เมทริกซ์ฮอดจ์ลาปลาเชียนแบบบรรทัดฐานและสเปกตรัม

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NORMALIZED HODGE LAPLACIAN MATRIX AND ITS SPECTRUM

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ให้ X เป็นซิมพลิเชียลคอมเพล็กซ์ ให้ $\partial_k : C_k \to C_{k-1}$ เป็นการส่งขอบเขต และ B_k เป็นเมทริกซ์ตัวแทนของ ∂_k เมทริกซ์ฮอดจ์ลาปลาเชียนที่ k บนซิมพลิเชียลคอมเพล็กซ์นิยามโดย $L_k = B_{k+1}B_{k+1}^T + B_k^T B_k$ เป็นนัยทั่วไปของเมทริกซ์ลาปลาเชียน L บนกราฟ ในงานวิจัยนี้ ผู้วิจัยศึกษาเมทริกซ์ฮอดจ์ลาปลาเชียนดังกล่าว จากนั้นจึงวางนัยทั่วไปบนเมทริกซ์ลาปลาเชียนแบบบรรทัดฐาน $\mathcal{L}$ บนกราฟให้เป็นเมทริกซ์ฮอดจ์ลาปลาเชียนแบบบรรทัดฐานที่ k (แทนด้วยสัญลักษณ์ $\mathcal{L}_k$) บนซิมพลิเชียลคอมเพล็กซ์ นั่นคือ $\mathcal{L}_0 = \mathcal{L}$ เมทริกซ์ที่นิยามขึ้นมีสมบัติเป็นเมทริกซ์ฮอดจ์ลาปลาเชียน ทำให้ได้สมบัติบางประการของเมทริกซ์ที่เป็นประโยชน์และค่าลักษณะเฉพาะตัวที่น้อยที่สุดของเมทริกซ์นี้สามารถบอกได้ว่าฮอมอโลยีและฮอมอโลยีร่วมของซิมพลิเชียลคอมเพล็กซ์เป็นศูนย์หรือไม่ ผู้วิจัยแสดงค่าลักษณะเฉพาะของเมทริกซ์ $\mathcal{L}_k$ สำหรับบางกรณีเฉพาะ และศึกษาความสัมพันธ์ระหว่างค่าลักษณะเฉพาะของเมทริกซ์กับผลรวมเวดจ์ของซิมเพล็กซ์ นอกจากนี้ ผู้วิจัยยังนำเมทริกซ์ที่นิยามขึ้นไปประยุกต์เกี่ยวกับแนวเดินแบบสุ่มบนซิมพลิเชียลคอมเพล็กซ์

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Let X be a simplicial complex, $\partial_k : C_k \to C_{k-1}$ a boundary map on X and B_k a matrix representation of ∂_k. A Hodge k-Laplacian matrix on simplicial complexes is defined by $L_k = B_{k+1}B_{k+1}^T + B_k^T B_k$ which is a generalization of a Laplacian matrix L on graphs. In this work, we study a Hodge k-Laplacian matrix and then generalize a normalized Laplacian matrix $\mathcal{L}$ on graphs to a normalized Hodge k-Laplacian matrix $\mathcal{L}_k$ (i.e. $\mathcal{L}_0 = \mathcal{L}$) on simplicial complexes. This matrix is also a Hodge Laplacian matrix and this fact leads some useful properties. We also obtain that the smallest eigenvalue of a normalized Hodge k-Laplacian indicates whether the homology (or cohomology) on a given simplicial complex is trivial. We demonstrate eigenvalues of $\mathcal{L}_k$ for some special cases and study a relation between its eigenvalues and q-wedge sum of simplices. We finally apply this matrix for random walks on simplicial complexes.

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CONTENTS

CHAPTER I
INTRODUCTION

There are many matrices in graph theory which represent some structures of graphs. Two of them, which are widely studied, are called *Laplacian matrix* $L = (l_{ij})$ which is defined by

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

and *normalized Laplacian matrix* $\mathcal{L} = (l'_{ij})$ which is defined by

$$l'_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\sqrt{d_i d_j}} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Some properties of graphs can be shown by eigenvalues of these matrices even though we know only their approximations. This is a powerful tool for applying in quantum physics, chemical quantum, and others. To study in this topic, spectral graph theory [4] introduced by F.R.K. Chung is recommended.

Every graph can be viewed as a 1-dimensional structure of the object called a *simplicial complex.* For a given simplicial complex, the Hodge Laplacian, also known as the Laplace–de Rham operator, is defined by

$$\Delta_k = \partial_{k+1}\partial_{k+1}^* + \partial_k^*\partial_k = \partial_{k+1}\delta_{k+1} + \delta_k\partial_k = \delta_{k+1}^*\delta_{k+1} + \delta_k\delta_k^*,$$

where $\partial_n : C_n \to C_{n-1}$ is a boundary map and δ_n is its dual map. This operator was first introduced to study some materials on manifolds, for more details, we refer the readers to study the topic *Hodge theory*. Its matrix representation whose eigenvalues can indicate some properties of simplicial complexes is defined for their k-dimensional structure namely *Hodge k-Laplacian matrix*. This matrix is defined by

$$L_k = B_{k+1}B_{k+1}^T + B_k^T B_k,$$

where B_k is a matrix representation of a boundary map $\partial_k : C_k \to C_{k-1}$. However, the Hodge Laplacian operator or Hodge Laplacian matrix is quite difficult to study especially for who is not familiar with tools in algebraic topology and differentiable manifolds.

In 2015, L.H. Lim [11] simplified the definition of Hodge k-Laplacian matrix to be under a condition in linear algebra. Let A be an $m \times n$ real matrix and B an $n \times p$ real matrix such that $AB = 0$. We call the matrix

$$A^*A + BB^*$$

a *Hodge Laplacian matrix*. As a composition of boundary maps is zero, Hodge k-Laplacian matrix can be viewed simply as a Hodge Laplacian matrix. Being a Hodge Laplacian matrix, this matrix can be applied in many fields, for example, to study random walks, ranking theory, data science and others.

If we consider a simplicial complex on its 0-dimensional structure and 1-dimensional structure (i.e. all of its points and all of its edges), then a Hodge 0-Laplacian matrix and a Laplacian matrix are coincide. In other words, a Hodge k-Laplacian matrix on simplicial complexes is a generalization of a Laplacian matrix on graphs. This fact leads us to define a normalized Hodge k-Laplacian matrix on simplicial complexes which is a generalization of a normalized Laplacian matrix on graphs. We also need a condition of being a Hodge Laplacian matrix to obtain some properties.

In 1993, F. Chung [5] defined a normalized Laplacian as $\partial\delta + \rho\delta\partial$, where ρ is the density of a given simplicial complex. In 2010, C. Taszus [19] defined a

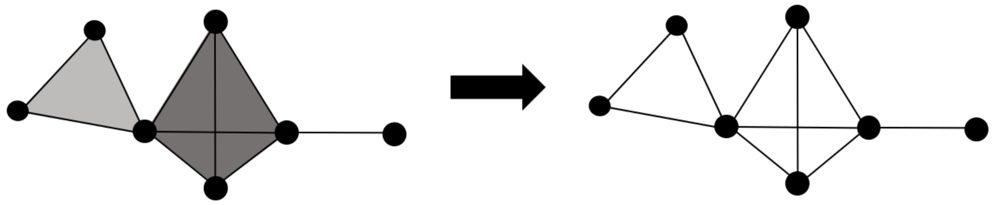


Figure 1.1: 1-dimensional structure of a simplicial complex can be considered as a graph.

normalized Laplacian matrix as $D^{-1/2}L_kD^{-1/2}$, where D is a diagonal matrix of L_k. In 2011, D. Horak [9] defined the normalized combinatorial Laplace operator in order to force an upper bound of the maximal eigenvalue of the operator to be a constant. In 2018, M. Schaub and others [16] defined a normalized Hodge 1-Laplacian to study random walks on edges. However, matrix representations of these operators and the matrices defined above are not Hodge Laplacian matrices.

In Chapter 2, we state some preliminaries in graph theory, linear algebra, algebraic topology and Hodge theory. We start Chapter 3 with analyzing Hodge k-Laplacian matrix on a simplicial complex and study the relation between its eigenvalues and homology on the simplicial complex. We give a proof of a well-known fact that the smallest eigenvalue of L_k can indicate whether the kth homology and the kth cohomology on a given simplicial complex are trivial. In Chapter 4, we define a normalized Hodge k-Laplacian matrix on a simplicial complex for any non-negative integer k which is a Hodge Laplacian matrix. Using some material in linear algebra and this fact, we obtain some properties of the matrix that we defined. We demonstrate eigenvalues of this matrix for some special cases of simplicial complexes. We obtain that the smallest eigenvalue of $\mathcal{L}_k$ can also indicate whether the kth homology and the kth cohomology on a given simplicial complex are trivial. Moreover, instead of finding eigenvalues of simplicial complexes, we study a method to find eigenvalues of the matrix by considering the simplicial complex as a wedge sum of simplices. Finally, in Chapter 5, we apply our matrix to be a transtion matrix on random walks for a simplicial complex.

CHAPTER II
PRELIMINARIES

In this chapter, we state some basic knowledge in linear algebra, graph theory, Hodge theory and algebraic topology which are needed in the next three chapters.

2.1 Linear Algebra

For this section, we state some basic tools in linear algebra. For more details, we recommend [17].

Definition 2.1.1. Let $T : V \to V$ be a linear operator on a vector space V over a field $\mathbb{F}$. A scalar $\lambda \in \mathbb{F}$ is called **eigenvalue** for T if there is a non-zero $v \in V$ such that $T(v) = \lambda v$. A non-zero vector v such that $T(v) = \lambda v$ is called an **eigenvector** corresponding to the eigenvalue λ. For each $\lambda \in \mathbb{F}$, define

$$V_\lambda = \{v \in V \mid T(v) = \lambda v\} = \operatorname{Ker}(T - \lambda I_V).$$

If λ is not an eigenvalue of T, then $V_\lambda = \{0\}$; otherwise, we call V_λ the **eigenspace** corresponding to the eigenvalue λ. Any non-zero vector in V_λ is an eigenvector corresponding to λ.

Remark that we can define an eigenvalue, an eigenvector and an eigenspace of matrix in analogous way, i.e., for any matrix $A \in M_n(\mathbb{F})$, a scalar $\lambda \in \mathbb{F}$ is called **eigenvalue** for A if there is a non-zero $v \in \mathbb{F}^n$ such that $Av = \lambda v$. A non-zero vector v such that $Av = \lambda v$ is called an **eigenvector** corresponding to the eigenvalue λ. For each $\lambda \in \mathbb{F}$, define

$$V_\lambda = \{v \in \mathbb{F}^n \mid Av = \lambda v\}.$$

If λ is not an eigenvalue of A, then $V_\lambda = \{0\}$; otherwise, we call V_λ the **eigenspace** corresponding to the eigenvalue λ. Any non-zero vector in V_λ is an eigenvector corresponding to λ.

Let A be an $n \times n$ matrix in $M_n(\mathbb{R})$ and V a vector space of dimensional n. Define $L_A : V \to V$ to be an operator such that $L_A(v) = Av$. It is easy to check that L_A is a linear operator and an eigenvalue (eigenvector, eigenspace) of a matrix A is an eigenvalue (eigenvector, eigenspace) of L_A. Thus, any results for a linear operator can be transferred analogously to results for a matrix as well. We next state the results in term of a linear operator and let readers keep in mind that these results hold for any matrix in $M_n(\mathbb{C})$.

In this work, for any $x, y \in \mathbb{C}^n$, the inner product of x and y is defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Moreover, for each $x \in \mathbb{C}^n$, we write

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Definition 2.1.2. Let V be a vector space.

(i) We say that $u, v \in V$ are **orthogonal** if $\langle u, v \rangle = 0$ and write $u \perp v$.

(ii) If $x \in V$ is orthogonal to every element of a subset W of V, then we say that x is **orthogonal** to W and write $x \perp W$.

(iii) if U, W are subsets of V and $u \perp w$ for all $u \in U$ and all $w \in W$, then we say that U and W are **orthogonal** and write $U \perp W$.

(iv) The set of all $x \in V$ orthogonal to a set W is denoted by $W^\perp$ and called the **orthogonal complement** of W:

$$W^\perp = \{x \in V \mid x \perp W\}.$$

Proposition 2.1.1. *Let T be a linear operation on $\mathbb{C}^n$. Then there is a unique linear operation T^* on V satisfying*

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x, y \in \mathbb{C}^n.$$

Definition 2.1.3. The linear operation T^* satisfying Proposition 2.1.1 is called the **adjoint of** T.

Theorem 2.1.2. *Let T, S be a linear operator on V. Then*

(i) $T^{**} = T$;

(ii) $(T+S)^* = T^* + S^*$;

(iii) $(TS)^* = S^*T^*$;

(iv) If T is invertible, then T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$.

Proof. Let T, S be linear operators on V and $x, y \in V$. Then

$$\langle x, T^{**}y\rangle = \langle T^*x, y\rangle = \langle x, Ty\rangle.$$

By Proposition 2.1.1, $T^{**} = T$. Consider

$$\langle x, (T+S)^*y\rangle = \langle (T+S)x, y\rangle = \langle Tx, y\rangle + \langle Sx, y\rangle = \langle x, T^*y\rangle + \langle x, S^*y\rangle = \langle x, (T^*+S^*)y\rangle.$$

This implies $(T+S)^* = T^* + S^*$. Since

$$\langle x, (TS)^*y\rangle = \langle TSx, y\rangle = \langle Sx, T^*y\rangle = \langle x, S^*T^*y\rangle,$$

$(TS)^* = S^*T^*$. Suppose that T is invertible. Then by (iii) $T^*(T^{-1})^* = (T^{-1}T)^* = I^* = I$ and $(T^{-1})^*T^* = (TT^{-1})^* = I^* = I$. Therefore $(T^*)^{-1} = (T^{-1})^*$. □

Definition 2.1.4. Let T be a linear operation on $\mathbb{C}^n$. Then T is said to be **self-adjoint** if $T^* = T$.

Recall that L_A is a linear operator on $\mathbb{C}^n$ such that $L_A(x) = Ax$, where $x \in \mathbb{C}^n$. Since

$$\langle L_A(x), y\rangle = \langle Ax, y\rangle = (Ax)^T\overline{y} = x^T A^T \overline{y} = \langle x, \overline{A}^T y\rangle,$$

we get

$$(L_A)^* = L_{A^*},$$

where $A^* = \overline{A}^T$.

Definition 2.1.5. Let $A \in M_n(\mathbb{C})$. Then A is said to be **self-adjoint** if $A^* = \overline{A}^T = A$. In particular, any real symmetric matrix is self-adjoint.

Proposition 2.1.3. *Let $A \in M_n(\mathbb{C})$ be a self-adjoint matrix. Then*

(i) Any eigenvalue of A is real.

(ii) If $v \in \mathbb{C}^n$ is an eigenvector of A corresponding to an eigenvalue λ, then $v \in \mathbb{C}^n$ is an eigenvector of A^* corresponding to an eigenvalue λ.

(iii) The eigenspaces associated with distinct eigenvalues are orthogonal.

Proof. To prove the first statement, let λ be an eigenvalue of A and v be an eigenvector corresponding to λ. Then

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Av, v\rangle = \langle v, A^*v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\langle v, v\rangle.$$

Since $v \neq 0$, this shows that $\lambda = \overline{\lambda}$ that is λ is a real number. To shows the second statement, we first note that for any self-adjoint matrix $B \in \mathbb{C}^n$ and $x \in \mathbb{C}^n$,

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle B^*x, B^*x\rangle = \|B^*x\|^2$$

which implies that $\|Bx\| = \|B^*x\|$. Since

$$(A - \lambda I)^* = A^* - \overline{\lambda}I^* = A - \lambda I,$$

$A - \lambda I$ is self-adjoint. Then by the note,

$$\|(A^* - \lambda I)v\| = \|(A - \lambda I)^* v\| = \|(A - \lambda I)v\| = 0.$$

Therefore, v is an eigenvector corresponding to eigenvalue λ. For the last statement, let λ and μ be two distinct eigenvalues of A. Note that λ and μ are real number by (i). Let u and v be eigenvectors corresponding to λ and μ, respectively. Then by (ii), $A^* v = \mu v$ and hence

$$\lambda\langle u, v\rangle = \langle \lambda u, v\rangle = \langle Au, v\rangle = \langle u, A^* v\rangle = \langle u, \mu v\rangle = \overline{\mu}\langle u, v\rangle = \mu\langle u, v\rangle.$$

Since $\lambda \neq \mu$, we have $\langle u, v\rangle = 0$ that is $u \perp v$. □

We now introduce an efficient tool for studying an eigenvalue problem called **Rayleigh's quotient**. To study more in this topic, [12] is recommended.

Definition 2.1.6. Let A be an $n \times n$ real symmetric matrix. The function **Rayleigh's quotient** R is defined by

$$R(A, x) = \frac{\langle x, Ax\rangle}{\langle x, x\rangle} = \frac{x^T A x}{x^T x} \in \mathbb{R},$$

for any $x \in \mathbb{R}^n - \{0\}$.

