Chapter 4

$SO(d,\mathbb{C})$-invariant holomorphic function spaces

In this chapter, we consider the subspace of the Segal-Bargmann space which is invariant under the action of the special complex orthogonal group. Our objective is to find an orthonormal basis and the reproducing kernel of this space. This will yield a pointwise bound of functions in this pace.

Lemma 4.1. *If $f, g \in \mathcal{H}(\mathbb{C}^d)$ are such that $f|_{\mathbb{R}^d} = g|_{\mathbb{R}^d}$, then $f = g$ on $\mathbb{C}^d$.*

Proof. It suffices to show the following statements:

$$\text{for any holomorphic functions } f \text{ on } \mathbb{C}^d\text{ , } f|_{\mathbb{R}^d} = 0 \text{ implies } f = 0 \text{ on } \mathbb{C}^d.$$

We prove this by induction on degree d. Let $P(d)$ denote the above statements. Then $P(1)$ holds by analytic continuation. Assume that $P(d-1)$ holds. We will show that $P(d)$ holds. Let $f \in \mathcal{H}(\mathbb{C}^d)$ be such that $f|_{\mathbb{R}^d} = 0$. For any $x \in \mathbb{R}$, define $f_x : \mathbb{C}^{d-1} \to \mathbb{C}$ by

$$f_x(x_1, \ldots, x_{d-1}) = f(x_1 \ldots, x_{d-1}, x)$$

for all $(x_1, \dots, x_{d-1}) \in \mathbb{C}^{d-1}$. Thus $f_x \in \mathcal{H}(\mathbb{C}^{d-1})$ and $f_x|_{\mathbb{R}^{d-1}} = 0$. By the induction hypothesis we have that $f_x = 0$. Hence, for any $x \in \mathbb{R}$, $f_x = 0$. For any $(x_1, \dots, x_{d-1}) \in \mathbb{C}^{d-1}$, define $f_{(x_1,\dots,x_{d-1})} : \mathbb{C} \to \mathbb{C}$ by

$$f_{(x_1,\dots,x_{d-1})}(x) = f(x_1, \dots, x_{d-1}, x)$$

for all $x \in \mathbb{C}$. Then $f_{(x_1,\dots,x_{d-1})} \in \mathcal{H}(\mathbb{C})$ and for each $x \in \mathbb{R}$,

$$f_{(x_1,\dots,x_{d-1})}(x) = f_x(x_1, \dots, x_{d-1}) = 0.$$

So $f_{(x_1,\dots,x_{d-1})}|_{\mathbb{R}} = 0$. By the induction hypothesis $f_{(x_1,\dots,x_{d-1})} = 0$, and hence $f_{(x_1,\dots,x_{d-1})} = 0$ for all $(x_1, \dots, x_{d-1}) \in \mathbb{C}^{d-1}$. Therefore, for any $(x_1, \dots, x_d) \in \mathbb{C}^d$

$$f(x_1, \dots, x_d) = f_{(x_1,\dots,x_{d-1})}(x_d) = 0.$$

Hence $f = 0$. □

Lemma 4.2. *Let $f \in \mathcal{H}(\mathbb{C}^d)$ be such that*

$$f(Ax) = f(x), \quad \text{for all } A \in SO(d) \text{ and all } x \in \mathbb{R}^d.$$

Then $f(Az) = f(z)$ for all $A \in SO(d, \mathbb{C})$ and all $z \in \mathbb{C}$.

