CHAPTER V

FUNDAMENTAL THEOREM OF CURVES IN EUCLIDEAN N-SPACE

The main purpose of this chapter is to generalize some theorems on curves in R^3 to curves in R^n, by using the curvatures defined in Chapter IV.

1. <u>Identical vanishing of the curvature, have geometrical interpretation as follows</u> :

5.1.1 <u>Theorem</u>. If $k_1(s) \equiv 0$, then the curve is a straight line.

<u>Proof</u>. Assume $k_1(s) \equiv 0$. By Theorem 4.1.6, we have

$$\frac{d}{dt} V_1(s) \equiv 0 .$$

By Theorem 2.2.16, $V_1(s) \equiv C$, where C is a constant vector.

Since $F'(s) = V_1(s) \equiv C$, therefore

$$F(s) = Cs + D, \quad D = \text{constant vector},$$

which is an equation of straight line. The proof is complete.

5.1.2 <u>Theorem</u>. If $k_r(s) = 0$, then the curve lies in an r-dimensional linear manifold, $r = 2,3,\ldots, n-1$.

<u>Proof</u>. Let $r \in \{2, \ldots, n-1\}$ and assume that $k_r(s) \equiv 0$. By Theorem 4.2.1, $F^{(r+1)}(s)$ happen to be linearly dependent upon $F'(s), F''(s), \ldots, F^{(r)}(s)$ for all $s \in I = [0.L]$.

Thus, $F^{(r+1)}(s)$ can be written as a linear combination

$$F^{(r+1)}(s) = \beta_1(s)F'(s) + \beta_2(s)F''(s) + \ldots + \beta_r(s)F^{(r)}(s) \quad \ldots\ldots.(1)$$

of the vectors $F'(s), F''(s), \ldots, F^{(r)}(s)$, where the functions β_i are real-valued functions defined on the interval I.

Equ.(1) is equivalent to the following n scalar equations

$$(A)\quad \begin{cases} f_1^{(r+1)}(s) = \beta_1(s)f_1'(s) + \beta_2(s)f_1''(s) + \ldots + \beta_r(s)f_1^{(r)}(s) & \ldots\ldots(1) \\ f_2^{(r+1)}(s) = \beta_1(s)f_2'(s) + \beta_2(s)f_2''(s) + \ldots + \beta_r(s)f_2^{(r)}(s) & \ldots\ldots(2) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ f_n^{(r+1)}(s) = \beta_1(s)f_n'(s) + \beta_2(s)f_n''(s) + \ldots + \beta_r(s)f_n^{(r)}(s), & \ldots\ldots(n) \end{cases}$$

where $F(s) = (f_1(s), f_2(s), \ldots, f_n(s))$.

The i th equation of the system (A) is a linear homogeneous equation of the function f_i', which by Theorem B.1 in the Appendix, has a solution of the form

$$f_i'(s) = w_{1i}g_1(s) + w_{2i}g_2(s) + \ldots + w_{ri}g_r(s),$$

for $i = 1,2,\ldots,n$. Or equivalently

$$(f_1'(s), f_2'(s), \ldots, f_n'(s)) = (w_{11}g_1(s)+w_{21}g_2(s)+\ldots+w_{r1}g_r(s), w_{12}g_1(s)+\ldots +w_{r2}g_r(s), \ldots, w_{1n}g_1(s)+w_{2n}g_2(s)+\ldots+w_{rn}g_r(s))$$

$$= W_1g_1(s)+W_2g_2(s)+\ldots+W_rg_r(s)$$

$$= F'(s) ,$$

where $W_i = (w_{i1}, w_{i2}, \ldots, w_{in})$, $i = 1,2,\ldots,r$,

are constant vectors, and the g_j, $j = 1,2,\ldots, r$ are the solutions of every differential equation of the system (A) determined by $\beta_1, \beta_2, \ldots, \beta_r$.

Since the function g_j is a solution of the differential equations of the system (A) and by assumption that $r \geq 2$, then g_j is clearly differentiable on the interval I, $j = 1,2,\ldots,r$.

Hence, the functions $g_1, g_2, \ldots, g_r$ are integrable on the interval interval I, and we have

$$F(s) = \int_0^s (W_1g_1(t)+W_2g_2(t)+\ldots+W_rg_r(t))dt+W_{r+1}$$

$$= (\int_0^s g_1(t)dt)W_1+(\int_0^s g_2(t)dt)W_2+\ldots+(\int_0^s g_r(t)dt)W_r+ W_{r+1}$$

Let $h_i(s) = \int_0^s g_i(t)dt$, $i = 1,2,\ldots, r$, thus

$$F(s) = h_1(s)W_1+h_2(s)W_2+\ldots+ h_r(s)W_r+ W_{r+1} ,$$

where W_{r+1} is a contant vector, and $s \in I$.

