim_{n\to\infty}\hat{A}_{n+1} = 1$. Thus, $A = \prod_{n=1}^{\infty}(1 + b_n(x)p(x)^{e_n})$.

For $n \geq 1$, let $P_n = \prod_{i=1}^{n}(1 + b_i(x)p(x)^{e_i})$. Then

$$A = P_n\hat{A}_{n+1} = P_n + b_{n+1}(x)p(x)^{e_{n+1}}A'_{n+1}P_n$$

and so follows the approximation

$$\nu\,(A - P_n) = \nu\left(b_{n+1}(x)p(x)^{e_{n+1}}A'_{n+1}P_n\right) = e_{n+1} \geq n+1 > n.$$

To show uniqueness, assume that

$$A = \prod_{i=1}^{\infty}(1 + b_i(x)p(x)^{e_i}) = \prod_{i=1}^{\infty}\left(1 + c_i(x)p(x)^{d_i}\right),$$

subject to the stated restrictions on the coefficients and exponents. Thus, by Lemma 4.9

$$A = 1 + b_1(x)p(x)^{e_1}A' = 1 + c_1(x)p(x)^{d_1}B',$$

with $A', B' \in P$, showing that $e_1 = d_1,\ b_1(x) = c_1(x)$. Cancelling out the first factor, we get

$$\prod_{i=2}^{\infty}(1 + b_i(x)p(x)^{e_i}) = \prod_{i=2}^{\infty}\left(1 + c_i(x)p(x)^{d_i}\right).$$

Proceeding in the same manner, we successively obtain $e_i = d_i, b_i(x) = c_i(x)$ $(i \geq 1)$. $\square$

Apart from those products of Type $\mathrm{I}_{p(x)}$, the problem of characterizing products of other types is still open. However, there are several sufficient conditions of rationality as witnessed, for example, in the following proposition.

Proposition 4.11. *Let* $A = \prod_{i=1}^{\infty}(1 + b_i(x)p(x)^{e_i}) \in P \setminus \{1\}$ *be a product of Type* $II_{p(x)}$. *If* $b_{i+1}(x) = b_i(x)^2$ *and* $e_{i+1} = 2e_i$ *for all* i *sufficiently large, then* $A \in \mathbb{F}_q(x)$.

Proof. Assume that there exists a positive integer N such that $b_{i+1}(x) = b_i(x)^2$ and $e_{i+1} = 2e_i$ $(i \geq N)$. Then

$$A = (1 + b_1(x)p(x)^{e_1}) \cdots (1 + b_{N-1}p(x)^{e_{N-1}})\,(1 + c(x))\left(1 + c(x)^2\right) \cdots \left(1 + c(x)^{2^{n-1}}\right) \cdots,$$

where $c(x) = b_N(x)p(x)^{e_N}$. For $n \geq 1$, since

$$\prod_{i=1}^{n}\left(1 + c(x)^{2^{i-1}}\right) = \frac{1 - c(x)^2}{1 - c(x)} \cdot \frac{1 - c(x)^4}{1 - c(x)^2} \cdots \frac{1 - c(x)^{2^n}}{1 - c(x)^{2^{n-1}}} = \frac{1 - c(x)^{2^n}}{1 - c(x)},$$

and $\nu\left(c(x)^{2^n}\right) = 2^n e_N \to \infty$, we get $\prod_{i=1}^{\infty}\left(1 + c(x)^{2^{i-1}}\right) = 1/\left(1 - c(x)\right)$ and so $A \in \mathbb{F}_q(x)$.

$\square$

For the field $\mathbf{F}^{1/x}$, as described earlier, analogous proof as in Lemma 4.9, Theorem 4.10 and Proposition 4.11 yield

Lemma 4.12. *Any product* $\prod_{i=1}^{\infty}\left(1 + c_i\,(1/x)^{d_i}\right)$ *with* $c_i \in \mathbb{F}_q\backslash\{0\}$ *such that* $d_i \in \mathbb{N}$ *and* $d_{i+1} > d_i$, *converges (relative to* $|\cdot|_{\infty}$*) to an element* $B' \neq 1$ *in* Q *with* $B' = 1 + c_1\,(1/x)^{d_1}\,B''$, $B'' \in Q$.

