ทอพอโลยีนัยทั่วไปที่ชักนำจากฟังก์ชันทางเดียวและคลาสพันธุกรรม

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GENERALIZED TOPOLOGIES INDUCED BY MONOTONIC MAPS AND HEREDITARY CLASSES

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จุฬาลงกรณ์มหาวิทยาลัย
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ทอพอโลยีนัยทั่วไปที่กำหนดมาให้ใดๆ จะสามารถนำไปสร้างทอพอโลยีนัยทั่วไปใหม่ได้โดยใช้คลาสพันธุกรรม และจากบทความ [5] ได้ศึกษาความต่อเนื่องบนทอพอโลยีนัยทั่วไปที่เกิดจากคลาสพันธุกรรม เรานิยามและศึกษาทอพอโลยีนัยทั่วไปที่ชักนำจากฟังก์ชันทางเดียวและคลาสพันธุกรรม อีกทั้งศึกษาความต่อเนื่องบนทอพอโลยีนัยทั่วไปที่ชักนำจากฟังก์ชันทางเดียวและคลาสพันธุกรรมอีกด้วย และสุดท้ายนี้ศึกษาสมบัติและลักษณะเฉพาะของทอพอโลยีนัยทั่วไปที่ชักนำจากฟังก์ชันทางเดียวซึ่งมีสมบัติพิเศษบางประการและคลาสพันธุกรรม

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It is shown in [3] that every generalized topology can be extended to a new one by using hereditary classes, called a generalized topology via a hereditary class. Following [5], the continuity on generalized topological spaces via hereditary classes was studied. In this thesis, we introduce the notion of generalized topological spaces induced by monotonic maps and hereditary classes and provide some of their properties. Also, we define and study the continuity on generalized topological spaces induced by monotonic maps and hereditary classes in various situations. Finally, we study some properties and characterizations of generalized topological spaces induced by monotonic maps having special properties and hereditary classes.

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CHAPTER I
INTRODUCTION

In 1997, Császár [1] presented the concept of open sets in a topological space via monotonic maps, called γ-open sets. He observed that for any monotonic map γ, the empty set is γ-open and any union of γ-open sets is also γ-open. From his observation, he defined a generalized topology which is a collection of subsets of a given nonempty set containing the empty set and arbitrary unions of members in this collection. In 1990, Janković and Hamlett [4] defined a hereditary class $\mathcal{H}$ on a topological space X, which is the collection of subsets of X such that every subset of elements in $\mathcal{H}$ is in $\mathcal{H}$. They also introduced the generalization of the closure in a topological space via the hereditary class $\mathcal{H}$. In 2007, Császár [3] defined a new generalized topology which contains an old generalized topology via hereditary classes. In 2018, Montagantirud and Thaikua [5] introduced a notion of the continuity on generalized topological spaces via hereditary classes in various situations. They also proved that the continuity between two generalized topological spaces can be preserved on generalized topological spaces via hereditary classes.

In this thesis, we shall use the motivation discussed above to study generalized topologies induced by monotonic maps and hereditary classes. Our thesis is organized as follows. In Chapter II, we recall the definition of monotonic maps, generalized topological spaces, hereditary classes and continuous maps on generalized topological spaces. We also provide some facts which follow from properties of a generalized topological space induced by a hereditary class. In Chapter III, we introduce a notion of a generalized topological space induced by a monotonic map and a hereditary class. Also, we provide some of their properties. Next, we obtain some results of the continuity on generalized topological spaces induced by monotonic maps and hereditary classes and construct a hereditary class $\mathcal{H}$ that makes a given function continuous on the generalized topological space via a given monotonic map and the set $\mathcal{H}$. Finally, we give some applications of generalized topological spaces induced by monotonic maps having particular properties and hereditary classes. We also investigate the relationships between generalized topological spaces via hereditary classes and generalized topological spaces induced by monotonic maps and hereditary classes.

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CHAPTER II
PRELIMINARIES

In this chapter, we give some definitions, notations and results which will be used for this dissertation.

2.1 Generalized topologies via hereditary classes

Let X be a nonempty set. We denote the power set of X by $\exp(X)$. In 1997, Császár [1] introduced a generalization of open sets, called γ-open sets.

Definition 2.1.1. The map $\gamma : \exp(X) \to \exp(X)$ is called **monotonic** if $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$ for all $A, B \in \exp(X)$. The set of all monotonic maps is denoted by $\Gamma(X)$.

Definition 2.1.2. Let γ be a monotonic map. A set $A \subseteq X$ is called **γ-open** if $A \subseteq \gamma(A)$. The collection of all γ-open sets is denoted by g_γ.

In [1], Császár observed that the collection of γ-open sets has some properties similar to the collection of open sets in a topological space. That is, for any monotonic map γ, the empty set is γ-open and any union of γ-open sets is γ-open. From his observation, he defined a generalized topological space.

Definition 2.1.3. Let X be a nonempty set. A collection μ of subsets of X is called a **generalized topology** on X if it satisfies the following conditions.

(1) The empty set is in μ.

(2) Any union of elements in μ is in μ.

The pair (X, μ) is called a **generalized topological space**, the elements of μ are called **μ-open sets**, and the complements of μ-open sets are called **μ-closed sets**. In particular, any topology is also a generalized topology.

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Remark 2.1.4. The set g_γ is a generalized topology. We are able to say that a monotonic map on $\exp(X)$ constitutes a generalized topology.

Definition 2.1.5. Let (X, μ) be a generalized topological space. For $A \subseteq X$, the **μ-interior** of A, denoted by $i_\mu(A)$, is the union of all μ-open subsets of A, and the **μ-closure** of A, denoted by $c_\mu(A)$, is the intersection of all μ-closed supersets of A. In particular, i_μ and c_μ can be regraded as monotonic maps.

In 1990, Janković and Hamlett [4] generalized the concept of the closure in a topological space X via a collection of a particular subset, called a hereditary class, $\mathcal{H}$ of X and they also obtained a new topology which contains an old topology by using this concept.

Definition 2.1.6. A collection $\mathcal{H}$ of subsets of X is said to be a **hereditary class** on X if for each $A, B \in \exp(X)$,

$$A \subseteq B \text{ and } B \in \mathcal{H} \text{ imply } A \in \mathcal{H}.$$

If a hereditary class $\mathcal{H}$ has a further property that for each $A, B \in \exp(X)$,

$$A, B \in \mathcal{H} \text{ imply } A \cup B \in \mathcal{H},$$

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then $\mathcal{H}$ is said to be an **ideal**. We call $(X, \mu, \mathcal{H})$ a generalized topological space (X, μ) together with a hereditary class $\mathcal{H}$.

Remark 2.1.7. $\varnothing \in \mathcal{H}$ and $\mathcal{H} = \exp(X)$ if $X \in \mathcal{H}$.

In 2007, Császár [3] introduced and studied the construction of a generalized topology via a hereditary class.

Definition 2.1.8. Let $(X, \mu, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For each $A \subseteq X$, we define

$$A^*_{\mu,\mathcal{H}} = \{x \in X \mid x \in M \in \mu \text{ implies } M \cap A \notin \mathcal{H}\}.$$

In particular, $A^*_{\mu,\{\varnothing\}} = c_\mu(A)$. If there is no ambiguity, then $A^*_{\mu,\mathcal{H}}$ will be denoted by A^*_μ.

The following are some properties of A^*_μ.

Proposition 2.1.9. *[3] Let $(X, \mu, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$ and $A, B, M \subseteq X$.*

(1) $A \subseteq B$ implies $A^*_\mu \subseteq B^*_\mu$;

(2) $A^*_\mu \subseteq c_\mu(A)$;

(3) If $M \in \mu$ and $M \cap A \in \mathcal{H}$, then $M \cap A^*_\mu = \varnothing$;

(4) A^*_μ is μ-closed;

(5) A is μ-closed implies $A^*_\mu \subseteq A$;

(6) $(A^*_\mu)^*_\mu \subseteq A^*_\mu$;

(7) $X = X^*_\mu$ if and only if $\mu \cap \mathcal{H} = \varnothing$.

Moreover, he observed that A^*_μ does not contain A. Therefore, A^*_μ would not be a generalization of the closure of A. This leads to the following definition.

Definition 2.1.10. Let $(X, \mu, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For each $A \subseteq X$,

$$c^*_{\mu,\mathcal{H}}(A) = A \cup A^*_{\mu,\mathcal{H}}.$$

If there is no ambiguity, then $c^*_{\mu,\mathcal{H}}(A)$ will be denoted by $c^*(A)$.

After that, Császár [3] proved that there is a generalized topology μ^* such that $c^*(A)$ is the intersection of all μ^*-closed supersets of A, that is, $M \in \mu^*$ if and only if $c^*(X - M) = X - M$. From this motivation, we obtain the following definition.

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Definition 2.1.11. Let $(X, \mu, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. Define a generalized topology on X via a hereditary class $\mathcal{H}$ by

$$\mu^*_{\mathcal{H}} = \{M \subseteq X \mid c^*(X - M) = X - M\}.$$

The elements of $\mu^*_{\mathcal{H}}$ are called $\boldsymbol{\mu^*_{\mathcal{H}}}$**-open sets** and the complements of $\mu^*_{\mathcal{H}}$-open sets are called $\boldsymbol{\mu^*_{\mathcal{H}}}$**-closed sets**. If there is no ambiguity, then $\mu^*_{\mathcal{H}}$ will be denoted by μ^*.

The following are some properties of the generalized topology μ^*.

Proposition 2.1.12. *[3] Let $(X, \mu, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. Then:*

(1) If $\mathcal{H} = \{\varnothing\}$, then $\mu^* = \mu$;

(2) F is μ^*-closed if and only if $F^*_\mu \subseteq F$;

(3) $\mu \subseteq \mu^*$.

Definition 2.1.13. Let (X, μ) be a generalized topological space. The collection $\mathcal{B}$ is a **base** for μ if and only if $\mathcal{B} \subseteq \mu$ and every $M \in \mu$ is a union of elements of $\mathcal{B}$.

Theorem 2.1.14. *[3] Let $(X, \mu, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. The set*

$$\{M - H \subseteq X \mid M \in \mu \text{ and } H \in \mathcal{H}\}$$

constitutes a base for μ^*.

2.2 Continuous maps on generalized topological spaces via hereditary classes

In a generalized topological space (X, μ), we can define a continuous map, an open map and a homeomorphism in the same way as in a topological space.

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Definition 2.2.1. Let (X, μ) and (Y, μ') be generalized topological spaces. Then $f : X \to Y$ is said to be $\boldsymbol{(\mu, \mu')}$**-continuous** if $U \in \mu'$ implies $f^{-1}(U) \in \mu$.

Definition 2.2.2. Let (X, μ) and (Y, μ') be generalized topological spaces. Then $f : X \to Y$ is said to be $\boldsymbol{(\mu, \mu')}$**-open** if $U \in \mu$ implies $f(U) \in \mu'$.

Theorem 2.2.3. *[5] Let (X, μ) and (Y, μ') be generalized topological spaces and f a bijective function from X onto Y. Then f is (μ, μ')-open if and only if f^{-1} is (μ', μ)-continuous.*

Definition 2.2.4. Let (X, μ) and (Y, μ') be generalized topological spaces. Then $f : X \to Y$ is said to be a $\boldsymbol{(\mu, \mu')}$**-homeomorphism** if f is a (μ, μ')-continuous bijection and f^{-1} is (μ', μ)-continuous.

Theorem 2.2.5. *[5] Let (X, μ) and (Y, μ') be generalized topological spaces and f a bijection from X onto Y. Then f is a (μ, μ')-homeomorphism if and only if f is (μ, μ')-continuous and (μ, μ')-open.*

Following [5], we obtain some results of the continuity on generalized topological spaces via hereditary classes.

Theorem 2.2.6. *[5] Let (X, μ) and $(Y, \nu, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. If $f : X \to Y$ is a (μ, ν)-continuous injection, then for the hereditary class on X defined by*

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$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is (μ^*, ν^*)-continuous.

Theorem 2.2.7. *[5] Let $(X, \mu, \mathcal{H}_X)$ and (Y, ν) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. If $f : X \to Y$ is a (μ, ν)-continuous bijection, then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (μ^*, ν^*)-continuous.

Corollary 2.2.8. *[5] Let* (X, μ) *and* $(Y, \nu, \mathcal{H}_Y)$ *be generalized topological spaces and* $\mathcal{H}_Y$ *a hereditary class on* Y*. If* $f : X \to Y$ *is a* (μ, ν)*-open bijection, then there is a hereditary class* $\mathcal{H}_X$ *on* X *such that* f *is* (μ^*, ν^*)*-open.*

Corollary 2.2.9. *[5] Let* $(X, \mu, \mathcal{H}_X)$ *and* (Y, ν) *be generalized topological spaces and* $\mathcal{H}_X$ *a hereditary class on* X*. If* $f : X \to Y$ *is a* (μ, ν)*-open bijection, then there is a hereditary class* $\mathcal{H}_Y$ *on* Y *such that* f *is* (μ^*, ν^*)*-open.*

Theorem 2.2.10. *[5] Let* (X, μ) *and* $(Y, \nu, \mathcal{H}_Y)$ *be generalized topological spaces and* $\mathcal{H}_Y$ *a hereditary class on* Y*. If* $f : X \to Y$ *is a* (μ, ν)*-homeomorphism, then there is a hereditary class* $\mathcal{H}_X$ *on* X *such that* f *is a* (μ^*, ν^*)*-homeomorphism.*

Corollary 2.2.11. *[5] Let* $(X, \mu, \mathcal{H}_X)$ *and* (Y, ν) *be generalized topological spaces and* $\mathcal{H}_X$ *a hereditary class on* X*. If* $f : X \to Y$ *is a* (μ, ν)*-homeomorphism, then there is a hereditary class* $\mathcal{H}_Y$ *on* Y *such that* f *is a* (μ^*, ν^*)*-homeomorphism.*

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CHAPTER III
GENERALIZED TOPOLOGICAL SPACES INDUCED BY MONOTONIC MAPS AND HEREDITARY CLASSES

Let γ be a monotonic map on a set X. Following Remark 2.1.4, we obtain the generalized topological space (X, g_γ) which is simply denoted by (X, γ).