Proof. Let $f \in \mathcal{H}(\mathbb{C}^d)$ be such that

$$f(Ax) = f(x) \quad \text{for all } A \in SO(d) \text{ and all } x \in \mathbb{R}^d.$$

For any $A \in SO(d)$, define $\phi_A : \mathbb{C}^d \to \mathbb{C}$ by

$$\phi_A(z) = f(Az) \quad \text{for all } z \in \mathbb{C}^d.$$

Then ϕ_A is holomorphic. By the assumption, we have that $\phi_A|_{\mathbb{R}^d} = f|_{\mathbb{R}^d}$. Hence, by Lemma 4.1, $\phi_A = f$ for all $A \in SO(d)$. So we have that

$$f(Az) = f(z) \quad \text{for all } A \in SO(d) \text{ and all } z \in \mathbb{C}^d.$$

For any $z \in \mathbb{C}^d$, define $g_z, h_z \colon SO(d,\mathbb{C}) \to \mathbb{C}$ by

$$g_z(A) = f(Az) \text{ and } h_z(A) = f(z)$$

for all $A \in SO(d,\mathbb{C})$. Then g_z and h_z are holomorphic and $g_z|_{SO(d)} = h_z|_{SO(d)}$. From[[H1], Lemma 5, p-111], it follows that $g_z = h_z$ for all $z \in \mathbb{C}^d$. Hence, $f(Az) = f(z)$, for all $A \in SO(d,\mathbb{C})$ and all $z \in \mathbb{C}$. □

Definition 4.3. Let F be a holomorphic function on $\mathbb{C}^d$. We say that F is $SO(d,\mathbb{C})$-**invariant** if

$$F(Az) = F(z) \quad \text{for all } A \in SO(d,\mathbb{C}) \text{ and all } z \in \mathbb{C}^d.$$

Notation. Define $\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ to be the set of all $SO(d,\mathbb{C})$-invariant holomorphic functions on $\mathbb{C}^d$, i.e.,

$$\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} = \{ f \in \mathcal{H}(\mathbb{C}^d) \mid f \text{ is } SO(d,\mathbb{C})\text{-invariant} \}.$$

Then it is a linear subspace of $\mathcal{H}(\mathbb{C}^d)$. We write

$$\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})} = \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \cap L^2(\mathbb{C}^d,\mu_t).$$

Theorem 4.4. *$\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ is a closed subspace of $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)$.*

Proof. We will show that $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ is a subspace of $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)$. Let $f, g \in \mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ and $\alpha \in \mathbb{C}$. Then $f+g$, $\alpha f \in \mathcal{H}L^2(\mathbb{C}^d,\mu_t)$. Moreover,

$$\begin{aligned}(f+g)(Az) &= f(Az) + g(Az) = f(z) + g(z) = (f+g)(z), \quad \text{and} \\ (\alpha f)(Az) &= \alpha f(Az) = \alpha f(z) = (\alpha f)(z).\end{aligned}$$

for all $A \in SO(d,\mathbb{C})$ and all $z \in \mathbb{C}^d$. Thus $f+g$, $\alpha f \in \mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$. Hence, $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ is a subspace of $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)$. Next, we will show that

$\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ is closed. Let (f_n) be a sequence in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ which is convergent to f in $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$. Then by Theorem 3.8 we have that,

$$| f_n(z) - f(z) |^2 \leq e^{|z|^2/t} \| f_n - f \|_2 \to 0 \quad \text{as } n \to \infty$$

for all $z \in \mathbb{C}^d$. Therefore $f_n(z) \to f(z)$ for all $z \in \mathbb{C}^d$. Let $A \in SO(d, \mathbb{C})$. For any $z \in \mathbb{C}^d$,

$$\begin{aligned} f(Az) &= \lim_{n\to\infty} f_n(Az) \\ &= \lim_{n\to\infty} f_n(z) \\ &= f(z). \end{aligned}$$

Hence, $f \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$. This implies that $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ is a closed subspace of $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$. $\square$

Notation. Define $\mathcal{H}(\mathbb{C})^{even}$ to be the space of all complex-even holomorphic functions on $\mathbb{C}$, i.e.,

$$\mathcal{H}(\mathbb{C})^{even} = \{ g \in \mathcal{H}(\mathbb{C}) \mid g(-z) = g(z) \text{ for all } z \in \mathbb{C} \}.$$

Form now on, we fix the dimension $d \geq 2$

Theorem 4.5. *The map* $\phi \colon \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})} \to \mathcal{H}(\mathbb{C})^{even}$ *defined by*

$$\phi(f)(x) = f(x, 0, \ldots, 0)$$

for all $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ *and all* $x \in \mathbb{C}$, *is a linear isomorphism whose inverse is given by*

$$\psi(g)(z) = g\left(\sqrt{(z, z)}\right)$$

for all $g \in \mathcal{H}(\mathbb{C})^{even}$ *and all* $z \in \mathbb{C}^d$.

