CHAPTER IV

THE FOKKER-PLANCK EQUATION

4.1 Introduction (6)

In this chapter, we will emphasize the essential stochastic nature of the phenomenon and seek a description in terms of the probability distributions of position and/or velocity at a later time starting from given initial distributions. Thus, in our discussion of the Brownian movement of a free particle we obtain explicitly the distribution functions $W(\vec{u},t;\vec{u}_0)$, $W(\vec{r},t;\vec{r}_0,\vec{u}_0)$ and $W(\vec{r},\vec{u},t;\vec{r}_0,\vec{u}_0)$ for given initial values of $\vec{r}_0$ and $\vec{u}_0$; similarly, we determined the distributions $W(u,t;x_0,u_0)$, $W(x,t;x_0,u_0)$ and $W(x,u,t;x_0,u_0)$ for a harmonically bound particle describing Brownian motion. In deriving these distributions we started with the Langevin equation [Eq.(9) in the field-free case, and Eq.(63) when an external field is present] and solved it in a manner appropriate to the problem. We shall now consider the question whether we cannot reduce the determination of these distribution functions to appropriate boundary value problems of suitably chosen partial differential equations, of which the solution can be obtained as solutions of boundary-value problems long familiar in the theory of diffusion or conduction of heat (8).

It is clear that for the solutions of the most general problem we require the density function $W(\vec{r},\vec{u},t)$; in other words, we should really consider the problem in the six-dimensional phase space. But before we proceed to establish such a general theorem it will be instructive to consider the simplest problem of the Brownian motion of a free particle in the velocity space and obtain a differential equation for $W(\vec{u},t)$; this leads us to the discussion of the Fokker-Planck equation in its most familiar form.

4.2 The derivation of Fokker-Planck equation (6,8)

(1) The Fokker-Planck equation in velocity space to describe the Brownian motion of a free particle (6)

Let Δt denote an interval of time long compared to the periods of fluctuations of the acceleration $\vec{A}(t)$ occurring in the Langevin equation but short compared to intervals during which the velocity of a Brownian particle changes by appreciable amounts. Under these circumstances we expect to derive the distribution $W(\vec{u},t+\Delta t)$ at time $t+\Delta t$ and a knowledge of the transition probability $\psi(\vec{u};\Delta\vec{u})$ that $\vec{u}$ suffers an increment $\Delta\vec{u}$ in time Δt . More particularly, we expect the relation

$$W(\vec{u},t+\Delta t)=\int W(\vec{u}-\Delta\vec{u},t)\,\psi(\vec{u}-\Delta\vec{u};\Delta\vec{u})\,d(\Delta\vec{u}) \tag{95}$$

to be valid. We may remark that in expecting this integral equation between $W(\vec{u},t+\Delta t)$ and $W(\vec{u},t)$ to be true we are supposing that the course which a Brownian particle will take depends only on the instantaneous values of its physical parameters and is entirely independent of its whole previous history. In general probability theory, a stochastic process which has this characteristic, namely, that what happens at a given instant of time t depends only on the state of the system at time t is said to be a Markoff process (6,8). But we should be careful not to conclude that every stochastic process is necessarily of the Markoff type. For, it can happen that the future course of a system is conditioned by its past history: i. e., what happens at a given instant of time t may depend on what has already happened during all time preceding t.

Returning to Eq.(95), for the case under discussion we have

$$\psi(\vec{u};\Delta\vec{u}) = \frac{1}{(4\pi q\Delta t)^{3/2}} \exp\left(-|\Delta\vec{u}+\beta\vec{u}\Delta t|^2/4q\Delta t\right);\ (q=\beta kT/m) \tag{96}$$

For, according to the Langevin equation [cf. Eq.(19)], integrate

$$\Delta\vec{u} = -\beta\vec{u}\Delta t + \vec{B}(\Delta t) \tag{97}$$

where $\vec{B}(\Delta t)$ denotes the net acceleration arising from fluctuations which a Brownian particle suffers in time Δt; and, since the distribution of $\vec{B}(\Delta t)$ is given by Eq.(21), the transition probability (95) follows at once.