Definition 3.0.12. For each $A \subseteq X$, we denote

$$\begin{aligned}
i_\gamma(A) &= \{x \in X \mid \gamma(M) \subseteq A \text{ for some } M \in g_\gamma \text{ containing } x\},\\
cl_\gamma(A) &= \{x \in X \mid x \in M \in g_\gamma \text{ implies } \gamma(M) \cap A \neq \varnothing\},\\
\iota_\gamma(A) &= \{x \in X \mid M \subseteq A \text{ for some } M \in g_\gamma \text{ containing } x\},\\
c_\gamma(A) &= \{x \in X \mid x \in M \in g_\gamma \text{ implies } M \cap A \neq \varnothing\}.
\end{aligned}$$

We observe that $i_\gamma(A) \subseteq \iota_\gamma(A) \subseteq A$ and $A \subseteq c_\gamma(A) \subseteq cl_\gamma(A)$ for all $A \subseteq X$. In addition, it is easy to see that $i_\gamma, cl_\gamma, \iota_\gamma$ and c_γ are monotonic maps. Following the definition 3.0.1, we provide some basic results of $i_\gamma, cl_\gamma, \iota_\gamma$ and c_γ.

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Proposition 3.0.13. *For each $A \subseteq X$, we have*

(1) $i_\gamma(A) = X - cl_\gamma(X - A)$;

(2) $cl_\gamma(A) = X - i_\gamma(X - A)$;

(3) $\iota_\gamma(A) = X - c_\gamma(X - A)$;

(4) $c_\gamma(A) = X - \iota_\gamma(X - A)$.

Proof. (1):

$$\begin{aligned} x \in i_\gamma(A) &\Leftrightarrow \gamma(M) \subseteq A \text{ for some } M \in g_\gamma \text{ containing } x \\ &\Leftrightarrow \gamma(M) \cap (X - A) = \varnothing \text{ for some } M \in g_\gamma \text{ containing } x \\ &\Leftrightarrow x \notin cl_\gamma(X - A) \\ &\Leftrightarrow x \in X - cl_\gamma(X - A). \end{aligned}$$

(2): By (1), we obtain that $cl_\gamma(A) = cl_\gamma(X - (X - A)) = X - i_\gamma(X - A)$.

(3):

$$\begin{aligned} x \in \iota_\gamma(A) &\Leftrightarrow M \subseteq A \text{ for some } M \in g_\gamma \text{ containing } x \\ &\Leftrightarrow M \cap (X - A) = \varnothing \text{ for some } M \in g_\gamma \text{ containing } x \\ &\Leftrightarrow x \notin c_\gamma(X - A) \\ &\Leftrightarrow x \in X - c_\gamma(X - A). \end{aligned}$$

(4): By (3), we obtain that $c_\gamma(A) = c_\gamma(X - (X - A)) = X - \iota_\gamma(X - A)$.

$\square$

Proposition 3.0.14. *For each $A \subseteq X$, we have*

(1) A is γ-open if and only if $\iota_\gamma(A) = A$;

(2) A is γ-closed if and only if $c_\gamma(A) = A$.

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Proof.

(1): It suffices to show that the converse is hold. Assume that $\iota_\gamma(A) = A$. Let $x \in A$. By the assumption, there is $M \in g_\gamma$ containing x such that $M \subseteq A$. This implies that $x \in M \subseteq \gamma(M) \subseteq \gamma(A)$. Hence, $A \subseteq \gamma(A)$, i.e. A is γ-open.

(2): We conclude that

$$\begin{aligned} A \text{ is } \gamma\text{-closed} &\Leftrightarrow X - A \text{ is } \gamma\text{-open} \\ &\Leftrightarrow \iota_\gamma(X - A) = X - A && \text{(By (1))} \\ &\Leftrightarrow c_\gamma(A) = X - \iota_\gamma(X - A) = A. && \text{(By Proposition 3.0.2 (4))} \end{aligned}$$

□

3.1 The set $A^*_{\gamma,\mathcal{H}}$

Let $\mathcal{H}$ denote a hereditary class on a generalized topological space X. In 2007, Császár [3] introduced and studied the construction of a generalized topology via a hereditary class. Similarly, we define a generalized topology induced by a monotonic map and a hereditary class as follows. Firstly, we study a generalization of the closure in a generalized topological space by using a monotonic map γ and a hereditary class $\mathcal{H}$.

Definition 3.1.1. Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For each $A \subseteq X$, we define

$$A^*_{\gamma,\mathcal{H}} = \{x \in X \mid x \in M \in g_\gamma \text{ implies } \gamma(M) \cap A \notin \mathcal{H}\}.$$

In particular, $A^*_{\gamma,\{\varnothing\}} = cl_\gamma(A)$. If there is no ambiguity, then $A^*_{\gamma,\mathcal{H}}$ will be denoted by A^*_γ.

The following are some properties of A^*_γ.

Proposition 3.1.2. *Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For each $A, B \subseteq X$, we have*

(1) $A^*_\gamma \subseteq B^*_\gamma$ if $A \subseteq B$;

(2) $A^*_\gamma \subseteq cl_\gamma(A)$;

(3) if $M \in g_\gamma$ and $\gamma(M) \cap A \in \mathcal{H}$, then $M \cap A^*_\gamma = \varnothing$;

(4) $c_\gamma(A^*_\gamma) = A^*_\gamma$;

(5) A^*_γ is γ-closed;

(6) if $cl_\gamma(A) \subseteq A$, then $A^*_\gamma \subseteq A$.

Proof. (1): Assume that $A \subseteq B$. We will show that $X - B^*_\gamma \subseteq X - A^*_\gamma$. Let $x \notin B^*_\gamma$. There exists an $M \in g_\gamma$ containing x such that $\gamma(M) \cap B \in \mathcal{H}$. Since $A \subseteq B$, we obtain $\gamma(M) \cap A \subseteq \gamma(M) \cap B \in \mathcal{H}$. We have $\gamma(M) \cap A \in \mathcal{H}$ because $\mathcal{H}$ is a hereditary class. Hence, $x \notin A^*_\gamma$.

(2): Let $x \notin cl_\gamma(A)$. There exists $M \in g_\gamma$ such that $x \in M$ and $\gamma(M) \cap A = \varnothing \in \mathcal{H}$. Hence, $x \notin A^*_\gamma$.

(3): Assume that $M \cap A^*_\gamma \neq \varnothing$ and $M \in g_\gamma$. There exists a point $x \in M \cap A^*_\gamma$. Since $x \in A^*_\gamma$ and $M \in g_\gamma$ containing x, we obtain $\gamma(M) \cap A \notin \mathcal{H}$.

(4): It is enough to show that $c_\gamma(A^*_\gamma) \subseteq A^*_\gamma$. Let $x \notin A^*_\gamma$. There exists $M \in g_\gamma$ such that $x \in M$ and $\gamma(M) \cap A \in \mathcal{H}$. By (3), $M \cap A^*_\gamma = \varnothing$. Therefore, $x \notin c_\gamma(A^*_\gamma)$.

(5): It follows from Proposition 3.0.3 and (4).

(6): Assume that $cl_\gamma(A) \subseteq A$. By (2), $A^*_\gamma \subseteq cl_\gamma(A) \subseteq A$.

□

Proposition 3.1.3. *Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For each $M, M' \subseteq X$, the following statements are equivalent.*

(i) If $M \in g_\gamma$, then $M \subseteq M^*_\gamma$.

(ii) If $M, M' \in g_\gamma$ and $M \cap \gamma(M') \in \mathcal{H}$, then $M \cap M' = \varnothing$.

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Proof. Firstly, we prove that (i) implies (ii). Let $M, M' \in g_\gamma$ and $M \cap M' \neq \varnothing$. Assume that the statement (i) holds. There is a point $x \in M \cap M'$. We obtain $x \in M' \in g_\gamma$. Since $M \in g_\gamma$, by the assumption, $x \in M \subseteq M^*_\gamma$. Thus, $x \in M^*_\gamma$. It follows that $\gamma(M') \cap M \notin \mathcal{H}$. Conversely, we assume the statement (ii) holds and $M \in g_\gamma$. Let $x \in M$ and $M' \in g_\gamma$ containing x. Thus, $x \in M \cap M'$, i.e., $M \cap M' \neq \varnothing$. So, we obtain $\gamma(M') \cap M \notin \mathcal{H}$. Hence, $x \in M^*_\gamma$. □

Definition 3.1.4. [3] A hereditary class $\mathcal{H}$ is said to be **γ-codense** if $X = X^*_\gamma$.

Proposition 3.1.5. *Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For $M \subseteq X$, the following statements are equivalent.*

(i) $\mathcal{H}$ is γ-codense.

(ii) If $M \in g_\gamma$ and $\gamma(M) \in \mathcal{H}$, then $M = \varnothing$.

Proof. Now, we will show that (i) implies (ii). Let $\varnothing \neq M \in g_\gamma$. Assume that $\mathcal{H}$ is γ-codense. We have $x \in M$ for some $x \in X$. Then $x \in X = X^*_\gamma$. Since $x \in M \in g_\gamma$, we obtain $\gamma(M) = \gamma(M) \cap X \notin \mathcal{H}$. Conversely, we assume the statement (ii) holds. It is enough to show that $X \subseteq X^*_\gamma$. Let $x \in X$ and $M \in g_\gamma$ containing x. By the assumption, we conclude $\gamma(M) \notin \mathcal{H}$. That is, $\gamma(M) \cap X = \gamma(M) \notin \mathcal{H}$. Hence, $x \in X^*_\gamma$. $\square$

Remark 3.1.6. The fact that for each $M, M' \subseteq X$, $M, M' \in g_\gamma$ and $M \cap \gamma(M') \in \mathcal{H}$ imply $M \cap M' = \varnothing$ obviously implies that $\mathcal{H}$ is γ-codense.

The following example shows that A^*_γ does not contain A.

Example 3.1.7. Let $X = \{a, b\}$ and $\mathcal{H} = \{\varnothing, \{a\}\}$. We define a monotonic map $\gamma : \exp(X) \to \exp(X)$ by $\gamma(\varnothing) = \varnothing, \gamma(\{a\}) = \{b\}, \gamma(\{b\}) = \{b\}$, and $\gamma(X) = X$. Then we have $g_\gamma = \{\varnothing, \{b\}, X\}$ and $\{a\}^*_\gamma = \varnothing$. Therefore, $\{a\} \nsubseteq \varnothing = \{a\}^*_\gamma$.

3.2 Generalized topological spaces induced by monotonic maps and hereditary classes

In 2007, Császár [3] defined a new generalized topology, called a generalized topology via a hereditary class, which contains the old generalized topology. Likewise, we provide the definition of a new generalized topology which contains g_γ by using a monotonic map γ and a hereditary class $\mathcal{H}$. This leads to the concept of generalized topological spaces induced by monotonic maps and hereditary classes.

Definition 3.2.1. Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. We define

$$g^*_{\gamma,\mathcal{H}} = \{M \subseteq X \mid M \cap (X - \gamma(M))^*_\gamma = \varnothing\}.$$

If there is no ambiguity, then $g^*_{\gamma,\mathcal{H}}$ will be denoted by g^*_γ.

Theorem 3.2.2. *g^*_γ is a generalized topology.*

Proof. It is clear that $\varnothing \in g^*_\gamma$. Let $\{M_\alpha\}_{\alpha\in\Lambda}$ be a collection of elements in g^*_γ. Let $x \in \bigcup_{\alpha\in\Lambda} M_\alpha$. Then $x \in M_\alpha$ for some $\alpha \in \Lambda$, i.e., $x \notin (X - \gamma(M_\alpha))^*_\gamma$. There exists $N \in g_\gamma$ such that $x \in N$ and $\gamma(N) \cap (X - \gamma(M_\alpha)) \in \mathcal{H}$. Since γ is monotonic, $\gamma(M_\alpha) \subseteq \gamma(\bigcup_{\alpha\in\Lambda} M_\alpha)$, i.e., $X - \gamma(\bigcup_{\alpha\in\Lambda} M_\alpha) \subseteq X - \gamma(M_\alpha)$. We obtain $\gamma(N) \cap (X - \gamma(\bigcup_{\alpha\in\Lambda} M_\alpha)) \subseteq \gamma(N) \cap (X - \gamma(M_\alpha)) \in \mathcal{H}$. By the definition of a hereditary class, we have $\gamma(N) \cap (X - \gamma(\bigcup_{\alpha\in\Lambda} M_\alpha)) \in \mathcal{H}$. Therefore, $x \notin (X - \gamma(\bigcup_{\alpha\in\Lambda} M_\alpha))^*_\gamma$. It follows that $\bigcup_{\alpha\in\Lambda} M_\alpha \cap (X - \gamma(\bigcup_{\alpha\in\Lambda} M_\alpha))^*_\gamma = \varnothing$. Hence, $\bigcup_{\alpha\in\Lambda} M_\alpha \in g^*_\gamma$. $\square$

By the above theorem, we call g^*_γ a **generalized topology induced by $\boldsymbol{\gamma}$ and $\boldsymbol{\mathcal{H}}$**.

Theorem 3.2.3. $g_\gamma \subseteq g^*_\gamma$.