Note that since $z \mapsto \sqrt{z}$ is a double-valued function, we have to make a choice of the branch we use. However, this is irrelevant in our situation because we are interested in complex-even function. We may and will choose the branch with a larger argument.

Proof. First, we will show that ϕ is well-defined. Let $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. It is clear that $\phi(f) \in \mathcal{H}(\mathbb{C})$. Moreover, for any $x \in \mathbb{C}$,

$$\begin{aligned}\phi(f)(-x) &= f(-x, 0, \ldots, 0) \\ &= f(A(-x, 0, \ldots, 0)) \\ &= f(x, 0, \ldots, 0)\end{aligned}$$

where $A = \operatorname{diag}(-1, -1, 1, 1, \ldots, 1)$. Thus ϕ is well-defined. Next, we will show that ϕ is a linear map. Let $f_1, f_2 \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and $\alpha \in \mathbb{C}$. Then $f_1 + f_2$, $\alpha f_1 \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. For any $x \in \mathbb{C}$, we have

$$\begin{aligned}\phi(f_1 + f_2)(x) &= (f_1 + f_2)(x, 0, \ldots, 0) \\ &= f_1(x, 0, \ldots, 0) + f_2(x, 0, \ldots, 0) \\ &= \phi(f_1)(x) + \phi(f_2)(x)\end{aligned}$$

and

$$\begin{aligned}\phi(\alpha f_1)(x) &= (\alpha f_1)(x, 0, \ldots, 0) \\ &= \alpha f_1(x, 0, \ldots, 0) \\ &= \alpha\phi(f_1)(x).\end{aligned}$$

So $\phi(f_1 + f_2) = \phi(f_1) + \phi(f_2)$ and $\phi(\alpha f_1) = \alpha\phi(f_1)$. Hence, ϕ is linear map. Define $\psi : \mathcal{H}(\mathbb{C})^{even} \to \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ by

$$\psi(g)(z) = g\left(\sqrt{(z, z)}\right)$$

for all $g \in \mathcal{H}(\mathbb{C})^{even}$ and all $z \in \mathbb{C}^d$. Then for each $g \in \mathcal{H}(\mathbb{C})^{even}$, we have $\psi(g) \in \mathcal{H}(\mathbb{C}^d)$ and

$$\begin{aligned} \psi(g)(Az) &= g\left(\sqrt{(Az, Az)}\right) \\ &= g\left(\sqrt{(z, z)}\right) \\ &= \psi(g)(z) \end{aligned}$$

for all $A \in SO(d, \mathbb{C})$ and $z \in \mathbb{C}^d$. Thus ψ is well-defined. To show that ψ is a linear map, let $g_1, g_2 \in \mathcal{H}(\mathbb{C})^{even}$ and $\beta \in \mathbb{C}$. Then $g_1 + g_2$, $\beta g_1 \in \mathcal{H}(\mathbb{C})^{even}$. For any $z \in \mathbb{C}^d$,

$$\begin{aligned} \psi(g_1 + g_2)(z) &= (g_1 + g_2)\left(\sqrt{(z, z)}\right) \\ &= g_1\left(\sqrt{(z, z)}\right) + g_2\left(\sqrt{(z, z)}\right) \\ &= \psi(g_1)(z) + \psi(g_2)(z) \end{aligned}$$

and

$$\begin{aligned} \psi(\beta g_1)(z) &= (\beta g_1)\left(\sqrt{(z, z)}\right) \\ &= \beta g_1\left(\sqrt{(z, z)}\right) \\ &= \beta\psi(g_1)(z). \end{aligned}$$