By Remark 3.3.2, $F(s)$ lies in the linear manifold, which is of dimension less than or equal to r.

Thus the proof is complete.

2. Fundamental existence theorem for curves in R^n

5.2.1 Theorem. Let n be an integer such that $n \geq 2$, and suppose that we are given n-1 real-valued function

$$k_1(s) > 0,\ k_2(s) > 0, \ldots,\ k_{n-1}(s) > 0$$

defined on an interval $[0,L]$.

Assume that the functions $k_i(s)$ are of class C^{n-i-1}, $i = 1,2,\ldots,n-1$ then there exists a curve F in R^n for which $k_1(s), k_2(s), \ldots, k_{n-1}(s)$ are the first, second,...,(n-1)th curvatures at the point $F(s)$ and s is the arc length measured from some suitable base point. Such a curve is uniquely determined up to a Euclidean motion.

Proof. Existence :

Consider the differential equations

$$(B)\quad \begin{cases} \dfrac{d}{ds} y_1(s) = k_1(s)y_2(s) \\ \dfrac{d}{ds} y_i(s) = -k_{i-1}(s)y_{i-1}(s)+k_i(s)y_{i+1}(s), \quad i = 2,3,\ldots,n-1 \\ \dfrac{d}{ds} y_n(s) = -k_{n-1}(s)y_{n-1}(s) . \end{cases}$$

It is proved in the Appendix that these equations admit a unique set of C^1-solutions which assume prescribed values $y_{01}, y_{02}, \ldots, y_{0n}$ when $s = 0$.

In particular there is a unique set $v_{1i}, v_{2i}, \ldots, v_{ni}$ which assume initial values $\delta_{i1}, \delta_{i2}, \ldots, \delta_{in}$ when $s = 0$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$, for $j = 1,2,\ldots,n$, $i = 1,2,\ldots, n$.

We now prove that for all value of s and for $i = 1,2,\ldots,n$,

$$v_{1i}^2(s) + v_{2i}^2(s) + \ldots + v_{ni}^2(s) = 1 .$$

Evidently

$$\frac{d}{ds}(v_{1i}^2(s)+v_{2i}^2(s)+\ldots+ v_{ni}^2(s))$$
$$= 2\,(v_{1i}(s)v_{1i}'(s)+v_{2i}(s)v_{2i}'(s)+\ldots+v_{ni}(s)v_{ni}'(s)).$$

Claim that the right-hand member of the previous equation vanishes identically.

To see this, since $\{v_{1i}, v_{2i}, \ldots, v_{ni}\}$ is a solution of (B), we can write the system of differential equations,

$$(C)\quad\left\{\begin{array}{ll} v_{1i}'(s) = k_1(s)v_{2i}(s) & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1) \\ v_{ji}'(s) = -k_{j-1}(s)v_{j-1i}(s)+k_j(s)v_{j+1i}(s), & \ldots\ldots\ldots\ (j) \\ j = 2,3,\ldots, n-1 & \\ v_{ni}'(s) = -k_{n-1}(s)v_{n-1i}(s) & \ldots\ldots\ldots\ldots\ldots\ldots\ (n) \end{array}\right.$$

Multiply the j th equation of the system (C) by $v_{mi}(s)$, $m = 1,2,\ldots, n$ we get

$$v_{1i}(s)v_{1i}'(s) = k_1(s)v_{1i}(s)v_{2i}(s)$$

$$v_{2i}(s)v_{2i}'(s) = -k_1(s)v_{1i}(s)v_{2i}(s)+k_2(s)v_{2i}(s)v_{3i}(s)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_{n-1i}(s)v_{n-1i}'(s) = -k_{n-2}(s)v_{n-2i}(s)v_{n-1i}(s)+k_{n-1}(s)v_{n-1i}(s)v_{ni}(s)$$

$$v_{ni}(s)v_{ni}'(s) = -k_{n-1}(s)v_{n-1i}(s)v_{ni}(s) .$$

Adding all these equations gives

$$\begin{aligned} v_{1i}(s)v_{1i}'(s)+v_{2i}(s)v_{2i}'(s)+\ldots+v_{ni}(s)v_{ni}'(s) &= (k_1(s)v_{1i}(s)v_{2i}(s)+k_2(s)v_{2i}(s)v_{3i}(s)+ \\ &\quad \ldots+k_{n-1}(s)v_{n-1i}(s)v_{ni}(s)) \\ &\quad -(k_1(s)v_{1i}(s)v_{2i}(s)+k_2(s)v_{2i}(s)v_{3i}(s)+ \\ &\quad \ldots+k_{n-1}(s)v_{n-1i}(s)v_{ni}(s)) \\ &\equiv 0 . \end{aligned}$$

Thus our claim is proved.