Lemma 2.1.4. *Let A be an $n \times n$ real symmetric matrix and $\lambda_n \leq \cdots \leq \lambda_2 \leq \lambda_1$ be eigenvalues of A. Then* $\displaystyle\min_{x \in \mathbb{R}^n - \{0\}} R(A, x) = \lambda_n$.

Proof. We first note that

$$\min_x R(A, x) = \min_{\|x\|=1} R(A, x).$$

Since A is a symmetric matrix, $A = P^T D P$ for some orthogonal matrix P and a diagonal matrix D. Note that $\|P^T x\| = \|x\|$. So, for any $x \in \mathbb{R}^n - \{0\}$ such that $\|x\| = 1$,

$$R(A, x) = x^T A x = (Px)^T D (Px).$$

Let $y = P^T x$, we have

$$R(A, y) = R(A, P^T x) = (PP^T x)^T D(PP^T x) = x^T Dx = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2.$$

Then, by choosing $x = (0, 0, \ldots, 0, 1)$, we get $\min_{\|x\|=1} R(A, x) = \lambda_n$. $\square$

Let $T : V \to W$. Define

$$\operatorname{Ker}(T) = \{v \in V \mid Tv = 0\} \text{ and } \operatorname{Im}(T) = \{w \in W \mid \exists v \in V, Tv = w\}.$$

Similarly, for a matrix $A \in M_{n\times m}(\mathbb{C})$. Define

$$\operatorname{Ker}(A) = \{v \in \mathbb{C}^m \mid Av = 0\} \text{ and } \operatorname{Im}(A) = \{w \in \mathbb{C}^n \mid \exists v \in \mathbb{C}^m, Av = w\}.$$

Note that $\operatorname{Ker}(A)$ and $\operatorname{Im}(A)$ that we defined are exactly $\operatorname{Ker}(L_A)$ and $\operatorname{Im}(L_A)$, respectively, where $L_A(v) = Av$ for any $v \in \mathbb{C}^n$.

Proposition 2.1.5. *[11] Let $A \in M_{n\times m}(\mathbb{R})$. Then the followings hold;*

(i) $\operatorname{Ker}(A^*A) = \operatorname{Ker}(A)$,

(ii) $\operatorname{Ker}(A^*) = \operatorname{Im}(A)^\perp$,

(iii) $\operatorname{Im}(A^*) = \operatorname{Ker}(A)^\perp$.

(iv) $\mathbb{R}^n = \operatorname{Ker}(A) \oplus \operatorname{Im}(A^*)$

Proof. (i) It is clear that $\operatorname{Ker}(A) \subset \operatorname{Ker}(A^*A)$. Let $x \in \mathbb{R}^n$ such that $A^*Ax = 0$. Then,

$$\|Ax\|^2 = \langle Ax, Ax\rangle = \langle x, A^*Ax\rangle = 0$$

which implies that $Ax = 0$. Then $x \in \operatorname{Ker}(A)$. (ii) Let $x \in \operatorname{Ker}(A^*)$. Then

$$0 = \langle A^*x, y\rangle = \langle x, Ay\rangle$$

for any $y \in \mathbb{R}^n$ that is $x \in \operatorname{Im}(A)^\perp$. Let $y \in \operatorname{Im}(A)^\perp$. Then,

$$0 = \langle y, Az \rangle = \langle A^* y, z \rangle$$

for any $z \in \mathbb{R}^n$. Then $A^* y = 0$ that is $y \in \operatorname{Ker}(A^*)$. (iii) By (ii), $\operatorname{Im}(A^*)^\perp = \operatorname{Ker}(A^{**}) = \operatorname{Ker}(A)$ and then $\operatorname{Im}(A^*) = \operatorname{Ker}(A)^\perp$. (iv) Consider $\mathbb{R}^n = \operatorname{Ker}(A) \oplus \operatorname{Ker}(A)^\perp = \operatorname{Ker}(A) \oplus \operatorname{Im}(A^*)$ by (iii). □

Definition 2.1.7. Let A and B be square matrices. We say that A **is similar to** B if there is an invertible matrix P such that $B = PAP^{-1}$.

Proposition 2.1.6. *Let A and B be square matrices such that A is similar to B. Then all eigenvalues of A and B with their multiplicities are equal.*

Proof. Suppose that A is similar to B i.e. there exists an invertible matrix P such that $B = PAP^{-1}$. Let v be an eigenvector of B corresponding to eigenvalue λ. Then, $\lambda v = Bv = PAP^{-1}v$ and hence $\lambda(P^{-1}v) = A(P^{-1}v)$. This shows that $P^{-1}v$ is an eigenvector of A corresponding to eigenvalue λ. Since λ is an arbitrary eigenvalue of B, every eigenvalue of B is an eigenvalue of A. Similarly, we can show that every eigenvalue of A is an eigenvalue of B. □

Theorem 2.1.7. *(Primary Decomposition) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Assume that the minimal polynomial $m_T(x)$ can be written as*

$$m_T(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_k)^{m_k},$$

where $\lambda_1, \dots, \lambda_k$ are distinct element in $\mathbb{F}$. Define

$$V_i = \operatorname{Ker}(T - \lambda_i I)^{m_i}, \ i = 1, \dots, k.$$

Then each V_i is a nonzero, T-invariant subspace of $\mathbb{R}^n$ and

$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k.$$

Lemma 2.1.8. *Let $T : V \to W$, $S : W \to U$ and $R : U \to V$ be bijective linear maps on finite dimensional spaces. Then*

$$\mathrm{Ker}(ST) \cong \mathrm{Ker}(T) \cong \mathrm{Ker}(TR).$$

Proof. This follows from the fact that T, S and R are injective. □

Lemma 2.1.9. *Let*

$$P = \begin{pmatrix} A_{n\times n} & \boldsymbol{O} \\ \boldsymbol{O} & B_{m\times m} \end{pmatrix}$$

be a block matrix, where $\boldsymbol{O}$ is the zero matrix. Then λ is an eigenvalue of P if and only if λ is an eigenvalue of A or B.

Proof. Suppose that λ be an eigenvalues of A or B. Then

$$0 = \det(A - \lambda I)\det(B - \lambda I) = \det(P - \lambda I). \tag{2.1}$$

This shows that λ is an eigenvalue of P. Conversely, suppose that λ be an eigenvalue of P. Then by (2.1), $\det(A - \lambda I) = 0$ or $\det(B - \lambda I) = 0$ which implies that λ is an eigenvalue of A or B. □

2.2 Graph Theory

In this section, we briefly state some basic definitions in graph theory, see [21] for more details.

Let $G = (V, E)$ be a graph of order n with the vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and the edge set $E(G)$. We simply write V and E instead of $V(G)$ and $E(G)$ if G is clear from the context. Let $v_i \in V(G)$. We said that v_i is **adjacent** to v_j if there is an edge v_iv_j between them. We write $v_i \sim v_j$ to denote that v_i is adjacent to v_j and define the **degree of** v_i, denoted by $\deg v_i$, to be the number of vertices which are adjacent to v_i. The graph without loops and multiple edges between any pairs of vertices is called a **simple graph**. A graph which contains no edge is called the **trivial graph**, otherwise it will be called a **nontrivial graph**. A vertex of degree 0 is referred as an **isolated vertex**. A **directed graph** is a pair (V, E) of disjoint sets together with two maps $I : E \to V$ and $T : E \to V$ assigning to every edge e an initial vertex $I(e)$ and a terminated vertex $T(e)$, respectively. A graph which is not a directed graph is called **undirected graph**. A graph together with a function mapping each edge to a real number is called a **weighted graph**.

There are many matrices in graph theory which represent some structures of graphs. Moreover, some properties of graph can be shown by eigenvalues of these matrices. To study in this topic, spectral graph theory [4] which is introduced by F. Chung, is recommended.

We first state definition of a simple matrix called adjacency matrix and then use it to defines the others.

Let G be a simple undirected graph of order n with a vertex set $V = \{v_1, v_2, ..., v_n\}$ and let d_i be the degree of vertex v_i for all $i \in \{1, 2, ..., n\}$.

Definition 2.2.1. The **adjacency matrix** $A = A(G) = (a_{ij})_{n\times n}$ of G is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2.2. A **degree matrix** $D = D(G) = (d_{ij})_{n\times n}$ is a diagonal matrix whose entries are of vertex degrees of graph G, i.e., $d_{ii} = d_i$ and $d_{ij} = 0$ if $i \neq j$.

Definition 2.2.3. The **Laplacian matrix** $L = L(G) = (l_{ij})_{n\times n}$ of G is defined by $L = D - A$ which can be written as

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } v_iv_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2.4. The **normalized Laplacian matrix** $\mathcal{L} = \mathcal{L}(G) = (l'_{ij})_{n\times n}$ of G is defined by $\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ where $(D^{-\frac{1}{2}})_{ii} = \dfrac{1}{\sqrt{d_i}}$ if $d_i \neq 0$ and 0 otherwise.

More precisely, $\mathcal{L}$ can be written as

$$l'_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -\dfrac{1}{\sqrt{d_i d_j}} & \text{if } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2.5. The **random walk normalized Laplacian matrix** $\mathcal{L}^{rw} = \mathcal{L}^{rw}(G) = (l^{rw}_{ij})_{n\times n}$ of G is defined by $\mathcal{L}^{rw} = D^{-1}L$ where $(D^{-1})_{ii} = \dfrac{1}{d_i}$ if $d_i \neq 0$ and 0 otherwise. More precisely, $\mathcal{L}^{rw}$ can be written as

$$l^{rw}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } \deg v_i \neq 0, \\ -\dfrac{1}{\deg v_i} & \text{if } i \neq j \text{ and } v_i v_j \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

In this work, we consider only simple nontrivial undirected graphs. Then an adjacency matrix, a Laplacian matrix and a normalized Laplacian matrix are then real symmetric matrices and hence self-adjoint matrices. Therefore all eigenvalues of $A(G)$, $L(G)$ and $\mathcal{L}(G)$ are real numbers for any graph G throughout this work.

The Laplacian matrix (or Kirchhoff matrix) is a discrete version of of the Laplacian operator in multivariable calculus. For a given graph G, the second smallest eigenvalue of the Laplacian matrix is known as *algebraic connectivity of the graph.* This eigenvalue can indicate whether the graph is connected. In fact, multiplicity of zero as an eigenvalue of Laplacian matrix is the number of connected components of graph.

Theorem 2.2.1. *Let G be a graph. The multiplicity of 0 as an eigenvalue of $L(G)$ is the number of connected components of graph.*

Proof. Note that for any square matrix A, the multiplicity of 0 as an eigenvalue of A is equal to $\dim(\mathrm{Ker}(A))$ by Theorem 2.1.7. We first consider a case that G is connected. Let G be a connected graph with a vertex set $\{v_1, v_2, \dots, v_n\}$. Let x

be a nonzero vector such that $L(x) = 0$. Then

$$0 = \langle x, Lx \rangle = \langle x, Dx - Ax \rangle = \sum_{v_i \sim v_j, i<j} -2x_i x_j + \sum_i d_i x_i^2 = \sum_{v_i \sim v_j, i<j} (x_i - x_j)^2.$$

Since G is connected, this shows that $x_1 = x_2 = \cdots = x_n$. Therefore, the dimension of $\mathrm{Ker}(L) = 0$ that is the multiplicity of eigenvalue 0 is 1.

Next, assume that a graph G has connected components $G_1, G_2, \ldots, G_m$. Then, we can reindex vertices in G and obtain that

$$L(G) = \begin{pmatrix} L(G_1) & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & L(G_2) & \mathbf{O} & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & L(G_m) \end{pmatrix},$$

where $\mathbf{O}$ denotes the zero matrix. Then

$$\mathrm{Ker}(L(G)) = \mathrm{Ker}(L(G_1)) \oplus \mathrm{Ker}(L(G_2)) \oplus \cdots \oplus \mathrm{Ker}(L(G_m)).$$

Since each G_i is connected,

$$\dim \mathrm{Ker}(L(G)) = \sum_{i=1}^{m} \dim \mathrm{Ker}(L(G_i)) = m. \qquad \square$$

Corollary 2.2.2. *Let G be a graph. The multiplicity of 0 as an eigenvalue of $\mathcal{L}(G)$ is the number of connected components of graph.*

Proof. Let G be a graph. Assume that a G has connected components $G_1, G_2, \ldots, G_m$. Note that $\mathcal{L} = D^{-1/2} L D^{-1/2}$. For each connected component G_i of G, $D^{-1/2}(G_i)$ is an invertible matrix. Then by Proposition 2.1.8, $\mathrm{Ker}(L(G_i)) \cong \mathrm{Ker}(\mathcal{L}(G_i))$ for each i. Therefore, $\dim \mathrm{Ker}(L(G_i)) = \dim \mathrm{Ker}(\mathcal{L}(G_i))$. Similar to the proof of

Theorem 2.2.1, we can obtain that

$$\mathrm{Ker}(\mathcal{L}(G)) = \mathrm{Ker}(\mathcal{L}(G_1)) \oplus \mathrm{Ker}(\mathcal{L}(G_2)) \oplus \cdots \oplus \mathrm{Ker}(\mathcal{L}(G_m)).$$

Then

$$\dim \mathrm{Ker}(\mathcal{L}(G)) = \sum_{i=1}^{m} \dim \mathrm{Ker}(\mathcal{L}(G_i)) = \sum_{i=1}^{m} \dim \mathrm{Ker}(L(G_i)) = m. \qquad \square$$

Definition 2.2.6. A simple graph in which each pair of distinct vertices are adjacent is a **complete graph**. We denote the complete graph on n vertices by K_n.

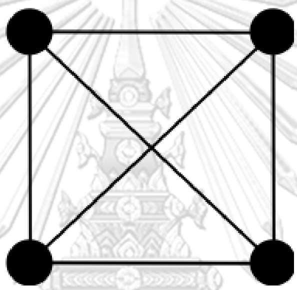


Figure 2.1: K_4

Proposition 2.2.3. *Let K_n be a complete graph of order n. Then eigenvalues of $\mathcal{L}(K_n)$ are 0 (with multiplicities 1) and $\dfrac{n}{n-1}$ (with multiplicities $n-1$).*

Proof. Let n be a positive integer. Then the normalized Laplacian matrix of a complete graph K_n canbe written as

$$\mathcal{L}(K_n) = \begin{pmatrix} 1 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & 1 & -\frac{1}{n-1} & \cdots & -\frac{1}{n-1} \\ -\frac{1}{n-1} & -\frac{1}{n-1} & 1 & \cdots & -\frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & 1 \end{pmatrix} = -\frac{1}{n-1}[-nI_n + J_n],$$

where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ all-ones matrix. Since K_n

is connected, by Corollary 2.2.1, 0 is an eigenvalue of $\mathcal{L}(K_n)$ with multiplicities 1. Note that $(1, 1, \dots, 1)$ is an eigenvector corresponding to eigenvalue 0. Let λ be a nonzero eigenvalue of $\mathcal{L}(K_n)$. Let v be an eigenvector corresponding to eigenvalue λ. Then by Proposition 2.1.3, v is orthogonal to $(1, 1, \dots, 1)$. Therefore, $\mathcal{L}(K_n)v = -\dfrac{1}{n-1}[-nI_n + J_n]v = \dfrac{n}{n-1}v$. This shows that $\dfrac{n}{n-1}$ is an eigenvalue of $\mathcal{L}(K_n)$ with multiplicities $n-1$. $\square$

2.3 Algebraic Topology

In this section, we briefly introduce some materials in algebraic topology. As they are quite abstract, we recommend [7] and [8] for the readers who are not familiar with these objects.

2.3.1 Simplicial Complexes

Definition 2.3.1. The smallest convex set in Euclidean space $\mathbb{R}^m$ containing $n+1$ points $v_0, v_1, \dots, v_n \in V$ such that $v_1 - v_0, \dots, v_n - v_0$ are linearly independent is called ***n*-simplex** denoted by $[v_0, v_1, \dots v_n]$ and n is called the **dimension** of $[v_0, v_1, \dots v_n]$. A subset of $[v_0, v_1, \dots v_n]$ of cardinality k is called a **$(k-1)$-face of** $[v_0, v_1, \dots v_n]$ and the union of all $(n-1)$-faces of $[v_0, v_1, \dots v_n]$ is called the **boundary of** $[v_0, v_1, \dots v_n]$. Note that for a 0-simplex we define (-1)-face to be the empty set. See Figure 2.2 for examples.

Definition 2.3.2. A **simplicial complex** is a finite collection X of simplices such that

- Any face of a simplex from X is also in X.
- The intersection of two simplices is a face of both simplices.

A dimension of a simplicial complex is defined to be the highest dimension of simplices in X. If a simplicial complex X has dimension k, we can call X a **simplicial *k*-complex** to specify the dimension of X. We also let a subset

$X^k \subset X$ be a set containing all k-simplices in X. Figure 2.3 is an example of simplicial complexes but Figure 2.4 is not a simplicial complex.

For example, the set of vertices V can be viewed as a set of 0-simplices X^0 and the set of edges E can be view as a set of 1-simplices X^1. In this work, all simplicial complexes that we consider are supposed to be finite i.e. containing finite vertices.

