Appendix

Appendix A

The material of this part is drawn from reference [7].

Existence theorem on linear differential equations.

A.1 Theorem. The differential equations

$$\frac{d}{ds} u_i(s) = \sum_{k=1}^{P} c_{ik}(s) u_k(s) , \quad i = 1,2,\ldots, P, \qquad \ldots.(1)$$

where c_{ik} are continuous functions in the interval $0 \le s \le r$, has a set of C^1-solutions which assume prescribed value u_i^0 when $s = 0$.

Proof. To prove this, let

$$\left.\begin{aligned} u_i^1(s) &= u_i^0 + \int_0^s \sum_{k=1}^{P} c_{ik}(t) u_k^0 \, dt \\ u_i^2(s) &= u_i^0 + \int_0^s \sum_{k=1}^{P} c_{ik}(t) u_k^1(t) dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ u_i^n(s) &= u_i^0 + \int_0^s \sum_{k=1}^{P} c_{ik}(t) u_k^{n-1}(t) dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \qquad (2)$$

Since c_{ik} are continuous functions on a compact set they are bounded i.e., there exist a constant C such that

$$|c_{ik}(s)| < C/_P \quad \text{for all} \quad s \in [0,r] \quad \text{and for}$$

$i = 1,2,\ldots, P, \; k = 1,2,\ldots, P.$

We assume that $\left|u_i^0\right| \leq K$ for $i = 1,2,\ldots, P$.

Then

$$\left|u_i^1(s) - u_i^0\right| = \left|\int_0^s \sum_{k=1}^{P} c_{ik}(t)u_k^0 \, dt\right|$$

$$\leq \sum_{k=1}^{P} \int_0^s \left|c_{ik}(t)u_k^0\right| dt$$

$$\leq K \sum_{k=1}^{P} \int_0^s \left|c_{ik}(t)\right| dt$$

$$KCs \, .$$

Also $u_i^2(s) - u_i^1(s) = \int_0^s \sum_{k=1}^{P} c_{ik}(t)(u_k^1(t) - u_k^0) \, dt$

and also

$$\left|u_i^2(s) - u_i^1(s)\right| \leq \sum_{k=1} \int_0^s \left|c_{ik}(t)\right| \left|u_k^1(t) - u_k^0 \, dt\right| dt$$

$$< (KC)(C/_n) \sum_{k=1}^{P} \int_0^s t \, dt$$

$$= KC^2 \frac{s^2}{2}.$$

In similar way it follows that

$$\left|u_i^n(s) - u_i^{n-1}(s)\right| < KC^n \frac{s^n}{n!} \leq KC^n \frac{r^n}{n!} \; .$$

Hence, by the Weierstrass M-test, the sequence $\left\{ u_i^n(s) \right\}$ converges uniform in the interval $0 \leq s \leq r$ and a continuous function $u_i(s)$ can be defined by the equation.

$$\lim_{n \to \infty} u_i^n(s) = u_i(s) .$$

Also, from the uniformity of convergence, it follows from (2), that, as $n \to \infty$,

$$u_i(s) = u_i^0 + \int_0^s \sum_{k=1}^{P} c_{ik}(t) u_k(t) dt.$$

Hence $\frac{d}{ds} u_i(s) = \sum_{k=1}^{P} c_{ik}(s) u_k(s)$ and $u_i(0) = u_i^0$.

This complete the proof of the existence theorem.

Appendix B

The material of this part is drawn from reference [5].

The solution of homogeneous linear differential equations of order n.

The general homogeneous linear equation of the n th order is

$$L(y) = a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \quad \ldots\ldots(1)$$

We use $L(y)$ to denote the result of substitute any any function y in the left member of Equ.(1). Since multiplying y by a constant multiplies each term by a constant,

$$L(cy_1) = cL(y_1), \text{ and } L(cy_1) = 0, \text{ if } L(y_1) = 0 .$$

Again, replacing y by $y_1 + y_2$ replaces each term in y by the sum of two similar terms, one in y_1 and one in y_2. Hence

$$L(y_1 + y_2) = L(y_1) + L(y_2) \text{ and } L(y_1 + y_2) = 0, \text{ if}$$

$L(y_1) = 0$, $L(y_2) = 0$.

Consequently the sum of two solutions, or the product of one solution by a constant, is again a solution of the homogeneous equation (1). Hence we have proven the first part of the following theorem.

B.1 Theorem. The solution of the homogeneous equation (1) form a vector space W. Furthermore $\dim(W) \leq n$.

Proof. The first part of this theorem follows at once from the above remark. To prove the second part, let $u_1, u_2, \ldots, u_k$ be k linearly independent solution of (1). Thus the associated vector-valued functions $U_1, U_2, \ldots, U_k$ would be linearly independent, where

$$U_i(x) = (u_i(x), u_i'(x) \ldots, u_i^{(n-1)}(x)), \quad i = 1,2,\ldots,k.$$

To see this, if $U_1, U_2, \ldots, U_k$ are linearly dependent vector-valued function, then there are scalars $c_1, c_2, \ldots, c_k$ not all zero, so that

$$c_1 U_1(x) + c_2 U_2(x) + \ldots + c_k U_k(x) \equiv \theta ,$$

where θ = zero vector.

