CHAPTER I

FACTORIZABLE SEMIGROUPS

In this chapter, various general properties of factorizable semigroups are introduced. In particular, it is shown that every factorizable semigroup is a regular semigroup and an ideal of a factorizable semigroup is not necessarily factorizable. Necessary and sufficient conditions of an ideal of a factorizable semigroup to be factorizable are given.

The first theorem shows that every factorizable semigroup is a regular semigroup. The following Lemma is required :

1.1 Lemma. Let a semigroup S be factorizable as GE. Then the identity of G is a left identity of S.

Proof : Let e be the identity of the group G. Let $x \in S$. Then $x = gf$ for some $g \in G$, $f \in E$. Therefore $eg = g$, and so $ex = e(gf) = (eg)f = gf = x$. This shows that e is a left identity of S, as required. #

1.2 Corollary. Let S be a semigroup which is factorizable as GE. Then the identity of G belongs to E.

Proof : Let e be the identity of G. Because S = GE, $e = gf$ for some $g \in G$, $f \in E$. By Lemma 1.1, e is a left identity of S. Thus $f = ef = g^{-1}gf = g^{-1}e = g^{-1} \in G$. Therefore f is an idempotent in the group G, so f is the identity of G. Hence $f = e$ which implies $e \in E$. #

1.3 *Theorem*. Every factorizable semigroup is a regular semigroup.

Proof : Let S be a factorizable semigroup. Then $S = GE$ for some subgroup G of S and some subset E of $E(S)$. Let e be the identity of G. By Lemma 1.1, e is a left identity of S. Let $x \in S$. Then $x = gf$ for some $g \in G$, $f \in E$. Because e is a left identity of S, $ef = f$. Therefore $xg^{-1}x = (gf)g^{-1}(gf) = gfef = gff = gf = x$, so x is regular. This shows that S is a regular semigroup. Hence, the theorem is proved. #

Let S be a semigroup. For $a \in S$, let H_a denote the $\mathcal{H}$-class containing a. If $e \in E(S)$, then H_e is a maximal subgroup of S or the maximum subgroup of S having e as its identity, and

$$H_e = \{a \in S \mid ae = ea = a \text{ and } aa' = e = a'a \text{ for some } a' \;\; S\}.$$

Then if S has an identity 1, then

$$H_1 = \{a \in S \mid aa' = 1 = a'a \text{ for some } a' \in S\}$$

and it is the group of units of S.

We show in the next proposition that if a semigroup S is factorizable as GE, then G is a maximal subgroup of S.

1.4 *Proposition*. Let a semigroup S be factorizable as GE. Then $G = H_e$, where e is the identity of G.

Proof : Because H_e is the maximum subgroup of S having e as its identity, $G \subseteq H_e$. To show $H_e \subseteq G$, let $x \in H_e$. Then $x = gf$

for some $g \in G$, $f \in E$. Therefore $g \in H_e$. Because e is a left identity of S by Lemma 1.1, ef = f. Thus $f = ef = g^{-1}gf = g^{-1}x \in H_e$ since $g, x \in H_e$. It then follows that f = e, so $x = ge = g \in G$. Hence $G = H_e$. #

A factorizable semigroup need not have an identity and need not be an inverse semigroup. An example is given as follows :

Let S be a nontrivial right zero semigroup; that is, $|S| > 1$ and xy = y for all $x, y \in S$. Then S has no identity. Because E(S) = S and any two distinct elements of S do not commute with each other, S is not an inverse semigroup. It is clear that for each $a \in S$, $\{a\}$ is a subgroup of S and $S = \{a\}S = \{a\}E(S)$. Hence S is factorizable.

The next theorem shows that if a semigroup S is factorizable as GE and S has an identity, then G is the group of units of S. To prove this, the following Lemma is required :

1.5 Lemma. Let a semigroup S be factorizable as GE. Let S have an identity 1. Then $1 \in G$.

Proof : Let e be the identity of G. By Lemma 1.1, e is a left identity of S. Thus $1 = e1 = e \in G$. #

The following corollary follows directly from Lemma 1.5 and Corollary 1.2.

1.6 Corollary. Let a semigroup S be factorizable as GE. If S has an identity 1, then $1 \in E$.

1.7 Theorem. Let S be a semigroup which is factorizable as GE. If S has an identity, then G is the group of units of S.

Proof : Let 1 be the identity of S. By Lemma 1.5, $1 \in G$. Then $G = H_1$ by Proposition 1.4. Hence G is the group of units of S. #

If S is a factorizable semigroup which factors as GE, it is clear that $S = GE(S)$.