Hence

$$\begin{aligned} v_{1i}^2(s)+v_{2i}^2(s)+\ldots+v_{ni}^2(s) &\equiv \text{constant} \\ &= ((v_{1i}^2(s)+v_{2i}^2(s)+\ldots+v_{ni}^2(s)))_{s = 0} \\ &= 1 , \end{aligned}$$

giving the require result.

From above result we can show that

$$\begin{pmatrix} v_{11}(s) & v_{21}(s)\ldots v_{n1}(s) \\ v_{12}(s) & v_{22}(s)\ldots v_{n2}(s) \\ \cdots & \cdots \\ v_{1n}(s) & v_{2n}(s)\ldots v_{nn}(s) \end{pmatrix} \begin{pmatrix} v_{11}(s) & v_{12}(s)\ldots v_{1n}(s) \\ v_{21}(s) & v_{22}(s)\ldots v_{2n}(s) \\ \cdots & \cdots \\ v_{n1}(s) & v_{n2}(s)\ldots v_{nn}(s) \end{pmatrix} = \mathrm{Id},$$

where Id is an n by n identity matrix.

To prove this it sufficies to show that

$$\frac{d}{ds}\left(\sum_{i=1}^{n} v_{ij}(s)v_{ik}(s)\right) \equiv 0 , \quad \text{if } j \neq k ,$$

or equivalently

$$\sum_{i=1}^{n} (v_{ij}(s)v'_{ik}(s)+v_{ik}(s)v'_{ij}(s)) \equiv 0 .$$

Consider the two system of differential equations ....(1)

$$(D) \quad \begin{cases} v'_{1k}(s) = k_1(s)v_{2k}(s) \\ v'_{ik}(s) = -k_{i-1}(s)v_{i-1k}(s)+k_i(s)v_{i+1k}(s), \quad .....(i) \\ i = 2,3,\ldots\ n-1 \\ v'_{nk}(s) = -k_{n-1}(s)v_{n-1k}(s) \quad ....(n) \end{cases}$$

and

$$(E) \quad \begin{cases} v'_{1j}(s) = k_1(s)v_{2j}(s) \quad .....(1) \\ v'_{ij}(s) = -k_{i-1}(s)v_{i-1j}(s)+k_i(s)v_{i+1j}(s), \quad .....(i) \\ i = 2,3,\ldots, n-1 \\ v'_{nj}(s) = -k_{n-1}(s)v_{n-1j}(s) \quad ....(n) \end{cases}$$

Multiply the m th equation of the system (D) and (E) by $v_{mj}(s)$ and $v_{mk}(s)$ respectively for m = 1,2,..., n, and then adding all there equations gives

$$\sum_{i=1}^{n}(v_{ij}(s)v_{ik}'(s)+v_{ik}(s)v_{ij}'(s))$$

$$= (k_1(s)v_{1j}(s)v_{2k}(s)+k_2(s)v_{2j}(s)v_{3k}(s)+\ldots+$$

$$k_{n-1}(s)v_{n-1j}(s)v_{nk}(s)+k_1(s)v_{1k}(s)v_{2j}(s)+$$

$$k_2(s)v_{2k}(s)v_{3j}(s)+\ldots+k_{n-1}(s)v_{n-1k}(s)v_{nj}(s))$$

$$-(k_1(s)v_{1k}(s)v_{2j}(s)+k_2(s)v_{2k}(s)v_{3j}(s)+\ldots+$$

$$k_{n-1}(s)v_{n-1k}(s)v_{nj}(s)+k_1(s)v_{1j}(s)v_{2k}(s)+$$

$$k_2(s)v_{2j}(s)v_{3k}(s)+\ldots+k_{n-1}(s)v_{n-1j}(s)v_{nk}(s))$$

$$\equiv 0.$$

Set

$$A(s) = \begin{pmatrix} v_{11}(s) & v_{21}(s)\ldots & v_{n1}(s) \\ v_{12}(s) & v_{22}(s)\ldots & v_{n2}(s) \\ \cdots & \cdots & \cdots \\ v_{1n}(s) & v_{2n}(s)\ldots & v_{nn}(s) \end{pmatrix}$$

where $s \in [0,L]$.

