CHAPTER IV.

Two Limiting Cases.

It is now a matter of routine method to check our result obtained in the preceeding chapter. The subject is that we take the small value of the magnetic field and the nonlocal field when absent and present of the external force field subsequently.

IV.1 Limiting Caess in the Absence of an External Force Field

We now consider the propagator when the external force field is zero. From Eqs.(III.26) and (III.27), we have

$$
\begin{aligned}
K_0(a,b) = & \left[\frac{m\nu^2 t}{8\pi i\hbar \sin([\omega + \frac{\Omega}{2}]t/2)\sin([\omega - \frac{\Omega}{2}]t/2)} \right] \\
& \times \exp\left\{ \frac{i}{\hbar}\left[\frac{m\omega}{2\sin(\omega t)} \left[\cos(\omega t)\left(\gamma_b^2 + \gamma_A^2\right) - 2\gamma_A^T e^{J\frac{\Omega}{2}t}\gamma_b \right] \right.\right. \\
& + \frac{m\omega}{4\sin(\omega t)} \left\{ [\cos(\tfrac{\Omega}{2}t) - \cos(\omega t)](\gamma_b + \gamma_a)^2 \right. \\
& - \frac{4}{\omega}[\omega\sin(\tfrac{\Omega}{2}t) - \tfrac{\Omega}{2}\sin(\omega t)]\gamma_b^T J\gamma_A \\
& \left.\left.\left. + \frac{1}{\omega^2}\frac{[\omega\sin(\frac{\Omega}{2}t) - \frac{\Omega}{2}\sin(\omega t)]^2}{[\cos(\frac{\Omega}{2}t) - \cos(\omega t)]}(\gamma_b - \gamma_a)^2 \right\} \right] \right\}
\end{aligned}
\qquad (\text{IV}.1)
$$

By using the identity $\exp(\pm J\phi) = \cos\phi \pm J\sin\phi$ and the representations of γ_A and γ_b, Eq. (IV.1) can be rewritten in the usual representation as

$$K_0(a,b) = \left[\frac{m\nu^2 t}{8\pi i\hbar \sin([\omega+\frac{\Omega}{2}]t/2)\sin([\omega-\frac{\Omega}{2}]t/2)} \right]$$

$$\times \exp\left\{ \frac{i}{\hbar}\left\{ \frac{m\omega}{2\sin(\omega t)} \left[\cos(\omega t)\left([x_a^2+x_b^2]+[y_a^2+y_b^2]\right) \right.\right.\right.$$

$$\left. + 2\left\{ \sin(\tfrac{\Omega}{2}t)[x_b y_b - x_a y_a] - \cos(\tfrac{\Omega}{2}t)[y_a y_b - x_b y_a] \right\} \right]$$

$$+ \frac{m\omega}{4\sin(\omega t)} \left\{ [\cos(\tfrac{\Omega}{2}t) - \cos(\omega t)]\left([x_b+x_a]^2+[y_b+y_a]^2\right) \right.$$

$$+ \frac{4}{\omega}\left[\omega\sin(\tfrac{\Omega}{2}t) - \tfrac{\Omega}{2}\sin(\omega t)\right](x_b y_a - x_a y_b)$$

$$+ \frac{1}{\omega^2}\frac{[\omega\sin(\frac{\Omega}{2}t) - \frac{\Omega}{2}\sin(\omega t)]^2}{[\cos(\frac{\Omega}{2}t) - \cos(\omega t)]}$$

$$\left.\left.\left. \times \left([x_b-x_a]^2+[y_b-y_a]^2\right) \right\} \right\} \right\} . \qquad (\text{IV}.2)$$

It is the exact propagator of an electron moving in two dimensions under the influence of a transverse magnetic field and a nonlocal harmonic oscillator potential. We consider the two cases.

IV.1.1 Nonlocal Field Going to Zero.