Theorem 4.13. (Product of Type $\mathbf{II}_{\infty}$) *Every* $B \in Q \setminus \{1\}$ *has a unique product representation of the form*

$$B = \prod_{i=1}^{\infty}\left(1 + b_i\left(\frac{1}{x}\right)^{e_i}\right),$$

where $b_i \in \mathbb{F}_q \backslash \{0\}$, *and* $e_i \in \mathbb{N},\ e_{i+1} > e_i$. *Further,*

$$\mu\,(B - Q_n) \geq e_{n+1} > n,$$

where $Q_n = \prod_{i=1}^{n} (1 + b_i\,(1/x)^{e_i})$.

Proposition 4.14. *Let* $B = \prod_{i=1}^{\infty} (1 + b_i\,(1/x)^{e_i}) \in Q \setminus \{1\}$ *be a product of type* II_∞. *If* $b_{i+1} = b_i^2$ *and* $e_{i+1} = 2e_i$, *for all* i *sufficiently large, then* $B \in \mathbb{F}_q(x)$.

4.3 Products of Type III$_{p(x)}$ and Type III$_\infty$

Let

$$P' = \left\{ 1 + \sum_{i=1}^{\infty} c_i p(x)^i;\ c_i \in \{0, 1, \ldots, p-1\} \right\} \subset \mathbf{F}^p.$$

and

$$Q' = \left\{ 1 + \sum_{i=1}^{\infty} c_i x^{-i};\ c_i \in \{0, 1, \ldots, p-1\} \right\} \subset \mathbf{F}^{1/x}.$$

For convergence and uniqueness, we need the following lemma :

Lemma 4.15. *Any product*

$$\prod_{i=1}^{\infty} (1 + p(x)^{s_i})^{d_i}$$

with $1 \leq d_i \leq p-1$, $s_i \in \mathbb{N}$ *and* $s_{i+1} > s_i$, *converges (relative to* $|\cdot|_{p(x)}$*) to an element* $B' \neq 1$ *in* P' *with* $B' = 1 + d_1 p(x)^{s_1} B''$, $B'' \in P'$.

Proof. For $n \geq 1$, let $Q_n = \prod_{i=1}^{n} (1 + p(x)^{s_i})^{d_i}$. Then, for $k \geq 1$,

$$Q_{n+k} - Q_n = Q_n \left\{ \prod_{i=n+1}^{n+k} (1 + p(x)^{s_i})^{d_i} - 1 \right\} = Q_n R_{n,k},$$

where

$$\begin{aligned} R_{n,k} &= \prod_{i=n+1}^{n+k} (1 + p(x)^{s_i})^{d_i} - 1 \\ &= \left\{1 + d_{n+1} p(x)^{s_{n+1}} + \cdots + p(x)^{d_{n+1}s_{n+1}}\right\} \left\{1 + d_{n+2} p(x)^{s_{n+2}} + \cdots + p(x)^{d_{n+2}s_{n+2}}\right\} \cdots \\ &\qquad \left\{1 + d_{n+k} p(x)^{s_{n+k}} + \cdots + p(x)^{d_{n+k}s_{n+k}}\right\} - 1 \\ &= d_{n+1} p(x)^{s_{n+1}} R'_{n,k} \end{aligned}$$

with $R'_{n,k} \in P'$, because (s_n) is a strictly increasing sequence. It is easy to see that $\nu(Q_n) = 0$ and so

$$\nu(Q_{n+k} - Q_n) = \nu(R_{n,k}) = s_{n+1} \geq s_n + 1 \geq \ldots \geq s_1 + n \geq n + 1.$$