Expanding $W(\vec{u},t+\Delta t)$, $W(\vec{u}-\Delta\vec{u},t)$ and $\psi(\vec{u}-\Delta\vec{u};\Delta\vec{u})$ in Eq.(95) in the form of Taylor series, we obtain

$$\begin{aligned} W(\vec{u},t) + \frac{\partial W}{\partial t}\Delta t + O(\Delta t^2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{W(\vec{u},t) - \sum_i \frac{\partial W}{\partial u_i}\Delta u_i + \frac{1}{2}\sum_i \frac{\partial^2 W}{\partial u_i^2}\Delta u_i^2 + \sum_{i<j}\frac{\partial^2 W}{\partial u_i \partial u_j}\Delta u_i \Delta u_j + \dots\right\} \\ &\times\left\{\psi(\vec{u};\Delta\vec{u}) - \sum_i \frac{\partial \psi}{\partial u_i}\Delta u_i + \frac{1}{2}\sum_i \frac{\partial^2 \psi}{\partial u_i^2}\Delta u_i^2 + \sum_{i<j}\frac{\partial^2 W}{\partial u_i \partial u_j}\Delta u_i \Delta u_j + \dots\right\} d(\Delta u_1)\,d(\Delta u_2)\,d(\Delta u_3) \end{aligned} \tag{98}$$

writing

$$\langle \Delta u_i \rangle = \int_{-\infty}^{\infty} \Delta u_i \psi(\vec{u};\Delta\vec{u})\, d(\Delta\vec{u}) \tag{99.1}$$

$$\langle \Delta u_i^2 \rangle = \int_{-\infty}^{\infty} \Delta u_i^2 \psi(\vec{u};\Delta\vec{u})\, d(\Delta\vec{u}) \tag{99.2}$$

$$\langle \Delta u_i \Delta u_j \rangle = \int_{-\infty}^{\infty} \Delta u_i \Delta u_j \psi(\vec{u};\Delta\vec{u})\, d(\Delta\vec{u}) \tag{99.3}$$

we have therefore $\frac{\partial W}{\partial t}\Delta t + O(\Delta t^2) = -\sum_i \frac{\partial W}{\partial u_i}\langle \Delta u_i \rangle + \frac{1}{2}\sum_i \frac{\partial^2 W}{\partial u_i^2}\langle \Delta u_i^2 \rangle$

$$+\sum_{i<j} \frac{\partial^2 W}{\partial u_i \partial u_j}\langle \Delta u_i \Delta u_j \rangle - \sum_i W \frac{\partial}{\partial u_i}\langle \Delta u_i \rangle + \sum_i \frac{\partial}{\partial u_i}\langle \Delta u_i^2 \rangle \frac{\partial W}{\partial u_i} \qquad (100)$$

$$+\sum_{i \neq j} \frac{\partial W}{\partial u_i}\frac{\partial}{\partial u_j}\langle \Delta u_i \Delta u_j \rangle + \frac{1}{2}\sum_i \frac{\partial^2}{\partial u_i^2}\langle \Delta u_i^2 \rangle W + \sum_{i<j} W \frac{\partial^2}{\partial u_i \partial u_j}\langle \Delta u_i \Delta u_j \rangle$$

$$+ O(\langle \Delta u_i \Delta u_j \Delta u_k \rangle)$$

where the remainder term involves the averages of the quantities

$$\Delta u_i^3, \ \Delta u_i^2 \Delta u_j \quad \text{and} \quad \Delta u_i \Delta u_j \Delta u_k, \ (i, j, k = 1, 2, 3)$$

Therefore, Eq(100) can be written as $\frac{\partial W}{\partial t}\Delta t + O(\Delta t^2)$

$$= -\sum_i \frac{\partial}{\partial u_i}(W\langle \Delta u_i \rangle) + \frac{1}{2}\sum_i \frac{\partial^2}{\partial u_i^2}(W\langle \Delta u_i^2 \rangle) + \sum_{i<j} \frac{\partial^2}{\partial u_i \partial u_j}(W\langle \Delta u_i \Delta u_j \rangle) \qquad (101)$$

$$+ O(\langle \Delta u_i \Delta u_j \Delta u_k \rangle)$$

which is the Fokker-Planck equation in its most general form.