But this implies

$$c_1 u_1(x) + c_2 u_2(x) + \ldots + c_k u_k(x) \equiv 0 ,$$

therefore $u_1, u_2, \ldots, u_k$ are linearly dependent.

Now since $U_1, U_2, \ldots, U_k$ are linearly independent then $U_1(x_0), U_2(x_0), \ldots, U_k(x_0)$ would be linear independent in V_n. But there can be no more than n linearly independent vectors in V_n, since dim $(V_n) = n$. Hence k cannot exceed n and $\dim(W) \leq n$.

Thus the theorem is proved.

Appendix C

A Proof of Remark 4.1.7.

Let $F : J = [0,L] \longrightarrow R^n$ be a C^k-parametrization by arc length. Let $s_0 \in J$ and assume that the vectors $F'(s_0)$, $F''(s_0)$, $F'''(s_0), \ldots, F^{(r+1)}(s_0)(r < k)$ are linearly independent then the vectors $F'(s), \ldots, F^{(r+1)}$ are linearly independent in some neighborhood U of s_0 in J.

Proof. Let $M((r+1)\times n)$ be the set of all $(r+1)\times n$ matrices with real entries $(r+1 \leqslant n)$. Since there exists a function f mapping $M((r+1)\times n)$ onto $R^{(r+1)n}$ in a one-to-one onto way, then $M((r+1)\times n)$ can be endowed with the same topology as that of $R^{(r+1)n}$. We denote by $M((r+1)\times n,\ r+1)$ the subset of $M((r+1)\times n)$ which consists of these matrices of rank r+1. Claim that $M((r+1)\times n,\ r+1)$ is an open subset of $M((r+1)\times n)$. To prove this we note that $M((r+1)\times n,\ r+1)$ is a submanifold of $M((r+1)\times n)$ of dimension $(r+1)n$ (see [2] on page 109). Thus it sufficies to show that if M is a manifold of dimension n and M' is a submanifold of M of the same dimension then M' is open in M. Let $a \in M'$, since M' is a submanifold of M of the same dimension, then there exists an open subset $V \ni a$ of M and a function $\emptyset$ such that $\emptyset$ is a homeomorphism of V onto some open subset W of R^n, and $\emptyset$ is a homeomorphism of $V \cap M$ onto some open subset $W' \subset W$ of R^n. Because W' is open in R^n, therefore W' and $\emptyset^{-1}(W') = V \cap M'$ is open in W and V respectively under the usual relative topology. Thus $V \cap M' = P \cap V$ for some open subset P of M. This implies that $V \cap M'$ is open in M. Hence a is an interior point of M , but a is arbitrary we conclude that M is open.

To finish the proof of Remark 4.1.7, we define a new function $\Psi : J \to M((r+1) \times n)$ as follows :

$$\Psi(s) = \begin{pmatrix} F'(s) \\ F''(s) \\ \cdot \\ \cdot \\ \cdot \\ F^{(r+1)}(s) \end{pmatrix}$$

By the continuity of the functions $F', F'', \ldots, F^{(r+1)}$, we have that Ψ is also contimuous on J. Clearly $\Psi(s_0) \in M((r+1)\times n, r+1)$, but $M((r+1)\times n, r+1)$ is open in $M((r+1)\times n)$, then there is an open subset $U' \ni \Psi(s_0)$ of $M((r+1)\times n)$ and $U' \subset M((r+1)\times n, r+1)$.

Furthermore $\Psi^{-1}(U)$ is open and contains the points s_0 in J because Ψ is continuous. Hence there exists a neighborhood U of s_0 in I such that $\Psi(U) \subset U'$, so U is the required neighborhood. The proof is complete.

References

[1] Apostol, Tom M., Mathematical Analysis. 5 th ed. Reading : Addison-Wesley, 1957.

[2] Auslander, L., and Mackenzie, R.E. Introduction to Differentiable Manifolds. Mc Graw-Hill, 1963.

[3] Birkoff, G. and Mac Lane, S., A Survey of Modern Algebra. 3rd. New York : Macmillan, 1965.

[4] Buck, R.C., Advanced Calculus. 2nd. ed. New York : MacGraw-Hill, 1965.

[5] Gluck, Herman, Higher Curvature of Curves in Euclidean Space. The American Mathematical Monthly, Vol.73, No.7, 1966.

[6] Kaplan, Wilfred, and Lewis, Donald J. Calculus and Linear Algebra New York : John Willey and Sons, Inc., 1970-1971.

[7] Komogorov, A.N. and Fomin, S.V., Introductory Real Analysis. transl. and ed. by Silverman, Richard A. Englewood Cliffs : Prentice-Hall, 1970.

[8] Willmore, T,J., Differential Geometry. Oxford University Press, 1959.

[9] Struik, D.J., Lectures on Classical Differential Geometry. Reading : Addison-Wesley, 1950.

[10] Chern, S.S., Notes on Differential Geometry. U. of Cal., Berkeley lecture notes.

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