In order to do a computation, we need to define an orientation of edges. For an edge $[v_i, v_j] \in E$, we simply choose the reference orientation of $[v_i, v_j]$ according to increasing subscripts. For a higher dimensional simplices, the orientations are defined base on the orientation of their edges.

In particular, an orientation of k-simplex S^k $(k > 0)$ is an equivalence class of orderings of its vertices, where two orderings are equivalent if they differ by an even permutation. For a convenience, we denote each k-simplex with $[v_{i_0}, \dots, v_{i_k}]$ where $i_o < i_1 < \dots < i_k$.

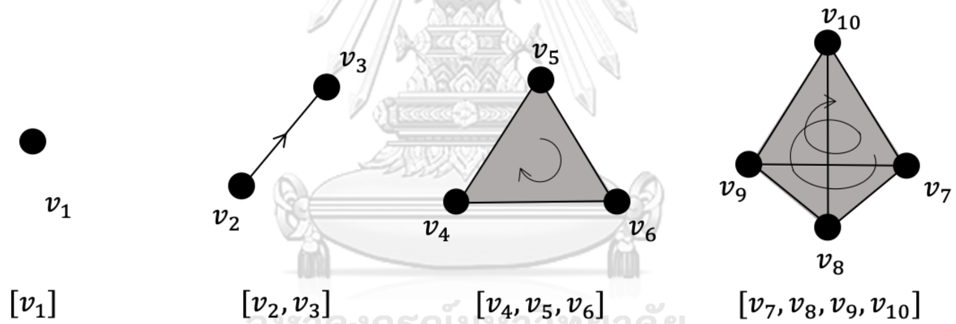


Figure 2.2: A 0-simplex, a 1-simplex, a 2-simplex and a 3-simplex.

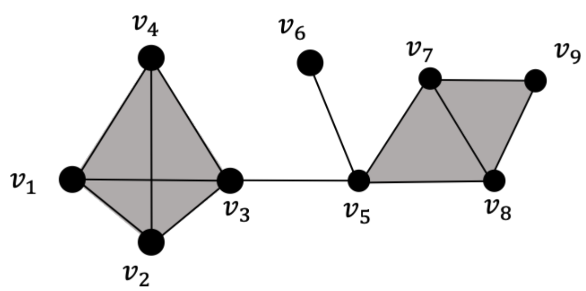


Figure 2.3: A simplicial 3-complex.

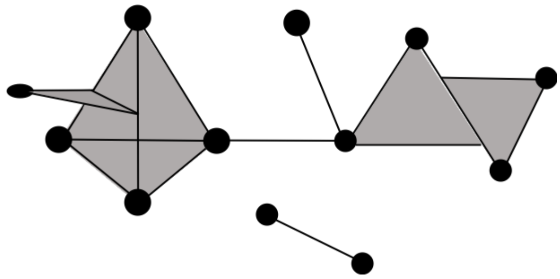


Figure 2.4: This object is not a simplicial complex.

Definition 2.3.3. If $\sigma, \tau \in X^k$ are both faces of the same $(k+1)$-simplex, we said that they are **upper adjacent**, denoted by $\sigma \sim_u \tau$. If $\sigma, \tau \in X^k$ both have a common face, we said that they are **lower adjacent**, denoted by $\sigma \sim_l \tau$.

Example 2.3.1. Consider a simplicial complex in Figure 2.3. We obtain that

- $[v_1, v_2, v_3] \sim_u [v_2, v_3, v_4]$ as $[v_1, v_2, v_3]$ and $[v_2, v_3, v_4]$ are both faces of $[v_1, v_2, v_3, v_4]$;
- $[v_5, v_7] \sim_u [v_7, v_8]$ as $[v_5, v_7]$ and $[v_7, v_8]$ are both faces of $[v_5, v_7, v_8]$;
- $[v_7] \sim_u [v_8]$ as $[v_7]$ and $[v_8]$ are both faces of $[v_7, v_8]$;
- $[v_1, v_4] \sim_l [v_3, v_4]$ as $[v_1, v_4]$ and $[v_3, v_4]$ have a common face $[v_4]$;
- $[v_1, v_2, v_3] \sim_l [v_1, v_3, v_4]$ as $[v_1, v_2, v_3]$ and $[v_1, v_3, v_4]$ have a common face $[v_1, v_3]$.

Definition 2.3.4. Let X be a simplicial complex and k be a nonnegative integer. Let $C_k(X)$ (or simply C_k) be the finite-dimensional vector space with real coefficient, whose basis elements are the oriented simplices $s_i^k \in X^k$. An element $c_k \in C_k$ is called a k-**chain**. More precise, $c_k = \sum_i \alpha_i s_i^k$, where $\alpha_i \in \mathbb{R}$.

By the above definition, we can represent $c_k = \sum_i \alpha_i s_i^k \in C_k$ with a vector $\mathbf{c} = (\alpha_1, \dots, \alpha_{n_k})^T$, where $n_k = |X^k|$. Thus, C_k is isomorphic to $\mathbb{R}^{n_k}$, so we determine a chain in C_k as a vector in $\mathbb{R}^{n_k}$ through this work. Moreover, a change of the orientation of the basis element s_i^k makes a change in the sign of the coefficient α_i. For a further works, we equip each C_k with the standard l^2 inner product

$$< c_1, c_2 >= c_1^T c_2.$$

This leads that each C_k has the structure of finite-dimensional Hilbert space.

Definition 2.3.5. Let X be a simplicial complex and k be a nonnegative integer. The dual space of $C_k(X)$, the set of linear map between $C_k(X)$ and $\mathbb{R}$, is called a space of cochains denoted by $C^k(X)$ (or simply C^k) and elements in C^k are called **k-cochain**.

Remark that we can view a chain C_k as a free abelian group whose basis are elements in X^k and a cochain C^k as a group of homomorphisms from C_k to $\mathbb{R}$.

Since we consider only finite simplicial complexes, C_k is finite dimensional for each k. We have

$$\dim(C^k) = \dim(\mathcal{L}(C_k, \mathbb{R})) = \dim(C_k) \times \dim(\mathbb{R}) = \dim(C_k) \times 1 = \dim(C_k).$$

Hence, C^k is isomorphic to C_k for each k. Similarly to C_k, we can determine elements C^k as vectors in $\mathbb{R}^{n_k}$, where $n_k = |X^k|$. We use this fact to work on $\mathbb{R}^{n_k}$ instead of C^k through this work.

2.3.2 Boundary and Coboundary Maps

For each $k \geq 0$, we define a map $\partial_k : X_k \to X_{k-1}$ by

$$\partial_k([v_0, v_1, \dots, v_k]) = \sum_{j=0}^{k} (-1)^j [v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k].$$

It is obvious that this map is a linear map for each k. Hence, we can attend this map linearly to each C_k.

Definition 2.3.6. For each $k \geq 0$, a map $\partial_k : C_k \to C_{k-1}$ defined on their basis elements by

$$\partial_k([v_0, v_1, \dots, v_k]) = \sum_{j=0}^{k} (-1)^j [v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k]$$

is called a **boundary map**.

We sometimes write ∂ or ∂_* instead of ∂_k if its domain is clear from the context or the dimension of domain does not matter. Moreover, for a $(k-1)$-simplex $\sigma = [v_0, v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_k]$ which is a face of k-simplex $\overline{\sigma} = [v_0, v_1, \dots, v_k]$, we denote the sign of σ with respect to $\partial\overline{\sigma}$ with $\operatorname{sgn}(\sigma, \partial\overline{\sigma}) = (-1)^j$. According to Definition 2.3.1, we remark that we also use $\partial\overline{\sigma}$ to denote the set of all $(k-1)$-faces of $\overline{\sigma}$.

Proposition 2.3.1. *[8] The composition* $C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2}$ *is zero for any* $n \geq 2$.

Proof. Let $\sigma = [v_0, v_1, \dots, v_n] \in X^n$. Then

$$\partial_n(\sigma) = \sum_{i=1}^{n} (-1)^i [v_0, v_1, \dots., v_{i-1}, v_{i+1}, \dots, v_n]$$

and

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \partial_{n-1}\left(\sum_{i=1}^{n} (-1)^i [v_0, v_1, \dots., v_{i-1}, v_{i+1}, \dots, v_n]\right) \\
&= \sum_{j<i} (-1)^i (-1)^j [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{i-1}, v_{i+1}, \dots, v_n] \\
&\quad + \sum_{j>i} (-1)^i (-1)^{j-1} [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_{j-1}, v_{j+1}, \dots, v_n].
\end{aligned}$$

By switching indexes i and j, the second sum is the negative of the first. Hence $\partial_{n-1}\partial_n(\sigma) = 0$ as desired. □

With this definition, we now have a sequence of linear maps

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

which is called a **chain complex** together with the property $\partial_n\partial_{n+1} = 0$ or simply $\partial\partial = 0$. In other words, we could say that **a boundary of boundary is zero**. Note that we extended the sequence by a 0 at the right end with $\partial_0 = 0$.

Moreover, if we replace each chain group C_k with its dual cochain group C^k and replace each boundary map $\partial_n : C_n \to C_{n-1}$ by its dual $\delta_n = \partial_n^* : C^{n-1} \to C^n$, we then obtain

$$\cdots \leftarrow C^{n+1} \xleftarrow{\delta_{n+1}} C^n \xleftarrow{\delta_n} C^{n-1} \leftarrow \cdots \leftarrow C^1 \xleftarrow{\delta^1} C_0 \xleftarrow{\delta_0} 0.$$

Definition 2.3.7. The map $\delta_n : C^{n-1} \to C^n$ defined above is called a **coboundary map** and the sequence is called a **cochain complex**.

We sometimes write δ or δ_* instead of δ_k. By Proposition 2.3.1, we directly obtain that $\delta\delta = 0$ that is **a coboundary of coboundary is zero**.

2.3.3 Homology and Cohomology

Since $\partial_k \partial_{k+1} = 0$ and $\delta_{k+1}\delta_k = 0$, we have $\mathrm{Ker}(\partial_k) \subset \mathrm{Im}(\partial_{k+1})$ and $\mathrm{Ker}(\delta_{k+1}) \subset \mathrm{Im}(\delta_k)$. We now define two groups which are our main points in this work using these properties as follows:

Definition 2.3.8. Let X be a simplicial complex and k be a nonnegative integer. Elements in $\mathrm{Ker}(\partial)$ and $\mathrm{Im}(\partial)$ are called **cycles** and **boundaries**, respectively. A k**th homology of** X is defined by $H_k(X) = {\mathrm{Ker}(\partial_k)}\big/{\mathrm{Im}(\partial_{k+1})}$. Similarly, elements in $\mathrm{Ker}(\delta)$ and $\mathrm{Im}(\delta)$ are called **cocycles** and **coboundaries**, respectively. A k**th cohomology of** X is defined by $H^k(X) = {\mathrm{Ker}(\delta_{k+1})}\big/{\mathrm{Im}(\delta_k)}$.

Note that some other books may use the notations $H_k(X, \mathbb{F})$ and $H^k(X, \mathbb{F})$ instead of $H_k(X)$ and $H^k(X)$ to emphasize that they are working on a field $\mathbb{F}$ as the coefficients of chains and cochains. However, we neglect this point and remind the readers that we are working on $\mathbb{R}$ as we define our chains and cochains at the beginning of this section.

Remark 1. Since the space of k-chain is defined on real number and simplicial complexes that we focus on are finite, we have

$$H_k(X) \cong \mathrm{Hom}(H_k(X), \mathbb{R}) \cong H^k(X),$$

for any simplicial complex X and for all k. In fact, for general, $H_k(X) \cong H^k(X)$ if X is finite.

2.3.4 Matrix Representations

Let G be a graph and X^k be a set of k-simplices on G. Recall that for each k, C_k and C^k are isomorphic to $\mathbb{R}^{n_k}$, where $n_k = |X^k|$. Since ∂ and δ are linear maps, we can represent them by matrix representations with an appropriate basis. Let B_k be a matrix representation of ∂_k. We automatically obtain that B_k^T is a matrix representation for $\partial_k^* = \delta_k$. Note that $B_0 = 0$ since ∂_0 is set to be a zero map. There is a well-known fact that the Laplacian matrix L can be written as

$$L = B_1 B_1^T$$

and hence the normalized Laplacian matrix can be written as

$$\mathcal{L} = D^{-1/2} L D^{-1/2} = D^{-1/2} B_1 B_1^T D^{-1/2},$$

where $D^{-1/2}$ is a diagonal matrix and $(d_{ii})^{-1/2} = \dfrac{1}{\sqrt{d_i}}$ if $d_i \neq 0$ and 0 otherwise.

Recall that for a simplicial complex X,

$$H_k(X) = {}^{\textstyle\mathrm{Ker}(\partial_k)}\big/_{\textstyle\mathrm{Im}(\partial_{k+1})}$$

and

$$H^k(X) = {}^{\textstyle\mathrm{Ker}(\delta_{k+1})}\big/_{\textstyle\mathrm{Im}(\delta_k)},$$

where ∂_k and δ_k are a boundary map and a coboundary map, respectively.

Remark that, from now, we consider elements in C_k as vector in $\mathbb{R}^{|X^k|}$ and a matrix B_k as a map $L_{B_k} : \mathbb{R}^{|X^k|} \to \mathbb{R}^{|X^{k-1}|}$ defined by

$$L_{B_k}(v) = B_k v.$$

By Proposition 2.3.1, for each k,

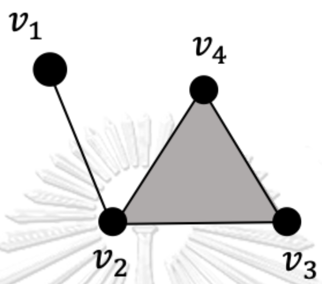
$$B_k B_{k+1} = 0 \text{ and } B_{k+1}^T B_k^T = 0. \tag{2.2}$$

Then, we can write $H_k(X)$ and $H^k(X)$ as

$$H_k(X) = {}^{\mathrm{Ker}(B_k)}\!\big/_{\mathrm{Im}(B_{k+1})} \text{ and } H^k(X) = {}^{\mathrm{Ker}(B_{k+1}^T)}\!\big/_{\mathrm{Im}(B_k^T)},$$

where B_k is a matrix representation of $\partial_k : C_k \to C_{k-1}$.

Example 2.3.2. Consider a simplicial 2-complex We get

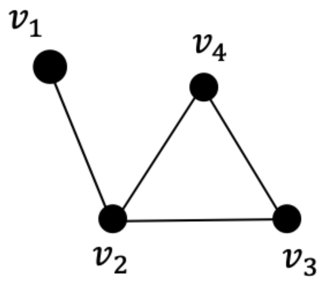


$$B_1 = \begin{array}{c} \\ [v_1] \\ [v_2] \\ [v_3] \\ [v_4] \end{array} \begin{array}{c} \begin{array}{cccc} [v_1,v_2] & [v_2,v_3] & [v_3,v_4] & [v_2,v_4] \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \end{array}$$

Then

$$L_0 = B_1 B_1^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

which is the Laplacian matrix of the following graph;



2.4 Hodge Theory

For this section, we discuss an efficient tool in Mathematics, the Hodge theory. This theory, introduced by William Vallance Douglas Hodge in 1930s, is a tool for studying cohomology and differentiable manifolds. It has applied for various applications in various fields such as the Hodge theory on metric spaces [2], Hodge Laplacian on graphs [11], ranking theory, game theory, neuroscience and others. As this theory is first introduced for applying in algebraic geometry, there are many notations which are difficult to study.

In fact, the Hodge Laplacian only requires two matrices (or linear operators) whose composition is zero, i.e., for a matrix $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$, the assumption for applying Hodge theory is $AB = 0$. We recommend [11] written by L.H. Lim for more details.

Definition 2.4.1. An $n \times n$ matrix that can be written as $A^*A + BB^*$ where $AB = 0$, $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$ is called a **Hodge Laplacian matrix**.

Lemma 2.4.1. *Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$. Then all eigenvalues of $A^*A + BB^*$ are nonnegative.*

Proof. Let $x \in \mathbb{R}^n - \{0\}$. Then $\langle x, (A^*A + BB^*)x\rangle = \langle x, A^*Ax + BB^*x\rangle = \langle x, A^*Ax\rangle + \langle x, BB^*x\rangle = \langle Ax, Ax\rangle + \langle B^*x, B^*x\rangle = \|Ax\|^2 + \|B^*x\|^2 \geq 0$. Then by Lemma 2.1.4,

$$R(A^*A + BB^*, x) = \frac{\langle x, (A^*A + BB^*)x\rangle}{\|x\|^2} \geq 0$$

and the proof is done. $\square$

Let $\operatorname{Spec}(A)$ denote a set of eigenvalues of A and $\operatorname{Spec}^*(A)$ a set of nonzero eigenvalues of A.