For the transition probability (96) and from Eq(99), we get

$$\langle \Delta u_i \rangle = -\beta u_i \Delta t \ ; \langle \Delta u_i \Delta u_j \rangle = O(t^2) \ ; \langle \Delta u_i^2 \rangle = 2q\Delta t + O(\Delta t^2) \qquad (102)$$

Hence, Eq.(101) reduces to

$$\frac{\partial W}{\partial t}\Delta t + O(\Delta t^2) = \left\{ \beta \, \mathrm{div}_{\vec{u}}(W\vec{u}) + q \nabla_{\vec{u}}^2 W \right\} \Delta t + O(\Delta t^2) \qquad (103)$$

for the limit $\Delta t = 0$ we have

$$\partial W / \partial t = \beta \, \mathrm{div}_{\vec{u}}(W\vec{u}) + q \nabla_{\vec{u}}^2 W \qquad (104)$$

We shall now show that the distribution function $W(\vec{u}, t; \vec{u}_0)$ obtained in Eq.(38) is the fundamental solution of the Fokker-Planck

Eq.(104) in the sense that this is the solution which tends to the δ function

$$\delta(u_1 - u_{1,0})\,\delta(u_2 - u_{2,0})\,\delta(u_3 - u_{3,0}) \tag{105}$$

as $t \rightarrow 0$. To prove this, we first note that but for the Laplacian term, Eq.(104) is a linear partial differential equation of the first order. Hence, it is natural to expect that the general solution of Eq.(104) will be connected with that of the associated first-order equation

$$\partial W/\partial t - \beta\, \mathrm{div}_{\vec{u}}(W\vec{u}) = 0 \tag{106}$$

The general solution of this first-order equation (106) involves the three first integrals of

$$d\vec{u}/dt = -\beta\vec{u} \tag{107}$$

The required first integrals are

$$\vec{u}\, e^{\beta t} = \vec{u}_0 = \text{constant} \tag{108}$$

Accordingly, for solving Eq.(104) we introduce a vector $\vec{\rho}$ defined by

$$\vec{\rho} = (\xi, \eta, \zeta) = \vec{u}\, e^{\beta t} \tag{109}$$

Equation (104) now becomes

$$\partial W/\partial t = 3\beta W + q\, e^{2\beta t}\, \nabla_{\vec{\rho}}^{2} W \tag{110}$$

Introducing the variable

$$\chi = W e^{-3\beta t} \tag{111}$$

We have from Eqs.(110) and (111)

$$\frac{\partial \chi}{\partial t} = q\, e^{2\beta t}\left(\frac{\partial^2 \chi}{\partial \xi^2} + \frac{\partial^2 \chi}{\partial \eta^2} + \frac{\partial^2 \chi}{\partial \zeta^2} \right) \tag{112}$$

The solution of this equation can be obtained using the following lemma :

Lemma I. If $\phi(t)$ is an arbitrary function of time, the solution of the partial differential equation

$$\partial \chi / \partial t = \phi^2(t) \nabla^2_{\vec{\rho}} \chi \tag{113}$$

where $\vec{\rho} = \vec{\rho}_0$ at time $t = 0$ is

$$\chi = \frac{1}{\left[4\pi \int_0^t \phi^2(t)\, dt\right]^{3/2}} \exp\left(-|\vec{\rho} - \vec{\rho}_0|^2 / 4 \int_0^t \phi^2(t)\, dt\right) \tag{114}$$

Applying this lemma to Eq.(112) we have the solution

$$\chi = \frac{1}{\left[4\pi q \int_0^t e^{2\beta t}\, dt\right]^{3/2}} \exp\left(-|\vec{u} e^{\beta t} - \vec{u}_0|^2 / 4q \int_0^t e^{2\beta t}\, dt\right) \tag{115}$$

Then according to Eq.(111) we have

$$W(\vec{u}, t; \vec{u}_0) = \frac{1}{\left\{2\pi q (1 - e^{-2\beta t}) / \beta\right\}^{3/2}} \exp\left[\frac{-\beta |\vec{u} - \vec{u}_0 e^{-\beta t}|^2}{2q(1 - e^{-2\beta t})}\right] \tag{116}$$

which agrees with the result in Eq.(38)

(2) The generalization of the Fokker-Planck equation(6)

We shall now consider the general problem of a particle describing Brownian motion and under the influence of an external field of force.