When the nonlocal harmonic oscillator potential going to zero, the system of interest corresponds to the case when $\nu \to 0$ or $\omega \to \frac{\Omega}{2}$. First we consider the pre-exponential factor in Eq. (IV.2) ;

$$F(1,0) = \frac{m\nu^2 t}{8\pi i\hbar \sin([\omega+\frac{\Omega}{2}]t/2)\sin([\omega-\frac{\Omega}{2}]t/2)}$$

$$= \frac{mt\,(\omega-\frac{\Omega}{2})(\omega+\frac{\Omega}{2})}{8\pi i\hbar \sin([\omega-\frac{\Omega}{2}]t/2)\sin([\omega+\frac{\Omega}{2}]t/2)} \qquad (\text{IV}.3)$$

when we use the identity

$$\nu^2 = \omega^2 - \frac{\Omega^2}{4} = (\omega - \tfrac{\Omega}{2})(\omega + \tfrac{\Omega}{2})$$

and the trigonometric relation

$$\cos A - \cos B = 2\sin([A+B]\tfrac{1}{2})\sin([A-B]\tfrac{1}{2})$$

If we take the limit $\omega \to \frac{\Omega}{2}$, then we get

$$\lim_{\omega \to \frac{\Omega}{2}} F(t,0) = \left[\frac{m\Omega}{4\pi i \sin(\frac{\Omega}{2}t)}\right]$$

$$= \left[\frac{m}{2\pi i \hbar t}\right]\left[\frac{\Omega t}{2\sin(\frac{\Omega}{2}t)}\right] \qquad (\text{IV}.4)$$

Now let us take our attention on the limiting behavior of the exponential terms of Eq. (IV.2) ;

$$\lim_{\omega \to \frac{\Omega}{2}} S_{el}(a,b) = \frac{m(\frac{\Omega}{2})}{2\sin(\frac{\Omega}{2}t)}\Big\{\cos(\tfrac{\Omega}{2}t)\left[(x_b^2+x_a^2)+(y_a^2+y_b^2)\right]$$

$$+2\left[\sin(\tfrac{\Omega}{2}t)(x_b y_b - x_a y_a) - \cos(\tfrac{\Omega}{2}t)(x_a y_b - x_b y_a)\right]\Big\}.$$

$$(\text{IV}.5)$$

Thus from Eqs. (IV.4) and (IV.5) , we have

$$\lim_{\nu \to 0} K_0(a,b) = \left[\frac{m}{2\pi i \hbar t}\right]\left[\frac{\Omega t}{2\sin(\frac{\Omega}{2}t)}\right]$$

$$\times \exp\Big\{\frac{im}{2\hbar}\Big[\frac{\Omega}{2}\cot(\tfrac{\Omega}{2}t)\left[(x_b-x_a)^2+(y_b-y_a)^2\right]$$

$$+\Omega[x_b y_a - x_a y_b]\Big]\Big\} \qquad (\text{IV}.6).$$

This is the propagator of a charged particle confined in two dimensions in the presence of a transverse magnetic field.

IV.1.2 Magnetic Field Going to Zero.

This limiting case corresponds to $\Omega \to 0$ or $\omega \to \nu$. We first consider the pre-exponential factor in Eq. (IV.2) ;

$$\lim_{\Omega \to 0} F(t,0) = \lim_{\Omega \to 0} \frac{m\nu^2 t}{8\pi i\hbar \sin([\omega + \frac{\Omega}{2}]t/2)\sin([\omega - \frac{\Omega}{2}]t/2)}$$

$$= \frac{m\nu^2 t}{8\pi i\hbar \sin(\frac{\nu}{2}t)\sin(\frac{\nu}{2}t)}$$

$$= \left[\frac{m}{2\pi i\hbar t}\right]\left[\frac{\nu t}{2\sin(\frac{\nu}{2}t)}\right]^2 \qquad (IV.7)$$