From the above result, for each $s \in [0,L]$ the matrix A(s) is orthogonal, i.e., $A(s)\,A'(s) = Id$ where $A'(s)$ is the transpose of A(s). But the equation

$A(s)\,A'(s) = Id$ implies $A'(s) = A^{-1}(s)$ and so

$A'(s)A(s) = Id$, which is equivalent to

$$\begin{pmatrix} v_{11}(s) & v_{12}(s)\ldots v_{1n}(s) \\ v_{21}(s) & v_{22}(S)\ldots v_{2n}(s) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ v_{n1}(s) & v_{n2}(s)\ldots v_{nn}(s) \end{pmatrix} \begin{pmatrix} v_{11}(s) & v_{21}(s)\ldots v_{n1}(s) \\ v_{12}(s) & v_{22}(s)\ldots v_{n2}(s) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ v_{1n}(s) & v_{2n}(s)\ldots v_{nn}(s) \end{pmatrix} = Id ,$$

or explicitly

$$\sum_{j=1}^{n} v_{ij}(s)v_{kj}(s) = \delta_{ik} ,$$

where $$\delta_{ik} = \begin{cases} 0 , & \text{if } i \neq k \\ 1 , & \text{if } i = k . \end{cases}$$

It follows that there are n mutually orthogonal units vector $V_i(s) = (v_{i1}(s), v_{i2}(s),\ldots, v_{in}(s)),\ i = 1,2,\ldots,n.$

Since the system of differential equations (C) admit a set of C^1-solutions then the function $V_1, V_2,\ldots,V_n$ are of class C^1, consequently $\int_0^s V_1(t)dt$ exists (Theorem 2.4.3) for all $s \in [0,L]$.

If $F(s) = \int_0^s V_1(t)dt , \quad s \in [0,L].$ ....(2)

Then claim that the function F determines a curve not only having $V_1(s)$ as its unit tangent vector at the point F(s), but also having $V_1(s), V_2(s),\ldots, V_n(s)$ as its Frenet frame, $k_1(s), k_2(s),\ldots, k_{n-1}(s)$ as its curvature at the point F(s), and s as its arc length.

To prove this claim, we must show that

(i) The curve F is a C^n-parametrization by arc length

(ii) For each $s \in [0,L]$, the vectors

$F'(s), F''(s),\ldots, F^{(n)}(s)$

are linearly independent

(iii) If $(U_1(s), U_2(s),\ldots, U_n(s))$ is the Frenet frame of F at the point F(s) then,

$$(U_1(s),U_2(s),\ldots,U_n(s)) = (V_1(s),V_2(s),\ldots,V_n(s))$$

(iv) If $h_1(s), h_2(s),\ldots, h_{n-1}(s)$ are the curvatures of F at the point F(s), then

$$h_1(s) = k_1(s),\ h_2(s) = k_2(s),\ldots,\ h_{n-1}(s) = k_{n-1}(s).$$

For the proof of this claim, we shall first prove of the following proposition.

5.2.2 <u>Proposition</u>. For each i = 2,...,n, the vector-valued function F defined by Equ.(2) is of class C^i on $[0,L]$. Furthermore the function $F^{(i)}$ can be written in of the form

$$F^{(i)}(s) = h_{i1}(s)V_1(s)+h_{i2}(s)V_2(s)+\ldots+h_{ii}(s)V_i(s),\ldots\ldots\ldots(3)$$

where $s \in [0,L]$; h_{im} is of class a_i on $[0,L]$ (see for the definition below), m = 1,2,...,i, and $h_{ii}(s) > 0$ for all $s \in [0.L]$.

5.2.3 <u>Definition</u>. A real-valued function f defined on $[0,L]$ is said to be of class a_i if it can be built up by addition, substraction, and multiplication from the function in the set A_i, where

$$A_i = \left\{ k_1, k_1',\ldots, k_1^{(i-2)}, k_2,k_2',\ldots, k_2^{(i-3)},\ldots,k_{i-1} \right\}.$$

<u>Note</u>. The definition of addition, substraction, and multiplication of two real-valued function have already been defined in Theorem 2.2.12.

Proof. (of Proposition 5.2.2)

The existence of $F'(s)$ follows from Equ.(2) and because of the system of differential equation (C), we obtain

$$F''(s) = V_1'(s) = k_1(s)\, V_2(s) .$$

By the hypothesis of this theorem, we know that $k_1(s) > 0$ for all $s \in [0,L]$ then by letting $h_{21}(s) = 0$ and $h_{22}(s) = k_1(s)$, our proposition is true for $i = 2$.

Now assume that our proposition is true for $i = j < n$.