Let Δt again denote an interval of time which is compared to the periods of fluctuations of the acceleration $\vec{A}(t)$ occurring in the Langevin Eq. (63) but short compared to the intervals in which any of the physical parameters change appreciably. Then, the increments $\Delta\vec{r}$ and $\Delta\vec{u}$ in position and velocity which the particle suffers during Δt are

$$\Delta\vec{r} = \vec{u}\Delta t \ ; \quad \Delta\vec{u} = -(\beta\vec{u} - \vec{K})\Delta t + \vec{B}(\Delta t) \tag{117}$$

where $\vec{K}$ denotes the acceleration per unit mass caused by the external field of force and $\vec{B}(\Delta t)$ the net acceleration arising from fluctuations which the particle suffers in time Δt. The distribution of $\vec{B}(\Delta t)$ is again given by Eq.(21)

Assuming as before that the Brownian movement can be idealized as a Markoff process the probability distribution $W(\vec{r},\vec{u},t+\Delta t)$ in phase space at time $t+\Delta t$ can be derived from the distribution $W(\vec{r},\vec{u},t)$ at the earlier time t by means of the integral equation

$$W(\vec{r},\vec{u},t+\Delta t)=\iint W(\vec{r}-\Delta\vec{r},\vec{u}-\Delta\vec{u},t)\Psi(\vec{r}-\Delta\vec{r},\vec{u}-\Delta\vec{u};\Delta\vec{r},\Delta\vec{u}) \times d(\Delta\vec{r})\,d(\Delta\vec{u}) \tag{118}$$

According to the Eqs. (117)

$$\Psi(\vec{r},\vec{u};\Delta\vec{r},\Delta\vec{u}) = \psi(\vec{r},\vec{u};\Delta\vec{u})\,\delta(\Delta x-u_1\Delta t)\,\delta(\Delta y-u_2\Delta t)\,\delta(\Delta z-u_3\Delta t) \tag{119}$$

where the δ's denote Dirac's delta functions and $\psi(\vec{r},\vec{u};\Delta\vec{u})$ the transition probability in the velocity space. With this form for the transition probability in the phase space the integration over $\Delta\vec{r}$ in Eq.(118) is immediately performed and we get

$$W(\vec{r},\vec{u},t+\Delta t)=\int W(\vec{r}-\vec{u}\Delta t,\vec{u}-\Delta\vec{u},t)\,\psi(\vec{r}-\vec{u}\Delta t,\vec{u}-\Delta\vec{u};\Delta\vec{u})\,d(\Delta\vec{u}) \tag{120}$$

Alternatively, we can write

$$W(\vec{r}+\vec{u}\Delta t,\vec{u},t+\Delta t)=\int W(\vec{r},\vec{u}-\Delta\vec{u},\Delta t)\,\psi(\vec{r},\vec{u}-\Delta\vec{u};\Delta\vec{u})\,d(\Delta\vec{u}) \tag{121}$$

Expanding the functions in the foregoing equation in the form of Taylor series and proceeding as in our derivation of the Fokker-Planck equation, we obtain [cf. Eq.(98)]

$$\left(\frac{\partial W}{\partial t}+\vec{u}.\mathrm{grad}_{\vec{r}} W\right)\Delta t+O(\Delta t^2)=-\sum_i \frac{\partial}{\partial u_i}\left(W\langle\Delta u_i\rangle\right)$$
$$+\frac{1}{2}\sum_i \frac{\partial^2}{\partial u_i^2}\left(W\langle\Delta u_i^2\rangle\right)+\sum_{i<j}\frac{\partial^2}{\partial u_i\partial u_j}\left(W\langle\Delta u_i\Delta u_j\rangle\right)+O(\langle\Delta u_i\Delta u_j\Delta u_k\rangle) \tag{122}$$

This is the analog in the phase space of the Fokker-Planck equation in the velocity space.

For the case (117) ,the transition probability $\psi(\vec{u};\Delta\vec{u})$ is given by [cf. Eq.(21)]

$$\psi(\vec{u};\Delta\vec{u})=\frac{1}{(4\pi q\Delta t)^{3/2}}\exp\left\{-\left|\Delta\vec{u}+(\beta\vec{u}-\vec{K})\Delta t\right|^2/4q\Delta t\right\} \tag{123}$$

And with this expression for the transition probability we have

$$\langle\Delta u_i\rangle=-(\beta u_i-K_i)\Delta t;\ \langle\Delta u_i^2\rangle=2q\Delta t+O(\Delta t^2);\ \langle\Delta u_i\Delta u_j\rangle=O(\Delta t^2) \tag{124}$$