Proof. Let $G \in g_\gamma$. Suppose that $G \cap (X - \gamma(G))^*_\gamma \neq \varnothing$. Let $x \in G \cap (X - \gamma(G))^*_\gamma$. We have $\varnothing = \gamma(G) \cap (X - \gamma(G)) \notin \mathcal{H}$. It contradicts the fact that $\varnothing \in \mathcal{H}$. $\square$

The following example shows that there is a monotonic map γ which makes $g_\gamma \neq g^*_\gamma$.

Example 3.2.4. Let $X = \{a, b, c\}$. We define a monotonic map $\gamma : \exp(X) \to \exp(X)$ by

$$\begin{aligned} \gamma(\varnothing) &= \varnothing, & \gamma(\{a\}) &= \{a\}, \\ \gamma(\{b\}) &= \{b\}, & \gamma(\{c\}) &= \varnothing, \\ \gamma(\{a,b\}) &= \{a,b\}, & \gamma(\{a,c\}) &= \{a\}, \\ \gamma(\{b,c\}) &= \{b\}, & \gamma(X) &= X. \end{aligned}$$

We have $g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$. Let $\mathcal{H} = \exp(X)$. We have $A^*_\gamma = \varnothing$ for all $A \in \exp(X)$. Thus, $g^*_\gamma = \exp(X)$. We obtain $g_\gamma \subsetneq g^*_\gamma$.

From the above example, we need to find a sufficient condition that makes $g_\gamma = g^*_\gamma$.

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Theorem 3.2.5. *Let γ be a monotonic map on* $\exp(X)$ *and* $\mathcal{H}$ *a hereditary class on* X*. Assume that* $X \notin \mathcal{H} \subseteq g_\gamma$ *and for each* $M \notin g_\gamma$*,* $(M - \gamma(M)) \cap \bigcup_{G \in g_\gamma - \{X\}} G = \varnothing$*. Then* $g_\gamma = g_\gamma^*$*.*

Proof. Let $M \notin g_\gamma$. Then $M \nsubseteq \gamma(M)$. There is $x \in M$ and $x \notin \gamma(M)$. We have $x \notin \bigcup_{G \in g_\gamma - \{X\}} G$. We claim that $x \in (X - \gamma(M))_\gamma^*$. Let G be a γ-open set which contains x. If $x \in \gamma(G) \cap (X - \gamma(M)) \in \mathcal{H} \subseteq g_\gamma$, then $\gamma(G) \cap (X - \gamma(M)) \neq X$ and $x \in \bigcup_{G \in g_\gamma - \{X\}} G$. It is impossible. We conclude that $\gamma(G) \cap (X - \gamma(M)) \notin \mathcal{H}$. Thus, $x \in (X - \gamma(M))_\gamma^*$. Therefore, $M \cap (X - \gamma(M))_\gamma^* \neq \varnothing$. Hence, $M \notin g_\gamma^*$. $\square$

Example 3.2.6. Let $X = \{a, b, c\}$ and $\mathcal{H} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. We define a monotonic map γ in the same way as in Example 3.2.4. We obtain $g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\bigcup_{G \in g_\gamma - \{X\}} G = \{a, b\}$. We observe that for each $M \notin g_\gamma$, $M - \gamma(M) = \{c\}$. By the above theorem, we obtain $g_\gamma^* = g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$.

Next, we give some conditions that make g_γ^* equal to $\exp(X)$.

Proposition 3.2.7. *Let γ be a monotonic map on* $\exp(X)$ *and* $\mathcal{H}$ *a hereditary class on* X*. Assume that* $\mathcal{H} = \exp(X)$ *and for any* $x \in X$*, there exists a* γ*-open set* G *such that* G *contains* x*. Then* $A_\gamma^* = \varnothing$ *for all* $A \in \exp(X)$*. Moreover,* $g_\gamma^* = \exp(X)$*.*

Proof. Let $A \in \exp(X)$. Let $x \in X$. There exists $G \in g_\gamma$ such that $x \in G$. Then $\gamma(G) \cap A \in \exp(X) = \mathcal{H}$. This implies that $x \notin A_\gamma^*$. $\square$

3.2.1 (g_γ^*, g_β^*)-continuous maps

Definition 3.2.8. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is $\boldsymbol{(g_\gamma, g_\beta)}$**-continuous** if $f^{-1}(G)$ is γ-open, for each β-open set G.

From Theorem 3.2.3, we obtain the following theorems.

Theorem 3.2.9. *Let (X, γ) and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that f is a (g_γ, g_β)-continuous function from X to Y. Then f is (g^*_γ, g_β)-continuous.*

Proof. For each $G \in g_\beta$, we have $f^{-1}(G) \in g_\gamma \subseteq g^*_\gamma$ because f is (g_γ, g_β)-continuous. $\square$

Theorem 3.2.10. *Let (X, γ) and (Y, β) be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that f is a (g_γ, g^*_β)-continuous function from X to Y. Then f is (g_γ, g_β)-continuous.*

Proof. For each $G \in g_\beta \subseteq g^*_\beta$, we have $f^{-1}(G) \in g_\gamma$ because f is (g_γ, g^*_β)-continuous. $\square$

Example 3.2.11. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. We define monotonic maps $\gamma : \exp(X) \to \exp(X)$ and $\beta : \exp(Y) \to \exp(Y)$ by

$$\begin{array}{rclrcl}
\gamma(\varnothing) & = & \varnothing, & \beta(\varnothing) & = & \varnothing, \\
\gamma(\{a\}) & = & \{a\}, & \beta(\{x\}) & = & \{x\}, \\
\gamma(\{b\}) & = & \{b\}, & \beta(\{y\}) & = & \{y\}, \\
\gamma(\{c\}) & = & \varnothing, & \beta(\{z\}) & = & \varnothing, \\
\gamma(\{a, b\}) & = & \{a, b\}, & \beta(\{x, y\}) & = & \{x, y\}, \\
\gamma(\{a, c\}) & = & \{a\}, & \beta(\{x, z\}) & = & \{x\}, \\
\gamma(\{b, c\}) & = & \{b\}, & \beta(\{y, z\}) & = & \{y\}, \\
\gamma(X) & = & X, & \beta(Y) & = & Y.
\end{array}$$

It is clear that $g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $g_\beta = \{\varnothing, \{x\}, \{y\}, \{x, y\}, Y\}$. Let $\mathcal{H}_X = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{H}_Y = \exp(Y)$. We conclude that

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$$
\begin{aligned}
\varnothing^*_{\gamma,\mathcal{H}_X} &= \varnothing, & \varnothing^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
\{a\}^*_{\gamma,\mathcal{H}_X} &= \varnothing, & \{x\}^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
\{b\}^*_{\gamma,\mathcal{H}_X} &= \varnothing, & \{y\}^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
\{c\}^*_{\gamma,\mathcal{H}_X} &= \{c\}, & \{z\}^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
\{a,b\}^*_{\gamma,\mathcal{H}_X} &= \varnothing, & \{x,y\}^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
\{a,c\}^*_{\gamma,\mathcal{H}_X} &= \{c\}, & \{x,z\}^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
\{b,c\}^*_{\gamma,\mathcal{H}_X} &= \{c\}, & \{y,z\}^*_{\beta,\mathcal{H}_Y} &= \varnothing,\\
X^*_{\gamma,\mathcal{H}_X} &= \{c\}, & Y^*_{\beta,\mathcal{H}_Y} &= \varnothing.
\end{aligned}
$$

We obtain $g^*_{\gamma,\mathcal{H}_X} = \{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$ and $g^*_{\beta,\mathcal{H}_Y} = \exp(Y)$. Let $f : X \to Y$ be defined by $f(a) = x, f(b) = z$, and $f(c) = z$. Therefore, f is (g_γ, g_β)-continuous. However, f is not $(g^*_{\gamma,\mathcal{H}_X}, g^*_{\beta,\mathcal{H}_Y})$-continuous because $f^{-1}(\{z\}) = \{b,c\} \notin g^*_{\gamma,\mathcal{H}_X}$.

It implies that not every (g_γ, g_β)-continuous function is (g^*_γ, g^*_β)-continuous. This motivates us to the notion of the (γ,β)-continuity and the strongly (γ,β)-continuity.

Definition 3.2.12. Let (X,γ) and (Y,β) be generalized topological spaces. A function f from X to Y is **(γ,β)-continuous** if $f^{-1}(\beta(B)) \subseteq \gamma(f^{-1}(B))$ for all $B \subseteq Y$.

Proposition 3.2.13. *Let (X,γ) and (Y,β) be generalized topological spaces. Assume that $f : X \to Y$ is (γ,β)-continuous. Then f is (g_γ, g_β)-continuous.*

Proof. Let G be β-open. Then $G \subseteq \beta(G)$. We obtain that $f^{-1}(G) \subseteq f^{-1}(\beta(G)) \subseteq \gamma(f^{-1}(G))$. Therefore, $f^{-1}(G)$ is γ-open. $\square$

Example 3.2.14. Let $X = \{a,b,c\}$ and $Y = \{x,y,z\}$. We define $f : X \to Y$ such that $f(a) = x, f(b) = z$, and $f(c) = z$. In the setting of γ and β in Example 3.2.11, we also obtain that f is (γ,β)-continuous but f is not $(g^*_{\gamma,\mathcal{H}_X}, g^*_{\beta,\mathcal{H}_Y})$-continuous.

Definition 3.2.15. Let (X,γ) and (Y,β) be generalized topological spaces. A function f from X to Y is **strongly (γ,β)-continuous** if $f^{-1}(\beta(B)) = \gamma(f^{-1}(B))$ for all $B \subseteq Y$.

Remark 3.2.16. From the above definitions, we have the following relations but the reverse relations may not be true in general.

$$\text{strongly } (\gamma,\beta)\text{-continuous} \Rightarrow (\gamma,\beta)\text{-continuous} \Rightarrow (g_\gamma, g_\beta)\text{-continuous}.$$

The following theorems show that for a given hereditary class on either X or Y, we can find another hereditary class that makes a given strongly (γ,β)- continuous bijective map (g^*_γ, g^*_β)-continuous.

Theorem 3.2.17. *Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X,γ) and $(Y,\beta,\mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a strongly (γ,β)-continuous injection. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is (g^*_γ, g^*_β)-continuous.

Proof. Firstly, we will show that $\mathcal{H}_X$ is a hereditary class on X. Let $A \subseteq f^{-1}(H) \subseteq X$ where $H \in \mathcal{H}_Y$. Then $f(A) \subseteq H$. We have $f(A) \in \mathcal{H}_Y$. Since f is injective, $A = f^{-1}[f(A)] \in \mathcal{H}_X$. It remains to show that f is (g^*_γ, g^*_β)-continuous. Let $G \in g^*_\beta$. Then $G \cap (Y - \beta(G))^*_\beta = \varnothing$, i.e., $G \subseteq Y - (Y - \beta(G))^*_\beta$. It follows that $f^{-1}(G) \subseteq X - f^{-1}[(Y-\beta(G))^*_\beta]$. Let $x \in f^{-1}(G)$. That is, $f(x) \in G$. Then $f(x) \notin (Y-\beta(G))^*_\beta$. There exists $B \in g_\beta$ such that $f(x) \in B$ and $\beta(B) \cap (Y-\beta(G)) \in \mathcal{H}_Y$. This implies that

$$\begin{aligned}\gamma(f^{-1}(B)) \cap [X - \gamma(f^{-1}(G))] &= f^{-1}(\beta(B)) \cap [X - f^{-1}(\beta(G))] \\ &= f^{-1}[\beta(B) \cap (Y - \beta(G))] \in \mathcal{H}_X.\end{aligned}$$

By Proposition 3.2.13, $x \in f^{-1}(B) \in g_\gamma$. Thus, $x \notin (X-\gamma(f^{-1}(G)))^*_\gamma$, i.e., $f^{-1}(G) \cap (X - \gamma(f^{-1}(G)))^*_\gamma = \varnothing$. This leads to the conclusion $f^{-1}(G) \in g^*_\gamma$. Hence, f is (g^*_γ, g^*_β)-continuous. □

The following example shows that the set $\mathcal{H}_X$ of pre-images of all elements in $\mathcal{H}_Y$ may not be a hereditary class if a function f is not injective.

Example 3.2.18. Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$. We define monotonic maps $\gamma : \exp(X) \to \exp(X)$ and $\beta : \exp(Y) \to \exp(Y)$ by

$$\begin{array}{rclrclrcl} \gamma(\varnothing) &=& \varnothing, & \gamma(\{a,b\}) &=& \varnothing, & \beta(\varnothing) &=& \varnothing, \\ \gamma(\{a\}) &=& \varnothing, & \gamma(\{a,c\}) &=& \{a,c\}, & \beta(\{1\}) &=& \varnothing, \\ \gamma(\{b\}) &=& \varnothing, & \gamma(\{b,c\}) &=& \varnothing, & \beta(\{2\}) &=& \varnothing, \\ \gamma(\{c\}) &=& \varnothing, & \gamma(X) &=& X, & \beta(Y) &=& Y. \end{array}$$

Let $f : X \to Y$ be defined by $f(a) = 1$ and $f(b) = 2 = f(c)$. We obtain f is strongly (γ, β)-continuous but f is not injective. We define $\mathcal{H}_Y = \{\varnothing, \{2\}\}$ to be a hereditary class on Y. By the construction, we have $\mathcal{H}_X = \{\varnothing, \{b, c\}\}$. Therefore, $\mathcal{H}_X$ is not a hereditary class on X.

Remark 3.2.19. Let $f : (X, \gamma) \to (Y, \beta, \mathcal{H}_Y)$ be an injective function. There is a hereditary class $\mathcal{H}_X$ on X such that we have the following implications.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma, g_\beta)\text{-continuous} \\ \Downarrow & & & & \\ (g_\gamma^*, g_\beta^*)\text{-continuous.} & & & & \end{array}$$

Theorem 3.2.20. *Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a strongly (γ, β)-continuous bijection. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (g_γ^*, g_β^*)-continuous.