Thus $\psi(g_1 + g_2) = \psi(g_1 + g_2)$ and $\psi(\beta g_1) = \beta\psi(g_1)$. Hence, ψ is a linear map. Claim that $\phi \circ \psi = \mathrm{id}_{\mathcal{H}(\mathbb{C})^{even}}$ and $\psi \circ \phi = \mathrm{id}_{\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}}$. To prove the claim, let $g \in \mathcal{H}(\mathbb{C})^{even}$. Then for all $x \in \mathbb{C}$,

$$\begin{aligned} \phi \circ \psi(g)(x) &= \phi(\psi(g))(x) \\ &= \psi(g)(x, 0, \ldots, 0) \\ &= g\left(\sqrt{(\,(x, 0, \ldots, 0), (x, 0, \ldots, 0)\,)}\right) \\ &= g(x). \end{aligned}$$

So $\phi \circ \psi = \mathrm{id}_{\mathcal{H}(\mathbb{C})^{even}}$. Let $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$. Then for any $x \in \mathbb{R}^d$,

$$\begin{aligned} \psi \circ \phi(f)(x) &= \psi(\phi(f))(x) \\ &= \phi(f)\left(\sqrt{(x,x)}\right) \\ &= f\left(\sqrt{(x,x)}, 0, \ldots, 0\right) \\ &= f(x), \qquad \text{by Lemma 2.8.} \end{aligned}$$

Hence, $\psi \circ \phi(f)|_{\mathbb{R}^d} = f|_{\mathbb{R}^d}$. By Lemma 4.1, $\psi \circ \phi(f) = f$. So $\psi \circ \phi = \mathrm{id}_{\mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}}$.

□

Corollary 4.6. *For each $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and each $r \in \mathbb{C}$, if x, $y \in S_r(d,\mathbb{C})$, then $f(x) = f(y)$.*

Proof. Let $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ and $r \in \mathbb{C}$. By Theorem 4.5, there exists a function $g \in \mathcal{H}(\mathbb{C})^{even}$ such that

$$f(z) = g\left(\sqrt{(z,z)}\right) \qquad \text{for all } z \in \mathbb{C}^d.$$

Hence, for all $x, y \in S_r(d,\mathbb{C})$,

$$f(x) = g\left(\sqrt{(x,x)}\right) = g\left(\sqrt{(y,y)}\right) = f(y).$$

□

Denote by $\mathcal{B}_d$ the Borel σ-algebra in $\mathbb{C}^d$ and by $\mathcal{B}$ the Borel σ-algebra in $\mathbb{C}$. Define $\Phi\colon (\mathbb{C}^d, \mathcal{B}_d, \mu_t) \to (\mathbb{C}, \mathcal{B})$ to be the branch of $\sqrt{(z,z)}$ with a larger argument. For each $E \in \mathcal{B}$, define

$$\lambda(E) = \mu_t(\Phi^{-1}(E)).$$

Then λ is a measure on $(\mathbb{C}, \mathcal{B})$ and for any measurable function g,

$$\int_E g\, d\lambda = \int_{\Phi^{-1}(E)} g \circ \Phi\, d\mu_t.$$

Notation. We write

$$\mathcal{H}L^2(\mathbb{C},\lambda)^{even} = \mathcal{H}(\mathbb{C})^{even} \cap L^2(\mathbb{C},\lambda).$$

It is easy to see that it is a closed subspace of $\mathcal{H}L^2(\mathbb{C},\lambda)$.