Accordingly Eq.(122) simplifies to

$$\left\{\frac{\partial W}{\partial t}+\vec{u}.\mathrm{grad}_{\vec{r}} W\right\}\Delta t+O(\Delta t^2)=\left\{\sum_i\frac{\partial}{\partial u_i}\left((\beta u_i-K_i)W\right)+q\sum_i\frac{\partial^2 W}{\partial u_i^2}\right\}\Delta t+O(\Delta t^2) \tag{125}$$

and passing to the limit $\Delta t = 0$ and after some minor rearranging of the terms

$$\frac{\partial W}{\partial t}+\vec{u}.\mathrm{grad}_{\vec{r}} W+\vec{K}.\mathrm{grad}_{\vec{u}} W=\beta\,\mathrm{div}_{\vec{u}}(W\vec{u})+q\nabla_{\vec{u}}^2 W \tag{126}$$

This equation represents the generalization of the Fokker-Planck Eq.(104) to the phase space.

4.3 The solution for the field free case (6)

When no external field is present Eq.(126) becomes

$$\frac{\partial W}{\partial t} + \vec{u}\cdot \mathrm{grad}_{\vec{r}} W = 3\beta W + \beta\vec{u}\cdot \mathrm{grad}_{\vec{u}} W + q\nabla^2_{\vec{u}} W \tag{127}$$

To solve this equation we again note that the equation

$$\frac{\partial W}{\partial t} + \vec{u}\cdot \mathrm{grad}_{\vec{r}} W = 3\beta W + \beta\vec{u}\cdot \mathrm{grad}_{\vec{u}} W \tag{128}$$

derived from (127) by ignoring the Laplacian term $q\nabla^2_{\vec{u}} W$ is a linear homogeneous first-order partial differential equation. Accordingly, the general solution of Eq. (128) can be expressed in terms of any six independent integrals of

$$d\vec{u}/dt = -\beta\vec{u} \;; \qquad d\vec{r}/dt = \vec{u} \tag{129}$$

Two vector integrals of Eq. (129) are

$$\vec{u}\, e^{\beta t} = \vec{I}_1 \;; \qquad \vec{r} + \vec{u}/\beta = \vec{I}_2 \tag{130}$$

Introduce the new variables

$$\vec{\rho} = (\xi, \eta, \zeta) = \vec{u}\, e^{\beta t} ; \quad \vec{P} = (X, Y, Z) = \vec{r} + \vec{u}/\beta \tag{131}$$

Then, for this transformation of the variables we have

$$\begin{aligned} \partial W/\partial t &= \partial W(\vec{\rho}, \vec{P}, t)/\partial t + \beta\vec{\rho}\cdot \mathrm{grad}_{\vec{\rho}} W \\ \mathrm{grad}_{\vec{r}} W &= \mathrm{grad}_{\vec{P}} W \\ \mathrm{grad}_{\vec{u}} W &= e^{\beta t}\, \mathrm{grad}_{\vec{\rho}} W + \frac{1}{\beta}\, \mathrm{grad}_{\vec{P}} W . \end{aligned} \tag{132}$$

$$\text{and} \qquad \nabla_{\vec{r}}^2 W = e^{2\beta t} \nabla_{\vec{\rho}}^2 W + \frac{2}{\beta} e^{\beta t} \nabla_{\vec{\rho}} \cdot \nabla_{\vec{P}} W + \frac{1}{\beta^2} \nabla_{\vec{P}}^2 W$$

Substituting Eq.(132) in Eq.(127) we obtain

$$\frac{\partial W}{\partial t} = 3\beta W + q\left\{ e^{2\beta t} \nabla_{\vec{\rho}}^2 W + \frac{2}{\beta} e^{\beta t} \nabla_{\vec{\rho}} \cdot \nabla_{\vec{P}} W + \frac{1}{\beta^2} \nabla_{\vec{P}}^2 W \right\} \tag{133}$$

introduce the variable

$$\chi = W e^{-3\beta t} \tag{134}$$

Equation (133) reduces to

$$\frac{\partial \chi}{\partial t} = q\left\{ e^{2\beta t} \nabla_{\vec{\rho}}^2 \chi + \frac{2}{\beta} e^{\beta t} \nabla_{\vec{\rho}} \cdot \nabla_{\vec{P}} \chi + \frac{1}{\beta^2} \nabla_{\vec{P}}^2 \chi \right\} \tag{135}$$

