Chapter 2

Some properties of complex orthogonal groups

In this chapter, we consider briefly fundamental definitions and some properties of the complex orthogonal group and special complex orthogonal group that will be used in later chapters.

Let $d \in \mathbb{N}$. Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Denote by $M_d(\mathbb{F})$ the set of all $d \times d$ matrices with entries in $\mathbb{F}$ and by $GL(d, \mathbb{F})$ the set of all invertible $d \times d$ matrices with entries in $\mathbb{F}$. Then $GL(d, \mathbb{F})$ is a group under multiplication. We can regard $M_d(\mathbb{F})$ as the vector space $\mathbb{F}^{d^2}$, so it has an inherited topology from the usual topology on $\mathbb{F}^{d^2}$.

Definition 2.1. Let (A_n) be a sequence of $d \times d$ matrices in $M_d(\mathbb{F})$. We say that (A_n) converges to a matrix A if and only if each entry of A_n converges to the corresponding entry of A in $\mathbb{F}$, i.e., if $(A_n)_{ij}$ converges to A_{ij} for all $1 \leq i, j \leq d$.

Consider the map $(\,\cdot\,,\,\cdot\,) : \mathbb{F}^d \times \mathbb{F}^d \to \mathbb{F}$ given by

$$(x, y) = x_1 y_1 + \cdots + x_d y_d$$

for all $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, x_d) \in \mathbb{F}^d$. This map is a symmetric bilinear form. If $\mathbb{F} = \mathbb{R}$, then this defines a positive inner product on $\mathbb{R}^d$. However, if $\mathbb{F} = \mathbb{C}$, this is not an inner product because

$$\big((1, i, 0 \dots, 0), (1, i, 0 \dots, 0)\big) = 0,$$

but $(1, i, 0 \dots, 0) \neq 0$. When $\mathbb{F} = \mathbb{C}$, we define a hermitian inner product on $\mathbb{C}^d$ by

$$\langle x, y\rangle = \sum_{i=1}^{d} \bar{x}_i y_i \quad \text{for all } x,\ y \in \mathbb{C}^d.$$

If we write $x \in \mathbb{F}^d$ as a column matrix, then we have

$$(x, y) = x^t y \quad \text{for all } x,\ y \in M_{d\times 1}(\mathbb{F}) \cong \mathbb{F}^d.$$

From this, we have

$$(Ax, y) = (Ax)^t y = x^t A^t y = (x, A^t y)$$

for all $x, y \in \mathbb{F}^d$ and all $A \in M_d(\mathbb{F})$.

Lemma 2.2. *The bilinear form* $(\cdot, \cdot)$ *is non-degenerate, i.e.,* $(x, y) = 0$ *for all* $y \in \mathbb{C}^d$ *implies* $x = 0$.

Proof. Let $x \in \mathbb{C}^d$ be such that $(x, y) = 0$ for all $y \in \mathbb{C}^d$. Then for all $i = 1, \dots, d$,

$$x_i = (x, e_i) = 0,$$

where $\{e_1, \dots, e_d\}$ is the standard basis for $\mathbb{C}^d$. Hence $x = 0$. $\square$

Definition 2.3. An invertible $d \times d$ matrix A which preserves the bilinear form $(\cdot, \cdot)$, i.e.

$$(Ax, Ay) = (x, y) \qquad \text{for all } x, y \in \mathbb{F}^d.$$

is called an **orthogonal matrix**. Denote by $O(d, \mathbb{F})$ the set of all $d \times d$ orthogonal matrices and by $SO(d, \mathbb{F})$ the set of all A in $O(d, \mathbb{F})$ with $\det A = 1$.

Lemma 2.4. *Let $A \in GL(d, \mathbb{F})$. Then A is an orthogonal matrix if and only if $A^t = A^{-1}$.*

Proof. ($\Rightarrow$) Assume that A is an orthogonal matrix. Then

$$(x, y) = (Ax, Ay) = (x, A^tAy)$$

for all $x, y \in \mathbb{F}^d$. It follows that $I = A^tA$, and that $A^t = A^{-1}$.

($\Leftarrow$) Assume that $A^t = A^{-1}$. Then

$$(Ax, Ay) = (x, A^tAy) = (x, y)$$

for all $x, y \in \mathbb{F}^d$. Hence, A is an orthogonal matrix. $\square$

Lemma 2.5. *$O(d, \mathbb{C})$ and $SO(d, \mathbb{C})$ are closed subgroups of $GL(d, \mathbb{C})$.*

Proof. First, we will prove that $O(d, \mathbb{C})$ is a subgroup of $GL(d, \mathbb{C})$. It is clear that $I \in O(d, \mathbb{C})$. Let $A, B \in O(d, \mathbb{C})$. Then $AB \in GL(d, \mathbb{C})$ and $A^t = A^{-1}$, $B^t = B^{-1}$. Moreover,

$$(AB)^t = B^tA^t = B^{-1}A^{-1} = (AB)^{-1} \qquad \text{and}$$
$$(A^{-1})^t = (A^t)^t = A = (A^{-1})^{-1}.$$