It has been shown by Chen and Hsieh in [3] that if an inverse semigroup S is factorizable as GE, then $E = E(S)$. The following example shows that any factorizable semigroup need not have this property :

Example. Let $X = \{1, 2, 3\}$ and T_X be the partial transformation semigroup on the set X. It is shown in the last chapter that any partial transformation semigroup on a finite set is factorizable. Then T_X is factorizable and by Theorem 1.7, $T_X = G_X E(T_X)$ where G_X is the permutation group on X; that is,
$G_X = \{1_G, (12), (13), (23), (123), (132)\}$ where 1_G denotes the identity map on X. Let α, β, γ denote the partial transformations on X defined by $\Delta\alpha = \{1, 2\}$, $\nabla\alpha = \{1\}$, $\Delta\beta = \{1, 3\}$, $\nabla\beta = \{1\}$, $\Delta\gamma = \{2, 3\}$ and $\nabla\gamma = \{1\}$. Then $\alpha, \beta \in E(T_X)$. Moreover, $G_X\beta = \{\beta, \gamma, \alpha\}$ and $\alpha = 1_G\alpha$, $\beta = (23)\alpha$, $\gamma = (132)\alpha$. This shows that
$G_X E(T_X) = G_X(E(T_X) \setminus \{\beta\})$. Let $E = E(T_X) \setminus \{\beta\}$. Hence $E \neq E(T_X)$ and $T_X = G_X E$. #

Let S be a semigroup. Assume that the semigroup S is factorizable as GE. Then $|S| \leq |G||E|$. Let e denote the identity of the group G. If the cardinality of E is one, then, by Corollary 1.2, $E = \{e\}$, and therefore $S = G$ which implies that $E = \{e\} = E(S)$. If S is a finite semigroup and $|E| = |S|$, then $S = E$ and hence $E = E(S) = S$.

The next proposition shows that a finite factorizable semigroup S with $|S| < 4$ has the following property : Let a semigroup S be factorizable as GE. If $|S| < 4$, then E coincides with E(S).

1.8 <u>Proposition</u>. Let a semigroup S be factorizable as GE. If $|S| < 4$, then $E = E(S)$.

<u>Proof</u> : As the above mention, if $|E| = 1$ or $|E| = |S|$, then $E = E(S)$.

Assume that $1 < |E| < |S|$ for the remaining of the proof. Let e denote the identity of G. Then e is a left identity of S by Lemma 1.1. Thus, if $|G| = 1$, then $S = E$, so $|E| = |S|$, and if $|G| = |S|$, then $G = S$ which implies $E(S) = \{e\} = E$, so $|E| = 1$. Then neither $|G| = 1$ nor $|G| = |S|$. Hence $1 < |G| < |S|$. Since $1 < |E| < |S|$ and $|S| < 4$, it follows that S has exactly three elements. Because $1 < |G| < |S|$ and $1 < |E| < |S|$, $|G| = 2$ and $|E| = 2$. Then there exists $g \in G$ such that $g \notin E(S)$. Therefore $2 = |E| \leq |E(S)| \leq |S| - 1 = 3-1 = 2$ and hence $E = E(S)$.

Therefore the proposition is completely proved. #

Let A be an ideal of a factorizable semigroup S which factors as GE. Then either $A \cap G = \phi$ or $A = S$. To prove this, suppose that $A \cap G \neq \phi$. Then there exists an element g of S such that $g \in A \cap G$. Because A is an ideal of S and $g \in G$, $gg^{-1} = e \in A$ where e is the identity of G. By Lemma 1.1, e is a left identity of S. Hence for each $x \in S$, $x = ex \in A$. Therefore $A = S$.

We note that, from the above proof, it is clearly seen that if R is a right ideal of a factorizable semigroup S which factors as GE, we also have that either $R \cap G = \phi$ or $R = S$.