Then $\nu(Q_{n+k} - Q_n) \to \infty$ and so $Q_{n+k} - Q_n \to 0 \quad (n \to \infty)$. Therefore, if the product is infinite, then (Q_n) forms a Cauchy sequence and so converges to an element B' of the complete metric space $\mathbf{F}^p$. Hence $Q_n - B' \to 0$ and so $\nu(B' - Q_n) \to \infty \quad (n \to \infty)$. Thus $\nu(B' - Q_N) > s_1$ for N sufficiently large and so $B' = 1 + d_1 p(x)^{s_1} B''$, $B'' \in P'$, because $Q_n = \prod_{i=1}^{n} (1 + p(x)^{s_i})^{d_i} = 1 + d_1 p(x)^{s_1} D'$, $D' \in P'$. $\square$

Theorem 4.16. (Product of Type $\mathbf{III}_{p(x)}$) *Every $A \in P' \setminus \{1\}$ has a unique product representation of the form*

$$A = \prod_{i=1}^{\infty} (1 + p(x)^{r_i})^{b_i},$$

where $1 \leq b_i \leq p - 1$, $r_i \in \mathbb{N}$ and $r_{i+1} > r_i$. Further

$$\nu(A - P_n) \geq r_{n+1} > n,$$

where $P_n = \prod_{i=1}^{n} (1 + p(x)^{r_i})^{b_i} \in \mathbb{N}$.

Proof. Let $\hat{A}_1 := A \in P' \setminus \{1\}$. Then we can write

$$\hat{A}_1 = 1 + b_1 p(x)^{r_1} + \cdots, \quad \text{with } r_1 \in \mathbb{N},\ 1 \leq b_1 \leq p - 1.$$

For $n \geq 1$, if $\hat{A}_n = 1 + b_n p(x)^{r_n} + \cdots$ has already been defined with $\hat{A}_n \in P' \setminus \{1\}$, $1 \leq b_n \leq p - 1$ and $r_n \in \mathbb{N}$, then set

$$\begin{aligned}
\hat{A}_{n+1} :&= (1 + p(x)^{r_n})^{-b_n} \hat{A}_n \\
&= \{1 - b_n p(x)^{r_n} + \cdots\} \{1 + b_n p(x)^{r_n} + \cdots\} \\
&= 1 + b_{n+1} p(x)^{r_{n+1}} + \cdots,
\end{aligned}$$

where $r_{n+1} > r_n$ and $1 \leq b_{n+1} \leq p-1$ if $\hat{A}_{n+1} \neq 1$. If $\hat{A}_{n+1} = 1$, then the algorithm terminates. Now

$$A = \hat{A}_1 = (1 + p(x)^{r_1})^{b_1} \hat{A}_2 = \ldots = \hat{A}_{n+1} \prod_{i=1}^{n} (1 + p(x)^{r_i})^{b_i}.$$

If the procedure does not terminate, then $\nu\left(\hat{A}_{n+1}-1\right)=r_{n+1}\geq 1+r_n\geq ...\geq n+r_1\geq n+1$, and so $A=\prod_{i=1}^{\infty}\left(1+p(x)^{r_i}\right)^{b_i}$. The proof of uniqueness is similar to that in Theorem 4.10 using Lemma 4.15.

From the definition of P_n, we can write $A=\hat{A}_{n+1}P_n$. Thus,

$$\nu\left(A-P_n\right)=\nu\left(\hat{A}_{n+1}-1\right)+\nu\left(P_n\right)=r_{n+1}>n.$$

□

A sufficient condition for rationality similar to the one in [6] is given in the next proposition.

Proposition 4.17. *Let* $A=\prod_{i=1}^{\infty}\left(1+p(x)^{r_i}\right)^{b_i}\in P'\setminus\{1\}$ *be a product of type* $III_{p(x)}$. *If* $b_i=b\in\{1,2,\dots,p-1\}$ *and* $r_{i+1}=2r_i$, *for all* i *sufficiently large, then* $A\in\mathbb{F}_q(x)$.

