CHAPTER II

PRELIMINARIES

This chapter will give some definitions and theorems which will be need in our investigation.

The materials of this chapter are drawn from reference. [1] and [2].

1. Vector in R^n.

2.1.1 Definition. Let $n > 0$ be an integer. An ordered set of n real numbers $(u_1, u_2,...,u_n)$ is called a vector with n components and will be denoted by a capital letter ; for example, $U = (u_1, u_2,...,u_n)$. The number u_k is called the k th component of the vector U. The set of all vectors with n components is called n - space and is denoted by E_n.

2.1.2 Definition. Let $U = (u_1, u_2,...,u_n)$ and $V = (v_1, v_2,...,v_n)$ be vectors in E_n. We define :

(a) Equality :

$U = V$ if, and only if, $u_1 = v_1$ $u_2 = v_2$, ..., $u_n = v_n$.

(b) Sum :

$$U + V = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

(c) Multiplication by scalars (scalar = a real number) :

$$aU = (au_1, au_2, \ldots, au_n) \quad (\text{a real}).$$

(d) Difference :

$$U - V = U + (-1)V .$$

(e) Zero vector or origin :

$$\theta = (0, 0, \ldots, 0) .$$

2.1.3 Definition. Let $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$ be vectors in E_n . The dot product of U and V, denoted by U.V is the real number $U.V = u_1v_1 + u_2v_2 + \ldots + u_nv_n$.

The vector U and V are said to be orthogonal if their dot product is zero, i.e., $U.V = 0$.

2.1.4 Definition. The space E_n with above operations of vector addition, scalar multiplication and dot product is called Euclidean n - space, and is denoted by R^n .

2.1.5 Definition. Let $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$ be vectors in R^n . We define

(a) The absolute value or norm of U by :

$$|U| = \left(\sum_{i=1}^{n} u_i^2 \right)^{1/2} = \sqrt{(U, V)} .$$

(b) The distance ρ between U and V :

$$\rho(U,V) = |U - V| = \left(\sum_{i=1}^{n} (u_i - v_i)^2 \right)^{1/2}$$

A vector E in R^n is said to be a unit vector if its norm is 1 .

2.1.6 Lemma. (Cauchy - Schwarz inequality). If $U = (u_1, u_2, \ldots, u_n)$ and $V = (v_1, v_2, \ldots, v_n)$ are arbitrary vectors in R^n, we have

$$(U.V) \leq |U|^2 \; |V|^2 .$$

Proof. A sum of squares can never be negative. Hence we have

$$\sum_{i=1}^{n} (u_k x + v_k)^2 \geq 0$$

for every real x. This inequality can be written in the form

$Ax^2 + 2Bx + C \geq 0$,

where

$$A = \sum_{k=1}^{n} u_k^2 \quad , \quad B = \sum_{k=1}^{n} u_k v_k \quad , \quad C = \sum_{k=1}^{n} v_k^2 .$$

If $A > 0$, put $x = -\frac{B}{A}$ to obtain $B^2 - AC \leqslant 0$, which is the desired inequality. If $A = 0$, the proof is trivial.

Thus the lemma is proved,

2.1.7 Theorem. Let U and V denote vectors in R^n. Then we have

(a) $|U| \geqslant 0$, and $|U| = o$ if, and only if, $U = 0$.

(b) $|U - V| = |V - U|$.

(c) $|U + V| \leqslant |U| + |V|$.

Proof. Statements (a) and (b) are immediate from definition. To prove (c) we make use of the Cauchy - Schwarz inequality which can now be written as

$$\left(\sum_{k=1}^{n} u_k v_k \right)^2 \leqslant |U|^2 |V|^2 .$$

Since we have

$$|U + V|^2 = \sum_{k=1}^{n} (u_k + v_k)^2 = \sum_{k=1}^{n} (u_k + 2u_k v_k + v_k^2)$$

$$= |U|^2 + |V|^2 + 2\sum_{k=1}^{n} u_k v_k$$

$$\leq |U|^2 + |V|^2 + 2|U||V| = (|U| + |V|)^2 ,$$

Property (c) follows at once. The proof is complete.