A homomorphic image of a factorizable semigroup is clearly a factorizable semigroup. An ideal of a regular semigroup S is a regular subsemigroup of S. However, an ideal of a factorizable semigroup is not necessarily factorizable. An example is given as follows :

Example. Let $X = \{a, b\}$ and I_X be the symmetric inverse semigroup on the set X. Let 0 and 1 denote the zero and the identity of I_X, respectively, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ be one-to-one partial transformations on X defined by $\Delta\alpha_1 = \nabla\alpha_1 = \{a\}$, $\Delta\alpha_2 = \nabla\alpha_2 = \{b\}$, $\Delta\alpha_3 = \{a\}$, $\nabla\alpha_3 = \{b\}$, $\Delta\alpha_4 = \{b\}$, $\nabla\alpha_4 = \{a\}$, and $\Delta\alpha_5 = \nabla\alpha_5 = \{a, b\}$ such that $a\alpha_5 = b$, $b\alpha_5 = a$. Then $I_X = \{0, 1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$

and the multiplication on I_X is as follows :

$\cdot$	0	α_1	α_2	α_3	α_4	α_5	1
0	0	0	0	0	0	0	0
α_1	0	α_1	0	α_3	0	α_3	α_1
α_2	0	0	α_2	0	α_4	α_4	α_2
α_3	0	0	α_3	0	α_1	α_1	α_3
α_4	0	α_4	0	α_2	0	α_2	α_4
α_5	0	α_4	α_3	α_2	α_1	1	α_5
1	0	α_1	α_2	α_3	α_4	α_5	1

From the table G_X (the permutation group on X) is $\{1, \alpha_5\}$ and $E(I_X) = \{0, \alpha_1, \alpha_2, 1\}$. Because X is finite, by Corollary of Theorem 3.1 in [3], I_X is a factorizable semigroup. Let $A = \{0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then A is an ideal of I_X. Moreover, all subgroups of A are $\{0\}$, $\{\alpha_1\}$ and $\{\alpha_2\}$ and $E(A) = \{0, \alpha_1, \alpha_2\}$. But $\{0\}E(A) = \{0\} \neq A$, $\{\alpha_1\}E(A) = \{0, \alpha_1\} \neq A$ and $\{\alpha_2\}E(A) = \{0, \alpha_2\} \neq A$. Hence A is not factorizable. #

The following proposition shows the form of an ideal of a factorizable semigroup :

1.9 Proposition. Let A be an ideal of a factorizable semigroup which factors as GE. Then $A = GE(A)$.

Proof : Because A is an ideal of S, $GE(A) \subseteq A$. Next, let $a \in A$. Since $S = GE$, $a = gf$ for some $g \in G$, $f \in E$. By Lemma 1.1, $f = g^{-1}gf$. But $a = gf \in A$ which is an ideal of S. Then $f \in A$ and so $f \in E(A)$. Thus $a \in Gf \subseteq GE(A)$.

Therefore, $A = GE(A)$ as required. #

Necessary and sufficient conditions of an ideal of a factorizable semigroup to be factorizable are given as follows :

1.10 Theorem. An ideal A of a factorizable semigroup S is factorizable if and only if A has a left identity.

Proof : Assume S is factorizable as GE and let e denote the identity of G. By Lemma 1.1, e is a left identity of S. Suppose the ideal A has a left identity, say $\bar{e}$. Then $\bar{e}.E(A) = E(A)$ and $G\bar{e} \subseteq A$. By Proposition 1.9, $A = GE(A) = (G\bar{e})E(A)$. Next, we show $G\bar{e}$ is a subgroup of A. Let g and $h \in G$. Because $h\bar{e} \in A$ and $\bar{e}$ is a left identity of A, $(g\bar{e})(h\bar{e}) = g(\bar{e}(h\bar{e})) = g(h\bar{e}) = (gh)\bar{e} \in G\bar{e}$. Also, $(g\bar{e})(e\bar{e}) = g(\bar{e}(e\bar{e})) = g(e\bar{e}) = g\bar{e}$ since $e\bar{e} \in A$ and e is a left identity of S. Moreover, $g^{-1}\bar{e} \in A$ and $(g\bar{e})(g^{-1}\bar{e}) = g(\bar{e}(g^{-1}\bar{e}) = g(g^{-1}\bar{e}) = e\bar{e}$. This proves that $G\bar{e}$ is a subgroup of A. Therefore A is factorizable.

The converse follows directly from Lemma 1.1. #

1.11 Corollary. Let A be an ideal of a factorizable semigroup. If A has an identity, then A is factorizable.

It has been proved in [3] that every factorizable inverse semigroup has an identity.

Let S be a factorizable inverse semigroup. Let A be an ideal of S. Then A is an inverse subsemigroup of S. If A has an identity, then, by Corollary 1.11, A is factorizable. Therefore, we have

1.12 Corollary. Let A be an ideal of a factorizable inverse semigroup. Then A is factorizable if and only if A has an identity.