Proof. Firstly, we prove that $\mathcal{H}_Y$ is a hereditary class on Y. Let $A \subseteq f(H) \subseteq Y$ where $H \in \mathcal{H}_X$. Then $f^{-1}(A) \subseteq f^{-1}[f(H)] = H \in \mathcal{H}_X$ because f is injective. We obtain $f^{-1}(A) \in \mathcal{H}_X$. Since f is surjective, $A = f[f^{-1}(A)]$. Thus, $A \in \mathcal{H}_Y$. Next, we show that f is (g_γ^*, g_β^*)-continuous. Let $G \in g_\beta^*$. Then $G \cap (Y - \beta(G))_\beta^* = \varnothing$, i.e., $G \subseteq Y - (Y - \beta(G))_\beta^*$. It follows that $f^{-1}(G) \subseteq X - f^{-1}[(Y - \beta(G))_\beta^*]$. Let

$x \in f^{-1}(G)$. That is, $f(x) \in G$. Then $f(x) \notin (Y - \beta(G))^*_\beta$. There exists $B \in g_\beta$ such that $f(x) \in B$ and $\beta(B) \cap (Y - \beta(G)) \in \mathcal{H}_Y$. Thus, $\beta(B) \cap (Y - \beta(G)) = f(A)$ for some $A \in \mathcal{H}_X$. Note that

$$\begin{aligned} \gamma(f^{-1}(B)) \cap [X - \gamma(f^{-1}(G))] &= f^{-1}(\beta(B)) \cap [X - f^{-1}(\beta(G))] \\ &= f^{-1}[\beta(B) \cap (Y - \beta(G))] \\ &= f^{-1}[f(A)] \\ &= A \in \mathcal{H}_X. \end{aligned}$$

By Proposition 3.2.13, $x \in f^{-1}(B) \in g_\gamma$. Thus, $x \notin (X - \gamma(f^{-1}(G)))^*_\gamma$, i.e., $f^{-1}(G) \cap (X - \gamma(f^{-1}(G)))^*_\gamma = \varnothing$. This leads to the conclusion $f^{-1}(G) \in g^*_\gamma$. Hence, f is (g^*_γ, g^*_β)-continuous. $\square$

Remark 3.2.21. Let $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ be a bijective function. There is a hereditary class $\mathcal{H}_Y$ on Y such that we have the following implications.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma, g_\beta)\text{-continuous} \\ \Downarrow & & & & \\ (g^*_\gamma, g^*_\beta)\text{-continuous.} & & & & \end{array}$$

Likewise, we study the composition of continuous functions in the following theorems.

Theorem 3.2.22. *Let $(X, \gamma), (Y, \beta)$ and (Z, α) be generalized topological spaces. Assume that $f : X \to Y$ is (g_γ, g_β)-continuous and $g : Y \to Z$ is (g_β, g_α)-continuous. Then $g \circ f : X \to Z$ is (g_γ, g_α)-continuous.*

Proof. Let $G \in g_\alpha$. Since g is (g_β, g_α)-continuous, we obtain $g^{-1}(G) \in g_\beta$. Similarly, $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G)) \in g_\gamma$ because f is (g_γ, g_β)-continuous. $\square$

Theorem 3.2.23. *Let $(X, \gamma), (Y, \beta)$ and (Z, α) be generalized topological spaces. Assume that $f : X \to Y$ is (γ, β)-continuous and $g : Y \to Z$ is (β, α)-continuous. Then $g \circ f : X \to Z$ is (γ, α)-continuous.*

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Proof. Let $G \subseteq Z$. We observe that

$$\begin{aligned}(g \circ f)^{-1}(\alpha(G)) &= f^{-1}[g^{-1}(\alpha(G))] \\ &\subseteq f^{-1}[\beta(g^{-1}(G))] \\ &\subseteq \gamma[f^{-1}(g^{-1}(G))] \\ &= \gamma[(g \circ f)^{-1}(G)].\end{aligned}$$

Hence, $g \circ f : X \to Z$ is (γ, α)-continuous. □

Theorem 3.2.24. *Let $(X, \gamma), (Y, \beta)$ and (Z, α) be generalized topological spaces. Assume that $f : X \to Y$ is strongly (γ, β)-continuous and $g : Y \to Z$ is strongly (β, α)-continuous. Then $g \circ f : X \to Z$ is strongly (γ, α)-continuous.*

Proof. The proof is similar to the proof of Theorem 3.2.23 by replacing $\subseteq$ with $=$. That is, for each $G \subseteq Z$, we obtain

$$\begin{aligned}(g \circ f)^{-1}(\alpha(G)) &= f^{-1}[g^{-1}(\alpha(G))] \\ &= f^{-1}[\beta(g^{-1}(G))] \\ &= \gamma[f^{-1}(g^{-1}(G))] \\ &= \gamma[(g \circ f)^{-1}(G)].\end{aligned}$$

Hence, $g \circ f : X \to Z$ is (γ, α)-continuous. □

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Corollary 3.2.25. *Let $(X, \gamma), (Y, \beta)$ and (Z, α) be generalized topological spaces. Assume that $f : X \to Y$ is a strongly (γ, β)-continuous bijective function and $g : Y \to Z$ is a strongly (β, α)-continuous bijective function. The following statements are satisfied.*

(1) If $\mathcal{H}_X$ is a hereditary class on X, then there are the hereditary classes $\mathcal{H}_Y$ on Y and $\mathcal{H}_Z$ on Z such that $g \circ f$ is (g^*_γ, g^*_α)-continuous.

(2) If $\mathcal{H}_Y$ is a hereditary class on Y, then there are the hereditary classes $\mathcal{H}_X$ on X and $\mathcal{H}_Z$ on Z such that $g \circ f$ is (g^*_γ, g^*_α)-continuous.

(3) If $\mathcal{H}_Z$ is a hereditary class on Z, then there are the hereditary classes $\mathcal{H}_X$ on X and $\mathcal{H}_Y$ on Y such that $g \circ f$ is (g^*_γ, g^*_α)-continuous.

3.2.2 (g^*_γ, g^*_β)-open maps

Definition 3.2.26. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is $\boldsymbol{(g_\gamma, g_\beta)}$**-open** if $f(G)$ is β-open, for each γ-open set G.

From Theorem 3.2.3, it is easy to prove the following theorems.

Theorem 3.2.27. *Let (X, γ) and (Y, β) be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that f is a (g_γ, g_β)-open function from X to Y. Then f is (g_γ, g^*_β)-open.*

Proof. For each $G \in g_\gamma$, we have $f(G) \in g_\beta \subseteq g^*_\beta$ because f is (g_γ, g_β)-open. □

Theorem 3.2.28. *Let (X, γ) and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that f is a (g^*_γ, g_β)-open function from X to Y. Then f is (g_γ, g_β)-open.*

Proof. For each $G \in g_\gamma \subseteq g^*_\gamma$, we have $f(G) \in g_\beta$ because f is (g^*_γ, g_β)-open. □

Definition 3.2.29. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is $\boldsymbol{(\gamma, \beta)}$**-open** if $f(\gamma(A)) \subseteq \beta(f(A))$ for all $A \subseteq X$.

Definition 3.2.30. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is **strongly** $\boldsymbol{(\gamma, \beta)}$**-open** if $f(\gamma(A)) = \beta(f(A))$ for all $A \subseteq X$.

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Proposition 3.2.31. *Let (X, γ) and (Y, β) be generalized topological spaces. Assume that $f : X \to Y$ is (γ, β)-open. Then f is (g_γ, g_β)-open.*

Proof. Let G be γ-open. Then $G \subseteq \gamma(G)$. We obtain that $f(G) \subseteq f(\gamma(G)) \subseteq \beta(f(G))$. Therefore, $f(G)$ is β-open. □

Remark 3.2.32. From the above definitions, we have the following implications but the reverse may not be true in general.

$$\text{strongly } (\gamma, \beta)\text{-open} \Rightarrow (\gamma, \beta)\text{-open} \Rightarrow (g_\gamma, g_\beta)\text{-open}.$$

Proposition 3.2.33. *Let* (X, γ) *and* (Y, β) *be generalized topological spaces and* f *a bijective function from* X *onto* Y. *Then* f *is* (g_γ, g_β)*-open if and only if* f^{-1} *is* (g_β, g_γ)*-continuous.*

Proof. It follows from Theorem 2.2.3. □

Proposition 3.2.34. *Let* (X, γ) *and* (Y, β) *be generalized topological spaces and* f *a bijective function from* X *onto* Y. *Then* f *is* (γ, β)*-open if and only if* f^{-1} *is* (β, γ)*-continuous.*

Proof.

$$\begin{aligned} f \text{ is } (\gamma, \beta)\text{-open} &\Leftrightarrow f(\gamma(A)) \subseteq \beta(f(A)) \text{ for all } A \subseteq X \\ &\Leftrightarrow (f^{-1})^{-1}(\gamma(A)) \subseteq \beta((f^{-1})^{-1}(A)) \text{ for all } A \subseteq X \\ &\Leftrightarrow f^{-1} \text{ is } (\beta, \gamma)\text{-continuous.} \end{aligned}$$

□

Proposition 3.2.35. *Let* (X, γ) *and* (Y, β) *be generalized topological spaces and* f *a bijective function from* X *onto* Y. *Then* f *is strongly* (γ, β)*-open if and only if* f^{-1} *is strongly* (β, γ)*-continuous.*

Proof. The proof is similar to the proof of Proposition 3.2.34 by replacing $\subseteq$ with $=$. That is,

$$\begin{aligned} f \text{ is strongly } (\gamma, \beta)\text{-open} &\Leftrightarrow f(\gamma(A)) = \beta(f(A)) \text{ for all } A \subseteq X \\ &\Leftrightarrow (f^{-1})^{-1}(\gamma(A)) = \beta((f^{-1})^{-1}(A)) \text{ for all } A \subseteq X \\ &\Leftrightarrow f^{-1} \text{ is strongly } (\beta, \gamma)\text{-continuous.} \end{aligned}$$

□

Similarly, we show that a given hereditary class on either X or Y under some conditions on the function f, we can find a hereditary class $\mathcal{H}$ on the another space that makes f is open on the generalized topological space induced by a given monotonic map and the hereditary class $\mathcal{H}$.

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Corollary 3.2.36. *Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a strongly (γ, β)-open bijection. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is (g^*_γ, g^*_β)-open.

Proof. Similar to Theorem 3.2.17, it suffices to show that f is (g^*_γ, g^*_β)-open.

$$\begin{aligned} f \text{ is strongly } (\gamma,\beta)\text{-open} &\Rightarrow f^{-1} \text{ is strongly } (\beta,\gamma)\text{-continuous} && \text{(By Proposition 3.2.35)} \\ &\Rightarrow f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Theorem 3.2.20)} \\ &\Rightarrow f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

Remark 3.2.37. Let $f : (X, \gamma) \to (Y, \beta, \mathcal{H}_Y)$ be an open bijection. By the above corollary, there is a hereditary class $\mathcal{H}_X$ on X such that we have the following implications.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\ \Downarrow & & & & \\ (g^*_\gamma, g^*_\beta)\text{-open.} & & & & \end{array}$$

Corollary 3.2.38. *Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a strongly (γ, β)-open bijection. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (g^*_γ, g^*_β)-open.

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Proof. Similar to Theorem 3.2.20, it suffices to show that f is (g^*_γ, g^*_β)-open.

$$\begin{aligned} f \text{ is strongly } (\gamma,\beta)\text{-open} &\Rightarrow f^{-1} \text{ is strongly } (\beta,\gamma)\text{-continuous} && \text{(By Proposition 3.2.35)} \\ &\Rightarrow f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Theorem 3.2.17)} \\ &\Rightarrow f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

Remark 3.2.39. Let $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ be an open bijection. By the above corollary, there is a hereditary class $\mathcal{H}_Y$ on Y such that we have the following implications.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\ \Downarrow & & & & \\ (g^*_\gamma, g^*_\beta)\text{-open.} & & & & \end{array}$$

In the following definitions, we can define homeomorphisms via monotonic maps.

Definition 3.2.40. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is a $\boldsymbol{(g_\gamma, g_\beta)}$**-homeomorphism** if f is a (g_γ, g_β)-continuous bijection and f^{-1} is (g_β, g_γ)-continuous.

Definition 3.2.41. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is a $\boldsymbol{(\gamma, \beta)}$**-homeomorphism** if f is a (γ, β)-continuous bijection and f^{-1} is (β, γ)-continuous.

Definition 3.2.42. Let (X, γ) and (Y, β) be generalized topological spaces. A function f from X to Y is a **strongly** $\boldsymbol{(\gamma, \beta)}$**-homeomorphism** if f is a strongly (γ, β)-continuous bijection and f^{-1} is strongly (β, γ)-continuous.

Remark 3.2.43. Similarly, we have some implications of homeomorphisms but the reverse relations may not be true in general.

$$\begin{array}{ccc} \text{a strongly } (\gamma,\beta)\text{-homeomorphism} & \Longrightarrow & \text{a } (\gamma,\beta)\text{-homeomorphism} \\ & & \Downarrow \\ & & \text{a } (g_\gamma, g_\beta)\text{-homeomorphism.} \end{array}$$

Corollary 3.2.44. *Let (X, γ) and (Y, β) be generalized topological spaces and f a bijection from X onto Y. Then f is a (g_γ, g_β)-homeomorphism if and only if f is (g_γ, g_β)-continuous and (g_γ, g_β)-open.*

Proof. Apply Corollary 2.2.5. □

Corollary 3.2.45. *Let (X, γ) and (Y, β) be generalized topological spaces and f a bijection from X onto Y. Then f is a (γ, β)-homeomorphism if and only if f is (γ, β)-continuous and (γ, β)-open.*

Proof. Apply Proposition 3.2.34. □

Corollary 3.2.46. *Let (X, γ) and (Y, β) be generalized topological spaces and f a bijection from X onto Y. Then f is a strongly (γ, β)-homeomorphism if and only if f is strongly (γ, β)-continuous and strongly (γ, β)-open.*

Proof. Apply Proposition 3.2.35. □

Now, it is easy to prove the following corollaries for the homeomorphism function.