Then we can write

$$F^{(j)}(s) = h_{j1}(s)V_1(s)+h_{j2}(s)V_2(s)+\ldots+h_{jj}(s)V_j(s) ,$$

where h_{jm} is of class a_j, $m = 1,2,\ldots, j$.

By assumption that k_m is of class C^{n-m-1} $(m = 1,2,\ldots, n-1)$, together with the condition that $j < n$ and Theorem 2.2.15, we see that every function which is of class a_j is at least of class C^1. It follows that h_{jm} is at least of class C^1. From the system of differential equation (C), we see that V_m is at least of class C^1, $m = 1,2,\ldots, j$. Hence by Theorem 2.2.15,

$F^{(j)}$ is of class C^1, and moreover

$$\begin{aligned} F^{(j+1)}(s) &= h_{j1}(s)V_1'(s)+h_{j1}'(s)V_1(s)+h_{j2}(s)V_2'(s)+h_{j2}'(s)V_2(s)+ \\ &\quad \ldots+ h_{jj}(s)V_j'(s)+h_{jj}'(s)V_j(s). \\ &= (h_{j1}'(s)V_1(s)+h_{j2}'(s)V_2(s)+\ldots+h_{jj}'(s)V_j(s)) \\ &\quad +(h_{j1}(s)V_1'(s)+h_{j2}(s)V_2'(s)+\ldots+h_{jj}(s)V_j'(s)) \end{aligned}$$

$$= (h'_{j1}(s)V_1(s) + h'_{j2}(s)V_2(s)+\dots+ h'_{jj}(s)V_j(s))$$

$$+\left[h_{j1}(s)(k_1(s)V_2(s)) + h_{j2}(s)(-k_1(s)V_1(s) + k_2(s)V_3(s))\right.$$

$$\left.+\dots+ h_{jj}(s)(-k_{j-1}(s)V_{j-1}(s) + k_j(s)V_{j+1}(s))\right]$$

$$= \left[h'_{j1}(s) - h_{j2}k_1(s)\right] V_1(s) + \left[h'_{j2}(s) + (h_{j1}(s)k_1(s)-k_2(s)h_{j3}(s))\right] V_2(s)$$

$$+\dots+ \left[h_{jj}(s)k_j(s)\right] V_{j+1}(s) \qquad \dots\dots\dots\dots\dots\dots (4)$$

Let $h_{j+11}(s) = h'_{j1}(s) - h_{j2}(s)k_1(s)$

$$h_{j+1m}(s) = h'_{jm}(s) + (h_{jm-1}(s)k_{m-1}(s) - k_m(s)h_{jm+1}(s)),$$

$m = 2,3,\dots, j-1$ and

$$h_{j+1j}(s) = h'_{jj}(s) + k_{j-1}(s)h_{jj-1}(s)$$

$$h_{j+1j+1}(s) = h_{jj}(s)k_j(s) .$$

Hence

$$F^{(j+1)}(s) = h_{j+11}(s)V_1(s)+h_{j+12}(s)V_2(s)+\dots+h_{j+1j+1}(s)V_{j+1}(s).$$

By the induction hypothesis and Equ.(4), it is easily seen that h_{j+1m} is of class a_{j+1}, and $h_{j+1j+1}(s)$ is obviously greater than zero for all s, since $h_{jj}(s) > 0$ by the induction hypothesis and $k_j(s) > 0$ by assumption.

Thus our proposition now follows by Mathematical induction.

Now we shall prove our claim.

To prove (i), the fact that F is of class C^n follows immediately from Proposition 5.2.2. Therefore F is rectifiable (Theorem 2.3.3).

Since $|F'(s)| = |V_1(s)| \equiv 1$, then by Theorem 2.3.5, F is parametrization by arc length. This prove (i).

To prove (ii) : Because of Equ.(2), we have

$$F'(s) = V_1(s) \neq \theta \quad \text{for all} \quad s \in [0,L] .$$

Thus for i = 1, part (ii) is proved.