The expontial terms in Eq. (IV.2) becomes

$$\lim_{\Omega \to 0} S_{cl}(a,b) = \frac{m}{2}\left[\frac{\nu}{2}\right]\cot(\frac{\nu}{2}t)\left[(x_b - x_a)^2 + (y_b - y_a)^2\right]. \qquad (IV.8)$$

when we use the trigonometric relations

$$1 + \cos(\nu t) = 1 + \cos^2(\frac{\nu}{2}t) - \sin^2(\frac{\nu}{2}t)$$

$$= 1 + 2\cos^2(\frac{\nu}{2}t) - 1$$

$$= 2\cos^2(\frac{\nu}{2}t)$$

and

$$\sin(\nu t) = 2\sin(\frac{\nu}{2}t)\cos(\frac{\nu}{2}t).$$

From Eqs. (IV.7) and (IV.8), it follows that

$$\lim_{\Omega \to 0} K_0(a,b) = \left[\frac{m}{2\pi i \hbar t}\right]\left[\frac{\nu t}{2\sin(\frac{\nu}{2}t)}\right]^2$$
$$\times \exp\left\{\frac{im}{\hbar 2}\left[\frac{\nu}{2}\right]\cot\left(\frac{\nu}{2}t\right)\left[(x_b-x_a)^2+(y_b-y_a)^2\right]\right\}$$
(IV.9)

This is the propagator of a particle confined in two dimensions under the influence of a nonlocal harmonic oscillator potential.

IV.2 Limiting Cases in the Presence of an External Force Field.

We now go back to Eq. (III.26) and consider the two limiting cases when $\nu \to 0$ and $\Omega \to 0$ in the presence of an external force field. From Eq. (III.26), we have

$$K(a,b) = F(t,0)\exp\left\{\frac{i}{\hbar}\left[S^0_{el}(a,b) + \int_0^t R^{cT}_{el}(\tau)f(\tau)d\tau + \frac{1}{2m}\int_0^t\int_0^t f^T(\tau)\,G(\tau,\tau')f(\tau')d\tau d\tau'\right]\right\}. \quad \text{(III.26)}$$

when the notation $S^0_{el}(a,b)$, $R^{cT}_{el}(\tau)$ and $G(\tau,\tau')$ are found in Eqs. (III.27), (III.28) and (III.29) respectively.

IV.2.1 Nonlocal Field Going to Zero

If we take the limit ν approach to zero, one can verify that the second term of $S^0_{el}(a,b)$ in Eq. (III.27) vanish:

$$\lim_{\nu \to 0} S^0_{el}(a,b) = S^{0*}_{el}(a,b) \qquad \text{(IV.10a)}$$

and also

$$\lim_{\nu \to 0} R_{cl}^{cT}(\tau) = r_{cl}^{cT}(\tau)\Big|_{\nu=0} \qquad (IV.10b)$$

and
$$\lim_{\nu \to 0} \mathcal{G}(\tau,\tau') = G(\tau,\tau')\Big|_{\nu=0} \qquad (IV.10c)$$

from Eqs. (III.28) and (III.29) respectively. Then Eq. (III.26) will reduce to

$$\lim_{\nu \to 0} K(a,b) = \left[\frac{m}{2\pi i\hbar t}\right]\left[\frac{\Omega}{2\sin(\frac{\Omega}{2}t)}\right]$$

$$\times \exp\left\{\frac{i}{\hbar}\left\{\left[\frac{m\Omega}{4\sin(\frac{\Omega}{2}t)}\left[\cos(\frac{\Omega}{2}t)\left[r_b^2+r_a^2\right] - 2r_a^T e^{J\frac{\Omega}{2}t} r_b\right]\right.\right.\right.$$

$$+ \frac{1}{\sin(\frac{\Omega}{2}t)}\int_0^t \left[\sin(\tfrac{\Omega}{2}[t-\tau])\, r_a^T + \sin(\tfrac{\Omega}{2}\tau)\, r_b^T e^{-J\frac{\Omega}{2}t}\right] e^{J\frac{\Omega}{2}\tau} f(\tau)d\tau$$

$$- \frac{2}{m\Omega\sin(\frac{\Omega}{2}t)}\int_0^t\!\!\int_0^\tau f^T(\tau)\, e^{-J\frac{\Omega}{2}[\tau-\tau']} f(\tau')\sin(\tfrac{\Omega}{2}[t-\tau])\sin(\tfrac{\Omega}{2}\tau') \ldots$$

$$\ldots \times d\tau d\tau' . \qquad (IV.11).$$

It is the propagator of an electron moving in two dimensions subject to a transverse magnetic field and the external force field.