Let U and V be two nonzero vectors in R^n, then by Lemma 2.1.6 ,

$$-|U|\,|V| \leq U.V \leq |U|\,|V|$$

or

$$-1 \leq \frac{U.V}{|U|\,|V|} \leq 1 \,.$$

Hence we can define the angle β between U and V by

$$\cos \beta = \frac{U.V}{|U||V|} \qquad 0 \leq \beta \leq \pi \,.$$

Because of the restriction $0 \leq \beta \leq \pi$, the angle β is unique and write

$$\beta = \measuredangle (U,V) \,.$$

2 Vector - valued functions .

2.2.1 Definition. By a vector - valued function we shall mean a function from some closed interval $[a, b]$ into R^n.

A vector-valued function will be denoted by a capital letter ; for example F. Since each value F(x) is then a vector in R^n and thus we can write

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), \text{ if } x \in [a, b] \,,$$

where each component function f_i is a real - valued function on $[a, b]$.

I assume that the reader is familiar with the basic theorems of differential calculus of real - valued function of a real variable. We now give a brief discussion of some theorems on the differential calculus of vector - valued functions of a real variable.

2.2.2 <u>Definition.</u> Let F be a vector - valued function from some closed interval $[a, b]$ into R^n. If $c \in [a, b]$ and if $A \in R^n$, then we write

$$\lim_{x \to c} F(x) = A$$

to mean that for each $\epsilon > 0$, there exist $\delta > 0$ such that

$$x \in ((c - \delta, c + \delta) - \{c\}) \cap [a, b] \text{ implies } |F(x) - A| < \epsilon$$

2.2.3 <u>Lemma</u>. Let F be a vector - valued function from $[a, b]$ into R^n. Let $c \in [a, b]$ and assume that we have

$$\lim_{x \to c} F(x) = A,$$

where $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ and $A = (a_1, a_2, \ldots, a_n)$

Then $\lim_{x \to c} f_1(x) = a_1$, $\lim_{x \to c} f_2(x) = a_2, \ldots,$ $\lim_{x \to c} f_n(x) = a_n$ and conversely.

Proof. Assume $\lim_{x \to c} F(x) = A$, and let $\epsilon > 0$ be given.

There exists a $\delta > 0$ such that

$$x \in ((c - \delta, c + \delta) - \{c\}) \cap [a, b] \text{ implies } |F(x) - A| < \epsilon$$

Since $|f_i(x) - a_i| \leq |F(x) - A|$ for each $i = 1, 2, \ldots, n$,

We thus have for each i

$$x \in ((c - \delta, c + \delta) - \{c\}) \cap [a, b] \text{ implies } |f_i(x) - a_i| < \epsilon$$

Therefore

$$\lim_{x \to c} f_i(x) = a_i ,$$

for all $i = 1, 2, \ldots, n$.

To prove the converse, assume that $\lim_{x \to c} f_i(x) = a_i$

$(i = 1, 2, \ldots, n)$.

Then given $\epsilon > 0$, there exist $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_n > 0$

such that for each i

$$x \in ((c - \delta_i, c + \delta_i) - \{c\}) \cap [a, b] \text{ implies}$$

$|f_i(x) - a_i| < \frac{\epsilon}{\sqrt{n}}$. Let

$$\delta = \min \{ \delta_1, \delta_2, \ldots, \delta_n \} .$$

Thus we have

$$x \in ((c - \delta, c + \delta) - \{c\}) \cap [a, b] \text{ implies}$$

$$|f_i(x) - a_i| < \frac{\epsilon}{\sqrt{n}}, \quad i = 1, 2, \ldots, n .$$

Hence $\lim\limits_{x \to c} F(x) = A$ and the proof is now complete.

2.2.4 Theorem. Let F and G be two vector - valued functions from $[a, b]$ into R^n. Let $c \in [a,b]$ and assume that we have

$$\lim_{x \to c} F(x) = A , \quad \lim_{x \to c} G(x) = B .$$

Then we have

$$\text{(i)} \quad \lim_{x \to c} (F(x) \pm G(x)) = A \pm B ,$$

$$\text{(ii)} \quad \lim_{x \to c} (F(x).G(x)) = A.B .$$

Also if $\phi(x)$ is any real - valued function defined on $[a,b]$ such that

$$\lim_{x \to c} \phi(x) = d, \text{ then}$$

$$\text{(iii)} \quad \lim_{x \to c} \phi(x)F(x) = dA .$$

Proof. Let

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x)) , \quad G(x) = (g_1(x), g_2(x), \ldots, g_n(x)),$$

$$A = (a_1, a_2, \ldots, a_n) \text{ and } B = (b_1, b_2, \ldots, b_n) .$$

First, we prove(i). Using the fact that

$$\lim_{x \to c} f_i(x) = a_i \quad \text{and} \quad \lim_{x \to c} g_i(x) = b_i$$

implies

$$\lim_{x \to c} f_i(x) \pm g_i(x) = a_i \pm b_i \, , \quad i = 1, 2, \ldots n \, ,$$

and applying the converse of Lemma 2.2.3, proves (i).