or, written out explicitly

$$\frac{\partial \chi}{\partial t} = q\left\{ e^{2\beta t}\left(\frac{\partial^2 \chi}{\partial \xi^2} + \frac{\partial^2 \chi}{\partial \eta^2} + \frac{\partial^2 \chi}{\partial \varphi^2} \right) + \frac{2}{\beta} e^{\beta t}\left(\frac{\partial^2 \chi}{\partial \xi \partial X} + \frac{\partial^2 \chi}{\partial \eta \partial Y} + \frac{\partial^2 \chi}{\partial \varphi \partial Z} \right) + \frac{1}{\beta^2}\left(\frac{\partial^2 \chi}{\partial X^2} + \frac{\partial^2 \chi}{\partial Y^2} + \frac{\partial^2 \chi}{\partial Z^2} \right) \right\} \tag{136}$$

To solve this equation we first prove the following lemma:

Lemma II Let $\phi(t)$ and $\psi(t)$ be two arbitrary functions of time. The solution of the differential equation

$$\frac{\partial \chi}{\partial t} = \phi^2(t) \frac{\partial^2 \chi}{\partial \xi^2} + 2\phi(t)\psi(t) \frac{\partial^2 \chi}{\partial \xi \partial X} + \psi^2(t) \frac{\partial^2 \chi}{\partial X^2} \tag{137}$$

when $\xi = X = 0$ at t=0 is

$$\chi = \frac{1}{2\pi\Delta^{1/2}}\, exp\left\{-\left(a\xi^2 + 2h\xi X + bX^2\right)/2\Delta\right\} \tag{138}$$

where

$$a = 2\int_0^t \psi^2(t)\,dt\ ; \quad h = -2\int_0^t \phi(t)\psi(t)\,dt\ ; \quad b = 2\int_0^t \phi^2(t)\,dt \tag{139}$$

and

$$\Delta = ab - h^2 \tag{140}$$

To prove this lemma we substitute for χ according to Eq. (138) in Eq. (137) we find

$$\frac{1}{\Delta}\frac{d\Delta}{dt} + \xi^2\frac{da_1}{dt} + 2\xi X\frac{dh_1}{dt} + X^2\frac{db_1}{dt} + 2\phi^2(a_1^2\xi^2 + 2a_1h_1\xi X + h_1^2X^2 - a_1) + 4\phi\psi(a_1h_1\xi^2 + h_1b_1X^2 + \xi X \times\{h_1^2 + a_1b_1\} - h_1) + 2\psi^2(h_1^2\xi^2 + 2h_1b_1\xi X + b_1^2X^2 - b_1) = 0 \tag{141}$$

where

$$a_1 = a/\Delta\ ; \quad h_1 = h/\Delta\ ; \quad b_1 = b/\Delta \tag{142}$$

Equating the coefficients of ξ^2, ξX and X^2 in (141), we obtain

$$da_1/dt = -2(a_1\phi + h_1\psi)^2\ ; \quad db_1/dt = -2(h_1\phi + b_1\psi)^2$$
$$dh_1/dt = -2(a_1\phi + h_1\psi)(h_1\phi + b_1\psi) \tag{143}$$

and

$$d\Delta/dt = 2\Delta(a_1\phi^2 + 2h_1\phi\psi + b_1\psi^2) \tag{144}$$

From Eq. (142)

$$da/dt = \Delta(da_1/dt) + a_1(d\Delta/dt) \tag{145}$$

we have according to Eqs. (143), (144), and (145)

$$\begin{aligned} da/dt &= -2\Delta(a_1\phi + h_1\psi)^2 + 2\Delta(a_1^2\phi^2 + 2a_1h_1\phi\psi + a_1b_1\psi^2) \\ &= 2\Delta(a_1b_1 - h_1^2)\psi^2 \end{aligned} \tag{146}$$

From Eqs.(140) and (146), we have

$$da/dt = 2\psi^2 \tag{147}$$

Similarly we can prove that

$$db/dt = 2\phi^2 ; \quad dh/dt = -2\phi\psi \tag{148}$$

From Eqs.(147) and(148) , integrate ; Hence

$$a = 2\int_0^t \psi^2 dt ; \quad h = -2\int_0^t \phi\psi\, dt ; \quad b = 2\int_0^t \phi^2 dt \tag{149}$$

The lemma is then proved.