Corollary 3.2.47. *Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a strongly (γ, β)-homeomorphism. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is a (g^*_γ, g^*_β)-homeomorphism.

Proof. Similar to Theorem 3.2.17, it suffices to show that f is a (g^*_γ, g^*_β)-homeomorphism. By Corollary 3.2.46, it implies that f is strongly (γ, β)-continuous and strongly (γ, β)-open. By Theorem 3.2.17 and Corollary 3.2.36, f is (g^*_γ, g^*_β)-continuous and (g^*_γ, g^*_β)-open. It follows from Corollary 3.2.44 that f is a (g^*_γ, g^*_β)-homeomorphism. □

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Corollary 3.2.48. *Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a strongly (γ, β)-homeomorphism. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is a (g^*_γ, g^*_β)-homeomorphism.

Proof. Consider f^{-1} and apply Corollary 3.2.47. $\square$

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CHAPTER IV

GENERALIZED TOPOLOGICAL SPACES INDUCED BY MONOTONIC MAPS HAVING PARTICULAR PROPERTIES AND HEREDITARY CLASSES

In this chapter, we study a notion of a generalized topological space induced by a monotonic map and a hereditary class when the monotonic map has some particular properties. In 1997, Császár [1] introduced and studied the properties of monotonic maps in the following definitions.

Definition 4.0.49. Let γ be a monotonic map. A map γ is called **enlarging** if $A \subseteq \gamma(A)$ for all $A \subseteq X$.

Definition 4.0.50. Let γ be a monotonic map. A map γ is called **restricting** if $\gamma(A) \subseteq A$ for all $A \subseteq X$.

Remark 4.0.51. If γ is enlarging, then $g_\gamma = \exp(X)$.

From the above remark, we will study a generalized topology $g^*_{\gamma,\mathcal{H}}$ when γ is restricting.

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4.1 Restricting maps

Remark 4.1.1. Let γ be a restricting map on $\exp(X)$.

1. A is γ-open if and only if $A = \gamma(A)$.

2. We obtain that $i_\gamma = \iota_\gamma$ and $c_\gamma = cl_\gamma$.

Following [3], Császár defined a new generalized topological space via given generalized topology and hereditary class. It is well-known that g_γ is a generalized topology induced by the monotonic map γ. Likewise, we also obtain some results via a monotonic map in the following definitions and theorems.

Definition 4.1.2. Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. For each $A \subseteq X$, we define

$$A^*_{g_\gamma,\mathcal{H}} = \{x \in X \mid x \in M \in g_\gamma \text{ implies } M \cap A \notin \mathcal{H}\}.$$

In particular, $A^*_{g_\gamma,\{\varnothing\}} = c_\gamma(A)$. If there is no ambiguity, then $A^*_{g_\gamma,\mathcal{H}}$ will be denoted by $A^*_{g_\gamma}$.

Proposition 4.1.3. *Let γ be a restricting map on $\exp(X)$ and $(X, \gamma, \mathcal{H})$ a generalized topological space together with a hereditary class $\mathcal{H}$. For each $A \subseteq X$, we have $A^*_\gamma = A^*_{g_\gamma}$.*

Proof.

$$\begin{aligned} x \in A^*_\gamma &\Leftrightarrow \gamma(M) \cap A \notin \mathcal{H} \text{ for all } M \in g_\gamma \text{ containing } x \\ &\Leftrightarrow M \cap A \notin \mathcal{H} \text{ for all } M \in g_\gamma \text{ containing } x \\ &\Leftrightarrow x \in A^*_{g_\gamma}. \end{aligned}$$

□

By using Propositions 2.1.9 and 4.1.3, we easily obtain the following theorem.

Theorem 4.1.4. *Let γ be a restricting map on $\exp(X)$ and $(X, \gamma, \mathcal{H})$ a generalized topological space together with a hereditary class $\mathcal{H}$. Let $A, B, M \subseteq X$.*

(1) $A \subseteq B$ implies $A^*_\gamma \subseteq B^*_\gamma$;

(2) $A^*_\gamma \subseteq c_\gamma(A)$;

(3) If $M \in \mu$ and $M \cap A \in \mathcal{H}$, then $M \cap A^*_\gamma = \varnothing$;

(4) A^*_γ is γ-closed;

(5) A is γ-closed implies $A^*_\gamma \subseteq A$;

(6) $(A^*_\gamma)^*_\gamma \subseteq A^*_\gamma$ when $A \subseteq X$;

(7) $X = X^*_\gamma$ *if and only if* $g_\gamma \cap \mathcal{H} = \{\varnothing\}$.

Remark 4.1.5. Following Proposition 4.1.3, the set A^*_γ and $A^*_{g_\gamma}$ can be denoted by A^* if γ is restricting.

We have known that g_γ is a generalized topology. Following [3], we use g_γ to define a new generalized topology via a hereditary class.

Definition 4.1.6. Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. We define

$$g^*_{g_\gamma,\mathcal{H}} = \{M \subseteq X \mid c^*(X - M) = X - M\}.$$

If there is no ambiguity, then $g^*_{g_\gamma,\mathcal{H}}$ will be denoted by $g^*_{g_\gamma}$.

Remark 4.1.7. By the definition of c^* in Chapter II, we have

$$g^*_{g_\gamma,\mathcal{H}} = \{M \subseteq X \mid M \cap (X - M)^*_{g_\gamma} = \varnothing\}.$$

Theorem 4.1.8 ([3])**.** *$g^*_{g_\gamma}$ is a generalized topology and $g_\gamma \subseteq g^*_{g_\gamma}$.*

Theorem 4.1.9. *Let γ be a restricting map on* $\exp(X)$ *and $(X, \gamma, \mathcal{H})$ a generalized topological space together with a hereditary class $\mathcal{H}$. Then $g^*_\gamma \subseteq g^*_{g_\gamma}$.*

Proof. It follows from $M \cap (X - M)^* = M \cap (X - M)^* \subseteq M \cap (X - \gamma(M))^*$ for all $M \subseteq X$. $\square$

Remark 4.1.10. $g_\gamma \subseteq g^*_\gamma \subseteq g^*_{g_\gamma}$ if γ is a restricting map.

Theorem 4.1.11 ([3])**.** *Let $(X, \gamma, \mathcal{H})$ be a generalized topological space together with a hereditary class $\mathcal{H}$. The set*

$$\{M - H \subseteq X \mid M \in g_\gamma \text{ and } H \in \mathcal{H}\}$$

constitutes a base for $g^*_{g_\gamma}$.

Corollary 4.1.12. *For each $G \in g_\gamma^*$, $G = \bigcup_{\alpha\in\Lambda}\{M_\alpha - H_\alpha \mid M_\alpha \in g_\gamma$ and $H_\alpha \in \mathcal{H}\}$ where Λ is an index set.*

Proof. It follows from Theorems 4.1.9 and 4.1.11. $\square$

The following example shows that there is a monotonic map γ which makes $g_\gamma^* \neq g_{g_\gamma}^*$.

Example 4.1.13. Let $X = \{a, b, c\}$ and $\mathcal{H} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. In the setting of a restricting map γ in Example 3.2.11, we get $g_\gamma^* = g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$. We can see that $\{c\} = X - \{a, b\} \in g_{g_\gamma}^*$ but $\{c\} \notin g_\gamma^*$. Hence, $g_\gamma^* \subsetneq g_{g_\gamma}^*$.

Proposition 4.1.14. *Let γ be a restricting map on $\exp(X)$ and $(X, \gamma, \mathcal{H})$ a generalized topological space together with a hereditary class $\mathcal{H}$. We obtain $g_\gamma^* = g_{g_\gamma}^*$ if and only if $M - H \in g_\gamma^*$ for all $M \in g_\gamma$ and $H \in \mathcal{H}$.*

Proof. It follows from Theorem 4.1.11. $\square$

Remark 4.1.15. Let γ be a restricting map on $\exp(X)$ and $(X, \gamma, \mathcal{H})$ a generalized topological space together with a hereditary class $\mathcal{H}$. If $M - H$ is γ-open for all $M \in g_\gamma$ and $H \in \mathcal{H}$, then $g_\gamma = g_\gamma^* = g_{g_\gamma}^*$.

Example 4.1.16. Let $X = \{a, b, c\}$. Define $\gamma : \exp(X) \to \exp(X)$ by

$$\begin{array}{rclrcl}
\gamma(\varnothing) & = & \varnothing, & \gamma(\{a,b\}) & = & \{a,b\}, \\
\gamma(\{a\}) & = & \{a\}, & \gamma(\{a,c\}) & = & \{a\}, \\
\gamma(\{b\}) & = & \{b\}, & \gamma(\{b,c\}) & = & \{b\}, \\
\gamma(\{c\}) & = & \varnothing, & \gamma(X) & = & \{a,b\}.
\end{array}$$

We can see that γ is restricting on $\exp(X)$ and $g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Let $\mathcal{H} = \{\varnothing, \{a\}\}$ be a hereditary class on X. We consider $\{M - H \mid M \in g_\gamma$ and $H \in \mathcal{H}\} = g_\gamma$. By the above proposition, we get $g_\gamma = g_\gamma^* = g_{g_\gamma}^*$.

So, we easily obtain that $g_\gamma^* = g_{g_\gamma}^*$ when γ is the trivial map. Next, we introduce the property of the hereditary class $\mathcal{H}$ that makes $g_\gamma^* = g_{g_\gamma}^*$.

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Proposition 4.1.17. *Let γ be a restricting map on* $\exp(X)$ *and* $(X, \gamma, \mathcal{H})$ *a generalized topological space together with a hereditary class $\mathcal{H}$. Assume that $M \cap H = \varnothing$ for all $M \in g_\gamma$ and $H \in \mathcal{H}$. Then $g^*_\gamma = g^*_{g_\gamma}$.*

Proof. By the assumption, we obtain $M - H = M \in g_\gamma$ for all $M \in g_\gamma$ and $H \in \mathcal{H}$. It follows from Remark 4.1.15. Hence, $g_\gamma = g^*_\gamma = g^*_{g_\gamma}$. □

4.1.1 (g^*_γ, g^*_β)-continuous maps

Let (X, γ) and (Y, β) be generalized topological spaces. Let $f : X \to Y$ be a function. Similar to the previous section, we can prove that f is (g^*_γ, g^*_β)-continuous when f is (γ, β)-continuous and γ is restricting.

Proposition 4.1.18. *Let γ be a restricting map and β an enlarging map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let (X, γ) and (Y, β) be generalized topological spaces. If $f : X \to Y$ is (γ, β)-continuous, then f is strongly (γ, β)-continuous.*

Proof. It follows from $\gamma(f^{-1}(B)) \subseteq f^{-1}(B) \subseteq f^{-1}(\beta(B))$ for all $B \subseteq X$. □

Remark 4.1.19. Let γ be a restricting map on $\exp(X)$ and β an enlarging map $\exp(Y)$. By Remark 3.2.16, we have the following relations.

$$\text{strongly } (\gamma, \beta)\text{-continuous} \Leftrightarrow (\gamma, \beta)\text{-continuous} \Rightarrow (g_\gamma, g_\beta)\text{-continuous}.$$

Proposition 4.1.20. *Let γ be an enlarging map and β a restricting map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Then $f : X \to Y$ is (γ, β)-continuous. In particular, $f : X \to Y$ is (g_γ, g_β)-continuous.*

Proof. Observe that $f^{-1}(\beta(B)) \subseteq f^{-1}(B) \subseteq \gamma(f^{-1}(B))$ for all $B \subseteq X$. □

By Proposition 4.1.18, we obtain the following Corollaries.

Corollary 4.1.21. *Let γ be a restricting map and β an enlarging map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces* *and* $\mathcal{H}_Y$ *a hereditary class on* Y*. Assume that* $f : X \to Y$ *is a* (γ, β)*-continuous injection. Then for the hereditary class on* X *defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f *is* (g^*_γ, g^*_β)*-continuous.*

Proof. Apply Proposition 4.1.18 and Theorem 3.2.17. □

Corollary 4.1.22. *Let* γ *be a restricting map and* β *an enlarging map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let* $(X, \gamma, \mathcal{H}_X)$ *and* (Y, β) *be generalized topological spaces and* $\mathcal{H}_X$ *a hereditary class on* X*. Assume that* $f : X \to Y$ *is a* (γ, β)*-continuous bijection. Then for the hereditary class on* Y *defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f *is* (g^*_γ, g^*_β)*-continuous.*

Proof. Apply Proposition 4.1.18 and Theorem 3.2.20. □

We observed that $g_\gamma = \exp(X)$ if γ is enlarging. Thus, $g^*_\gamma = \exp(X)$.

Remark 4.1.23. Let γ be an enlarging map and β a monotonic map on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces. Then $f : X \to Y$ is (g^*_γ, g^*_β)-continuous.

The following theorems show that we can only assume f is (γ, β)-continuous where γ is restricting in the above corollaries. So, we obtain the same results that f is (g^*_γ, g^*_β)-continuous.