To prove (ii) , Using the fact that

$$\lim_{x \to c} f_i(x) = a_i \quad \text{and} \quad \lim_{x \to c} g_i(x) = b_i$$

implies

$$\lim_{x \to c} f_i(x)\, g_i(x) = a_i b_i \, , \quad i = 1, 2, \ldots, n \, .$$

Thus

$$\lim_{x \to c} F(x).G(x) = \lim_{x \to c} \left(\sum_{i=1}^{n} f_i(x)\, g_i(x) \right) \quad \text{exist} \, ,$$

and moreover

$$\begin{aligned} \lim_{x \to c} \left(\sum_{i=1}^{n} f_i(x) g_i(x) \right) &= \sum_{i=1}^{n} \left(\lim_{x \to c} f_i(x)\, g_i(x) \right) \\ &= \sum_{i=1}^{n} a_i b_i \\ &= A.B \end{aligned}$$

Hence part (ii) is proved.

We prove (iii), part (iii) is proved in the similar way that part (i) is. This completes the proof.

2.2.5 Definition. It F be a vector- valued function from [a,b] into R^n. The function F is said to be continuous at a point $c \in [a, b]$ if.

(i) F is defined at c,

(ii) $\lim_{x \to c} F(x) = F(c)$.

The function F is said to be continuous on [a, b] if it is continuous at every point on [a, b].

2.2.6 Theorem. Let F and G be two vector - valued functions from [a, b] into R^n. Let $c \in [a,b]$ and assume that F and G are continuous at c, Then we also have F + G, F - G and F.G are continuous at c. If, in addition, $\emptyset$ a real - valued function defined on [a, b], is continuous at c then $\emptyset F$ is also continuous at c.

Proof. Apply Theorem 2.2.4, and we are done.

2.2.7 Definition. Let F be a vector - valued function from [a, b] into R^n. The function F is said to have a derivative at $c \in [a, b]$ if the limit

$$\lim_{x \to c} \frac{F(x) - F(c)}{x - c}$$

exists. This limit, denoted by $F'(c)$, is called the derivative of F at c.

The function F is said to be differentiable on [a, b] if it has a derivative at every point x on [a, b] .

2.2.8 Theorem. Let F be a vector - valued function from [a, b] into R^n . Let $c \in [a,b]$ and assume that F has a derivative at c, then F is continuous at c.

proof. If $x \in [a, b]$, $x \neq c$, we can write

$$F(x) - F(c) = (x-c) \frac{F(x) - F(c)}{x-c}$$

Applying Theorem 2.2.4 (iii), we find $\lim_{x \to c} F(x) = F(c)$. This prover the assertion.

2.2.9 Theorem. Let $F = (f_1, f_2, \ldots, f_n)$ be a vector - valued function from [a,b] into R^n and assume that F has a derivative at a point $c \in [a, b]$. Then the function f_i also has a derivative at c, for each i = 1, 2, ..., n and conversely .

Proof. It $x \in [a,b]$, $x \neq c$, we can write

$$\frac{F(x) - F(c)}{x-c} = \left(\frac{f_1(x) - f_1(c)}{x - c} , \frac{f_2(x) - f_2(c)}{x - c} , \ldots , \frac{f_n(x)-f_n(c)}{x-c} \right)$$

By Theorem 2.2.3, $\lim_{x \to c} \frac{F(x) - F(c)}{x - c}$ exists if, and only if, $\lim_{x \to c} \frac{f_i(x)-f_i(c)}{x - c}$ exists, for each i = 1, 2, ..., n and thus we have

$$F'(c) = (f_1'(c),\ f_2'(c),\ \ldots\ f_n'(c)\ .$$

This proves the theorem.

2.2.10 Theorem. Let F and G be two vector - valued functions, each defined on an interval $[a,b]$, with function values in R^n. Let $c \in [a,b]$ and assume that F and G have a derivative at the point c, then the function F + G, F - G, and F.G also have a derirative at c, If, in addition, $\emptyset$ a real - valued function defined on $[a,b]$, has a derivative at c then $\emptyset F$ also has a derivative at c. These derivatives are given by the following formulae :

$$\text{(i)}\quad (F \pm G)' = F' \pm G' ,$$

$$\text{(ii)}\quad (F.G) = F.G' + F'.G ,$$

$$\text{(iii)}\quad (\emptyset F)' = \emptyset F' + \emptyset' F .$$

Proof. First apply the Product Theorem for derivatives of real - valued functions on $[a,b]$ to each component function of F and G, and then apply the converse of the theorem 2.2.9 . We are done.