Theorem 4.7. *$\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ and $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$ are unitarily equivalent.*

Proof. From Theorem 4.5 we have that the function

$$\psi\colon \mathcal{H}(\mathbb{C})^{even} \to \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$$

is a linear isomorphism. We consider the restriction of ψ to the space $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$. Let $g \in \mathcal{H}(\mathbb{C})^{even}$ and $f \in \mathcal{H}(\mathbb{C}^d)^{SO(d,\mathbb{C})}$ be such that $f = \psi(g)$. Thus

$$\begin{aligned}
\int_{\mathbb{C}} |g|^2 d\lambda &= \int_{\Phi^{-1}(\mathbb{C})} |g\circ\Phi(z)|^2\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} |g(\sqrt{(z,z)}\,)|^2\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} |\psi(g)\,(z)|^2\, \mu_t(z)\, dz \\
&= \int_{\mathbb{C}^d} |f(z)|^2\, \mu_t(z)\, dz.
\end{aligned}$$

So we have, $\|g\|_{L^2(\mathbb{C},\lambda)} = \|f\|_{L^2(\mathbb{C}^d,\mu_t)}$. Hence, $f \in \mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$ if and only if $g \in \mathcal{H}L^2(\mathbb{C},\lambda)^{even}$. This shows that ψ is a unitary map from $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$ onto $\mathcal{H}L^2(\mathbb{C}^d,\mu_t)^{SO(d,\mathbb{C})}$. □

Theorem 4.8. *The set $\{\, x^{2n}\,\}_{n=0}^{\infty}$ forms an orthogonal basis for $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$.*

Proof. For each $n \in \mathbb{N}\cup\{0\}$, define $g_n\colon \mathbb{C}\to\mathbb{C}$ by

$$g_n(x) = x^{2n} \qquad \text{for all } x \in \mathbb{C}.$$

It is clear that $g_n \in \mathcal{H}(\mathbb{C})$. Claim that $\{g_n\}_{n=0}^{\infty}$ is an orthogonal subset of $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$. To prove the claim, let σ be a positive real number. Define

$$D_\sigma := \{\, z \in \mathbb{C}^d \mid |z_j| \le \sigma \text{ for all } j = 1,\ldots,d\,\} \qquad \text{and}$$

$$D_\sigma^1 := \{\, x \in \mathbb{C} \mid |x| \le \sigma\,\}.$$

Next, let

$$\begin{aligned} M_\sigma(j,k) &= \int_{D_\sigma^1} x^{2j}\bar{x}^{2k}\, e^{-x^2/t}(\pi t)^{-1}\, dx \\ &= \int_0^{2\pi}\int_0^{\sigma} e^{2i\theta(j-k)} r^{2j+2k+1} e^{-r^2/t}(\pi t)^{-1}\, dr\, d\theta. \end{aligned}$$

It follows that

(i.) $M_\sigma(j,k) = 0$ if $j \neq k$,

(ii.) $M_\sigma(k,k) \to t^{2k}(2k)!$ as $\sigma \to \infty$.

In other words,

$$\int_{\mathbb{C}} x^{2j}\bar{x}^{2k}\, d\mu_t(z) = \delta_{jk} t^{2k}(2k)!.$$

For any $m, n \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned} \int_{\mathbb{C}} g_m \cdot \overline{g_n}\, d\lambda &= \int_{\mathbb{C}^d} (g_m \cdot \overline{g_n}) \circ \Phi(z)\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d} g_m\left(\sqrt{(z,z)}\right)\, \overline{g_n\left(\sqrt{(z,z)}\right)}\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d} (z,z)^m\, \overline{(z,z)^n}\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d} (z_1^2 + \cdots + z_d^2)^m\, (\bar{z}_1^2 + \cdots + \bar{z}_d^2)^n\, \mu_t(z)\, dz \\ &= \int_{\mathbb{C}^d}\left(\sum_{j_1+\cdots+j_d=m} z_1^{2j_1}\cdots z_d^{2j_d}\right)\left(\sum_{k_1+\cdots+k_d=n} \bar{z}_1^{2k_1}\cdots \bar{z}_d^{2k_d}\right) \mu_t(z)\, dz \\ &= \sum_{j_1+\cdots+j_d=m}\ \sum_{k_1+\cdots+k_d=n} \int_{\mathbb{C}^d} (z_1^{j_1}\bar{z}_1^{k_1})^2 \cdots (z_d^{j_d}\bar{z}_d^{k_d})^2\, \frac{e^{-|z|^2/t}}{(\pi t)^d}\, dz \\ &= \sum_{j_1+\cdots+j_d=m}\ \sum_{k_1+\cdots+k_d=n} M_\sigma(j_1,k_1)\cdots M_\sigma(j_d,k_d). \end{aligned}$$