Theorem 4.1.24. *Let* γ *be a restricting map and* β *a monotonic map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let* (X, γ) *and* $(Y, \beta, \mathcal{H}_Y)$ *be generalized topological spaces and* $\mathcal{H}_Y$ *a hereditary class on* Y*. Assume that* $f : X \to Y$ *is a* (γ, β)*-continuous injection. Then for the hereditary class on* X *defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

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the function f is (g^*_γ, g^*_β)-continuous.

Proof. Similar to proof Theorem 3.2.17, it remains to show that f is (g^*_γ, g^*_β)-continuous. Let $G \in g^*_\beta$. Then $G \cap (Y - \beta(G))^* = \varnothing$, i.e. $G \subseteq Y - (Y - \beta(G))^*$. It follows that $f^{-1}(G) \subseteq X - f^{-1}[(Y - \beta(G))^*]$. Let $x \in f^{-1}(G)$. Then $f(x) \notin (Y - \beta(G))^*$. There exists $B \in g_\beta$ such that $f(x) \in B$ and $\beta(B) \cap (Y - \beta(G)) \in \mathcal{H}_Y$. This implies that

$$\begin{aligned}\gamma(f^{-1}(B)) \cap [X - \gamma(f^{-1}(G))] \subseteq\ & f^{-1}(\beta(B)) \cap [X - f^{-1}(\beta(G))] \\ =\ & f^{-1}[\beta(B) \cap (Y - \beta(G))] \in \mathcal{H}_X.\end{aligned}$$

By Proposition 3.2.13, $x \in f^{-1}(B) \in g_\gamma$. Thus, $x \notin (X - \gamma(f^{-1}(G)))^*$, i.e. $f^{-1}(G) \cap (X - \gamma(f^{-1}(G)))^* = \varnothing$. This leads to $f^{-1}(G) \in g^*_\gamma$. Hence, f is (g^*_γ, g^*_β)-continuous. □

Remark 4.1.25. Let γ be a restricting map and β a monotonic map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma) \to (Y, \beta, \mathcal{H}_Y)$ is an injective function. By Theorems 2.2.6 and 4.1.24, there is a hereditary class $\mathcal{H}_X$ on X such that we have the following relationships.

$$\begin{tikzcd}[arrows=Rightarrow] \text{strongly } (\gamma,\beta)\text{-continuous} \arrow[r] & (\gamma,\beta)\text{-continuous} \arrow[r] \arrow[d] & (g_\gamma, g_\beta)\text{-continuous} \arrow[d] \\ & (g^*_\gamma, g^*_\beta)\text{-continuous} & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous.} \end{tikzcd}$$

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Theorem 4.1.26. *Let γ be a restricting map and β a monotonic map on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a (γ, β)-continuous bijection. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (g^*_γ, g^*_β)-continuous.

Proof. Similar to proof Theorem 3.2.20, we will show that f is (g^*_γ, g^*_β)-continuous. Let $G \in g^*_\beta$. Then $G \cap (Y - \beta(G))^* = \varnothing$, i.e. $G \subseteq Y - (Y - \beta(G))^*$. It follows that $f^{-1}(G) \subseteq X - f^{-1}[(Y - \beta(G))^*]$. Let $x \in f^{-1}(G)$. Then $f(x) \notin (Y - \beta(G))^*$. There exists $B \in g_\beta$ such that $f(x) \in B$ and $\beta(B) \cap (Y - \beta(G)) \in \mathcal{H}_Y$. Thus, $\beta(B) \cap (Y - \beta(G)) \in \mathcal{H}_Y = f(A)$ for some $A \in \mathcal{H}_X$.

$$
\begin{aligned}
\gamma(f^{-1}(B)) \cap [X - \gamma(f^{-1}(G))] &\subseteq f^{-1}(\beta(B)) \cap [X - f^{-1}(\beta(G))] \\
&= f^{-1}[\beta(B) \cap (Y - \beta(G))] \\
&= f^{-1}[f(A)] \\
&= A \in \mathcal{H}_X.
\end{aligned}
$$

By Proposition 3.2.13, $x \in f^{-1}(B) \in g_\gamma$. Thus, $x \notin (X - \gamma(f^{-1}(G)))^*$, i.e. $f^{-1}(G) \cap (X - \gamma(f^{-1}(G)))^* = \varnothing$. This leads to $f^{-1}(G) \in g^*_\gamma$. Hence, f is (g^*_γ, g^*_β)-continuous. $\square$

Remark 4.1.27. Let γ be a restricting map and β a monotonic map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ is a bijection. It follows from Theorems 2.2.7 and 4.1.26 that there is a hereditary class $\mathcal{H}_Y$ on Y such that we have the following relationships.

$$
\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma, g_\beta)\text{-continuous} \\
 & & \Downarrow & & \Downarrow \\
 & & (g^*_\gamma, g^*_\beta)\text{-continuous} & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous.}
\end{array}
$$

Corollary 4.1.28. *Let γ, β and α be monotonic maps such that γ and β are restricting. Let $(X, \gamma), (Y, \beta)$ and (Z, α) be generalized topological spaces. Assume that $f : X \to Y$ is a (γ, β)-continuous bijective function and $g : Y \to Z$ is a (β, α)-continuous bijective function. The following statements are satisfied.*

(1) If $\mathcal{H}_X$ is a hereditary class on X, then there are the hereditary classes $\mathcal{H}_Y$ on Y and $\mathcal{H}_Z$ on Z such that $g \circ f$ is (g^*_γ, g^*_α)-continuous.

(2) If $\mathcal{H}_Y$ is a hereditary class on Y, then there are the hereditary classes $\mathcal{H}_X$ on X and $\mathcal{H}_Z$ on Z such that $g \circ f$ is (g^*_γ, g^*_α)-continuous.

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(3) If $\mathcal{H}_Z$ is a hereditary class on Z, then there are the hereditary classes $\mathcal{H}_X$ on X and $\mathcal{H}_Y$ on Y such that $g \circ f$ is (g^*_γ, g^*_α)-continuous.

Proof. Apply Theorem 4.1.24 and Theorem 4.1.26. □

4.1.2 (g^*_γ, g^*_β)-open maps

Proposition 4.1.29. *Let γ be an enlarging map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and (Y, β) be generalized topological spaces. If $f : X \to Y$ is (γ, β)-open, then f is strongly (γ, β)-open.*

Proof. It follows from $\beta(f(A)) \subseteq f(A) \subseteq f(\gamma(A))$ for all $A \subseteq X$. □

Proposition 4.1.30. *Let γ be a restricting map and β an enlarging map on $\exp(X)$ and $\exp(Y)$, respectively. Then $f : X \to Y$ is (γ, β)-open. In particular, $f : X \to Y$ is (g_γ, g_β)-open.*

Proof. Observe that $f(\gamma(A)) \subseteq f(A) \subseteq \beta(f(A))$ for all $A \subseteq X$. □

Corollary 4.1.31. *Let γ be a monotonic map and β a restricting on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a (γ, β)-open bijection. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is (g^*_γ, g^*_β)-open.

Proof. Similar to Theorem 3.2.20, it suffices to show that f is (g^*_γ, g^*_β)-open.

$$\begin{aligned} f \text{ is } (\gamma,\beta)\text{-open} \Rightarrow\ & f^{-1} \text{ is } (\beta,\gamma)\text{-continuous} && \text{(By Proposition 3.2.34)} \\ \Rightarrow\ & f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Theorem 4.1.26)} \\ \Rightarrow\ & f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

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Remark 4.1.32. Let γ be a monotonic map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma) \to (Y, \beta, \mathcal{H}_Y)$ is a bijection. By Theorem 2.2.8 and the above corollary, there is a hereditary class $\mathcal{H}_X$ on X such that we have the following relationships.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\ & & \Downarrow & & \Downarrow \\ & & (g^*_\gamma, g^*_\beta)\text{-open} & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} \end{array}$$

Corollary 4.1.33. *Let γ be a monotonic map and β a restricting on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a (γ, β)-open bijection. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (g^*_γ, g^*_β)-open.

Proof. Similar to Theorem 3.2.17, it suffices to show that f is (g^*_γ, g^*_β)-open.

$$\begin{aligned} f \text{ is } (\gamma,\beta)\text{-open} \Rightarrow&\ f^{-1} \text{ is } (\beta,\gamma)\text{-continuous} && \text{(By Proposition 3.2.34)} \\ \Rightarrow&\ f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Theorem 4.1.24)} \\ \Rightarrow&\ f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

Remark 4.1.34. Let γ be a monotonic map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ is a bijection. By Theorem 2.2.9 and the above corollary, there is a hereditary class $\mathcal{H}_Y$ on Y such that we have the following relationships.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\ & & \Downarrow & & \Downarrow \\ & & (g^*_\gamma, g^*_\beta)\text{-open} & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} \end{array}$$

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Remark 4.1.35. Let γ be a monotonic map and β an enlarging map on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces. Then $f : X \to Y$ is (g^*_γ, g^*_β)-open.

Corollary 4.1.36. *Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a (γ, β)-homeomorphism. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is a (g^*_γ, g^*_β)-homeomorphism.

Proof. Similar to Theorem 3.2.17, it suffices to show that f is a (g^*_γ, g^*_β)-homeomorphism. By Corollary 3.2.45, it implies that f is (γ, β)-continuous and (γ, β)-open. By Theorem 4.1.24 and Corollary 4.1.31, f is (g^*_γ, g^*_β)-continuous and (g^*_γ, g^*_β)-open. It follows from Corollary 3.2.44 that f is a (g^*_γ, g^*_β)-homeomorphism. □

Corollary 4.1.37. *Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a (γ, β)-homeomorphism. Then for the hereditary class on X defined by*

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$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is a (g^*_γ, g^*_β)-homeomorphism.

Proof. Consider f^{-1} and apply Corollary 4.1.36. □

Remark 4.1.38. Let γ and β be enlarging maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces. Assume that $f : X \to Y$ is bijective. Then $f : X \to Y$ is a (g^*_γ, g^*_β)-homeomorphism.

4.1.3 Relations between (g^*_γ, g^*_β)-continuity and $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuity

Let γ and β are restricting maps. In this subsection, we discuss on the relations between (g^*_γ, g^*_β)-continuity and $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuity. Firstly, we begin with the following example.

Example 4.1.39. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. We define monotonic maps $\gamma : \exp(X) \to \exp(X)$ and $\beta : \exp(Y) \to \exp(Y)$ by

$$\gamma(A) = \begin{cases} A & \text{if } A = \{b\}, \{c\}, \{b, c\}, X, \\ \{b\} & \text{if } A = \{a, b\}, \\ \{c\} & \text{if } A = \{a, c\}, \\ \varnothing & \text{otherwise.} \end{cases}$$

$$\beta(B) = \begin{cases} B & \text{if } B = \{1\}, \{1, 2\}, \{1, 3, 4\}, Y, \\ \{1\} & \text{if } B = \{1, 3\}, \{1, 4\}, \\ \{1, 2\} & \text{if } B = \{1, 2, 3\}, \{1, 2, 4\} \\ \varnothing & \text{otherwise.} \end{cases}$$

It is clear that γ and β are restricting. We obtain that $g_\gamma = \{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$ and $g_\beta = \{\varnothing, \{1\}, \{1, 2\}, \{1, 3, 4\}, Y\}$. Let $\mathcal{H}_X = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, c\}\}$ and $\mathcal{H}_Y = \{\varnothing, \{1\}, \{2\}, \{4\}, \{2, 4\}\}$. Assume that $f : X \to Y$ is defined by $f(a) = 2, f(b) = 1$ and $f(c) = 4$. Then f is injective. By Theorem 4.1.11, we conclude that

$$g^*_{g_\gamma} = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\} \text{ and}$$

$$g^*_{g_\beta} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, Y\}.$$

Since $f^{-1}(\{2\}) = \{a\} \notin g^*_{g_\gamma}$, f is not $(g^*_{g_\gamma}, g^*_{g_\beta})$- continuous. By using Remark 4.1.10, we can compute

$$g^*_\gamma = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} \text{ and}$$

$$g^*_\beta = \{\varnothing, \{1\}, \{1, 2\}, \{1, 3, 4\}, Y\} = g_\beta.$$

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Hence, we can see that f is (g^*_γ, g^*_β)-continuous.

This implies that not every (g^*_γ, g^*_β)-continuous injection f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous when we assume that $\mathcal{H}_Y$ is a hereditary class on Y and $\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\}$ is a hereditary class on X. So, we can ask what conditions make an injective (g^*_γ, g^*_β)-continuous function to be $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous. This leads to the following theorems.

Theorem 4.1.40. *Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ an ideal on Y. Assume that $f : X \to Y$ is a (g^*_γ, g^*_β)-continuous injection. Then for the ideal on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous.