2.2.11 Definition. Let f be a real - valued function defined on $[a,b]$, f is said to be of class C^k on $[a,b]$, if f', f'', ..., f^k exist and are continuous for all x with $a \leq x \leq b$.

If f is merely continuous on $[a,b]$, then f is said to be of class C^0 on $[a,b]$.

2.2.12 Lemma. Let f and g betwo real - valued function defined on $[a,b]$ and assume that f and g are of class C^k on $[a,b]$ $k \geq 1$. Then f+ g, f- g, and fg are each of class C^k on $[a,b]$ The quotient f/g is also of class C^k on $[a,b]$, provided that $g(x) \neq 0$ for all $x \in [a,b]$

Note. We denote by f + g, f - g, fg, and f/g the function whose value at x is, respectively, f(x) + g(x), f(x) - g(x), f(x)g(x), and f(x) /g(x).

Proof. We shall only prove that fg is of class C^k by induction. The other part is proved in the similar way.

Let p(k) be the statement that "If f and g are of class C^k on $[a,b]$, then fg is also of class C^k on $[a,b]$." (k = 1, 2, ...).

Clearly, p(1) is true. Now assume that p(k) is true, to prove p(k+1) is true we can assume that f and g are of class C^{k+1} on $[a,b]$.

Let $h = fg$, thus

$$h' = f'g + fg'.$$

By assumption that f and g are of class C^{k+1}, f, g, f', and g' are at least of class C^k.

Therefore h is of class C^{k+1}, and our theorem is proved.

2.2.13 Corollary to lemma. Let f be a real-valued function defined on a closed interval $[a,b]$ and let $f([a,b])$ be the image of $[a,b]$ under f. Let g be a real-valued function defined on $f([a,b])$ and consider the composite function g o f defined for each x in a,b by $g \circ f(x) = g(f(x))$. Assume that f is of class C^k on $[a,b]$ and g is of class C^k on $f([a,b])$, $k \geq 1$. Then g o f is also of class C^k on $[a,b]$.

Proof. We shall prove that g o f is of class C^k on $[a,b]$ by by induction.

Let P(k) be the statement that " If f and g are of class C^k on $[a,b]$ and $f([a,b])$ respectively, then g o f is also of class C^k." $(k = 1,2,\ldots.)$.

Clearly P(1) is true. Now assume that P(k) is true, to prove P(k+1) is true we can assume that f and g are of class C^k on $[a,b]$ and $f([a,b])$ respectively.

Let

$$h = g \circ f, \quad \text{thus}$$

$$h' = (g' \circ f)(f')$$

By assumption that f and g are of class C^{k+1}, f, f', and g' are at least of class C^k. Hence by induction hypothesis $g' \circ f$ is of class C^k.

Therefore h' is of class C^k (by Theorem 2.2.12), so h is of class C^{k+1}. Thus the theorem is proved.

2.2.14 Definition. Let F be a vector-valued function from $[a,b]$ into R^n, F is said to be of class C^k on $[a,b]$ if each of its component functions is of class C^k on $[a,b]$, where $k = 0,1,2,\ldots$.

2.2.15 Theorem. Let F and G be two vector-valued function from $[a,b]$ into R^n. Assume that F and G are of class C^k on $[a,b]$ then $F+G$, $F-G$, $F\cdot G$, and ϕF are of class C^k on $[a,b]$, where ϕ is a C^k-real-valued function defined on $[a,b]$.

If, in addition, $F(x) \neq \theta$ for all $x \in [a,b]$ then $|F|$ is also of class C^k on $[a,b]$, where θ is the zero vector.

Proof. By the virtue of Theorem 2.2.6, this is true for $k = 0$. If $k \geq 1$, then the first part of this theorem follows immediately from Definition 2.2.14, and Lemma 2.2.12.

For the second part, assume that F is of class C^k and $F(x) \neq \theta$ for all $x \in [a,b]$.

Write $F = (f_1,f_2,\ldots,f_n)$ then each component function f_i is also of class C^k, and thus

$$|F|^2 = \sum_{i=1}^{n} f_i^2$$

is of class C^k.

Since the square root function is of class C^k on any compact subinterval of the open interval $(0,+\infty)$ and by assumption that $|F|^2(x) = \sum_{i=1}^{n} f_i^2(x) > 0$ for all x belong to [a,b].