Lemma 2.4.2. *Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$ such that $AB = 0$. Then*

$$\operatorname{Spec}^*(A^*A + BB^*) = \operatorname{Spec}^*(A^*A) \cup \operatorname{Spec}^*(BB^*).$$

Proof. Since $AB = 0$, we have

$$\operatorname{Im}(BB^*) \subset \operatorname{Ker}(A^*A) \tag{2.3}$$

and

$$\operatorname{Im}(A^*A) \subset \operatorname{Ker}(BB^*) \tag{2.4}$$

Suppose that there exists $c \neq 0$ such that

$$A^*Ac + BB^*c = \lambda c. \tag{2.5}$$

Assume that λ is not an eigenvalue of BB^*. Taking A^*A into (2.5) both sides, we get

$$A^*A(A^*Ac) + A^*A(BB^*c) = \lambda(A^*Ac).$$

By (2.3), we have $A^*A(BB^*c) = 0$ and hence

$$A^*A(A^*Ac) = \lambda(A^*Ac).$$

Suppose that $A^*Ac = 0$ by (2.5), $BB^*c = \lambda c$. This contradicts with λ is not an eigenvalue of BB^*. Therefore $A^*Ac \neq 0$ and hence λ is an eigenvalue of A^*A.

Conversely, suppose that there exists $c \neq 0$ such that $BB^*c = \lambda c$. Then $BB^*(BB^*c) = \lambda(BB^*c)$. By (2.3), $AA^*(BB^*c) = 0$ and hence

$$(A^*A + BB^*)(BB^*c) = A^*A(BB^*c) + BB^*(BB^*c) = BB^*(BB^*c) = \lambda(BB^*c).$$

Since c and λ are nonzero, $BB^*c = \lambda c \neq 0$. Then λ is an eigenvalue of $A^*A + BB^*$. $\square$

Lemma 2.4.3. *Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$. Assume that $AB = 0$. Then*

$$\operatorname{Ker}(A^*A + BB^*) = \operatorname{Ker}(A) \cap \operatorname{Ker}(B^*) \cong {}^{\operatorname{Ker}(A)}\!/\!_{\operatorname{Im}(B)}. \tag{2.6}$$

Proof. It is clear that $\operatorname{Ker}(A) \cap \operatorname{Ker}(B^*) \subset \operatorname{Ker}(A^*A + BB^*)$. To show the converse, let $c \in \operatorname{Ker}(A^*A + BB^*)$. Then

$$A^*Ac = -BB^*c. \tag{2.7}$$

By multiplying both sides of (2.7) with A, we get $-ABB^*c = AA^*Ac = 0$. Thus, $A^*Ac \in \operatorname{Ker}(A)$. By Proposition 2.1.5, we now have $A^*Ac \in \operatorname{Im}(A^*) = \operatorname{Ker}(A)^\perp$. So, $A^*Ac = 0$ and $c \in \operatorname{Ker}(A^*A) = \operatorname{Ker}(A)$. By multiplying both sides of (2.7) with B^*, we get $0 = B^*A^*Ac = -B^*BB^*c$. Then $BB^*c \in \operatorname{Ker}(B^*)$. By Proposition 2.1.5, $BB^*c \in \operatorname{Im}(B) = \operatorname{Ker}(B^*)^\perp$. So $BB^*c = 0$ and hence $c \in \operatorname{Ker}(BB^*) = \operatorname{Ker}(B^*)$. The proof of first equation is done.

Define $\phi : \operatorname{Ker}(A) \cap \operatorname{Ker}(B^*) \to {}^{\operatorname{Ker}(A)}\!/\!_{\operatorname{Im}(B)}$ by $x \mapsto x + \operatorname{Im}(B)$, for any $x \in \operatorname{Ker}(A) \cap \operatorname{Ker}(B^*)$. It is easy to see that ϕ is well-defined and linear. By Proposition 2.1.5 (ii), $\operatorname{Ker}(B^*) = \operatorname{Im}(B)^\perp$. Then

$$\begin{aligned}
\operatorname{Ker}\phi &= \{x \in \operatorname{Ker}(A) \cap \operatorname{Ker}(B^*) \mid \phi(x) = 0\} \\
&= \{x \in \operatorname{Ker}(A) \cap \operatorname{Ker}(B^*) \mid x \in \operatorname{Im}(B)\} \\
&= \{x \in \operatorname{Ker}(A) \cap \operatorname{Im}(B)^\perp \mid x \in \operatorname{Im}(B)\} \\
&= \{0\}.
\end{aligned}$$

This shows that ϕ is injective. Let $x + \operatorname{Im}(B) \in {}^{\operatorname{Ker}(A)}\!/\!_{\operatorname{Im}(B)}$, where $x \in \operatorname{Ker}(A)$. Write $x = v_1 + v_2$ where $v_1 \in \operatorname{Im}(B)$ and $v_2 \in \operatorname{Im}(B)^\perp = \operatorname{Ker}(B^*)$. Since $AB = 0$, we have $\operatorname{Im}(B) \subset \operatorname{Ker}(A)$. Then, $v_1 \in \operatorname{Ker}(A)$ and $0 = Ax = A(v_1 + v_2) = Av_1 + Av_2 = Av_2$. Then $v_2 \in \operatorname{Ker}(A)$. Consider $\phi(v_2) = v_2 + \operatorname{Im}(B) = x - v_1 + \operatorname{Im}(B) = x + \operatorname{Im}(B)$. This shows that ϕ is surjective. $\square$

Lemma 2.4.4. *Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times p}(\mathbb{R})$ such that $AB = 0$. Then*

(i) $\mathrm{Ker}(B^*) = \mathrm{Im}(A^*) \oplus \mathrm{Ker}(A^*A + BB^*)$;

(ii) $\mathbb{R}^n = \mathrm{Im}(A^*) \oplus \mathrm{Ker}(A^*A + BB^*) \oplus \mathrm{Im}(B)$.

Proof. (i) By Proposition 2.1.5 (iv), we get

$$\begin{aligned}
\mathrm{Ker}(B^*) &= \mathbb{R}^n \cap \mathrm{Ker}(B^*) \\
&= [\mathrm{Ker}(A) \oplus \mathrm{Im}(A^*)] \cap \mathrm{Ker}(B^*) \\
&= [\mathrm{Ker}(A) \cap \mathrm{Ker}(B^*)] \oplus [\mathrm{Im}(A^*) \cap \mathrm{Ker}(B^*)] \\
&= \mathrm{Ker}(A^*A + BB^*) \oplus [\mathrm{Im}(A^*) \cap \mathrm{Ker}(B^*)] \qquad \text{(By Proposition 2.4.3)} \\
&= \mathrm{Ker}(A^*A + BB^*) \oplus \mathrm{Im}(A^*) \qquad \text{(Since } B^*A^* = 0 \text{ i.e. } \mathrm{Im}(A^*) \subset \mathrm{Ker}(B^*)\text{)}.
\end{aligned}$$

(ii) By Proposition 2.1.5 (iv) and (i), we get

$$\begin{aligned}
\mathbb{R}^n &= \mathrm{Ker}(B^*) \oplus \mathrm{Im}(B) \\
&= \mathrm{Im}(A^*) \oplus \mathrm{Ker}(A^*A + BB^*) \oplus \mathrm{Im}(B),
\end{aligned}$$

and the proof is done. □

We emphasize that the statement (ii) of Lemma 2.4.4 is well-known as a *Hodge decomposition* which has various applications especially in ranking theory. See more in [11].

2.4.1 Hodge Laplacians on Simplicial Complex

Let X be a simplicial complex and X^k the set of k-simplices. From now on, we are going to write boundary maps and coboundary maps in terms of their matrix representations and elements in C_k and C^k as vectors on $\mathbb{R}^{n_k}$, where $n_k = |X^k|$. We now state the definition of a Hodge k-Laplacian matrix as follows:

Definition 2.4.2. Let B_k be a matrix representation of $\partial_k : C_k \to C_{k-1}$. The

Hodge k-Laplacian of X is defined to be

$$L_k = B_{k+1}B_{k+1}^T + B_k^T B_k,$$

Moreover, we define $L_k^{\mathrm{up}} := B_{k+1}B_{k+1}^T$ and $L_k^{\mathrm{down}} := B_k^T B_k$.

Remark 2. (i) Since $B_k B_{k+1} = 0$, a Hodge k-Laplacian matrix is a Hodge Laplacian matrix.

(ii) The definition above can be defined with the different notations. For example, we can write L_k in term of ∂ or δ or both;

$$L_k = \partial_{k+1}\partial_{k+1}^* + \partial_k^*\partial_k = \delta_k^*\delta_k + \delta_{k-1}\delta_{k-1}^* = \partial_{k+1}\delta_k + \delta_{k-1}\partial_k.$$

(iii) Since $B_0 = 0$, we have

$$L_0 = B_1B_1^T + B_0^T B_0 = B_1B_1^T = L$$

which is a Laplacian matrix.

Theorem 2.4.5. *Let X be a simplicial complex, L_k a Hodge k-Laplacian matrix on X and $n = |X^k|$. Then the followings hold.*

(i) All eigenvalues of L_k are nonnegative.

(ii) $\mathrm{Spec}^*(L_k) = \mathrm{Spec}^*(L_k^{\mathrm{up}}) \cup \mathrm{Spec}^*(L_k^{\mathrm{down}})$.

(iii) $\mathrm{Im}(B_k^T) \oplus \mathrm{Ker}(L_k) \oplus \mathrm{Im}(B_{k+1}) \cong \mathbb{R}^n$.

(iv) $H_k(X) \cong H^k(X) \cong \mathrm{Ker}(L_k)$.

(v) $\mathrm{Spec}(L_k^{\mathrm{up}}) = \mathrm{Spec}(L_{k+1}^{\mathrm{down}})$.

Proof. The proof of (i) to (iv) is done by replacing $A = B_k$ and $B = B_{k+1}$ into Lemma 2.4.1, Lemma 2.4.2, Lemma 2.4.4 (ii) and Lemma 2.4.3, respectively. For the last statement, we claim that for any two linear maps S and T, ST and TS

have the same set of eigenvalues. Let S, T be linear maps. Let v be an eigenvector of ST corresponding to eigenvalue λ. We first assume that $\lambda = 0$. Then S or T is not invertible, so is TS. This implies that λ is an eigenvalue of TS. Next, suppose that $\lambda \neq 0$. Then $STv = \lambda v$ and hence $TS(Tv) = \lambda(Tv)$. Since $\lambda \neq 0$, we have $Tv \neq 0$. Then Tv is an eigenvector corresponding to eigenvalue λ. Similarly, we can show that all eigenvalues of TS are eigenvalues of ST. So, the clam is done. Note that $L_k^{\text{up}} = B_{k+1}B_{k+1}^T$ and $L_{k+1}^{\text{down}} = B_{k+1}^T B_{k+1}$ and the last statement is done by the claim. $\square$

2.5 Random Walks on Graphs

In this section, we briefly introduce notations and definitions about a *Markov chain* and a *transition probability matrix*. We recommend [18] for more details.

A *stochastic process* $\{X(t), t \in T\}$ is a collection of random variables. The index T is often interpreted as time. If T is countable, the stochastic process is said to be a *discrete-time* process. We call $X(t)$ the *state* of the process at time t and if $X(t) = i$, then the process is said to be in state i at time t.

Definition 2.5.1. Let $\{X(t), t \in T\}$ be a discrete-time stochastic process. Suppose that whenever the process is in state i at time t, there is a fixed probability M_{ij} that the state at time $t+1$ is in state j. Then the stochastic process is called a **Markov chain** and M_{ij} is called a **transition probability**.

Definition 2.5.2. Let $\{X(t), t \in T\}$ be a Markov chain. Suppose that the set T is finite, then the matrix $M = (M_{ij})$, where M_{ij} is defined in Definition 2.5.1, is called a **transition probability matrix of a Markov chain** $\{X(t), t \in T\}$.

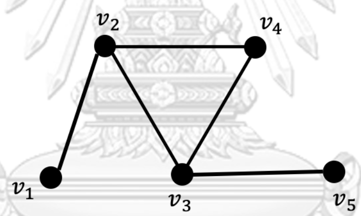
In other words, a Markov chain is a discrete-time stochastic process such that any future state X_{n+1} with given the past states $X_0, X_1, \dots, X_{n-1}, X_n$ depends only on the present state X_n. That is a transition probability does not depend upon the history of previous transitions. Note that $M_{ij} \in [0, 1]$ for all i, j and $\sum_j M_{ij} = 1$.

Let G be a connected simple graph with a vertex set $\{v_0, v_1, \dots, v_k, \dots, v_n\}$. A *random walk on graph* is a process of walking from the *root* v_0 along an edge by steps to a vertex v_k. Since the random walk picks a neighbor of a vertex each step randomly, random walk on graph is a Markov chain. In this work, all graphs that we consider are undirected and unweighted. For general studies, see [6].

Proposition 2.5.1. *Let $(D^{-1})_{ii} = \dfrac{1}{d_i}$ if $d_i \neq 0$ and 0 otherwise. Let A be an adjacency matrix of G. Then a transition probability matrix of a random walk on G is given by $M = D^{-1}A$.*

Proof. Suppose that the current state at time t is v_i. Then all states at time $t+1$ which are possible are v_j such that $v_i \sim v_j$. Therefore $M_{ij} = \dfrac{1}{d_i}$ if $d_i \neq 0$ and $M_{ij} = 0$ if $d_i = 0$. This shows that $M_{ij} = (D^{-1}A)_{ij}$. □

Example 2.5.1. Let G be the following graph;



A transition probability matrix of a random walk on this graph is given by

$$M = D^{-1}A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Proposition 2.5.2. *Let G be a connected graph, $\mathcal{L}^{\mathrm{rw}}$ a random walk normalized Laplacian matrix on G. Then $P = I - \mathcal{L}^{\mathrm{rw}}$ is a transition probability matrix of a random walk on G.*

Proof. Since $L = D - A$, where D is a degree matrix and A is an adjacency matrix on G,

$$P = I - \mathcal{L}^{\mathrm{rw}} = I - D^{-1}L = I - D^{-1}(D - A) = D^{-1}A.$$

By Proposition 2.5.1, the proof is done. $\square$

CHAPTER III
HODGE LAPLACIAN MATRIX

The Hodge theory on simplicial complexes can be applied in various fields for many applications. As shown in Section 2.4.1, the Hodge k-Laplacian is the higher dimensional forms of the Laplacian matrix for a given graph G.

In this chapter, we first generalize a degree matrix and an adjacency matrix on graphs to be a degree matrix and an adjacency matrix for any dimensions of simplicial complexes. Next, we analyze Hodge k-Laplacian matrix by writing it as a degree matrix and an adjacency matrix that we defined. This fact leads us to obtain the formula of Laplacian operator. We end this chapter with a proof of the beautiful fact that the kth homology and the kth cohomology of a simplicial complex are trivial if and only if the smallest eigenvalue of a Hodge k-Laplacian matrix is nonzero.

3.1 Hodge k-Laplacian

Recall that the Hodge k-Laplacian L_k is of the form

$$L_k = L_k^{\text{up}} + L_k^{\text{down}} = B_{k+1}B_{k+1}^T + B_k^T B_k,$$

where B_k is a matrix representation of $\partial_k : C_k \to C_{k-1}$. For the case that $k = 0$, L_k is exactly a Laplacian matrix $L = D - A$, where D is a degree matrix and A is an adjacency matrix.

3.1.1 Degree Matrices on Higher Dimensions

For a degree matrix $D = (D_{ij}) = d_i$ if $i = j$ and 0 otherwise, we observe that an entry d_i on its diagonal which is a degree of vertex v_i is a number of edges having

v_i as their face. Thus, we analogously define degree matrices on higher dimensions as followed.

Definition 3.1.1. For each k-simplex σ, define a **degree of** σ denoted by $\deg\sigma$ to be a number of its upper adjacent elements in $(k+1)$-simplex.

Definition 3.1.2. Let X^k be a set of all k-simplices of a simplicial complex X. A **degree matrix of** X^k is defined by $D'_{k+1} = (D'_{\sigma\tau})_{|X^k|\times|X^k|}$ where $D'_{\sigma\tau} = \deg\sigma$ if $\sigma = \tau$ and 0 otherwise.

For $k = 0$, D'_1 is exactly the same as the degree matrix defined in Definition 3.1.2. For $k = 1$, degree of an edge e is defined to be a number triangles which has e as their face. For $k = 2$, degree of an triangle t is defined to be a number tetrahedrals which has t as their face and so on.

Example 3.1.1. From the simplicial complex in Example 2.3.1, we get

$$D'_3 = \begin{array}{c} \\ {[v_1,v_2,v_3]} \\ {[v_1,v_2,v_4]} \\ {[v_1,v_3,v_4]} \\ {[v_2,v_3,v_4]} \\ {[v_5,v_7,v_8]} \\ {[v_7,v_8,v_9]} \end{array} \begin{array}{c} \begin{array}{cccccc} [v_1,v_2,v_3] & [v_1,v_2,v_4] & [v_1,v_3,v_4] & [v_2,v_3,v_4] & [v_5,v_7,v_8] & [v_7,v_8,v_9] \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and D'_4 is the zero matrix.

