CHAPTER II

MATHEMATICAL MODEL

2.1 Liquid Flow in Pipelines

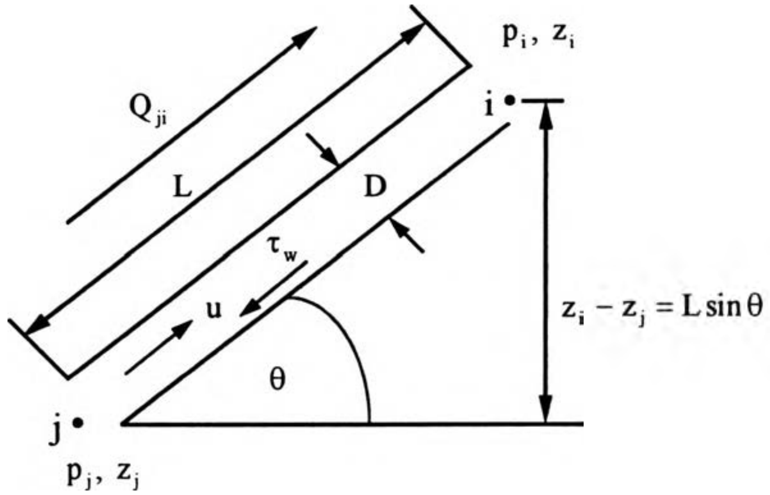


Fig. 2.1 Pressure drop in an inclined pipeline for flow from node j to node i.

A steady state momentum balance in the direction of flow from node j to node i in the pipeline gives:

$$\left(p_j - p_i\right)\frac{\pi D^2}{4} - \tau_w \pi D L - \frac{\pi D^2}{4}\rho L g \sin\theta = 0 \tag{2.1}$$

The Fanning friction factor is defined as:

$$f_F = \frac{\tau_w}{\frac{1}{2}\rho u^2} \tag{2.2}$$

Eqn. (2.1) can be rewritten as:

$$p_j - p_i = 2 f_F \rho u^2 \frac{L}{D} + \rho g\left(z_i - z_j\right) \tag{2.3}$$

Since:

$$u^2 = \frac{16Q^2}{\pi^2 D^4} \tag{2.4}$$

Eqn. (2.3) gives:

$$p_j - p_i = \frac{32 f_F \rho Q^2 L}{\pi^2 D^5} + \rho g(z_i - z_j) \tag{2.5}$$

Rearrangement of Eqn. (2.5) gives:

$$(p_j - p_i) + \rho g(z_j - z_i) = \frac{32 f_F \rho Q^2 L}{\pi^2 D^5} \tag{2.6}$$

Subscripts are now inserted to emphasize that the flow is from node j to node i, so Eqn. (2.6) becomes:

$$(p_j - p_i) + \rho g(z_j - z_i) = \frac{32 f_{F_{ji}} \rho Q_{ji}^2 L_{ji}}{\pi^2 D_{ji}^5} \tag{2.7}$$

The corresponding relation for flow from node i to node j is:

$$(p_j - p_i) + \rho g(z_j - z_i) = -\frac{32 f_{F_{ji}} \rho Q_{ij}^2 L_{ji}}{\pi^2 D_{ji}^5} \tag{2.8}$$

Note that we always consider node i as the second or receiving node since the simultaneous nonlinear equations are generated from nodal material balances on every node i in the whole network.

All constant quantities are in consistent units. Equation (2.7) can be rewritten as:

$$(p_j - p_i) + \beta(z_j - z_i) = \alpha_{ji} f_{F_{ji}} Q_{ji}^2 \tag{2.9}$$

Here:

$$\alpha_{ji} = \frac{32 \rho L_{ji}}{\pi^2 D_{ji}^5} \quad \text{and} \quad \beta = \rho g \tag{2.10}$$