We thus have $|F| = \left(\sum_{i=1}^{n} f_i^2\right)^{1/2}$ is also of class C^k on [a,b].

Our theorem is proved.

2.2.16 Theorem. Let F be a vector-valued from [a,b] into R^n. Assume that F is differentiable on [a,b] and $F'(x) = \theta$ for each $x \in (a,b)$, then F is constant through out [a,b].

Proof. Write

$$F(x) = (f_1(x), f_2(x), \ldots, f_n(x)), \text{ where } x \in [a,b] .$$

By assumption, we have

$$F'(x) = (f_1'(x), f_2'(x), \ldots, f_n'(x)) = \theta ,$$

for all $x \in (a,b)$. Hence

$$f_1'(x) = f_2'(x) = \ldots = f_n'(x) = 0 \text{ for all } x \in (a,b).$$

This implies that (see[1] on page 94)

$$f_1(x) \equiv c_1, \; f_2(x) \equiv c_1, \ldots, f_n(x) \equiv c_n ,$$

for some real constants $c_1, c_2, \ldots, c_n$. Therefore

$$F(x) \equiv C = (c_1, c_2, \ldots, c_n).$$

Hence the theorem is proved.

3. Curve in R^n

2.3.1 Definition. A parametrized curve in R^n is a continuous vector-valued function F from some closed interval [a,b] into R^n.

Consider a curve in R^n described by a continuous function $F = (f_1, f_2, \ldots, f_n)$ defined on [a,b] . For each partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b] we set

$$P_j = F(x_j)$$

The consecutive line segments joining P_0 to P_1, P_1 to $P_2, \ldots,$ and P_{n-1} to P_n form a polygon C. Since each P_j lies on F, we speak of C as inscribed in F.

For such polygons, we define length by

$$L(C) = |P_0 - P_1| + |P_1 - P_2| + \ldots + |P_{n-1} - P_n|$$
$$= \sum_{j=0}^{n-1} |F(x_{j+1}) - F(x_j)| .$$

2.3.2 Definition. The length of a continuous curve F is defined to be the least upperbound of the number L(C), where C ranges over all polygons inscribed in F.

When the set of numbers L(C) is not bounded, then we write $L(F) = +\infty$, and say that F has infinite length. If $L(F) < +\infty$, then F is said to be rectifiable.

2.3.3 Theorem. If F is a curve of class C^1 from $[a,b]$ into R^n then F is rectifiable, and L(F) is given by

$$L(F) = \int_a^b |F'(x)|\,dx.$$

For the proof of this theorem see e.g.[2] on page 321.

2.3.4 Definition. Let F be a curve in R^n defined on $[a,b]$. Then F is called parametrization by arc length if the arc length along the curve from $F(s_1)$ to $F(s_2)$ is $|s_1 - s_2|$ for all s_1, s_2 belong to $[a,b]$.

2.3.5 Theorem. Let $F : [a,b] \longrightarrow R^n$ be a C^1-parametrization by arc length. Then $|F'(s)| = 1$ for all $s \in [a,b]$ and conversely.

Proof. Suppose that F is a C^1-parametrization by arc length. Then, by Definition 2.3.4, and Theorem 2.3.3,

$$\int_{s_1}^{s_2} |F'(s)|\,ds = |s_1 - s_2|$$

for all $s_1, s_2 \in [a,b]$.

But $\int_{s_1}^{s_2} ds = |s_1 - s_2|$ for all $s_1\ s_2 \in [a,b]$, hence

$$\int_{s_1}^{s_2} (|F'(s)| - 1)\,ds = 0$$

for all $s_1, s_2 \in [a,b]$.

Claim that $(|F'(s)| - 1) \equiv 0$.

To prove this suppose that there exists an $s_0 \in [a,b]$ such that $(|F'(s_0)| - 1) \neq 0$.

Without loss of generality we may assume that $(|F'(s_0)| - 1) > 0$.

Since $F'(s)$ is continuous then $|F'|$ and also $|F'| - 1$ are continuous on $[a,b]$.

Therefore there exists a neighborhood U about s_0 and an $\epsilon > 0$ such that

$$(|F'(s)| - 1) > \epsilon$$

for all $s \in U \cap [a,b]$.