Proof. In first step, we will show that $\mathcal{H}_X$ is an ideal. It is enough to show that $f^{-1}(A) \cup f^{-1}(B) \in \mathcal{H}_X$ for all $A, B \in \mathcal{H}_Y$. Let $A, B \in \mathcal{H}_Y$. Then $A \cup B \in \mathcal{H}_Y$ because $\mathcal{H}_Y$ is an ideal on Y. So, $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \in \mathcal{H}_X$. Thus, $\mathcal{H}_X$ is an ideal on X. Next, we prove that f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous. Let $G \in g^*_{g_\beta}$. By Theorem 4.1.11, we have $G = \bigcup_{b \in \Lambda'}(M_b - H_b)$ where $M_b \in g_\beta$ and $H_b \in \mathcal{H}_Y$. Since $g_\beta \subseteq g^*_\beta$ and f is (g^*_γ, g^*_β)-continuous, $f^{-1}(M_b) \in g^*_\gamma \subseteq g^*_{g_\gamma}$ for all $b \in \Lambda'$. Then for each $b \in \Lambda'$, $f^{-1}(M_b) = \bigcup_{a \in \Lambda}(N_{ba} - H_{ba})$ where $N_{ba} \in g_\gamma$ and $H_{ba} \in \mathcal{H}_X$. Consider

$$\begin{aligned} f^{-1}(G) &= \textstyle\bigcup_{b \in \Lambda'}(f^{-1}(M_b) - f^{-1}(H_b)) \\ &= \textstyle\bigcup_{b \in \Lambda'}(\bigcup_{a \in \Lambda}(N_{ba} - H_{ba}) - f^{-1}(H_b)) \\ &= \textstyle\bigcup_{b \in \Lambda'} \bigcup_{a \in \Lambda}(N_{ba} - (H_{ba} \cup f^{-1}(H_b))). \end{aligned}$$

Since $\mathcal{H}_X$ is an ideal on X, $H_{ba} \cup f^{-1}(H_b) \in \mathcal{H}_X$ and so $N_{ba} - (H_{ba} \cup f^{-1}(H_b))$ belongs to the basis for $g^*_{g_\gamma}$ for all $b \in \Lambda'$ and $a \in \Lambda$. Thus, $f^{-1}(G) \in g^*_{g_\gamma}$. Hence, f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous. $\square$

Remark 4.1.41. Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f:(X,\gamma)\to(Y,\beta,\mathcal{H}_Y)$ is an injective function and $\mathcal{H}_Y$ is an ideal on Y. By Theorems 2.2.6 and 4.1.40, there is an ideal $\mathcal{H}_X$ on X such that we have the following relationships.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma,g_\beta)\text{-continuous} \\ & & \Downarrow & & \Downarrow \\ & & (g^*_\gamma,g^*_\beta)\text{-continuous} & \Longrightarrow & (g^*_{g_\gamma},g^*_{g_\beta})\text{-continuous.} \end{array}$$

Theorem 4.1.42. *Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X,\gamma,\mathcal{H}_X)$ and (Y,β) be generalized topological spaces and $\mathcal{H}_X$ an ideal on X. Assume that $f:X\to Y$ is a (g^*_γ,g^*_β)-continuous bijection. Then for the ideal on Y defined by*

$$\mathcal{H}_Y=\{f(H)\mid H\in\mathcal{H}_X\},$$

the function f is $(g^*_{g_\gamma},g^*_{g_\beta})$-continuous.

Proof. First, we will show that $\mathcal{H}_Y$ is an ideal. It is enough to show that $f(A)\cup f(B)\in\mathcal{H}_Y$ for all $A,B\in\mathcal{H}_X$. Let $A,B\in\mathcal{H}_X$. Then $A\cup B\in\mathcal{H}_X$ because $\mathcal{H}_X$ is an ideal on X. So, $f(A)\cup f(B)=f(A\cup B)\in\mathcal{H}_Y$. Thus, $\mathcal{H}_Y$ is an ideal on Y. Next, we prove that f is $(g^*_{g_\gamma},g^*_{g_\beta})$-continuous. Let $G\in g^*_{g_\beta}$. By Theorem 4.1.11, we have $G=\bigcup_{b\in\Lambda'}(M_b-f(H_b))$ where $M_b\in g_\beta$ and $H_b\in\mathcal{H}_X$. Since $g_\beta\subseteq g^*_\beta$ and f is (g^*_γ,g^*_β)-continuous, $f^{-1}(M_b)\in g^*_\gamma\subseteq g^*_{g_\gamma}$ for all $b\in\Lambda'$. Then for each $b\in\Lambda'$, $f^{-1}(M_b)=\bigcup_{a\in\Lambda}(N_{ba}-H_{ba})$ where $N_{ba}\in g_\gamma$ and $H_{ba}\in\mathcal{H}_X$. Consider

$$\begin{aligned} f^{-1}(G) &= \textstyle\bigcup_{b\in\Lambda'}(f^{-1}(M_b)-f^{-1}(f(H_b))) \\ &= \textstyle\bigcup_{b\in\Lambda'}(\bigcup_{a\in\Lambda}(N_{ba}-H_{ba})-H_b) \\ &= \textstyle\bigcup_{b\in\Lambda'}\bigcup_{a\in\Lambda}(N_{ba}-(H_{ba}\cup H_b)). \end{aligned}$$

Since $\mathcal{H}_X$ is an ideal on X, $H_{ba}\cup H_b\in\mathcal{H}_X$ and so $N_{ba}-(H_{ba}\cup H_b)$ belongs to the basis for $g^*_{g_\gamma}$ for all $b\in\Lambda'$ and $a\in\Lambda$. Thus, $f^{-1}(G)\in g^*_{g_\gamma}$. Hence, f is $(g^*_{g_\gamma},g^*_{g_\beta})$-continuous. $\square$

Remark 4.1.43. Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ is a bijection and $\mathcal{H}_X$ is an ideal on X. By Theorems 2.2.7 and 4.1.42, there is an ideal $\mathcal{H}_Y$ on Y such that we have the following relationships.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma, g_\beta)\text{-continuous} \\ & & \Downarrow & & \Downarrow \\ & & (g^*_\gamma, g^*_\beta)\text{-continuous} & \Longrightarrow & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous.} \end{array}$$

Moreover, we obtain the same results on open maps and homeomorphisms.

Corollary 4.1.44. *Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ an ideal on Y. Assume that $f : X \to Y$ is a (g^*_γ, g^*_β)-open bijection. Then for the ideal on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-open.

Proof. Similar to Theorem 4.1.40, it suffices to show that f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-open.

$$\begin{aligned} f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open} &\Rightarrow f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Proposition 3.2.33)} \\ &\Rightarrow f^{-1} \text{ is } (g^*_{g_\beta}, g^*_{g_\gamma})\text{-continuous} && \text{(By Theorem 4.1.42)} \\ &\Rightarrow f \text{ is } (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

Remark 4.1.45. Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma) \to (Y, \beta, \mathcal{H}_Y)$ is a bijection and $\mathcal{H}_Y$ is an ideal on Y. By Corollaries 2.2.8 and 4.1.44, there is an ideal $\mathcal{H}_X$ on X such that we have the following relations.

$$\begin{array}{ccccc} \text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\ & & \Downarrow & & \Downarrow \\ & & (g^*_\gamma, g^*_\beta)\text{-open} & \Longrightarrow & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} \end{array}$$

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Corollary 4.1.46. *Let γ and β be restricting maps on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ is an ideal on X. Assume that $f : X \to Y$ is a (g^*_γ, g^*_β)-open bijection. Then for the ideal on X defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-open.

Proof. Similar to Theorem 4.1.42, it suffices to show that f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-open.

$$\begin{aligned} f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open} &\Rightarrow f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Proposition 3.2.33)} \\ &\Rightarrow f^{-1} \text{ is } (g^*_{g_\beta}, g^*_{g_\gamma})\text{-continuous} && \text{(By Theorem 4.1.40)} \\ &\Rightarrow f \text{ is } (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

Remark 4.1.47. Let γ and β be restricting maps on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ is a bijection and $\mathcal{H}_X$ is an ideal on X. By Theorems 2.2.9 and 4.1.46, there is an ideal $\mathcal{H}_Y$ on Y such that we have the following relations.

$$\begin{array}{ccccc} \text{strongly } (\gamma, \beta)\text{-open} & \Longrightarrow & (\gamma, \beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\ & & \Downarrow & & \Downarrow \\ & & (g^*_\gamma, g^*_\beta)\text{-open} & \Longrightarrow & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} \end{array}$$

Corollary 4.1.48. *Let γ and β be restricting maps on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ an ideal on Y. Assume that $f : X \to Y$ is a (g^*_γ, g^*_β)-homeomorphism. Then for the ideal on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is a $(g^*_{g_\gamma}, g^*_{g_\beta})$-homeomorphism.

Proof. Similar to Theorem 4.1.40, it suffices to show that f is a $(g^*_{g_\gamma}, g^*_{g_\beta})$-homeomorphism. By Corollary 3.2.44, it implies that f is (g^*_γ, g^*_β)-continuous and (g^*_γ, g^*_β)-open. By

Theorem 4.1.40 and Corollary 4.1.44, f is $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous and $(g^*_{g_\gamma}, g^*_{g_\beta})$-open. It follows from Corollary 3.2.44 that f is a $(g^*_{g_\gamma}, g^*_{g_\beta})$-homeomorphism. □

Corollary 4.1.49. *Let γ and β be restricting maps on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ an ideal on X. Assume that $f : X \to Y$ is a (g^*_γ, g^*_β)-homeomorphism. Then for the ideal on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is a $(g^*_{g_\gamma}, g^*_{g_\beta})$-homeomorphism.

Proof. Consider f^{-1} and apply Corollary 4.1.48. □

4.2 Strong maps

Definition 4.2.1. Let γ be a monotonic map. A map γ is called **idempotent** if $\gamma(A) = \gamma(\gamma(A))$ for all $A \subseteq X$.

Definition 4.2.2. Let γ be a monotonic map. A map γ is called **strong** if γ is restricting and idempotent.

4.2.1 (g^*_γ, g^*_β)-continuous maps

Proposition 4.2.3. *Let γ be a monotonic map and β a strong map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let (X, γ) and (Y, β) be generalized topological spaces. If $f : X \to Y$ is (g_γ, g_β)-continuous, then f is (γ, β)-continuous.*

Proof. Let $B \subseteq Y$. Then $\beta(B) = \beta(\beta(B))$, i.e. $\beta(B)$ is β-open. By the assumption, we obtain $f^{-1}(\beta(B))$ is γ-open. It follows that $f^{-1}(\beta(B)) \subseteq \gamma[f^{-1}(\beta(B))]$. Thus, $f^{-1}(\beta(B)) \subseteq \gamma[f^{-1}(\beta(B))] \subseteq \gamma[f^{-1}(B)]$ because β is restricting. This is true for all $B \subseteq Y$. Hence, f is (γ, β)-continuous. □

Remark 4.2.4. Let γ be a monotonic map on $\exp(X)$ and β a strong map on $\exp(Y)$. By Remark 3.2.16, we have the following relations.

strongly (γ,β)-continuous $\Rightarrow$ (γ,β)-continuous $\Leftrightarrow$ (g_γ, g_β)-continuous.

Let (X,γ) and (Y,β) be generalized topological spaces. Let $f : X \to Y$ be a function. Similarly, the following corollaries show that f is (g^*_γ, g^*_β)-continuous when f is (g_γ, g_β)-continuous, γ is restricting, and β is strong.

Corollary 4.2.5. *Let γ be a restricting map and β a strong map on* $\exp(X)$ *and* $\exp(Y)$*, respectively. Let (X,γ) and $(Y,\beta,\mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a (g_γ, g_β)-continuous injection. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is (g^*_γ, g^*_β)-continuous.

Proof. Apply Proposition 4.2.3 and Theorem 4.1.24. □

Remark 4.2.6. Let γ be a restricting map and β a strong map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X,\gamma) \to (Y,\beta,\mathcal{H}_Y)$ is an injective function and $\mathcal{H}_Y$ is an ideal on Y. By Theorem 4.1.40 and Corollary 4.2.5, there is an ideal $\mathcal{H}_X$ on X such that we have the following implications.

$$\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & & \\
 & & \Updownarrow & & \\
 & & (g_\gamma, g_\beta)\text{-continuous} & \Longrightarrow & (g^*_\gamma, g^*_\beta)\text{-continuous} \\
 & & & & \Downarrow \\
 & & & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous.}
\end{array}$$

By the above corollary, the following example shows that a (g_γ, g_β)-continuous injection may not be (g^*_γ, g^*_β)-continuous if a monotonic map β is not strong.

Example 4.2.7. Let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. We define monotonic maps $\gamma : \exp(X) \to \exp(X)$ and $\beta : \exp(Y) \to \exp(Y)$ by

$$\begin{aligned}
\gamma(\varnothing) &= \varnothing, & \beta(\varnothing) &= \varnothing,\\
\gamma(\{a\}) &= \{a\}, & \beta(\{x\}) &= \varnothing,\\
\gamma(\{b\}) &= \{b\}, & \beta(\{y\}) &= \varnothing,\\
\gamma(\{c\}) &= \varnothing, & \beta(\{z\}) &= \varnothing,\\
\gamma(\{a,b\}) &= \{a,b\}, & \beta(\{x,y\}) &= \{x\},\\
\gamma(\{a,c\}) &= \{a\}, & \beta(\{x,z\}) &= \{z\},\\
\gamma(\{b,c\}) &= \{b\}, & \beta(\{y,z\}) &= \{y\},\\
\gamma(X) &= X, & \beta(Y) &= Y.
\end{aligned}$$

We obtain that $\beta(\beta(\{x, y\})) = \beta(\{x\}) = \varnothing \neq \{x\} = \beta(\{x, y\})$. Thus, β is not strong. It is clear that $g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $g_\beta = \{\varnothing, Y\}$. Let $f : X \to Y$ be defined by $f(a) = x, f(b) = y$ and $f(c) = z$. Therefore, f is a (g_γ, g_β)-continuous injection. Let $\mathcal{H}_Y = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}$ and $\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\} = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ be hereditary classes on $\exp(Y)$ and $\exp(X)$, respectively. Next, we observe that

$$\{x, z\} \cap (Y - \beta(\{x, z\}))^*_\beta = \{x, z\} \cap (\{x, y\})^*_\beta = \{x, z\} \cap \varnothing = \varnothing.$$

So, we have $g^*_{\gamma,\mathcal{H}_X} = g_\gamma = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\{x, z\} \in g^*_{\beta,\mathcal{H}_Y}$. Hence, f is not $(g^*_{\gamma,\mathcal{H}_X}, g^*_{\beta,\mathcal{H}_Y})$-continuous because $f^{-1}(\{x, z\}) = \{a, c\} \notin g^*_{\gamma,\mathcal{H}_X}$.