3.1.2 Adjacency Matrices on Higher Dimensions

Recall that an adjacency matrix on a graph G, denoted by $A(G)$, is defined by $(a_{ij}) = 1$ if v_i is adjacent to v_j and 0 otherwise. However, for any two simplices on higher dimensional, the word **adjacent** could be considered as **upper adjacent** or **lower adjacent** which is defined in Definition 2.3.3. We define adjacency matrices which are referred to upper adjacency and lower adjacency as follows:

Definition 3.1.3. Let X^k be a set of all k-simplices of a simplicial q-complex X and $0 \leq k \leq q$. The **upper adjacency matrix** of X^k is defined by $A_k^{\text{up}} = (a_{\sigma\tau}^{\text{up}})$ where

$$a_{\sigma\tau}^{\text{up}} = \begin{cases} \operatorname{sgn}(\sigma, \partial\overline{\sigma})\operatorname{sgn}(\tau, \partial\overline{\sigma}), & \text{if } \sigma \sim_u \tau \text{ and } \sigma, \tau \in \partial\overline{\sigma}, \\ 0, & \text{otherwise.} \end{cases}$$

The **lower adjacency matrix** of X^k is defined by $A_k^{\text{down}} = (a_{\sigma\tau}^{\text{down}})$ where

$$a_{\sigma\tau}^{\text{down}} = \begin{cases} \operatorname{sgn}((\sigma \cap \tau), \partial\sigma)\operatorname{sgn}((\sigma \cap \tau), \partial\tau), & \text{if } \sigma \sim_l \tau, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.1.2. From the simplicial complex in the example 2.3.1, we get

$$A_2^{\text{up}} = \begin{array}{c} \\ [v_1,v_2,v_3] \\ [v_1,v_2,v_4] \\ [v_1,v_3,v_4] \\ [v_2,v_3,v_4] \\ [v_5,v_7,v_8] \\ [v_7,v_8,v_9] \end{array} \begin{array}{c} \begin{array}{cccccc} [v_1,v_2,v_3] & [v_1,v_2,v_4] & [v_1,v_3,v_4] & [v_2,v_3,v_4] & [v_5,v_7,v_8] & [v_7,v_8,v_9] \end{array} \\ \begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

and

$$A_2^{\text{down}} = \begin{array}{c} \\ [v_1,v_2,v_3] \\ [v_1,v_2,v_4] \\ [v_1,v_3,v_4] \\ [v_2,v_3,v_4] \\ [v_5,v_7,v_8] \\ [v_7,v_8,v_9] \end{array} \begin{array}{c} \begin{array}{cccccc} [v_1,v_2,v_3] & [v_1,v_2,v_4] & [v_1,v_3,v_4] & [v_2,v_3,v_4] & [v_5,v_7,v_8] & [v_7,v_8,v_9] \end{array} \\ \begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Note that both A_k^{up} and A_k^{down} are $|X^k| \times |X^k|$ symmetric matrices whose entries are 1 or -1. For $k = 0$, since two adjacent vertices always have the different signs, A_0^{up} is $-A$, where A is an adjacency matrix defined in Definition 2.2.1.

The following proposition states L_k^{up} and L_k^{down} in terms of D'_{k+1}, A_k^{up} and A_k^{down}. As entries of A_k^{up} and A_k^{down} are -1 or 1 and D'_{k+1} is a diagonal matrix, writing L_k^{up} and L_k^{down} in this way can be useful for doing algebra on them.

Proposition 3.1.1. *For $k \geq 1$, the matrices L_k^{up} and L_k^{down} can be written as*

$$L_k^{\text{up}} = D'_{k+1} + A_k^{\text{up}} \textit{ and } L_k^{\text{down}} = (k+1)I_k + A_k^{\text{down}},$$

where I_k, D'_{k+1}, A_k^{up}, A_k^{down} are the $|X^k| \times |X^k|$ identity matrix, a degree matrix, an upper adjacency matrix and a lower adjacency matrix of simplicial k-complex, respectively.

Proof. Let X be a simplicial complex and $\sigma, \tau \in X^k$. If $\sigma = \tau$, then $(L_k^{\text{up}})_{\sigma\tau} = (B_{k+1}B_{k+1}^T)_{\sigma\tau} = \displaystyle\sum_{\overline{\sigma};\sigma\in\partial\overline{\sigma}} \text{sgn}(\sigma, \partial\overline{\sigma})^2 = \deg\sigma = (D_{k+1})_{\sigma\tau}$ and $(L_k^{\text{down}})_{\sigma\tau} = (B_k^T B_k)_{\sigma\tau} = \displaystyle\sum_{\mu\in\partial\sigma} (\text{sgn}(\mu, \partial\sigma))^2 = k+1$. Next, suppose that $\sigma \neq \tau$. If σ is not upper adjacent to τ, then $(L_k^{\text{up}})_{\sigma\tau} = 0 = (A_k^{\text{up}})_{\sigma\tau}$, otherwise, $(L_k^{\text{up}})_{\sigma\tau} = \text{sgn}(\sigma, \partial\overline{\sigma})\text{sgn}(\tau, \partial\overline{\sigma}) = (A_k^{\text{up}})_{\sigma\tau}$, where $\sigma, \tau \in \partial\overline{\sigma}$. If σ is not lower adjacent to τ, then $(L_k^{\text{down}})_{\sigma\tau} = 0 = (A_k^{\text{down}})_{\sigma\tau}$, otherwise, $(L_k^{\text{down}})_{\sigma\tau} = \text{sgn}(\sigma\cap\tau, \partial\sigma)\text{sgn}(\sigma\cap\tau, \partial\tau) = (A_k^{\text{down}})_{\sigma\tau}$. $\square$

Remark 3. For $k = 0$, $L_0^{\text{up}} = D'_1 + A_0^{\text{up}} = D - A = L$ where L is a Laplacian matrix. However, with this Proposition, $L_0^{\text{down}} = I_{|X^0|}$ which contradicts with $L_0^{\text{down}} = B_0 B_0^T = 0$. We avoid this confusion by adding the assumption that $k \geq 1$ and let the readers keep in mind that L_0^{down} is a zero matrix.

From Proposition 3.1.1, we obtain the formula of the Hodge Laplacian operator. Note that, in this work, we only consider unweighted simplicial complexes. For the formula of the Hodge Laplaician operator on weighted simplicial complexes, we recommend [9].

Proposition 3.1.2. *Let $k \geq 1$, $f \in C^k = Hom(C_k, \mathbb{R})$ and $\sigma \in X^k$. The operators*

$\Delta_k^{\text{up}} := \partial_{k+1}\delta_{k+1}$ and $\Delta_k^{\text{down}} := \delta_k\partial_k$ are given by

$$(\Delta_k^{\text{up}} f)(\sigma) = \deg(\sigma) f(\sigma) + \sum_{\substack{\sigma' \in X^k, \sigma \sim_u \sigma' \\ \sigma, \sigma' \in \partial\overline{\sigma}}} \operatorname{sgn}(\sigma, \partial\overline{\sigma}) \operatorname{sgn}(\sigma', \partial\overline{\sigma}) f(\sigma'),$$

$$(\Delta_k^{\text{down}} f)(\sigma) = (k+1) f(\sigma) + \sum_{\substack{\sigma' \in X^k \\ \sigma \sim_l \sigma'}} \operatorname{sgn}(\sigma \cap \sigma', \partial\sigma) \operatorname{sgn}(\sigma \cap \sigma', \partial\sigma') f(\sigma').$$

Proof. Let X be a simplicial complex, $f \in C^k$ and $n = |X^k|$. Since $X^k = \{\tau_1, \tau_2, \dots, \tau_n\}$ is a basis of C_k, we get $\{\tau^1, \tau^2, \dots, \tau^n\}$ is a basis of C^k, where

$$\tau^i(\tau_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Write $f = \sum_{i=1}^{n} \beta_i \tau^i$, where $\beta_i \in \mathbb{R}$ for each i. Note that $f \in C^k \cong C_k \cong \mathbb{R}^n$. It is easy to see that $\phi : C^k \to \mathbb{R}^n$ defined by $\left(\sum_{i=1}^{n} \beta_i \tau^i\right) \mapsto (\beta_1, \beta_2, \dots, \beta_n)^T$ is an isomorphism. Then f can be viewed as a column vector $(\beta_1, \beta_2, \dots, \beta_n)^T$. To avoid confusion, we write $[f] = (\beta_1, \beta_2, \dots, \beta_n)^T$. Note that matrix representations of Δ_k^{up} and Δ_k^{down} are L_k^{up} and L_k^{down}, respectively. By Proposition 3.1.1, $L_k^{\text{up}} = D'_{k+1} + A_k^{\text{up}}$. Consider

$$\begin{aligned} L_k^{\text{up}}[f] &= (D'_{k+1} + A_k^{\text{up}})[f] \\ &= \begin{bmatrix} (\deg \tau_1)\beta_1 + \sum\limits_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_1 \\ \tau_i, \tau_1 \in \partial\tau}} \operatorname{sgn}(\tau_1, \partial\tau)\operatorname{sgn}(\tau_i, \partial\tau) \cdot \beta_i \\ (\deg \tau_2)\beta_2 + \sum\limits_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_2 \\ \tau_i, \tau_2 \in \partial\tau}} \operatorname{sgn}(\tau_2, \partial\tau)\operatorname{sgn}(\tau_i, \partial\tau) \cdot \beta_i \\ \vdots \\ (\deg \tau_n)\beta_n + \sum\limits_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_n \\ \tau_i, \tau_n \in \partial\tau}} \operatorname{sgn}(\tau_n, \partial\tau)\operatorname{sgn}(\tau_i, \partial\tau) \cdot \beta_i \end{bmatrix}. \end{aligned}$$

This column vector corresponds to a map $\Delta_k^{\text{up}} f = \sum_{i=1}^{n} \gamma_i \tau^i$, where γ_i is the element in ith row of $L_k^{\mathrm{up}}[f]$. Let $\tau_i \in X^k$. Since $f(\tau_k) = \sum_{j=1}^{n} \beta_j \tau^j(\tau_k) = \beta_k$ for any k, we have

$$\begin{aligned}(\Delta_k^{\mathrm{up}})(\tau_i) &= \sum_{j=1}^{n} \gamma_j \tau^j(\tau_i) = \gamma_i \\ &= (\deg \tau_i)\beta_i + \sum_{\substack{\tau_j \in X^k, \tau_j \sim_u \tau_i \\ \tau_j, \tau_i \in \partial\tau}} \mathrm{sgn}(\tau_i, \partial\tau)\mathrm{sgn}(\tau_j, \partial\tau) \cdot \beta_j \\ &= (\deg \tau_i) f(\tau_i) + \sum_{\substack{\tau_j \in X^k, \tau_j \sim_u \tau_i \\ \tau_j, \tau_i \in \partial\tau}} \mathrm{sgn}(\tau_i, \partial\tau)\mathrm{sgn}(\tau_j, \partial\tau) \cdot f(\tau_j).\end{aligned}$$

By Proposition 3.1.1, $L_k^{\mathrm{down}} = (k+1)I_n + A_k^{\mathrm{down}}$. Consider

$$\begin{aligned}L_k^{\mathrm{down}}[f] &= ((k+1)I_n + A_k^{\mathrm{down}})[f] \\ &= \begin{bmatrix} (k+1)\beta_1 + \sum\limits_{\substack{\tau_i \in X^k \\ \tau_i \sim_l \tau_1}} \mathrm{sgn}(\tau_1 \cap \tau_i, \partial\tau_1)\mathrm{sgn}(\tau_1 \cap \tau_i, \partial\tau_i) \cdot \beta_i \\ (k+1)\beta_2 + \sum\limits_{\substack{\tau_i \in X^k \\ \tau_i \sim_l \tau_2}} \mathrm{sgn}(\tau_2 \cap \tau_i, \partial\tau_2)\mathrm{sgn}(\tau_2 \cap \tau_i, \partial\tau_i) \cdot \beta_i \\ \vdots \\ (k+1)\beta_n + \sum\limits_{\substack{\tau_i \in X^k \\ \tau_i \sim_l \tau_n}} \mathrm{sgn}(\tau_n \cap \tau_i, \partial\tau_n)\mathrm{sgn}(\tau_n \cap \tau_i, \partial\tau_i) \cdot \beta_i \end{bmatrix}.\end{aligned}$$

This column vector corresponds to a map $\Delta_k^{\mathrm{down}} f = \sum_{i=1}^{n} \alpha_i \tau^i$, where α_i is the element in ith row of $L_k^{\mathrm{down}}[f]$. Since $f(\tau_k) = \sum_{j=1}^{n} \beta_j \tau^j(\tau_k) = \beta_k$ for any k, we have

$$\begin{aligned}(\Delta_k^{\mathrm{down}})(\tau_i) &= \sum_{j=1}^{n} \alpha_j \tau^j(\tau_i) = \alpha_i \\ &= (k+1)\beta_i + \sum_{\substack{\tau_j \in X^k \\ \tau_j \sim_l \tau_i}} \mathrm{sgn}(\tau_i \cap \tau_j, \partial\tau_i)\mathrm{sgn}(\tau_i \cap \tau_j, \partial\tau_j) \cdot \beta_j \\ &= (k+1) f(\tau_i) + \sum_{\substack{\tau_j \in X^k \\ \tau_j \sim_l \tau_i}} \mathrm{sgn}(\tau_i \cap \tau_j, \partial\tau_i)\mathrm{sgn}(\tau_i \cap \tau_j, \partial\tau_j) f(\tau_j). \quad \square\end{aligned}$$

3.2 kth Homology on Simplicial Complex and the Smallest Eigenvalue of Hodge k-Laplacian Matrix

Sometimes, a chain complex (also a cochain)

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

is denoted by C_* or (C_*, ∂_*) (or C^* or (C^*, δ_*)). Moreover, for a simplicial complex X, we can denote a kth-homology on X (or a kth-cohomology on X) by $H_k(C_*)$ (or $H^k(C^*)$).

A chain complex (or cochian complexes) together with a map f satisfying

$$\partial f = f\partial \tag{3.1}$$

(or a map g satisfying $\delta g = g\delta$) can be viewed as a *direct sum of chain complexes* (or a direct sum of cochain complex). To see this, some tools in homological algebra are required, see more in [3], [10], [14], [15], [20].

Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \to & C_{n+1} & \xrightarrow{B_{n+1}} & C_n & \xrightarrow{B_n} & C_{n-1} & \to & \cdots \\
 & & \big\downarrow L_{n+1} & & \big\downarrow L_n & & \big\downarrow L_{n-1} & & \\
\cdots & \to & C_{n+1} & \xrightarrow{B_{n+1}} & C_n & \xrightarrow{B_n} & C_{n-1} & \to & \cdots
\end{array}$$

Since

$$B_{n+1}L_{n+1} = B_{n+1}B_{n+2}B_{n+2}^T + B_{n+1}B_{n+1}^T B_{n+1} = B_{n+1}B_{n+1}^T B_{n+1}$$

and

$$L_n B_{n+1} = B_{n+1}B_{n+1}^T B_{n+1} + B_n^T B_n B_{n+1} = B_{n+1}B_{n+1}^T B_{n+1},$$

this diagram is commute for any n, i.e. $B_*L_* = L_*B_*$. Let C_*^λ be an eigenspace corresponding eigenvalue λ of a Hodge Laplacian matrix. Then

$$C_* = \bigoplus_{\lambda} C_*^{\lambda}$$

and the direct sum $(\bigoplus_{\lambda} C_*^{\lambda}, \partial)$ is a chain complex with $\partial = \oplus_* \partial_*$. Moreover, this fact implies that a homology commutes with direct sums, i.e. for all n,

$$H_n\left(\bigoplus_{\lambda} C_*^{\lambda}\right) \cong \bigoplus_{\lambda} H_n(C_*^{\lambda}),$$

we recommend [20] for more details.

Theorem 3.2.1. *[1] Let X be a simplicial complex and L_k a Hodge k-Laplacian matrix on X. Let λ be the smallest eigenvalue of L_k and $H_k(X)$ a kth-homology on X. Then*

$$\lambda \neq 0 \textit{ if and only if } H_k(X) = 0.$$

Proof. Let $\lambda \neq 0$ be an eigenvalue of L_k and C_k^{λ} be an eigenspace corresponding to λ. Let $c \in C_k^{\lambda}$ such that $B_k c = 0$. Then,

$$c = \frac{1}{\lambda}(B_{k+1}B_{k+1}^T c) = B_{k+1}(\frac{1}{\lambda}B_{k+1}^T c).$$

That is each cycle of C_k^{λ} is a boundary. Since λ is the smallest eigenvalue of L_k, every eigenvalue of L_k is also nonzero. This implies that for any λ every cycle of C_k^{λ} is a boundary. Then,

$$H_k(X) = H_k\left(\bigoplus_{\lambda} C^{\lambda}\right) \cong \bigoplus_{\lambda} H_k(C^{\lambda}) = 0.$$

Conversely, let $c \in C_k^\lambda$ be an eigenvector corresponding to eigenvalue $\lambda = 0$. Then

$$\begin{aligned}
0 &= \langle L_k c, c\rangle \\
&= \langle B_{k+1}B_{k+1}^T c, c\rangle + \langle B_k^T B_k c, c\rangle \\
&= \langle B_{k+1}^T c, B_{k+1}^T c\rangle + \langle B_k c, B_k c\rangle \\
&= \|B_{k+1}^T c\|^2 + \|B_k c\|^2.
\end{aligned}$$

This implies that $H_k(C_*^\lambda) = C_k^\lambda$. Hence,

$$H_k(X) = H_k\left(\bigoplus_\lambda C_*^\lambda\right) \cong \bigoplus_\lambda H_k(C_*^\lambda) \neq 0. \qquad \square$$

From Remark 1, we obtain the following corollary.