In order to apply the foregoing lemma to Eq.(136) we first notice that the equation is separable into the pairs of variables (ξ, X), (η, Y) and (ζ, Z). Expressing therefore the solution in the form

$$\chi = \chi_1(\xi, X)\, \chi_2(\eta, Y)\, \chi_3(\zeta, Z) \tag{150}$$

we see that each of the functions χ_1, χ_2 and χ_3 satisfies Eq.(137) with

$$\phi(t) = q^{1/2} e^{\beta t} ; \quad \psi(t) = q^{1/2}/\beta \tag{151}$$

Hence, from lemma II, the solution of Eq.(136) with the boundary condition

$$\vec{\rho} = \vec{\rho}_0 , \quad \vec{P} = \vec{P}_0 , \quad at \quad t = 0 \tag{152}$$

is

$$\chi = \frac{1}{8\pi^3 \Delta^{3/2}} exp\left\{ -\left[a|\vec{\rho} - \vec{\rho}_0|^2 + 2h(\vec{\rho} - \vec{\rho}_0).(\vec{P} - \vec{P}_0) + b|\vec{P} - \vec{P}_0|^2 \right] / 2\Delta \right\} \tag{153}$$

where $\quad a = 2q\beta^{-2} \int_0^t dt = 2q\beta^{-2} t$

$$b = 2q\int_0^t e^{2\beta t}\,dt = q\beta^{-1}(e^{2\beta t}-1) \tag{154}$$
$$h = -2q\beta^{-1}\int_0^t e^{\beta t}\,dt = -2q\beta^{-2}(e^{\beta t}-1)$$

and
$$\vec{\rho}-\vec{\rho}_0 = e^{\beta t}\vec{u}-\vec{u}_0\ ,\quad \vec{P}-\vec{P}_0 = \vec{r}+\vec{u}/\beta-\vec{r}_0-\vec{u}_0/\beta \tag{155}$$

In Eq. (155) $\vec{r}_0$ and $\vec{u}_0$ are the position and velocity of the Brownian particle at time t = 0. Therefore

$$W = \frac{e^{3\beta t}}{8\pi^3\Delta^{3/2}}\exp\frac{1}{2\Delta}\left\{-a|\vec{\rho}-\vec{\rho}_0|^2+2h(\vec{\rho}-\vec{\rho}_0).(\vec{P}-\vec{P}_0)+b|\vec{P}-\vec{P}_0|^2\right\} \tag{156}$$

We will verify that the foregoing solution for W obtained as the fundamental solution of Eq. (127) agrees with what we obtained through a discussion of the Langevin equation:With $\vec{R}$ and $\vec{S}$ defined as in Eqs.(60) we have from Eq. (155)

$$\vec{\rho}-\vec{\rho}_0 = e^{\beta t}\vec{S}\ ;\quad \vec{P}-\vec{P}_0 = \vec{R}+(1/\beta)\vec{S} \tag{157}$$

Accordingly, from Eq (157), we obtain $a|\vec{\rho}-\vec{\rho}_0|^2+2h(\vec{\rho}-\vec{\rho}_0).(\vec{P}-\vec{P}_0)$

$$+b|\vec{P}-\vec{P}_0|^2 = ae^{2\beta t}|\vec{S}|^2+2he^{\beta t}(\vec{R}.\vec{S}+(1/\beta)|\vec{S}|^2)+b|\vec{R}+(1/\beta)\vec{S}|^2$$
$$= e^{2\beta t}(F|\vec{S}|^2-2H\vec{R}.\vec{S}+G|\vec{R}|^2) \tag{158}$$

where

$$F = a+2h\beta^{-1}e^{-\beta t}+b\beta^{-2}e^{-2\beta t}\ ;\ G = be^{-2\beta t}\ ;\ H = -(he^{-\beta t}+b\beta^{-1}e^{-2\beta t}) \tag{159}$$

With a, b and h as given by Eqs.(154) we find that

$$F = q\beta^{-3}(2\beta t-3+4e^{-\beta t}-e^{-2\beta t})\ ;\quad G = q\beta^{-1}(1-e^{-2\beta t})\ ;\ H = q\beta^{-2}(1-e^{-\beta t})^2 \tag{160}$$

Therefore
$$FG-H^2 = (ab-h^2)e^{-2\beta t} = \Delta e^{-2\beta t} \tag{161}$$

Thus the solution (156) can be expressed alternatively in the form

$$W = \frac{1}{8\pi^3(FG-H^2)^{3/2}} \exp\left\{-(F|\vec{S}|^2 - 2H\vec{R}.\vec{S} + G|\vec{R}|^2)/2(FG-H^2)\right\} \qquad (162)$$

Comparing Eqs(160) and (162) with Eqs(57), (61), (62) we see that the verification is complete.