Corollary 4.2.8. *Let γ be a restricting map and β a strong map on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a (g_γ, g_β)-continuous bijection. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (g^*_γ, g^*_β)-continuous.

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Proof. Apply Proposition 4.2.3 and Theorem 4.1.26. □

Remark 4.2.9. Let γ be a restricting map and β a strong map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f:(X,\gamma,\mathcal{H}_X)\to(Y,\beta)$ is a bijection and $\mathcal{H}_X$ is an ideal on X. By Theorem 4.1.42 and Corollary 4.2.8, there is an ideal $\mathcal{H}_Y$ on Y such that we have the following implications.

$$\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & & \\
 & & \Updownarrow & & \\
 & & (g_\gamma,g_\beta)\text{-continuous} & \Longrightarrow & (g^*_\gamma,g^*_\beta)\text{-continuous} \\
 & & & & \Downarrow \\
 & & & & (g^*_{g_\gamma},g^*_{g_\beta})\text{-continuous.}
\end{array}$$

Next, it is easy to obtain the following corollary for the composition.

Corollary 4.2.10. *Let γ,β and α be monotonic maps such that γ is restricting and β,α are strong. Let $(X,\gamma),(Y,\beta)$ and (Z,α) be generalized topological spaces. Assume that $f:X\to Y$ is a (g_γ,g_β)-continuous bijective function and $g:Y\to Z$ is a (g_β,g_α)-continuous bijective function. The following statements are satisfied.*

(1) If $\mathcal{H}_X$ is a hereditary class on X, then there are the hereditary classes $\mathcal{H}_Y$ on Y and $\mathcal{H}_Z$ on Z such that $g\circ f$ is (g^*_γ,g^*_α)-continuous.

(2) If $\mathcal{H}_Y$ is a hereditary class on Y, then there are the hereditary classes $\mathcal{H}_X$ on X and $\mathcal{H}_Z$ on Z such that $g\circ f$ is (g^*_γ,g^*_α)-continuous.

(3) If $\mathcal{H}_Z$ is a hereditary class on Z, then there are the hereditary classes $\mathcal{H}_X$ on X and $\mathcal{H}_Y$ on Y such that $g\circ f$ is (g^*_γ,g^*_α)-continuous.

Proof. Apply Theorems 4.2.5 and 4.2.8. □

4.2.2 (g^*_γ,g^*_β)-open maps

Similarly, we obtain the same results on open maps and homeomorphisms.

Proposition 4.2.11. *Let γ be a strong map and β a monotonic on $\exp(X)$ and $\exp(Y)$, respectively. Let (X,γ) and (Y,β) be generalized topological spaces. If $f : X \to Y$ is (g_γ, g_β)-open, then f is (γ,β)-open.*

Proof. Let $A \subseteq X$. Since γ is strong, $\gamma(A)$ is γ-open. It follows that $f(\gamma(A))$ is β-open. Thus, $f(\gamma(A)) \subseteq \beta(f(\gamma(A))) \subseteq \beta(f(A))$. Hence, f is (γ,β)-open. □

Corollary 4.2.12. *Let γ be a strong map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Let (X,γ) and $(Y,\beta,\mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a (g_γ, g_β)-open bijection. Then for the hereditary class on X defined by*

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is (g^*_γ, g^*_β)-open.

Proof. Similar to Theorem 3.2.20, it suffices to show that f is (g^*_γ, g^*_β)-open.

$$\begin{aligned}
f \text{ is } (g_\gamma, g_\beta)\text{-open} &\Rightarrow \ f^{-1} \text{ is } (g_\beta, g_\gamma)\text{-continuous} && \text{(By Proposition 3.2.33)}\\
&\Rightarrow \ f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Corollary 4.2.8)}\\
&\Rightarrow \ f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open.} && \text{(By Proposition 3.2.33)}
\end{aligned}$$

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□

Remark 4.2.13. Let γ be a strong map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X,\gamma) \to (Y,\beta,\mathcal{H}_Y)$ is a bijection and $\mathcal{H}_Y$ is an ideal on Y. By Corollaries 4.1.44 and 4.2.12, there is an ideal $\mathcal{H}_X$ on X such that we have the following relationships.

$$\begin{tikzcd}
\text{strongly } (\gamma,\beta)\text{-open} \arrow[r, Rightarrow] & (\gamma,\beta)\text{-open} \arrow[d, Leftrightarrow] & \\
& (g_\gamma, g_\beta)\text{-open} \arrow[r, Rightarrow] & (g^*_\gamma, g^*_\beta)\text{-open} \arrow[d, Rightarrow] \\
& & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.}
\end{tikzcd}$$

Corollary 4.2.14. *Let γ be a strong map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a (g_γ, g_β)-open bijection. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is (g^*_γ, g^*_β)-open.

Proof. Similar to Theorem 3.2.17, it suffices to show that f is (g^*_γ, g^*_β)-open.

$$\begin{aligned} f \text{ is } (g_\gamma, g_\beta)\text{-open} \Rightarrow\ & f^{-1} \text{ is } (g_\beta, g_\gamma)\text{-continuous} && \text{(By Proposition 3.2.33)} \\ \Rightarrow\ & f^{-1} \text{ is } (g^*_\beta, g^*_\gamma)\text{-continuous} && \text{(By Corollary 4.2.5)} \\ \Rightarrow\ & f \text{ is } (g^*_\gamma, g^*_\beta)\text{-open.} && \text{(By Proposition 3.2.33)} \end{aligned}$$

□

Remark 4.2.15. Let γ be a strong map and β a restricting map on $\exp(X)$ and $\exp(Y)$, respectively. Assume that $f : (X, \gamma, \mathcal{H}_X) \to (Y, \beta)$ is a bijection and $\mathcal{H}_X$ is an ideal on X. By Corollaries 4.1.46 and 4.2.14, there is an ideal $\mathcal{H}_Y$ on Y such that we have the following relationships.

$$\begin{tikzcd} \text{strongly } (\gamma, \beta)\text{-open} \arrow[r, Rightarrow] & (\gamma, \beta)\text{-open} \arrow[d, Leftrightarrow] & \\ & (g_\gamma, g_\beta)\text{-open} \arrow[r, Rightarrow] & (g^*_\gamma, g^*_\beta)\text{-open} \arrow[d, Rightarrow] \\ & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open.} \end{tikzcd}$$

We finish this section with the following corollaries on homeomorphism maps.

Corollary 4.2.16. *Let γ and β be strong maps on $\exp(X)$ and $\exp(Y)$, respectively. Let (X, γ) and $(Y, \beta, \mathcal{H}_Y)$ be generalized topological spaces and $\mathcal{H}_Y$ a hereditary class on Y. Assume that $f : X \to Y$ is a (g_γ, g_β)-homeomorphism. Then for the*

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hereditary class on X defined by

$$\mathcal{H}_X = \{f^{-1}(H) \mid H \in \mathcal{H}_Y\},$$

the function f is a (g^*_γ, g^*_β)-homeomorphism.

Proof. Similar to Theorem 3.2.17, it suffices to show that f is a (g^*_γ, g^*_β)-homeomorphism. By Corollary 3.2.44, it implies that f is (g_γ, g_β)-continuous and (g_γ, g_β)-open. By Corollary 4.2.5 and 4.2.12, f is (g^*_γ, g^*_β)-continuous and (g^*_γ, g^*_β)-open. It follows from Corollary 3.2.44 that f is a (g^*_γ, g^*_β)-homeomorphism. □

Corollary 4.2.17. *Let γ and β be strong maps on $\exp(X)$ and $\exp(Y)$, respectively. Let $(X, \gamma, \mathcal{H}_X)$ and (Y, β) be generalized topological spaces and $\mathcal{H}_X$ a hereditary class on X. Assume that $f : X \to Y$ is a (g_γ, g_β)-homeomorphism. Then for the hereditary class on Y defined by*

$$\mathcal{H}_Y = \{f(H) \mid H \in \mathcal{H}_X\},$$

the function f is a (g^*_γ, g^*_β)-homeomorphism.

Proof. Consider f^{-1} and apply Corollary 4.2.16. □

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CHAPTER V
CONCLUSION AND DISCUSSION

Let X and Y be nonempty sets. Let γ and β be monotonic maps on $\exp(X)$ and $\exp(Y)$, respectively, and $\mathcal{H}$ a hereditary class on X. In this thesis, we present the concepts of a generalized topology induced by a monotonic map γ and a hereditary class $\mathcal{H}$, denoted by $g^*_{\gamma,\mathcal{H}}$. Also, we show that $g^*_{\gamma,\mathcal{H}}$ contains the collection of all γ-open sets, denoted by g_γ. That is, $g_\gamma \subseteq g^*_{\gamma,\mathcal{H}}$. After that, we study a notion of the continuity on generalized topological spaces induced by monotonic maps and hereditary classes. We prove that the strongly (γ,β)-continuity between two generalized topological spaces (X,γ) and (Y,β) implies the continuity on generalized topological spaces induced by monotonic maps and hereditary classes in various situations. That is, given a hereditary class on either X or Y, we can find a hereditary class on the another space that makes a given strongly (γ,β)-continuous bijective map to be (g^*_γ, g^*_β)-continuous. By applying these theorem, we obtain the same results on open maps. So, we get the following implications on the bijective function:

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$$\begin{tikzcd}
\text{strongly } (\gamma,\beta)\text{-continuous} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\gamma,\beta)\text{-continuous} \arrow[r, Rightarrow] & (g_\gamma, g_\beta)\text{-continuous} \\
(g^*_\gamma, g^*_\beta)\text{-continuous} & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\text{strongly } (\gamma,\beta)\text{-open} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & (\gamma,\beta)\text{-open} \arrow[r, Rightarrow] & (g_\gamma, g_\beta)\text{-open} \\
(g^*_\gamma, g^*_\beta)\text{-open} & &
\end{tikzcd}$$

Let $g^*_{g_\gamma,\mathcal{H}}$ denote the generalized topology induced by a generalized topology g_γ and a hereditary class $\mathcal{H}$. In Chapter IV, we consider the properties of the monotonic maps in two cases.

1. On restricting maps

Let γ and β be restricting maps. We conclude that $g_\gamma \subseteq g^*_{\gamma,\mathcal{H}} \subseteq g^*_{g_\gamma,\mathcal{H}}$. We also obtain the continuity on the generalized topological spaces induced by monotonic maps and hereditary classes when we reduce some conditions by replacing the strongly (γ,β)-continuity by the (γ,β)-continuity. That is, given a hereditary class on either X or Y, we can construct a hereditary class on the another space that makes a given (γ,β)-continuous bijective map to be (g^*_γ, g^*_β)-continuous. Similarly, we obtain results concerning open maps. We have the following relationships on the bijective function:

$$\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma, g_\beta)\text{-continuous} \\
 & & \Downarrow & & \Downarrow \\
 & & (g^*_\gamma, g^*_\beta)\text{-continuous} & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous}
\end{array}$$

and

$$\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\
 & & \Downarrow & & \Downarrow \\
 & & (g^*_\gamma, g^*_\beta)\text{-open} & & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open}
\end{array}$$

Moreover, we prove that given an ideal on either X or Y, we can construct an ideal on the another space that makes a given (g^*_γ, g^*_β)-continuous bijective map to be $(g^*_{g_\gamma}, g^*_{g_\beta})$-continuous. Thus, we have the following relationships on the bijective function:

$$\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-continuous} & \Longrightarrow & (\gamma,\beta)\text{-continuous} & \Longrightarrow & (g_\gamma, g_\beta)\text{-continuous} \\
 & & \Downarrow & & \Downarrow \\
 & & (g^*_\gamma, g^*_\beta)\text{-continuous} & \Longrightarrow & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous}
\end{array}$$

and

$$\begin{array}{ccccc}
\text{strongly } (\gamma,\beta)\text{-open} & \Longrightarrow & (\gamma,\beta)\text{-open} & \Longrightarrow & (g_\gamma, g_\beta)\text{-open} \\
 & & \Downarrow & & \Downarrow \\
 & & (g^*_\gamma, g^*_\beta)\text{-open} & \Longrightarrow & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open}
\end{array}$$

2. On strong maps

Let γ and β be strong maps. We let a hereditary class on either X or Y. Similarly, we can define a hereditary class on the another space that makes a given (g_γ, g_β)-continuous bijective map to be (g^*_γ, g^*_β)-continuous. Similar to 1., since γ and β are restricting, we also get the following implications on the bijective function:

$$\begin{tikzcd}
\text{strongly } (\gamma,\beta)\text{-continuous} \arrow[r, Rightarrow] & (\gamma,\beta)\text{-continuous} \arrow[d, Leftrightarrow] & \\
& (g_\gamma, g_\beta)\text{-continuous} \arrow[r, Rightarrow] & (g^*_\gamma, g^*_\beta)\text{-continuous} \arrow[d, Rightarrow] \\
& & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-continuous}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\text{strongly } (\gamma,\beta)\text{-open} \arrow[r, Rightarrow] & (\gamma,\beta)\text{-open} \arrow[d, Leftrightarrow] & \\
& (g_\gamma, g_\beta)\text{-open} \arrow[r, Rightarrow] & (g^*_\gamma, g^*_\beta)\text{-open} \arrow[d, Rightarrow] \\
& & (g^*_{g_\gamma}, g^*_{g_\beta})\text{-open}
\end{tikzcd}$$

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Therefore, in this thesis, we define and study generalized topological spaces induced by monotonic maps and hereditary classes. After that, we study the continuity on generalized topological spaces induced by monotonic maps and hereditary classes. Finally, we obtain some results of generalized topological spaces induced by monotonic maps having particular properties and hereditary classes.

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