Assume that (ii) is true for $i = j < n$, and assume further that $s_0 \in [0,L]$ such that $F^{(j+1)}(s_0)$ is linearly dependent with respect to $F'(s_0)$, $F''(s_0)$,..., $F^{(j)}(s_0)$, then there exist $b_1, b_2, \ldots, b_j$ not all zero such that

$$F^{(j+1)}(s_0) = b_1F'(s_0)+b_2F''(s_0)+\ldots+b_jF^{(j)}(s_0).$$

But Equ.(3), allows us to write

$$F'(s_0) = V_1(s_0) = h_{11}(s_0)V_1(s_0) \; ; \text{ if } h_{11}(s_0) = 1$$

$$F''(s_0) = h_{21}(s_0)V_1(s_0)+h_{22}(s_0)V_2(s_0)$$

$$F'''(s_0) = h_{31}(s_0)V_1(s_0)+h_{32}(s_0)V_2(s_0)+h_{33}(s_0)V_3(s_0)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F^{(j+1)}(s_0) = h_{j+11}(s_0)V_1(s_0)+h_{j+12}(s_0)V_2(s_0)+\ldots+h_{j+1j+1}(s_0)V_{j+1}(s_0),$$

where $h_{j+1j+1}(s_0) > 0$.

Since

$$F^{(j+1)}(s_0) = b_1F'(s_0)+b_2F''(s_0)+\ldots+b_jF^{(j)}(s_0), \quad \text{thus}$$

$$\begin{aligned} h_{j+11}(s_0)V_1(s_0)+\ldots+h_{j+1j+1}(s_0)V_{j+1}(s_0) \\ &= b_1(h_{11}(s_0)V_1(s_0))+b_2(h_{21}(s_0)V_1(s_0)+h_{22}(s_0)V_2(s_0)) \\ &\quad +\ldots+ b_j(h_{j1}(s_0)V_1(s_0)+h_{j2}(s_0)V_2(s_0)+\ldots+h_{jj}(s_0)V_j(s_0)) \\ &= g_1(s_0)V_1(s_0)+g_2(s_0)V_2(s_0)+\ldots+g_j(s_0)V_j(s_0), \end{aligned}$$

where $g_m(s_0) = b_m\left(\sum_{k=m}^{j} h_{km}(s_0)\right)$, $m = 1,2,\ldots, j$.

Hence

$$h_{j+1j+1}(s_0)V_{j+1}(s_0) = (g_1(s_0)-h_{j+11}(s_0))V_1(s_0)+(g_2(s_0)-h_{j+12}(s_0))V_2(s_0)+\ldots + (g_j(s_0)-h_{j+1j}(s_0))V_j(s_0).$$

But $h_{j+1j+1}(s_0) > 0$, thus

$$V_{j+1}(s_0) = \frac{1}{h_{j+1j+1}(s_0)}(g_1(s_0)-h_{j+11}(s_0))V_1(s_0)+\ldots + \frac{1}{h_{j+1j+1}(s_0)}(g_j(s_0)-h_{j+1j}(s_0))V_j(s_0) ,$$

is a linear combination of $V_1(s_0)$, $V_2(s_0),\ldots, V_j(s_0)$, this contradicts the orthogonality of $V_1(s_0)$, $V_2(s_0),\ldots, V_{j+1}(s_0)$.

Therefore $F^{(j+1)}(s)$ is linearly independent with respect to $F'(s)$, $F''(s),\ldots, F^{(j)}(s)$ for all $s \in [0,L]$. The proof of part (ii) now follows by induction.

Next prove (iii), let $(U_1(s), U_2(s),\ldots, U_n(s))$ be the Frenet-frame of F at the point F(s).

Since, by the definition of Frenet frame $U_1(s) = F'(s)$ and from Equ.(2), $V_1(s) = F'(s)$, we thus have that

$$U_1(s) = V_1(s).$$

Now assume that $(U_1(s), U_2(s),\ldots, U_m(s)) = (V_1(s), V_2(s),\ldots, V_m(s))$ for some number $m < n$. Consider the vector

$$E_{m+1}(s) = F^{(m+1)}(s) - \sum_{j<m+1} \left[F^{(m+1)}(s)\cdot U_j(s)\right] U_j(s),$$

then by the induction hypothesis, we have

$$E_{m+1}(s) = F^{(m+1)}(s) - \sum_{j<m+1} \left[F^{(m+1)}(s)\cdot V_j(s)\right] V_j(s) \quad \ldots\ldots(5)$$

Replace Equ.(3) in Equ.(5), this gives

$$E_{m+1}(s) = \sum_{k=1}^{m+1} h_{m+1k}(s)V_k(s) - \sum_{j<m+1} \left[\sum_{k=1}^{m+1} h_{m+1k}(s)V_k(s)\cdot V_j(s)\right] V_j(s)$$

$$= \sum_{k=1}^{m+1} h_{m+1k}(s)V_k(s) - \sum_{k=1}^{m} h_{m+1k}(s)V_k(s)$$

$$= h_{m+1m+1}(s)V_{m+1}(s).$$

Since $U_{m+1}(s) = \dfrac{E_{m+1}(s)}{|E_{m+1}(s)|}$, and $h_{m+1\ m+1}(s) > 0$,

we can conclude that

$$U_{m+1}(s) = V_{m+1}(s).$$

Hence the proofs of part (iii) follows by induction.