Thus it follows from (i) that

$$\int_{\mathbb{C}} g_m \cdot \overline{g_n}\, d\lambda = 0, \qquad \text{if } m \neq n.$$

If $n = m$, we have

$$\begin{aligned}\int_{\mathbb{C}} |g_n|^2 \, d\lambda &= \sum_{k_1+\cdots+k_d=n} M_\sigma(k_1,k_1)\cdots M_\sigma(k_d,k_d) \\ &= \sum_{k_1+\cdots+k_d=n} t^{2k_1}(2k_1)!\cdots t^{2k_d}(2k_d)! \\ &= t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)!\cdots(2k_d)!.\end{aligned}$$

Thus for any $n \in \mathbb{N}\cup\{0\}$,

$$||g_n||^2_{L^2(\mathbb{C},\lambda)} = t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)!\cdots(2k_d)!. \tag{4.1}$$

Hence, $\{g_n\}_{n=0}^\infty$ is an orthogonal subset of $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$. Next, we will show that $\{g_n\}_{n=0}^\infty$ is an orthogonal basis of $\mathcal{H}L^2(\mathbb{C},\lambda)^{even}$. Let $g \in \mathcal{H}(\mathbb{C})^{even}$. Then $g(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ for each $x \in \mathbb{C}$, where $a_n \in \mathbb{C}$ for all $n \in \mathbb{N}\cup\{0\}$. But $g(x) = g(-x)$, so $a_n = 0$ for each odd positive integer n. Thus

$$g(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}.$$

Since this power series converges uniformly on the set D_σ, we have

$$\begin{aligned}&\int_{D_\sigma} |g(\sqrt{(z,z)})|^2 \mu_t(z)\,dz \\ &= \int_{D_\sigma} \sum_{n=0}^{\infty} a_{2n}(z,z)^n \sum_{m=0}^{\infty} \overline{a_{2m}(z,z)^m}\, \mu_t(z)\,dz \\ &= \int_{D_\sigma} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}}(z,z)^n\overline{(z,z)^m}\,\mu_t(z)\,dz \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_{D_\sigma} (z_1^2+\cdots+z_d^2)^n\,(\bar{z}_1^2+\cdots+\bar{z}_d^2)^m\,\mu_t(z)\,dz \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \int_{D_\sigma} \sum_{j_1+\cdots+j_d=n} z_1^{2j_1}\cdots z_d^{2j_d} \sum_{k_1+\cdots+k_d=m} \bar{z}_1^{2k_1}\cdots\bar{z}_d^{2k_d}\,\frac{e^{-|z|^2/t}}{(\pi t)^d}\,dz \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{2n}\overline{a_{2m}} \sum_{j_1+\cdots+j_d=n}\ \sum_{k_1+\cdots+k_d=m} \int_{D_\sigma} (z_1^{j_1}\bar{z}_1^{k_1})^2\cdots(z_d^{j_d}\bar{z}_d^{k_d})^2\,\frac{e^{-|z|^2/t}}{(\pi t)^d}\,dz \\ &= \sum_{n=0}^{\infty} |a_{2n}|^2 \sum_{k_1+\cdots+k_d=n} M_\sigma(k_1,k_1)\cdots M_\sigma(k_d,k_d).\end{aligned}$$