Corollary 3.2.2. *Let X be a simplicial complex and L_k a Hodge k-Laplacian matrix on X. Let λ be the smallest eigenvalue of L_k and $H^k(X)$ a kth-cohomology on X. Then*

$$\lambda \neq 0 \textit{ if and only if } H^k(X) = 0.$$

CHAPTER IV
NORMALIZED HODGE LAPLACIAN MATRIX

In this section, we define a normalized Hodge Laplacian matrix for arbitrary dimensions of simplicial complexes which is also a Hodge Laplacian matrix. Then, we state some properties of the matrix which are obtained directly by being Hodge Laplacian matrix. Using some results in Chapter 3, we obtain the formula of normalized Hodge Laplacian operator. We also indicate a spectrum of this matrix for the case that a simplicial complex is itself a simplex. For the second section, we show that the smallest eigenvalue of $\mathcal{L}_k$ can indicate whether the kth homology and the kth cohomology on a given simplicial complex are trivial. We end this chapter by showing a relation between a spectrum of $\mathcal{L}_*^{\text{up}}$ and k-wedge sum of simplices.

4.1 Definition and Basic Properties

There are several works of defining normalized Laplacian operator and normalized Laplacian matrix on a simplicial complex.

In 1993, F. Chung [5] defined a normalized Laplacian operator as

$$\partial\delta + \rho\delta\partial,$$

where ρ is the density of a given simplicial complex.

In 2010, C. Taszus [19] defined a normalized Laplacian matrix as

$$(D'_{k+1})^{-1/2} L_k (D'_{k+1})^{-1/2}.$$

We point out here that $(D'_{k+1})^{-1/2}$ needs not be invertible matrix.

In 2011, D. Horak [9] defined a normalized combinatorial Laplace operator in order to force an upper bound of the maximal eigenvalue of the operator to be a constant.

In 2018, M. Schaub et al [16] defined a normalized Hodge 1-Laplacian matrix to study random walks. The matrix is defined by

$$\mathcal{L}_1 = D_2 B_1^T D_1^{-1} B_1 + B_2 D_3 B_2^T D_2^{-1},$$

where $(D_2)_{[i,j],[i,j]} = \max\{\deg[i,j], 1\}$, $D_1 = 2 \times \operatorname{diag}(|B_1 D_2|\mathbf{1})$ and D_3 is the diagonal matrix with $\frac{1}{3}$ on the diagonal.

However, matrix representations of these operators and the matrix defined above are not Hodge Laplacian matrices. We now define a normalized Hodge Laplacian matrices for arbitrary dimensions of simplicial complexes.

Definition 4.1.1. Let C_k be a space of k-chain of simplicial complex X and X^k a set of all k-simplices on X. Let B_k be a matrix representation of a boundary map $\partial_k : C_k \to C_{k-1}$. The **normalized Hodge k-Laplacian matrix** $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k = D_{k+1}^{-1/2} B_{k+1} B_{k+1}^T D_{k+1}^{-1/2} + D_{k+1}^{1/2} B_k^T B_k D_{k+1}^{1/2},$$

where $D_{k+1}^{1/2}$ and $D_{k+1}^{-1/2}$ are $|X^k| \times |X^k|$ diagonal matrices defined by $(D_{k+1}^{1/2})_{\sigma\tau} = \max\{\sqrt{\deg \sigma}, 1\}$ if $\sigma = \tau$ and 0 otherwise, and $D_{k+1}^{-1/2}$ is the inverse of $D_{k+1}^{1/2}$. Moreover, we define

$$\mathcal{L}_k^{\mathrm{up}} := D_{k+1}^{-1/2} B_{k+1} B_{k+1}^T D_{k+1}^{-1/2},$$

$$\mathcal{L}_k^{\mathrm{down}} := D_{k+1}^{1/2} B_k^T B_k D_{k+1}^{1/2}.$$

According to this definition, we have

$$\mathcal{L}_0 = D_1^{-1/2} B_1 B_1^T D_1^{-1/2} + D_1^{1/2} B_0^T B_0 D_1^{1/2} = D_1^{-1/2} B_1 B_1^T D_1^{-1/2} = \mathcal{L}.$$

A purpose of putting a maximum on $D_{k+1}^{1/2}$ for each k is to guarantee that $D_{k+1}^{1/2}$ is invertible. This leads our definition of $\mathcal{L}_k$ preserving some properties obtained analogously to properties of Hodge k-Laplacian matrix shown in Theorem 4.1.1.

From the definition, one can see that this matrix is positive definite, real symmetric and all of its eigenvalues are real numbers.

Remark 4. Let $(D'_{k+1})^{-1/2}$ be a diagonal matrix defined by $(D'_{k+1})^{-1/2}_{\sigma\tau} = \dfrac{1}{\sqrt{\deg \sigma}}$ if $\sigma = \tau$, $\deg \sigma \neq 0$ and 0 otherwise. By Proposition 3.1.1, we get

$$(D'_{k+1})^{-1/2} L_k^{\mathrm{up}} (D'_{k+1})^{-1/2} = (D'_{k+1})^{-1/2} D'_{k+1} (D'_{k+1})^{-1/2} + (D'_{k+1})^{-1/2} A_k^{\mathrm{up}} (D'_{k+1})^{-1/2} \tag{4.1}$$

and

$$D_{k+1}^{-1/2} L_k^{\mathrm{up}} D_{k+1}^{-1/2} = D_{k+1}^{-1/2} D'_{k+1} D_{k+1}^{-1/2} + D_{k+1}^{-1/2} A_k^{\mathrm{up}} D_{k+1}^{-1/2}. \tag{4.2}$$

Let X be a simplicial complex. If every simplex in X^k has nonzero degree, then $D_{k+1}^{-1/2} = (D'_{k+1})^{-1/2}$ and hence $D_{k+1}^{-1/2} L_k^{\mathrm{up}} D_{k+1}^{-1/2} = (D'_{k+1})^{-1/2} L_k^{\mathrm{up}} (D'_{k+1})^{-1/2}$. Suppose that there is a simplex $\sigma \in X^k$ such that $\deg \sigma = 0$. Then for any $\tau \in X^k$, we have $((D'_{k+1})^{-1/2})_{\sigma\tau} = 0$ and $(A_k^{\mathrm{up}})_{\sigma\tau} = 0$. From the equations (4.1) and (4.2), $((D'_{k+1})^{-1/2} L_k^{\mathrm{up}} (D'_{k+1})^{-1/2})_{\sigma\tau} = 0 = (D_{k+1}^{-1/2} L_k^{\mathrm{up}} D_{k+1}^{-1/2})_{\sigma\tau}$ for any $\tau \in X^k$. Similarly, we can show that $((D'_{k+1})^{-1/2} L_k^{\mathrm{up}} (D'_{k+1})^{-1/2})_{\tau\sigma} = 0 = (D_{k+1}^{-1/2} L_k^{\mathrm{up}} D_{k+1}^{-1/2})_{\tau\sigma}$ for any $\tau \in X^k$. We now conclude that $D_{k+1}^{-1/2} L_k^{\mathrm{up}} D_{k+1}^{-1/2} = (D'_{k+1})^{-1/2} L_k^{\mathrm{up}} (D'_{k+1})^{-1/2}$. This shows that defining $\mathcal{L}_k$ by putting a maximum on a degree matrix instead of using a degree matrix does not change the meaning of $\mathcal{L}_k^{\mathrm{up}}$.

We give the next theorem to support our idea of defining $\mathcal{L}_k$ to be a Hodge Laplacian matrix and $D_{k+1}^{-1/2}$ to be an invertible matrix.

Theorem 4.1.1. *Let X be a simplicial complex, $\mathcal{L}_k$ a normalized Hodge k-Laplacian matrix on X and $n = |X^k|$. Then the followings hold.*

(i) *All eigenvalues of $\mathcal{L}_k$ are nonnegative.*

(ii) $\mathrm{Spec}^*(\mathcal{L}_k) = \mathrm{Spec}^*(\mathcal{L}_k^{\mathrm{up}}) \cup \mathrm{Spec}^*(\mathcal{L}_k^{\mathrm{down}})$.

(iii) $\mathrm{Im}(D_{k+1}^{1/2} B_k^T) \oplus \mathrm{Ker}(\mathcal{L}_k) \oplus \mathrm{Im}(D_{k+1}^{-1/2} B_{k+1}) \cong \mathbb{R}^n$.

(iv) $H_k(X) \cong H^k(X) \cong \mathrm{Ker}(\mathcal{L}_k)$.

Proof. The first, the second and the third statements are done by replacing $A = B_k D_{k+1}^{1/2}$ and $B = D_{k+1}^{-1/2} B_{k+1}$ into Lemma 2.4.1, Lemma 2.4.2 and Lemma 2.4.4 (ii), respectively. Note that from the definition of $\mathcal{L}_k$, we know that $D_{k+1}^{-1/2}$ and $D_{k+1}^{1/2}$ are invertible. In the other words, it can be considered as a representation of a bijective map. Then by Lemma 2.1.8, we have $\mathrm{Ker}(B_k D_{k+1}^{1/2}) \cong \mathrm{Ker}(B_k)$ and $\mathrm{Ker}(B_{k+1}^T D_{k+1}^{-1/2}) \cong \mathrm{Ker}(B_{k+1}^T)$. Therefore, by Lemma 2.1.8

$$\mathrm{Ker}(B_k D_{k+1}^{1/2}) \cap \mathrm{Ker}(B_{k+1}^T D_{k+1}^{-1/2}) \cong \mathrm{Ker}(B_k) \cap \mathrm{Ker}(B_{k+1}^T) \cong H_k(X)$$

and the last statement is done. $\square$

There is a well-known fact that eigenvalues of a normalized Laplacian matrix on a graph are nonnegative. Theorem 4.1.1 (i) shows that our definition remains this fact. By Theorem 4.1.1 (ii), we can calculate all of nonzero eigenvalues of $\mathcal{L}_k$ by calculating on $\mathcal{L}_k^{\mathrm{up}}$ and $\mathcal{L}_k^{\mathrm{down}}$ separately. From Theorem 4.1.1 (iv), for a given simplicial complex, we can calculate its homology and cohomology by considering the kernel of $\mathcal{L}_k$. It can be shown that $\mathrm{Im}(D_{k+1}^{1/2} B_k^T) \cong \mathrm{Im}(B_k^T)$ and $\mathrm{Im}(D_{k+1}^{-1/2} B_{k+1}) \cong \mathrm{Im}(B_{k+1})$. By Theorem 3.2.1 and Theorem 4.1.1 (iv), we obtain that $\mathrm{Ker}(L_k) = \mathrm{Ker}(\mathcal{L}_k)$ and hence the decomposition of $\mathbb{R}^n$ in Theorem 4.1.1 (iii) can be considered as a Hodge decomposition.

Proposition 4.1.2. *Let $k \geq 1$, $f \in C^k = Hom(C_k, \mathbb{R})$ and $\sigma \in X^k$. The operator $\widetilde{\Delta}_k^{\mathrm{up}}$ corresponding to $\mathcal{L}_k^{\mathrm{up}}$ is given by*

$$(\widetilde{\Delta}_k^{\mathrm{up}} f)(\sigma) = \begin{cases} f(\sigma) + \displaystyle\sum_{\substack{\sigma' \in X^k, \sigma \sim_u \sigma' \\ \sigma, \sigma' \in \partial\overline{\sigma}}} \frac{\mathrm{sgn}(\sigma, \partial\overline{\sigma})\mathrm{sgn}(\sigma', \partial\overline{\sigma})}{\sqrt{\deg\sigma \deg\sigma'}} f(\sigma'), & \deg\sigma \neq 0, \\ 0 & \deg\sigma = 0. \end{cases}$$

The operator $\widetilde{\Delta}_k^{\mathrm{down}}$ corresponding to $\mathcal{L}_k^{\mathrm{down}}$ is given by

$$(\widetilde{\Delta}_k^{\mathrm{down}} f)(\sigma) = (k+1)\max\{\deg\sigma, 1\} f(\sigma)$$

$$+ \sum_{\substack{\sigma' \in X^k, \sigma \sim_l \sigma' \\ \sigma \cap \sigma' = \tau}} \max\{\deg\sigma, 1\}\max\{\deg\sigma', 1\}\mathrm{sgn}(\tau, \partial\sigma)\mathrm{sgn}(\tau, \partial\sigma') f(\sigma').$$

Proof. Let X be a simplicial complex, $f \in C^k$ and $n = |X^k|$. Similar to the proof of Proposition 3.1.2, we write $f = \sum_{i=1}^{n} \beta_i \tau^i$. We write $[f] = (\beta_1, \beta_2, \dots, \beta_n)^T$. By Proposition 3.1.1, $L_k^{\mathrm{up}} = D'_{k+1} + A_k^{\mathrm{up}}$. Consider

$$\begin{aligned}
\mathcal{L}_k^{\mathrm{up}}[f] &= (D_{k+1}^{-1/2} L_k^{\mathrm{up}} D_{k+1}^{-1/2})[f] \\
&= (D_{k+1}^{-1/2}(D'_{k+1} + A_k^{\mathrm{up}}) D_{k+1}^{-1/2})[f] \\
&= (D_{k+1}^{-1/2} D'_{k+1} D_{k+1}^{-1/2} + D_{k+1}^{-1/2} A_k^{\mathrm{up}} D_{k+1}^{-1/2})[f] \\
&= \begin{bmatrix}
\delta_1\beta_1 + \sum\limits_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_1 \\ \tau_i, \tau_1 \in \partial\tau}} \dfrac{\mathrm{sgn}(\tau_1, \partial\tau)\mathrm{sgn}(\tau_i, \partial\tau)}{\sqrt{\deg\tau_1 \deg\tau_i}}\beta_i \\
\delta_2\beta_2 + \sum\limits_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_2 \\ \tau_i, \tau_2 \in \partial\tau}} \dfrac{\mathrm{sgn}(\tau_2, \partial\tau)\mathrm{sgn}(\tau_i, \partial\tau)}{\sqrt{\deg\tau_2 \deg\tau_i}}\beta_i \\
\vdots \\
\delta_n\beta_n + \sum\limits_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_n \\ \tau_i, \tau_n \in \partial\tau}} \dfrac{\mathrm{sgn}(\tau_n, \partial\tau)\mathrm{sgn}(\tau_i, \partial\tau)}{\sqrt{\deg\tau_n \deg\tau_i}}\beta_i
\end{bmatrix}
\end{aligned}$$

where

$$\delta_i = \begin{cases} 1, & \deg\sigma_i \neq 0, \\ 0, & \deg\sigma_i = 0. \end{cases}$$

This column vector corresponds to a map $\widetilde{\Delta}_k^{\mathrm{up}} f = \sum_{i=1}^{n} \gamma_i \tau^i$, where γ_i is the element in ith row of $\mathcal{L}_k^{\mathrm{up}}[f]$. Let $\tau_q \in X^k$. If $\deg\tau_q = 0$, then $(\widetilde{\Delta}_k^{\mathrm{up}} f)(\tau_q) = \gamma_q = 0$. Suppose that $\deg\tau_q \neq 0$. Since $f(\tau_i) = \sum_{j=1}^{n} \beta_j \tau^j(\tau_i) = \beta_i$ for any i, we have

$$\begin{aligned}
(\widetilde{\Delta}_k^{\text{up}} f)(\tau_q) &= \sum_{i=1}^{n} \gamma_i \tau^i(\tau_q) = \gamma_q \\
&= \beta_q + \sum_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_q \\ \tau_i, \tau_q \in \partial\tau}} \frac{\operatorname{sgn}(\tau_q, \partial\tau)\operatorname{sgn}(\tau_i, \partial\tau)}{\sqrt{\deg \tau_q \deg \tau_i}} \beta_i \\
&= f(\tau_q) + \sum_{\substack{\tau_i \in X^k, \tau_i \sim_u \tau_q \\ \tau_i, \tau_q \in \partial\tau}} \frac{\operatorname{sgn}(\tau_q, \partial\tau)\operatorname{sgn}(\tau_i, \partial\tau)}{\sqrt{\deg \tau_q \deg \tau_i}} f(\tau_i).
\end{aligned}$$