By Lemma 2.4, we have that $A^{-1}, AB \in O(d, \mathbb{C})$. Hence, $O(d, \mathbb{C})$ is a subgroup of $GL(d, \mathbb{C})$. Similarly, we have that $SO(d, \mathbb{C})$ is a subgroup of $GL(d, \mathbb{C})$. Next, we will show that $O(d, \mathbb{C})$ is closed in $GL(d, \mathbb{C})$. Define

$$\Psi : GL(d, \mathbb{C}) \to M_d(\mathbb{C}) \quad \text{by } \Psi(A) = AA^t$$

for all $A \in GL(d, \mathbb{C})$. Then Ψ is a continuous function. But

$$O(d, \mathbb{C}) = \{ A \in GL(d, \mathbb{C}) \mid AA^t = I \} = \Psi^{-1}(\{I\}).$$

Hence, $O(d,\mathbb{C})$ is a closed subgroup of $GL(d,\mathbb{C})$. Denote by

$$\det : M_d(\mathbb{C}) \to \mathbb{C}$$

the determinant function. Then det is a polynomial function, so it is continuous. Moreover,

$$SO(d,\mathbb{C}) = O(d,\mathbb{C}) \cap \det^{-1}(\{1\}).$$

Thus $SO(d,\mathbb{C})$ is a closed subgroup of $GL(d,\mathbb{C})$. $\square$

Definition 2.6. The set of all $d \times d$ complex orthogonal matrices is called the **complex orthogonal group** $O(d,\mathbb{C})$ and the set of all $d \times d$ complex orthogonal matrices with determinant one is called the **special complex orthogonal group** $SO(d,\mathbb{C})$. The set of $d \times d$ real orthogonal matrices is called the **(real)orthogonal group** $O(d)$. The set of $d \times d$ real orthogonal matrices with determinant one is called the **special (real)orthogonal group** $SO(d)$. Then $O(d)$ and $SO(d)$ are subgroups of $O(d,\mathbb{C})$, and hence of $GL(d,\mathbb{C})$. Moreover, they are closed subgroups of $GL(d,\mathbb{C})$. Geometrically, an element of $O(d)$ is either a rotation, or a combination of rotation and reflection. An element of $SO(d)$ is just a rotation.

Lemma 2.7. *Every element A of $O(2)$ is of one of the two forms*

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad \text{or}$$

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

If A is of the first form, then $\det A = 1$, *i.e.* $A \in SO(2)$; *if A is of the second form, then* $\det A = -1$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$. Then $A^{-1} = A^t$ and $\det(A) = \pm 1$. If $\det A = 1$, then

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A^{-1} = A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Thus $a = d$ and $c = -b$, i.e.,

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since $A^tA = I$, $a^2 + b^2 = 1$. So $a \in [-1, 1]$. Hence there exists $\theta \in [0, 2\pi]$ such that $a = \cos\theta$, i.e., $b = \sin\theta$. Then

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Similarly, if $\det A = -1$, we have

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

□

Notation.

$$S_r(d, \mathbb{C}) := \{ z \in \mathbb{C}^d \mid (z, z) = r \} \quad \text{for all } r \in \mathbb{C} \text{ and}$$

$$S_r(d) := \{ x \in \mathbb{R}^d \mid (x, x) = r \} \quad \text{for all } r \in \mathbb{R}^+ \cup \{0\}.$$

Lemma 2.8. *For each $d \geq 2$, $SO(d)$ acts transitively on $S_r(d)$ for all $r \in \mathbb{R}^+$.*

Proof. Let $r \in \mathbb{R}^+$. For all $A \in SO(d)$, $x \in S_r(d)$, we have

$$(Ax, Ax) = (x, A^tAx) = (x, x) = r.$$

This show that $SO(d)$ acts on $S_r(d)$ by left-multiplication. Next, we will show that this is a transitive action. First we will show that $SO(d)$ acts transitively on $S_1(d)$. Let v be a unit vector of $\mathbb{R}^d$. If v and e_1 are linearly dependent, then $v = \alpha e_1$ where $|\alpha| = 1$. If v and e_1 are linearly independent, let

$$e_1 = f_1, \text{ and } f_2 = v - (v, e_1).$$

Then $f_1 \perp f_2$ and $\text{span}\{ e_1, v \} = \text{span}\{ f_1, f_2 \}$. By Gram-Schmidt orthogonalization process, there exists an orthonormal basis $B_0 = \{ f_1, f_2, f_3, \ldots, f_d \}$ of $\mathbb{R}^d$. Let P be a matrix transition from the basis B_0 to the standard basis. Then $P \in SO(d)$ and $Pv = [v]_{B_0}$. So

$$P^{-1} \begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} Pv = e_1$$

where $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a matrix which rotates $[v]_{B_0}$ to $[e_1]_{B_0}$ in the $\{f_1, f_2\}$-plane. Therefore

$$P^{-1} \begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} P \in SO(d).$$

Hence, $SO(d)$ acts transitively on $S_1(d)$. If $r \neq 1$, then

$$P^{-1} \begin{pmatrix} A & 0 \\ 0 & I_{d-2} \end{pmatrix} Pv = \sqrt{r}\, e_1.$$

Hence, $SO(d)$ acts transitively on $S_r(d)$ for any $r \in \mathbb{R}^+$. $\square$