Next, using the monotone convergence theorem, we have

$$\begin{aligned}
\int_{\mathbb{C}} |g|^2 \, d\lambda &= \int_{\mathbb{C}^d} |g \circ \Phi(z)|^2 \, \mu_t(z) \, dz \\
&= \lim_{\sigma \to \infty} \int_{D_\sigma} |g(\sqrt{(z, z)})|^2 \, \mu_t(z) \, dz \\
&= \lim_{\sigma \to \infty} \sum_{n=0}^{\infty} |a_{2n}|^2 \sum_{k_1+\cdots+k_d=n} M_\sigma(k_1, k_1) \cdots M_\sigma(k_d, k_d) \\
&= \sum_{n=0}^{\infty} |a_{2n}|^2 \sum_{k_1+\cdots+k_d=n} (t^{2k_1}(2k_1)!) \cdots (t^{2k_d}(2k_d)!) \\
&= \sum_{n=0}^{\infty} \Big(|a_{2n}|^2 \, t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)! \Big).
\end{aligned}$$

Therefore, if g is square-integrable with respect to λ, then

$$\sum_{n=0}^{\infty} \Big(|a_{2n}|^2 \, t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)! \Big) = \int_{\mathbb{C}} |g|^2 \, d\lambda < \infty.$$

Define a sequence (F_n) to be

$$F_n(x) := \sum_{m=0}^{n} a_{2m} \, x^{2m}$$

for any $x \in \mathbb{C}$. Then (F_n) is a Cauchy sequence in $L^2(\mathbb{C}, \lambda)$. To see this, notice that for $m > n$,

$$\begin{aligned}
\|F_m - F_n\|_{L^2(\mathbb{C},\lambda)} &= \int_{\mathbb{C}} |F_m - F_n|^2 \, d\lambda \\
&= \int_{\mathbb{C}^d} \left| \sum_{k=n+1}^{m} a_{2k}(z, z)^k \right|^2 \mu_t(z) \, dz \\
&= \sum_{k=n+1}^{m} |a_{2k}|^2 \, t^{2k} \sum_{k_1+\cdots+k_d=k} (2k_1)! \cdots (2k_d)! \to 0 \qquad \text{as } m, n \to \infty.
\end{aligned}$$

Thus (F_n) converges in $L^2(\mathbb{C}, \lambda)$ to some function h, and hence it has a subsequence which converges pointwise almost everywhere to h . But (F_n) converges pointwise to g, so we have that $h = g$ a.e.$[\lambda]$. Therefore $h = g$ in $L^2(\mathbb{C}, \lambda)$, so (F_n) converges to g in $L^2(\mathbb{C}, \lambda)$. Thus $g \in \overline{\operatorname{span}\{g_n \mid n \in \mathbb{N}\}}$. This show that $\{x^{2n}\}_{n=0}^{\infty}$ forms an orthogonal basis for $\mathcal{H}L^2(\mathbb{C}, \lambda)$. $\square$

Corollary 4.9. *The following set*

$$\left\{ \frac{x^{2n}}{\left(t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)! \right)^{1/2}} \right\}_{n=0}^{\infty},$$

forms an orthonormal basis for $\mathcal{H}L^2(\mathbb{C}, \lambda)^{even}$.

Proof. This follows from Theorem 4.8 and identity (4.1). □

Corollary 4.10. *The following set*

$$\left\{ \frac{(z, z)^n}{\left(t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)! \right)^{1/2}} \right\}_{n=0}^{\infty}.$$

forms an orthonormal basis for $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.

Proof. This follows from Theorem 4.7 and Corollary 4.9. □

It is easy to see that a closed subspace of a reproducing kernel Hilbert space is also a reproducing kernel Hilbert space. Thus $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ has a reproducing kernel.

Corollary 4.11. *The reproducing kernel for the space* $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$ *is given by*

$$K(z, w) = \sum_{n=0}^{\infty} \frac{(z, z)^n \, \overline{(w, w)^n}}{t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)!}.$$

Hence we have the pointwise bound

$$|F(z)|^2 \leq \sum_{n=0}^{\infty} \frac{|(z, z)|^{2n}}{t^{2n} \sum_{k_1+\cdots+k_d=n} (2k_1)! \cdots (2k_d)!} ||F||^2$$

for any $F \in \mathcal{H}L^2(\mathbb{C}^d, \mu_t)^{SO(d,\mathbb{C})}$.

Proof. This follows from Theorem 3.3 and Corollary 4.10 □