Choose two distinct points s_1 and s_2 in $U \cap [a,b]$, we then have

$$\int_{s_1}^{s_2} (|F'(s)| - 1)ds \geq \int_{s_1}^{s_2} \epsilon \ ds = \epsilon|s_1 - s_2| > 0.$$

Thus contradicts the assumption that

$$\int_{s_1}^{s_2} (|F'(s)| - 1) \ ds = 0$$

for all $s_1, s_2 \in [a,b]$.

Hence our claim is proved, i.e., $|F'(s)| \equiv 1$.

Conversely, the hypothesis $|F'(s)| \equiv 1$ implies

$$\int_{s_1}^{s_2} |F'(s)|ds = |s_1 - s_2|$$

for all $s_1, s_2 \in [a,b]$.

Thus F is parametrized by arc length and the theorem is proved.

4. <u>Integral of Vector-valued functions</u>.

2.4.1 <u>Definition</u>. If $[a,b]$ is a finite interval, then a set of points

$$P = \{x_0, x_1, \ldots, x_n\}$$

satisfying the inequalities $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is called a partition of $[a,b]$.

2.4.2 <u>Definition</u>. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$ and assume that $t_k \in [x_{k-1}, x_k]$ is chosen $k = 1, \ldots, n$. If

$$F = (f_1, f_2, \ldots, f_n)$$

is a vector-valued function from $[a,b]$ into R^n, we form the sum

$$S(P,F) = \sum_{k=1}^{n} F(t_k)(x_k - x_{k-1}).$$

We say that F is <u>integrable</u> on $[a,b]$ if there is a vector $A \in R^n$ having the following property : for every $\epsilon > 0$, there is a partition P_ϵ of $[a,b]$ such that for every partition $P \supset P_\epsilon$ and for every choice of the point $t_k \in [x_{k-1}, x_k]$, we have

$$|S(P, F) - A| < \epsilon \quad .$$

When such a number A exists, clearly it is uniquely determined and is denoted by $\int_a^b F(x)dx$.

2.4.3 Theorem. Let $F = (f_1, f_2, \ldots, f_n)$ be a vector-valued function on $[a,b]$. Then we have

$$\int_a^b F(x)dx = \left(\int_a^b f_1(x)dx, \int_a^b f_2(x)dx, \ldots, \int_a^b f_n(x)dx\right)$$

whenever each integral on the right exists.

Proof. For each partition P of $[a,b]$ and for every choice of $t_k \in [x_{k-1}, x_k]$, we have

$$S(P, F) = \sum_{k=1}^{n} F(t_k)(x_k - x_{k-1}). \qquad \text{Let}$$

$$S(P,f_i) = \sum_{k=1}^{n} f_i(t_k)(x_k - x_{k-1}) \quad (i = 1,2,\ldots, n),$$

then

$$S(P, F) = (S(P,f_1), S(P,f_2),\ldots,S(P,f_n)) \ldots\ldots\ldots\ldots (1)$$

Assume that each function f_i is integrable on $[a,b]$, then there is a real number a_i correspond to the function f_i, having the property that, for given $\epsilon > 0$ there is a partition $P_{i\epsilon}$ of $[a,b]$ such that

$$S(P,f_i) - a_i < \epsilon/\sqrt{n} \qquad (i = 1,2,\ldots,n) .$$

This sum is independent of the partition $P \supset P_{i\epsilon}$ and of the choice $t_k \in [x_{k-1}, x_k]$.

If we let $P_\epsilon = \bigcup_{i=1}^{n} P_{i\epsilon}$, then

$$|S(P,f_i) - a_i| < \epsilon/\sqrt{n} \qquad (i = 1,2,\ldots,n) ,$$

for every partition $P \supset P_\epsilon$ and for every choice of $t_k \in [x_{k-1}, x_k]$. Which implies that

$$\sum_{i=1}^{n} \left| S(P, f_i) - a_i \right|^2 < \epsilon^2 .$$

Because of Equ.(1), we have

$$|S(P,F) - A| < \epsilon \qquad (A = (a_1, a_2, \ldots, a_n)) ;$$

for every partition P of [a,b] such that $P \supset P_\epsilon$ and for every choice $t_k \in [x_{k-1}, x_k]$. Hence the theorem is proved.

2.4.4 <u>Theorem</u>. Let F be a continuous vector-valued function from [a,b] into R^n, then F is integrable on [a,b].

<u>Proof</u>. Since F is continuous on [a,b], then each component function is also continuous on [a,b] (Lemma 2.2.3).

Therefore each component function of F is integrable on [a,b] (see [1] on page 211).

Hence by Theorem 2.4.3, F is integrable on [a,b]. The proof is complete.