IV.2.2 Magnetic Field Going to Zero.

For the absence of the magnetic field, $S_{cl}^{0}(a,b)$ in Eq. (III.27) reduces to

$$\lim_{\Omega \to 0} S_{cl}^{0}(a,b) = \frac{m\nu}{4}\cot(\tfrac{\nu}{2}t)\left[r_b - r_a\right]^2 . \qquad (IV.12)$$

The second term on the exponential term of Eq. (III.26) becomes

$$\lim_{\Omega \to 0}\int_0^t R_{cl}^{cT}(\tau) f(\tau)d\tau = \frac{1}{\sin(\nu t)}\int_0^t\left\{\sin(\nu[t-\tau])\, r_a^T + \sin(\nu t)\, r_b^T\right\} f(\tau)d\tau$$

$$- \frac{\sin(\frac{\nu}{2}t)}{\sin(\nu t)}\left[r_b^T + r_a^T\right]\int_0^t \sin(\tfrac{\nu}{2}[t-\tau])\sin(\tfrac{\nu}{2}\tau)\, d\tau$$

$$(IV.13)$$

and the last exponential term of Eq. (III.26) becomes

$$\lim_{\Omega\to 0} \frac{1}{2m} \int_0^t\int_0^t f^T(\tau)\, G(\tau,\tau')\, f(\tau')\,d\tau' d\tau$$

$$= -\frac{1}{m\nu\sin(\nu t)} \int_0^t\int_0^\tau f^T(\tau)\sin(\nu[t-\tau])\sin(\nu\tau')\, f(\tau')\,d\tau' d\tau$$

$$+ \frac{4}{m\nu\sin(\nu t)} \int_0^t\int_0^\tau f^T(\tau)\sin(\tfrac{\nu}{2}\tau)\sin(\tfrac{\nu}{2}[t-\tau])$$

$$\times \sin(\tfrac{\nu}{2}\tau')\sin(\tfrac{\nu}{2}[t-\tau'])\, f(\tau')\,d\tau d\tau' \; . \qquad (IV.14)$$

The details of these results in Eqs. (IV.12) , (IV.13) and (IV.14) are appeared in appendix H.

From Eqs. (IV.11) , (IV.12) , (IV.13) and (IV.14), we have the propagator;

$$\lim_{\Omega\to 0} K(a,b) = \left[\frac{m}{2\pi i\hbar t}\right]\left[\frac{\nu t}{2\sin(\frac{\nu}{2}t)}\right]^2 \exp\left\{\frac{i}{\hbar}\left\{\frac{m\nu}{4}\cot(\tfrac{\nu}{2}t)\left[\gamma_b-\gamma_a\right]^2\right.\right.$$

$$+ \frac{\gamma_b^T}{\sin(\nu t)}\int_0^t\left[\sin(\nu\tau) - 2\sin(\tfrac{\nu}{2}t)\sin(\tfrac{\nu}{2}\tau)\sin(\tfrac{\nu}{2}[t-\tau])\right] f(\tau)\,d\tau$$

$$+ \frac{\gamma_a^T}{\sin(\nu t)}\int_0^t\left[\sin(\nu[t-\tau]) - 2\sin(\tfrac{\nu}{2}t)\sin(\tfrac{\nu}{2}\tau)\sin(\tfrac{\nu}{2}[t-\tau])\right] f(\tau)\,d\tau$$

$$- \frac{1}{m\nu\sin(\nu t)}\int_0^t\int_0^\tau f^T(\tau) f(\tau')\left[\sin(\nu[t-\tau])\sin(\nu\tau')\right.$$

$$\left.\left.\left. - 4\sin(\tfrac{\nu}{2}\tau)\sin(\tfrac{\nu}{2}[t-\tau])\sin(\tfrac{\nu}{2}\tau')\sin(\tfrac{\nu}{2}[t-\tau'])\right] d\tau d\tau'\right\}\right\} \qquad (IV.15)$$

This is the forced nonlocal harmonic oscillator propagator in two dimensions and it is the same form as that obtained by Sa-yakanit (10) in the evaluation of the path integral theory of a model of disordered system.