Finally to prove (iv), by virtue of theorem 4.1.6, we obtain

$$h_i(s) = U_i'(s) \cdot U_{i+1}(s) \ , \quad i = 1,2,\ldots, n-1$$

and from the system of differential equation (C), we have

$$k_i(s) = V_i'(s)\cdot V_{i+1}(s) \ , \quad i = 1,2,\ldots, n-1 \ .$$

By (iii), $U_i(s) = V_i(s)$, $i = 1,2,\ldots, n$. Hence

$$h_i(s) \ = \ k_i(s) \ , \quad i = 1,2,\ldots, n-1$$

Therefore our claim is proved.

This prove the existence of the required curve.

Uniqueness :

Suppose we are given two curves F and G defined in terms of their respectively arc lengths $s \in [0,L]$, such that

$$k_1(s) = h_1(s),\ k_2(s) = h_2(s),\ldots,\ k_{n-1}(s) = h_{n-1}(s)$$

for all s, where $k_i(s)$ and $h_i(s)$ are the curvatures of F and G at the point F(s) and G(s) respectively, i = 1,2,..., n-1. We are going to show that there is a unique Euclidean motion T such that

$$T(F(s)) \ = \ G(s) \ , \quad \text{if } s \in [0,L] \ ,$$

Let $(V_1(s), V_2(s),\ldots, V_n(s))$ and $(W_1(s), W_2(s),\ldots, W_n(s))$ be the Frenet frame of F and G at the point F(s) and G(s) respectively.

By Theorem 3.4.5, there is a unique Euclidean motion T = R o A, where A is linear and R is a translation of R^n such that

$$T(F(0)) \ = \ G(0) \ , \quad \text{and}$$

$$(A(V_1(0)),A(V_2(0)),\ldots,A(V_n(0)) = (W_1(0),W_2(0),\ldots,W_n(0)).$$

By Theorem 4.1.6 and since A is linear, we have

$$\frac{d}{ds}(A(V_1(s))\cdot W_1(s)) = A(V_1(s))\cdot W_1'(s) + (A \circ V_1)'(s)\cdot W_1(s)$$

$$= A(V_1(s))\cdot(k_1(s)W_2(s)) + A(V_1'(s))\cdot W_1(s)$$

$$= k_1(s)(A(V_1(s))\cdot W_2(s)) + k_1(s)(A(V_2(s))\cdot W_1(s))$$

$$= k_1(s)(A(V_1(s))\cdot W_2(s) + A(V_2(s))\cdot W_1(s)),$$

similarly

$$\frac{d}{ds}(A(V_2(s))\cdot W_2(s)) = A(V_2(s))\cdot W_2'(s) + (A \circ V_2)'(S)\cdot W_2(s)$$

$$= -k_1(s)(A(V_2(s))\cdot W_1(s) + A(V_1(s))\cdot W_2(s))$$

$$+ k_2(s)(A(V_2(s))\cdot W_3(s) + A(V_3(s))\cdot W_2(s))$$

. . . .

$$\frac{d}{ds}(A(V_{n-1}(s))\cdot W_{n-1}(s)) = A(V_{n-1}(s))\cdot W_{n-1}'(s) + (A\circ V_{n-1})'(s)\cdot W_{n-1}(s)$$

$$= -k_{n-2}(s)(A(V_{n-1}(s))\cdot W_{n-2}(s)+A(V_{n-2}(s))\cdot W_{n-1}(s))$$

$$+ k_{n-1}(s)(A(V_{n-1}(s))\cdot W_n(s)+A(V_n(s))\cdot W_{n-1}(s))$$

$$\frac{d}{ds}(A(V_n(s))\cdot W_n(s) = A(V_n(s))\cdot W_n'(s) + (A\circ V_n)'(s)\cdot W_n(s)$$

$$= -k_{n-1}(s)(A(V_n(s))\cdot W_{n-1}(s)+A(V_{n-1}(s))\cdot W_n(s)).$$

It follows by addition that

$$\frac{d}{ds}(A(V_1(s)\cdot W_1(s)+A(V_2(s))\cdot W_2(s)+\ldots+A(V_n(s))\cdot W_n(s)) \equiv 0.$$

Therefore

$$(A(V_1(s))\cdot W_1(s)+A(V_2(s))\cdot W_2(s)+\ldots+A(V_n(s))\cdot W_n(s)) \equiv \text{constant}$$

But at $s = 0$, $A(V_1(0)) = W_1(0)$, $A(V_2(0)) = W_2(0),\ldots,A(V_n(0)) = W_n(0)$.