4.4 The solution for the case of a harmonically bound particle (6)

The method of solution is illustrated by considering the case of a one-dimensional oscillator describing Brownian motion. Equation. (126) then reduces to

$$\frac{\partial W}{\partial t} + u\frac{\partial W}{\partial x} - \omega^2 x\frac{\partial W}{\partial u} = \beta u\frac{\partial W}{\partial u} + \beta W + q\frac{\partial^2 W}{\partial u^2} \qquad (163)$$

Introduce as variables two first integrals of the associated subsidiary system:

$$dx/dt = u \; ; \quad du/dt = -\beta u - \omega^2 x \qquad (164)$$

From two independent first integrals of Eqs(164), we obtain

$$(x\mu_1 - u)\exp(-\mu_2 t) \quad \text{and} \quad (x\mu_2 - u)\exp(-\mu_1 t) \qquad (165)$$

where μ_1 and μ_2 have the same meanings as in Eqs.(67) and (68).

Accordingly we set

$$\xi = (x\mu_1 - u)\exp(-\mu_2 t) \; ; \; \eta = (x\mu_2 - u)\exp(-\mu_1 t) \qquad (166)$$

In these variables Eq.(163) becomes

$$\frac{\partial W}{\partial t} = \beta W + q\left(\exp(-2\mu_2 t)\frac{\partial^2 W}{\partial \xi^2} + 2\exp(-(\mu_1+\mu_2)t)\frac{\partial^2 W}{\partial \xi\,\partial \eta} + \exp(-2\mu_1 t)\frac{\partial^2 W}{\partial \eta^2}\right) \qquad (167)$$

Introducing the transformation

$$W = \chi\, e^{\beta t} \qquad (168)$$

we finally obtain from Eqs(167) and (168)

$$\frac{\partial \chi}{\partial t} = q\left(\exp(-2\mu_2 t)\frac{\partial^2 \chi}{\partial \xi^2} + 2\exp\{-(\mu_1+\mu_2)t\}\frac{\partial^2 \chi}{\partial \xi \partial \eta} + \exp(-2\mu_1 t)\frac{\partial^2 \chi}{\partial \eta^2}\right) \tag{169}$$

This equation is of the same form as Eq(137) in lemma II. Hence the solution of this equation which tends to $\delta(\xi-\xi_0)\delta(\eta-\eta_0)$ as $t \to 0$ is

$$\chi = \frac{1}{2\pi\Delta^{1/2}}\exp\frac{-1}{2\Delta}\left\{+a(\xi-\xi_0)^2 + 2h(\xi-\xi_0)(\eta-\eta_0) + b(\eta-\eta_0)^2\right\} \tag{170}$$

where

$$a = 2q\int_0^t \exp(-2\mu_1 t)\,dt = \frac{q}{\mu_1}\left\{1-\exp(-2\mu_1 t)\right\} \tag{171.1}$$

$$b = 2q\int_0^t \exp(-2\mu_2 t)\,dt = \frac{q}{\mu_2}\left\{1-\exp(-2\mu_2 t)\right\} \tag{171.2}$$

$$h = -2q\int_0^t \exp\{-(\mu_1+\mu_2)t\}\,dt = -\frac{2q}{\mu_1+\mu_2}\left\{1-\exp(-(\mu_1+\mu_2)t)\right\} \tag{171.3}$$

Further, from Eq.(166)

$$\xi_0 = x_0\mu_1 - u_0; \quad \eta_0 = x_0\mu_2 - u_0 \tag{172}$$

where x_0 and u_0 denote the position and velocity of the particle at time $t = 0$. From Eqs.(168) and (170), we can verify that

$$W = \frac{e^{\beta t}}{2\pi\Delta^{1/2}}\exp\left\{-\left[a(\xi-\xi_0)^2 + 2h(\xi-\xi_0)(\eta-\eta_0) + b(\eta-\eta_0)^2\right]/2\Delta\right\} \tag{173}$$

obtained as the solution of Eq.(163) which is in agreement with the distributions obtained through a discussion of the Langevin equation.