Proof. Assume that there exists a positive integer N such that $b_i=b\in\{1,2,\dots,p-1\}$ and $r_{i+1}=2r_i$ for $i\geq N$. Observe that

$$A=\left(1+p(x)^{r_1}\right)^{b_1}\cdots\left(1+p(x)^{r_{N-1}}\right)^{b_{N-1}}\left(1+p(x)^{r_N}\right)^{b}\left(1+p(x)^{2r_N}\right)^{b}\left(1+p(x)^{2^2r_N}\right)^{b}\cdots.$$

Let

$$C=\left(1+p(x)^{r_N}\right)^{b}\left(1+p(x)^{2r_N}\right)^{b}\left(1+p(x)^{2^2r_N}\right)^{b}\cdots,\quad \delta(x)=p(x)^{r_N}\quad\text{and}$$
$$C_n=\left(1+p(x)^{r_N}\right)^{b}\left(1+p(x)^{2r_N}\right)^{b}\left(1+p(x)^{2^2r_N}\right)^{b}\cdots\left(1+p(x)^{2^{n-1}r_N}\right)^{b}\quad(n\geq 1).$$

Then

$$C_n=\left(\left(1+\delta(x)\right)\left(1+\delta(x)^2\right)\left(1+\delta(x)^{2^2}\right)\cdots\left(1+\delta(x)^{2^{n-1}}\right)\right)^{b}=\left(\frac{1-\delta(x)^{2^n}}{1-\delta(x)}\right)^{b}.$$

Since $\nu\left(\delta(x)^{2^n}\right)=2^nr_N$ and $1\leq b\leq p-1$, we get $C=\lim_{n\to\infty}C_n=1/\left(1-\delta(x)\right)^{b}\in\mathbb{F}_q(x)$ and so

$$A=\left(1+p(x)^{r_1}\right)^{b_1}\cdots\left(1+p(x)^{r_{N-1}}\right)^{b_{N-1}}C\in\mathbb{F}_q(x).$$

□

For the field $\mathbf{F}^{1/x}$, similar proofs as in Lemma 4.15, Theorem 4.16 and Proposition 4.17 yield :

Lemma 4.18. *Any product*

$$\prod_{i=1}^{\infty}\left(1+\left(\frac{1}{x}\right)^{s_i}\right)^{d_i}$$

with $1 \leq d_i \leq p-1$, $s_i \in \mathbb{N}$ *and* $s_{i+1} > s_i$, *converges (relative to* $|\cdot|_{\infty}$*) to an element* $B' \neq 1$ *in* Q' *with* $B' = 1 + d_1 (1/x)^{s_1} B''$, $B'' \in Q'$.

Theorem 4.19. (Product of Type III$_{\infty}$) *Every* $B \in Q' \setminus \{1\}$ *has a unique product representation of the form*

$$B = \prod_{i=1}^{\infty}\left(1+\left(\frac{1}{x}\right)^{r_i}\right)^{b_i},$$

where $1 \leq b_i \leq p-1, r_i \in \mathbb{N}$ *and* $r_{i+1} > r_i$. *Further*

$$\mu\left(B - Q_n\right) \geq r_{n+1} > n,$$

where $Q_n = \prod_{i=1}^{n}\left(1 + (1/x)^{r_i}\right)^{b_i} \in \mathbb{N}$.

Proposition 4.20. *Let* $B = \prod_{i=1}^{\infty}\left(1 + (1/x)^{r_i}\right)^{b_i} \in Q' \setminus \{1\}$ *be a product representation of Type* III_{∞}. *If* $b_i = b \in \{1, 2, \ldots, p-1\}$ *and* $r_{i+1} = 2r_i$, *for all* i *sufficiently large, then* $B \in \mathbb{F}_q(x)$.

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VITA

Miss Narakorn Rompurk was born on December 7, 1975 in Karasin, Thailand. She graduated with a Bachelor of Science Degree in Mathematics from Khon Kaen University in 1997. Then she became an instructor at the Department of Mathematics, Khon Kaen University and taught there for one year. Later she went to study in Mathematics for a Master degree program at Chulalongkorn University with a scholarship from the University Development Commission (U.D.C) and graduated in 2001. At present, she is working towards a Ph.D. in Mathematics at the Faculty of Science, Chulalongkorn University, with a scholarship from the University Development Commission (U.D.C) again. This scholarship allowed her to go aboard for one year, from which she went to the Pensylvania State University as a visiting scholar from August 1, 2004 to May 1, 2005. After graduation she will continue her job at Khon Kaen University.