By Proposition 3.1.1, $L_k^{\text{down}} = (k+1)I_n + A_k^{\text{down}}$. Consider

$$\begin{aligned}
\mathcal{L}_k^{\text{down}}[f] &= (D_{k+1}^{1/2} L_k^{\text{down}} D_{k+1}^{1/2})[f] \\
&= (D_{k+1}^{1/2}((k+1)I_n + A_k^{\text{down}})D_{k+1}^{1/2})[f] \\
&= ((k+1)D_{k+1} + D_{k+1}^{1/2} A_k^{\text{down}} D_{k+1}^{1/2})[f] \\
&= \begin{bmatrix}
(k+1)\phi_1(\beta_1) + \displaystyle\sum_{\substack{\tau_i \in X^k, \tau_i \cap \tau_1 = \sigma \\ \tau_i \sim_l \tau_1}} \phi_1 \phi_i \operatorname{sgn}(\sigma, \partial\tau_1)\operatorname{sgn}(\sigma, \partial\tau_i)\beta_i \\
(k+1)\phi_2(\beta_2) + \displaystyle\sum_{\substack{\tau_i \in X^k, \tau_i \cap \tau_2 = \sigma \\ \tau_i \sim_l \tau_2}} \phi_2 \phi_i \operatorname{sgn}(\sigma, \partial\tau_2)\operatorname{sgn}(\sigma, \partial\tau_i)\beta_i \\
\vdots \\
(k+1)\phi_n(\beta_n) + \displaystyle\sum_{\substack{\tau_i \in X^k, \tau_i \cap \tau_n = \sigma \\ \tau_i \sim_l \tau_n}} \phi_n \phi_i \operatorname{sgn}(\sigma, \partial\tau_n)\operatorname{sgn}(\sigma, \partial\tau_i)\beta_i
\end{bmatrix}
\end{aligned}$$

where $\phi_i = \max\{\deg \tau_i, 1\}$. This column vector corresponds to a map $\widetilde{\Delta}_k^{\text{down}} f = \sum_{i=1}^{n} \alpha_i \tau^i$, where α_i is the element in ith row of $\mathcal{L}_k^{\text{down}}[f]$. Let $\tau_q \in X^k$. Since $f(\tau_i) = \sum_{j=1}^{n} \beta_j \tau^j(\tau_i) = \beta_i$ for any i, we have

$$\begin{aligned}
(\widetilde{\Delta}_k^{\text{down}} f)(\tau_q) &= \sum_{i=1}^{n} \alpha_i \tau^i(\tau_q) = \alpha_q \\
&= (k+1)\max\{\deg \tau_q, 1\}\beta_q \\
&\quad + \sum_{\substack{\tau_i \in X^k, \tau_i \cap \tau_q = \sigma \\ \tau_i \sim_l \tau_q}} \max\{\deg \tau_q, 1\}\max\{\deg \tau_i, 1\}\operatorname{sgn}(\sigma, \partial\tau_q)\operatorname{sgn}(\sigma, \partial\tau_i)\beta_i \\
&= (k+1)\max\{\deg \tau_q, 1\} f(\tau_q) \\
&\quad + \sum_{\substack{\tau_i \in X^k, \tau_i \cap \tau_q = \sigma \\ \tau_i \sim_l \tau_q}} \max\{\deg \tau_q, 1\}\max\{\deg \tau_i, 1\}\operatorname{sgn}(\sigma, \partial\tau_q)\operatorname{sgn}(\sigma, \partial\tau_i) f(\tau_i).
\end{aligned}$$

$\square$

Let A and B be $n \times n$ matrices. Recall that we denote the set of eigenvalues of A by $\mathrm{Spec}(A)$. Note that a multiset is a set that allows for multiple instances for each of its elements, for example $\{0,0,1,1,3\}$. Let $\mathcal{S}pec$ denote a multiset of eigenvalues of A with their multiplicities. We also denotes a union of multisets by $\sqcup$ and write $\mathcal{S}pec(A) \doteq \mathcal{S}pec(B)$ when these two multisets are equal.

Proposition 4.1.3. *Let X be a k-simplex. Then*

(i) $\mathcal{S}pec(\mathcal{L}_0^{\mathrm{up}}) \doteq \{0, \underbrace{\frac{k+1}{k}, \frac{k+1}{k}, \dots, \frac{k+1}{k}}_{k\ times}\} \doteq \mathcal{S}pec(L_0^{\mathrm{up}})$,

(ii) $\mathcal{S}pec(\mathcal{L}_{k-1}^{\mathrm{up}}) \doteq \{k+1, \underbrace{0, 0, \dots, 0}_{k\ times}\} \doteq \mathcal{S}pec(L_{k-1}^{\mathrm{up}})$,

(iii) $\mathcal{S}pec(\mathcal{L}_k^{\mathrm{down}}) \doteq \{k+1\} \doteq \mathcal{S}pec(L_k^{\mathrm{down}})$.

Proof. We first note that for a k-simplex X, $D_k^{-1/2} = D_k^{1/2} = I_{k+1}$. Then

$$\mathcal{S}pec(\mathcal{L}_{k-1}^{\mathrm{up}}) \doteq \mathcal{S}pec(L_{k-1}^{\mathrm{up}})$$

and

$$\mathcal{S}pec(\mathcal{L}_k^{\mathrm{down}}) \doteq \mathcal{S}pec(L_k^{\mathrm{down}}).$$

We remark that L_0^{up} and $\mathcal{L}_0^{\mathrm{up}}$ indicate a relation between vertices and edges. Then we can consider only 1-structure of the simplex which can be seen as a graph. Moreover, 1-structure of a k-simplex is indeed a complete graph K_{k+1}. Then by Proposition 2.2.3, the first statement is done.

Let $X = [\sigma_0, \sigma_1, \dots, \sigma_k]$ be a k-simplex. We index an $(i+1)$th row of B_k as $\hat{\sigma_i} := [\sigma_0, \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_k]$. Then we obtain that

$$B_k = \begin{array}{c c} & [\sigma_0, \sigma_1, \dots, \sigma_k] \\ \begin{array}{c} \hat{\sigma_0} \\ \hat{\sigma_1} \\ \vdots \\ \hat{\sigma_k} \end{array} & \begin{pmatrix} 1 \\ -1 \\ \vdots \\ (-1)^k \end{pmatrix} \end{array} \quad \text{and} \quad \mathcal{L}_{k-1}^{\mathrm{up}} = B_k B_k^T = \begin{pmatrix} 1 & -1 & \cdots & (-1)^k \\ -1 & 1 & \cdots & (-1)^{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^k & (-1)^{k+1} & \cdots & (-1)^{2k} \end{pmatrix}.$$

We observe that $\operatorname{rank}(\mathcal{L}_{k-1}^{\text{up}}) = 1$ and

$$\dim(\operatorname{Ker}(\mathcal{L}_{k-1}^{\text{up}})) = \operatorname{null}(\mathcal{L}_{k-1}^{\text{up}}) = (k+1) - \operatorname{rank}(\mathcal{L}_{k-1}^{\text{up}}) = (k+1) - 1 = k.$$

Therefore, the multiplicity of eigenvalue 0 is k. Moreover, $(1, -1, 1, \dots, (-1)^k)$ is an eigenvector corresponding to eigenvalue $k+1$. Then, the second statement is done.

For the last statement, consider

$$\mathcal{L}_k^{\text{down}} = B_k B_k^T = [k+1].$$

Therefore $\operatorname{Spec}(\mathcal{L}_k^{\text{down}}) = \{k+1\}$. □

4.2 kth Homology on Simplicial Complex and the Smallest Eigenvalue of normalized Hodge k-Laplacian Matrix

In Section 3.2, we state a relation between the smallest eigenvalue of Hodge k-Laplacian matrix on a simplicial complex and its kth homology. Unfortunately, the normalized Laplacian matrix that we defined does not satisfy (3.1). Then, the chain complex on a given simplicial complex may not be split as a direct sum of eigenspaces corresponding to eigenvalues of $\mathcal{L}_k$. However, the smallest eigenvalue of normalized Hodge k-Laplacian matrix can also tell whether the homology (or cohomology) of a given simplicial complex is trivial. We prove the following theorem by using some facts from the last section.

Theorem 4.2.1. *Let X be a simplicial complex and $\mathcal{L}_k$ a normalized Hodge k-Laplacian matrix on X. Let λ be the smallest eigenvalue of $\mathcal{L}_k$ and $H_k(X)$ a kth-homology on X. Then*

$$\lambda \neq 0 \text{ if and only if } H_k(X) = 0.$$

Proof. By Theorem 2.4.5 (iv) and Theorem 4.1.1 (iv), we obtain that $\operatorname{Ker}(L_k) \cong$

$\mathrm{Ker}(\mathcal{L}_k)$. Then $\dim(\mathrm{Ker}(L_k)) = \dim(\mathrm{Ker}(\mathcal{L}_k))$. This implies that the multiplicity of eigenvalue 0 of L_k and $\mathcal{L}_k$ are equal. Then, by Theorem 2.4.5 (i) and Theorem 4.1.1 (i), if the smallest eigenvalue of $\mathcal{L}_k$ is 0, so is the smallest eigenvalue of L_k. By Theorem 3.2.1 , the proof is done. □

Corollary 4.2.2. *Let X be a simplicial complex and $\mathcal{L}_k$ a normalized Hodge k-Laplacian matrix on X. Let λ be the smallest eigenvalue of $\mathcal{L}_k$ and $H^k(X)$ a kth-cohomology on X. Then*

$$\lambda \neq 0 \textit{ if and only if } H^k(X) = 0.$$

4.3 Spectrum on Normalized Laplacian Matrix of k-wedge Sum of Simplices

Definition 4.3.1. Given simplicial complexes X_1 and X_2 with chosen k-simplices $\sigma \in X_1^k$ and $\tau \in X_2^k$. Then, the **k-wedge sum** $X_1 \vee_k X_2$ is the quotient of the disjoint union of X_1 and X_2 obtained by identifying simplices σ and τ as a single simplex.

Remark 5. The definition of k-wedge sum is defined for any k such that

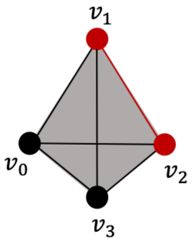
$$k \leq \min\{\dim(X_1), \dim(X_2)\}$$

since X_1^m and X_2^n are empty sets if $m > \dim(X_1)$ and $n > \dim(X_2)$.

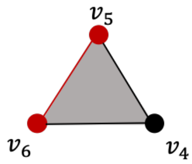
Example 4.3.1. Given simplicial complexes $X_1 = [v_0, v_1, v_2, v_3]$ and $X_2 = [v_4, v_5, v_6]$ with chosen 1-simplices $\sigma = [v_1, v_2]$ and $\tau = [v_5, v_6]$. Then, $X_1 \vee_1 X_2$ is shown in the following figure;

Theorem 4.3.1. *Let X_1 and X_2 be simplices. Let q be nonnegative integers. If $q < k$, then*

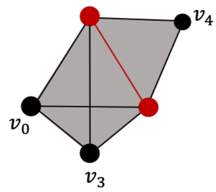
$$\mathcal{S}pec(L_k^{\mathrm{up}}(X_1 \vee_q X_2)) \doteq \mathcal{S}pec(L_k^{\mathrm{up}}(X_1)) \sqcup \mathcal{S}pec(L_k^{\mathrm{up}}(X_2)).$$



(a) X_1



(b) X_2



(c) $X_1 \vee_1 X_2$

Proof. We observe that if $q < k$

$$L_k^{\text{up}}(X_1 \vee_q X_2) = \begin{pmatrix} L_k^{\text{up}}(X_1) & \mathbf{O} \\ \mathbf{O} & L_k^{\text{up}}(X_2) \end{pmatrix},$$

where $\mathbf{O}$ is the zero matrix. Then, by Lemma 2.1.9, the proof is done. $\square$

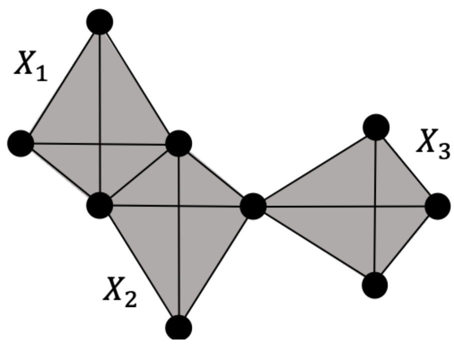
Corollary 4.3.2. *Let X_1 and X_2 be $(k+1)$-simplices. Let q be nonnegative integers. If $q < k$, then*

$$\mathcal{S}pec(\mathcal{L}_k^{\text{up}}(X_1 \vee_q X_2)) \doteq \mathcal{S}pec(\mathcal{L}_k^{\text{up}}(X_1)) \sqcup \mathcal{S}pec(\mathcal{L}_k^{\text{up}}(X_2)).$$

Proof. Note that for a $(k+1)$-simplex X, we have $D_{k+1}^{-1/2} = I_{|X^k|} = D_{k+1}^{1/2}$. Therefore, $L_k^{\text{up}} = \mathcal{L}_k^{\text{up}}$. By Theorem 4.3.1, the proof is done. $\square$

Example 4.3.2. Given the simplicial complex X;

We observe that



$$X = (X_1 \vee_1 X_2) \vee_0 X_3.$$

Then by Corollary 4.3.2,

$$\mathcal{S}pec(\mathcal{L}_2^{\text{up}}(X)) \doteq \mathcal{S}pec(\mathcal{L}_2^{\text{up}}(X_1)) \sqcup \mathcal{S}pec(\mathcal{L}_2^{\text{up}}(X_2)) \sqcup \mathcal{S}pec(\mathcal{L}_2^{\text{up}}(X_3)).$$

By Proposition 4.1.3,

$$\mathcal{S}pec(\mathcal{L}_2^{\text{up}}(X)) \doteq \{4,0,0,0\} \sqcup \{4,0,0,0\} \sqcup \{4,0,0,0\} \doteq \{4,4,4,0,0,0,0,0,0,0,0,0\}.$$

CHAPTER V
APPLICATIONS OF NORMALIZED HODGE k-LAPLACIAN MATRIX ON RAMDOM WALKS

In 2019, Schaub et al [16] defined a normalized Hodge 1-Laplacian matrix and applied the matrix on a random walk on edges. However, they consider simplicial complxes together with their given directions. The way we define and consider a random walk on a normalized Hodge 1-Laplacian matrix is much simpler.

Let $A = (a_{ij})$ be a matrix with real entries. Define $|A| = (|a_{ij}|)$ to guarantee that all entries of A are nonnegative. The matrix $|L_k|$ is well-known as a *signless k-Laplacian*. We next define a *random walk normalized Hodge k-Laplacian matrix*. Then use the sign $|\cdot|$ to do an application on random walks which means that we abandon the directions of a given simplicial complex. Note that, by now, all considered simplicial complexes are connected, i.e. there is a path connecting every pair of vertices.

5.1 Random Walk Normalized Hodge k-Laplacian Matrix

Definition 5.1.1. Let C_k be a space of k-chain of simplicial complex X and X^k a set of all k-simplices on X. Let B_k be a matrix representation of a boundary map $\partial_k : C_k \to C_{k-1}$. The **random walk normalized Hodge k-Laplacian matrix** $\mathcal{L}_k^{\mathrm{rw}}$ is defined by

$$\mathcal{L}_k^{\mathrm{rw}} = D_{k+1}^{-1/2}\mathcal{L}_k D_{k+1}^{1/2} = D_{k+1}^{-1}B_{k+1}B_{k+1}^T + B_k^T B_k D_{k+1},$$

where D_{k+1} and D_{k+1}^{-1} are $|X^k| \times |X^k|$ diagonal matrices defined by, for $\sigma, \tau \in X^k$, $(D_{k+1})_{\sigma\tau} = \max\{\deg \sigma, 1\}$ if $\sigma = \tau$ and 0 otherwise, and D_{k+1}^{-1} is the inverse of

D_{k+1}. Moreover, we define

$$\mathcal{L}_k^{\mathrm{rw(up)}} := D_{k+1}^{-1} B_{k+1} B_{k+1}^T,$$

$$\mathcal{L}_k^{\mathrm{rw(down)}} := B_k^T B_k D_{k+1}.$$

Remark 6. Recall that a random walk normalized Laplacian matrix $\mathcal{L}^{rw} = D^{-1}L$ where $(D^{-1})_{ii} = \dfrac{1}{d_i}$ if $d_i \neq 0$ and 0 otherwise.

Let $(D'_{k+1})^{-1}$ be a diagonal matrix defined by $(D'_{k+1})^{-1}_{\sigma\tau} = \dfrac{1}{\deg \sigma}$ if $\sigma = \tau$, $\deg \neq 0$ and 0 otherwise.

Similar to Remark 4, we can show that

$$D_{k+1}^{-1} L_k^{\mathrm{up}} = (D'_{k+1})^{-1} L_k^{\mathrm{up}}$$

for any k. Then,

$$\mathcal{L}_0^{\mathrm{rw}} = D_1^{-1} B_1 B_1^T + B_0^T B_0 D_1 = D^{-1} L = \mathcal{L}^{\mathrm{rw}}$$

since $D'_1 = D$ and $B_0 = 0$. This shows that $\mathcal{L}_k^{\mathrm{rw}}$ is a generalization of $\mathcal{L}^{\mathrm{rw}}$.