Thus at $s = 0$, and hence for all s

$$A(V_1(s))\cdot W_1(s)+A(V_1(s))\cdot W_2(s)+\ldots+A(V_n(s))\cdot W_n(s) \equiv n .$$

Now two units vectors, say $A(V_1(s))$ and $W_1(s)$, have the property that $-1 \leq A(V_1(s))\cdot W_1(s) = \cos \measuredangle (A(V_1(s))\cdot W_1(s)) \leq 1$. Hence if $A(V_1(s)\cdot W_1(s)+A(V_2(s))\cdot W_2(s)+\ldots+A(V_n(s))\cdot W_n(s) \equiv n$, then $A(V_1(s))\cdot W_1(s) \equiv 1$, $A(V_2(s))\cdot W_2(s) \equiv 1,\ldots,$ $A(V_n(s))\cdot W_n(s) \equiv 1$.

Since the angle between any two vectors $A(V_i(s))$ and $W_i(s)$ is zero, and they are of the same length, therefore they must be equal, so we have for all s

$$A(V_i(s)) = W_i(s), \qquad i = 1,2,\ldots,n.$$

Finally, since

$$V_1(s) = F'(s) ,$$

therefore

$$\begin{aligned} A(V_1(s)) &= A(F'(s)) \\ &= W_1(s) \\ &= G'(s). \end{aligned}$$

Hence

$$A(F(s)) = G(s) + C , \quad \text{and}$$

$$R \circ A(F(s)) = G(s) + D.$$

where C and D are some fixed constant vectors.

But at s = 0, R o A(F(0)) = G(0). Hence T(F(s)) = G(s) for all s, and it follows that F and G coincide. Whence F and G are identical to within a Euclidean motion.

The proof is complete.

4. Natural Equations.

When a curve is defined by an equation F = F(s), its form depends on the choice of the coordinate system. When a curve is moved without change in its shape, its equation with respect to the coordinate system changes. It is not always immediately obvious whether two equations represent the same curve except for its position with respect to coordinate system. The question therefore arises : Is it possible to characterize a curve by a relation independent of the coordinates?

This can be accomplished in a certain cases and such equation is called a natural equation.

If the curvatures $k_1 = k_1(s)$, $k_2 = k_2(s), \ldots, k_{n-1} = k_{n-1}(s)$ and the Frenet frame are analytic then they determine the natural equations of an analytic curve in Euclidean n-space.

To see this, suppose the curve is analytic then we can write, in the neighborhood of a point $s = s_0$, $h = s - s_0$:

$$F(s) = F(s_0) + \frac{h}{1} F'(s_0) + \frac{h^2}{2!} F''(s_0) + \frac{h^3}{3!} F'''(s_0) + \ldots\ldots,$$

and this series is convergent in a certain interval $s_1 < s_0 < s_2$. Then substituting for F', F'', F''', etc., their values with respect to the Frenet at $F(s_0)$, we obtain

$$F'(s_0) = V_1(s_0)$$

$$F''(s_0) = \dot{V}_1(s_0) = k_1(s_0)V_2(s_0)$$

$$F'''(s_0) = k_1(s_0)(-k_1(s_0)V_1(s_0) + k_2(s_0)V_3(s_0)) + k_1'(s_0)V_2(s_0)$$

$$= -k_1^2(s_0)V_1(s_0) + k_1'(s_0)V_2(s_0) + k_1(s_0)k_2(s_0)V_3(s_0)$$

$$F''''(s_0) = \ldots.,$$

so that

$$F(s) = F(s_0) + hV_1(s_0) + \frac{1}{2}k_1(s_0)h^2V_2(s_0) + \frac{1}{6}h^3(-k_1^2(s_0)V_1(s_0) + k_1'(s_0)V_2(s_0) + k_1(s_0)k_2(s_0)V_3(s_0)) + \ldots\ldots(6)$$

where all terms can be found by differenting the Frenet-formulae, and all successive derivatives of $k_1, k_2, \ldots, k_{n-1}$ as well as $V_1, V_2, \ldots, V_n$ at the point s_0 are assumed to exist because of the analytical character of F. (see also Theorem 4.1.6).

If we choose at an arbitrary point $F(s_0)$ and arbitrary set of n mutually perpendicular unit vectors and select them as $V_1, V_2, \ldots, V_n$, then Equ.(6) determines the curve uniquely (inside the interval of convergence) up to Euclidean motion.