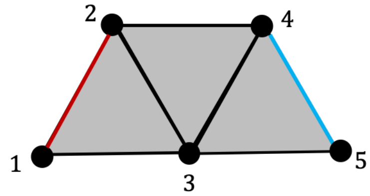
Lemma 5.1.1. *All eigenvalue (with their multiplicities) of $\mathcal{L}_k$ are the same with all eigenvalues (with their multiplicities) of $\mathcal{L}_k^{rw}$.*

Proof. Note that $\mathcal{L}_k^{\mathrm{rw}} = D^{-1/2} \mathcal{L}_k D^{1/2}$ and $D^{1/2}$ is an invertible matrix. Then, the proof is done by Proposition 2.1.6. □

For a random walk on graph, the word *walk* means walking from *vertex* to *vertex* through *edge*. The following example gives us a direction to define a *random walk on a simplicial complex*.

Example 5.1.1. From the following picture, we consider a process to move from the edge $[1, 2]$ to $[4, 5]$. We observe that we can move $[1, 2]$ to $[4, 5]$ through vertex or move through triangle in each step. If intermediary simplices are vertices, one of paths is

$$[1, 2] \xrightarrow{[2]} [2, 3] \xrightarrow{[3]} [3, 4] \xrightarrow{[4]} [4, 5].$$



If intermediary simplices are triangles, one of paths is

$$[1,2] \xrightarrow{[1,2,3]} [2,3] \xrightarrow{[2,3,4]} [3,4] \xrightarrow{[3,4,5]} [4,5].$$

5.2 Upper k-walk and Lower k-walk

From the idea of Example 5.1.1, we define an upper k-walk and a lower k-walk as follow.

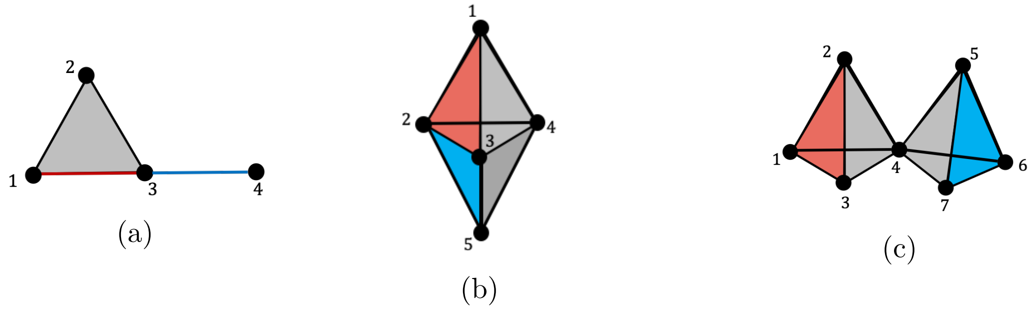
Definition 5.2.1. A finite sequence of distinct k-simplices $\{\sigma_0, \sigma_1, \dots, \sigma_n\}$ of path stating at σ_0, ending at σ_n and $\sigma_i \sim_u \sigma_{i+1}$ for all i whose intermediaries between two successive simplices are $(k+1)$-simplices is called an **upper k-walk**. The simplex σ_0 is called a **root**.

Definition 5.2.2. A finite sequence of distinct k-simplices $\{\sigma_0, \sigma_1, \dots, \sigma_n\}$ of path stating at σ_0, ending at σ_n and $\sigma_i \sim_l \sigma_{i+1}$ for all i whose intermediaries between two successive simplices are $(k-1)$-simplices is called a **lower k-walk**. The simplex σ_0 is called a **root**.

Analogously to a random walk on graph, we call a randomly-process of walking from the root simplex to another simplex a *random upper/lower k-walk*. Remark that we do not allow states at time t and $t+1$ to be the same. Moreover, since the random upper/lower k-walk picks one of adjacency simplices of the current state simplex each step randomly, random upper/lower k-walk is a Markov chain.

Example 5.2.1. From the following pictures, we can find an upper/lower k-walks (not need to be unique) from the red simplex to the blue one.

(a) A lower 1-walk $[1,3] \xrightarrow{[3]} [3,4]$ but there is no an upper 1-walk.



(b) A lower 2-walk $[1,2,3] \xrightarrow{[2,3]} [2,3,5]$ and
an upper 2-walk $[1,2,3] \xrightarrow{[1,2,3,4]} [2,3,4] \xrightarrow{[2,3,4,5]} [2,3,5]$.

(c) There is no a lower 2-walk and an upper 2-walk on this simplicial complex.

Theorem 5.2.1. *Let X be a simplicial complex. Suppose that there exists an upper k-walk on X. Then the matrix*

$$M_k^{\mathrm{up}} = \frac{1}{k+1}(|I_{|X^k|} - \mathcal{L}_k^{\mathrm{rw(up)}}|)$$

is a transition matrix of a random upper k-walk on X.

Proof. Let X be a simplicial complex. Let $X^k = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be the set of k-simplices on X and $n = |X^k|$. For each $(k+1)$-simplex $\overline{\sigma}$ which is a coface of k-simplex σ, since $\overline{\sigma}$ has $k+2$ k-faces, there are $k+1$ simplices which are upper adjacent to σ. Therefore, if we fix two distinct upper adjacent simplices σ_i and σ_j, a transition probability of upper k-walk from state σ_i at time t to state σ_j at time $t+1$ is $M_{ij} = \dfrac{1}{(k+1)\deg \sigma_i}$. Claim $M_k^{\mathrm{up}} = (M_{ij})$, where $M_{ij} = \dfrac{1}{(k+1)\deg \sigma_i}$ if $\sigma_i \sim_u \sigma_j$ and 0 otherwise. Note that $M_{ii} = 0$ since we do not allow states at time t and $t+1$ to be the same. Let $i, j \in \{1, 2, \dots, n\}$ be such that $i \neq j$. Then

$(I_n)_{ij} = 0$. Note that $(D'_{k+1})^{-1}L_k^{\text{up}} = D_{k+1}^{-1}L_k^{\text{up}}$ by Remark 6. Consider

$$
\begin{aligned}
M_{ij} &= \frac{1}{k+1}(|0 - \mathcal{L}_k^{\text{rw(up)}}|)_{ij} \\
&= \frac{1}{k+1}(|D_{k+1}^{-1}L_k^{\text{up}}|)_{ij} \\
&= \frac{1}{k+1}(|(D'_{k+1})^{-1}L_k^{\text{up}}|)_{ij} \\
&= \frac{1}{k+1}(|(D'_{k+1})^{-1}D'_{k+1} + (D'_{k+1})^{-1}A_k^{\text{up}}|)_{ij} \qquad \text{(by Proposition 3.1.1)} \\
&= \frac{1}{k+1}(|(D'_{k+1})^{-1}A_k^{\text{up}}|)_{ij}. \qquad (5.1)
\end{aligned}
$$

If σ_i is not upper adjacent to σ_j, then $(A_k^{\text{up}})_{ij} = 0$. From (5.1), $M_{ij}^{\text{up}} = 0$. Suppose that σ_i is upper adjacent to σ_j and $\sigma_i, \sigma_j \in \partial\overline{\sigma}$. Then $(A_k^{\text{up}})_{ij} = \text{sgn}(\sigma_i, \partial\overline{\sigma})\text{sgn}(\sigma_j, \partial\overline{\sigma})$ and $\deg \sigma \neq 0$. From (5.1),

$$
\begin{aligned}
M_{ij}^{\text{up}} &= \frac{1}{k+1}(|((D'_{k+1})^{-1}A_k^{\text{up}})_{ij}|) \\
&= \frac{1}{k+1}\left|\frac{\text{sgn}(\sigma_i, \partial\overline{\sigma})\text{sgn}(\sigma_j, \partial\overline{\sigma})}{\deg \sigma_i}\right| \\
&= \frac{1}{(k+1)\deg \sigma_i}
\end{aligned}
$$

and the proof is done. $\square$

From Theorem 5.2.1, for $k = 0$,

$$M_0^{\text{up}} = \frac{1}{0+1}(|I_{|V|} - \mathcal{L}_0^{\text{rw}}|) = I_{|V|} - \mathcal{L}^{\text{rw}} = P,$$

where P is the matrix stated in Proposition 2.5.2. That is we can consider M_k^{up} as a generalization of a transition matrix of random walk on graphs.

Unfortunately, $\mathcal{L}_k^{\text{rw(down)}}$ is not suitable for applying to a random lower k-walk. We use a term of Hodge k-Laplacian matrix L_{k+1}^{down} instead.

Theorem 5.2.2. *Let X be a simplicial complex and X^k the set of all k-simplices* *in X. For each k, define a $|X^k| \times |X^k|$ diagonal matrix $\overline{D}_{k+1}$ by, for $\sigma, \tau \in X^k$,*

$$(\overline{D}_{k+1})_{\sigma\tau} = \begin{cases} \dfrac{1}{\left(\displaystyle\sum_{\sigma' \in \partial\sigma} \deg \sigma'\right) - (k+1)}, & \text{if } \sigma = \tau; \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that there exists a lower k-walk on X. Then the matrix

$$M_k^{\text{down}} = \overline{D}_{k+1}(|L_k^{\text{down}}| - (k+1)I_{|X^k|})$$

is a transition matrix of a random lower k-walk on X.

Proof. Let X be a simplicial complex and k a nonnegative integer. Let $X^k = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ be the set of k-simplices on X and $n = |X^k|$. Define $m_\sigma = \left(\displaystyle\sum_{\sigma' \in \partial\sigma} \deg \sigma'\right) - (k+1)$. Note that, for a k-simplex σ, if we fix $(k-1)$-simplex $\sigma' \in \partial\sigma$, then $\deg \sigma'$ is a number of cofaces of σ' including σ. Since the number of $(k-1)$-faces of σ is $k+1$, $m_\sigma = \left(\displaystyle\sum_{\sigma' \in \partial\sigma} \deg \sigma'\right) - (k+1)$ is counting the number of lower adjacent simplices of σ. Therefore, if we fix two distinct lower adjacent simplices σ_i and σ_j, a transition probability of lower k-walk from state σ_i at time t to state σ_j at time $t+1$ is $M_{ij} = \dfrac{1}{m_{\sigma_i}}$. Claim $M_k^{\text{down}} = (M_{ij})$, where $M_{ij} = \dfrac{1}{m_{\sigma_i}}$ if $\sigma_i \sim_l \sigma_j$ and 0 otherwise. Note that $M_{ii} = 0$ since we do not allow states at time t and $t+1$ to be the same. Let $i, j \in \{1, 2, \dots, n\}$ be such that $i \neq j$. Then $(I_n)_{ij} = 0$. Consider

$$\begin{aligned} M_{ij} &= (\overline{D}_{k+1}(|L_k^{\text{down}}| - (k+1)I_n))_{ij} \\ &= (\overline{D}_{k+1}(|(k+1)I_n + A_k^{\text{down}}| - (k+1)I_n))_{ij} && \text{(by Proposition 3.1.1)} \\ &= (|\overline{D}_{k+1}A_k^{\text{down}}|)_{ij}. && (5.2) \end{aligned}$$

If σ_i is not lower adjacent to σ_j, then $(A_k^{\text{down}})_{ij} = 0$. By (5.2), $M_{ij} = 0$.

Suppose that $\sigma_i \sim_l \sigma_j$, then

$$
\begin{aligned}
M_{ij} &= (|\overline{D}_{k+1} A_k^{\text{down}}|)_{ij} \\
&= \left| \frac{1}{m_{\sigma_i}} \times \operatorname{sgn}((\sigma_i \cap \sigma_j), \partial\sigma_i) \operatorname{sgn}((\sigma_i \cap \sigma_j), \partial\sigma_j) \right| \\
&= \frac{1}{m_{\sigma_i}}.
\end{aligned}
$$

□

Example 5.2.2. From Example 5.2.1, a Markov chain of a random upper 1-walk and a Markov chain of a random lower 1-walk on the simplicial complex (b) are shown as follow:

$$
M_1^{\text{up}} = \frac{1}{2}(|I_{|E|} - \mathcal{L}_1^{\text{rw(up)}}|)
$$

$$
= \begin{array}{cc}
 & \begin{array}{ccccccccc} [1,2] & [1,3] & [1,4] & [2,3] & [2,4] & [3,4] & [2,5] & [3,5] & [4,5] \end{array} \\
\begin{array}{c} [1,2] \\ [1,3] \\ [1,4] \\ [2,3] \\ [2,4] \\ [3,4] \\ [2,5] \\ [3,5] \\ [4,5] \end{array} &
\begin{pmatrix}
0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 0 \\
1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 & 0 & 0 & 0 \\
1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 & 0 & 0 & 0 \\
1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\
1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 \\
0 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 \\
0 & 0 & 0 & 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\
0 & 0 & 0 & 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\
0 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0
\end{pmatrix}
\end{array}
$$

$$M_1^{\mathrm{down}} = \overline{D}_2(|L_1^{\mathrm{down}}| - 2I_{|E|})$$

$$= \begin{array}{c} \\ [1,2] \\ [1,3] \\ [1,4] \\ [2,3] \\ [2,4] \\ [3,4] \\ [2,5] \\ [3,5] \\ [4,5] \end{array} \begin{array}{c} \begin{array}{ccccccccc} [1,2] & [1,3] & [1,4] & [2,3] & [2,4] & [3,4] & [2,5] & [3,5] & [4,5] \end{array} \\ \begin{pmatrix} 0 & 1/5 & 1/5 & 1/5 & 1/5 & 0 & 1/5 & 0 & 0 \\ 1/5 & 0 & 1/5 & 1/5 & 0 & 1/5 & 0 & 1/5 & 0 \\ 1/5 & 1/5 & 0 & 0 & 1/5 & 1/5 & 0 & 0 & 1/5 \\ 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 0 \\ 1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 & 1/6 & 0 & 1/6 \\ 0 & 1/6 & 1/6 & 1/6 & 1/6 & 0 & 0 & 1/6 & 1/6 \\ 1/5 & 0 & 0 & 1/5 & 1/5 & 0 & 0 & 1/5 & 1/5 \\ 0 & 1/5 & 0 & 1/5 & 0 & 1/5 & 1/5 & 0 & 1/5 \\ 0 & 0 & 1/5 & 0 & 1/5 & 1/5 & 1/5 & 1/5 & 0 \end{pmatrix} \end{array}$$

CHAPTER VI
CONCLUSION

6.1 Conclusion and Discussion

Let X be a simplicial complex. Recall the definitions of a Hodge k-Laplacian matrix and a normalized Hodge k-Laplacian matrix as followed;

$$L_k = B_{k+1}B_{k+1}^T + B_k^T B_k$$

and

$$\mathcal{L}_k = D_{k+1}^{-1/2} B_{k+1} B_{k+1}^T D_{k+1}^{-1/2} + D_{k+1}^{1/2} B_k^T B_k D_{k+1}^{1/2},$$

where $D_{k+1}^{1/2}$ and $D_{k+1}^{-1/2}$ are $|X^k| \times |X^k|$ diagonal matrices defined by $(D_{k+1}^{1/2})_{\sigma\tau} = \max\{\sqrt{\deg\sigma},1\}$ if $\sigma = \tau$ and 0 otherwise, and $D_{k+1}^{-1/2}$ is the inverse of $D_{k+1}^{1/2}$.

Since 1-stucture of any simplicial complexes is a graph, a Hodge k-Laplacian matrix on simplicial complexes is a generalization of a Laplacian matrix on graphs and a normalized Hodge k-Laplacian matrix on simplicial complexes is a generalization of a normalized Laplacian matrix on graphs. These two matrices are Hodge Laplacian matrix and this fact leads us to many properties of them which could be applied for many applications. Moreover, we obtain that the smallest eigenvalue of both Hodge k-Laplacian and normalized Hodge k-Laplacian on a simplicial complex can indicate whether the homology (or cohomology) on a given simplicial complex is trivial. Finally, we obtain a general from of a Markov chain of random walks on graphs using the matrix that we defined.

6.2 Further Works

Let X be a simplicial complex and R a commutative ring. Let $w : X \to R$ satisfying that for any σ_1, σ_2 in X such that σ_1 is a face of σ_2, then $w(\sigma_1) \mid w(\sigma_2)$. Then a pair (X, w) is called a *weighted simplicial complex.* In this work, we only define a Hodge k-Laplacian matrix and a normalized Hodge k-Laplacian matrix on unweighted simplicial complex. We suggest the readers to general our results to work on weighted simplicial complexes. Moreover, an interesting point is to check that whether Theorem 3.2.1 and Theorem 4.1.1 (iv) hold for a weighted homology on weighted simplicial complex. For more details in weighted simplicial complex and weighted homology, see [13] and [22].

Recall that from Lemma 2.4.4 (ii), the composition is called a *Hodge decomposition* which has many applications in data analysis, ranking, game theory and others, see [11]. From Theorem 4.1.1 (iii), we obtain the decomposition

$$\mathrm{Im}(D_{k+1}^{1/2} B_k^T) \oplus \mathrm{Ker}(\mathcal{L}_k) \oplus \mathrm{Im}(D_{k+1}^{-1/2} B_{k+1}) \cong \mathbb{R}^n,$$

where n is the number of k-simplices of a given simplicial complex. We recommend to do some applications from this result.

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