If we define:

$$y = p_j - p_i + \beta(z_j - z_i) \tag{2.11}$$

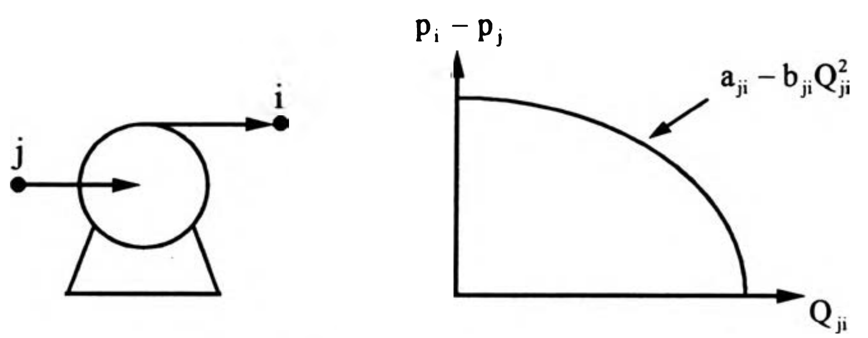
from Eqn. (2.9), the flow rate from node j to node i is given by:

$$Q_{ji} = \sqrt{y/\alpha_{ji} f_{F_{ji}}} \qquad \text{for} \qquad y > 0 \qquad (2.12)$$

and from Eqn. (2.8), the flow rate from node i to node j is given by:

$$Q_{ij} = -\sqrt{-y/\alpha_{ji} f_{F_{ji}}} \qquad \text{for} \qquad y < 0 \qquad (2.13)$$

2.2 Liquid Flow across a Pump with Elevation change



$$p_i - p_j = a_{ji} - b_{ji}Q_{ji}^2$$

Fig. 2.2 Centrifugal pump and performance curve.

There are two separate cases to be considered for each of three possibilities, as follows:

1. $Q_{ji} > 0$, for flow across the pump from node j to node i:

$$Q_{ji} = 0 \qquad \text{for } p_i + \beta z_i > p_j + \beta z_j + a_{ji} \qquad (2.14)$$

$$Q_{ji} = \sqrt{a_{ji}/b_{ji}} \qquad \text{for } p_j + \beta z_j > p_i + \beta z_i \qquad (2.15)$$

$$Q_{ji} = \sqrt{\left(p_j - p_i + a_{ji} + \beta(z_j - z_i)\right)/b_{ji}} \qquad \text{otherwise} \qquad (2.16)$$

Note that Q_{ji} can not be negative, even if $p_i > p_j + a_{ji}$ because the pump is equipped with a check valve.

2. $Q_{ij} < 0$, for flow across the pump from node i to node j:

$$Q_{ij} = 0 \qquad \text{for } p_j + \beta z_j > p_i + \beta z_i + a_{ij} \tag{2.17}$$

$$Q_{ij} = -\sqrt{a_{ij}/b_{ij}} \qquad \text{for } p_i + \beta z_i > p_j + \beta z_j \tag{2.18}$$

$$Q_{ij} = -\sqrt{\left(p_i - p_j + a_{ij} + \beta\left(z_i - z_j\right)\right)/b_{ij}} \qquad \text{otherwise} \tag{2.19}$$

2.3 Compressible Gas Flow in Pipelines

2.3.1 <u>Inclined Flow ($z_i \neq z_j$)</u>:

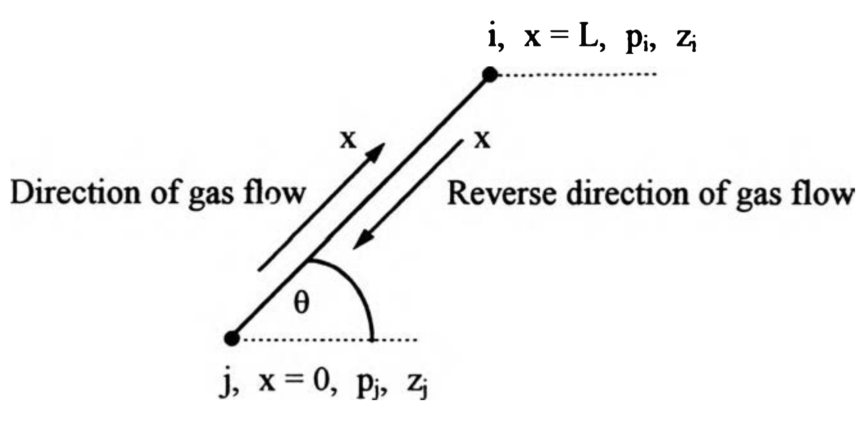


Fig. 2.3 Inclined compressible gas flow from node j to node i.

Consider the inclined steady flow of compressible gas in a long-distance pipeline of length L and diameter D, with inlet and outlet pressures and elevation change, flowing from node j to node i as shown in Fig. 2.3. If the flow is upward, both θ and $\sin\theta$ will be positive and if it is downward, both θ and $\sin\theta$ will be negative. The pipeline is assumed to be sufficiently long in relation to its diameter so that it comes into thermal equilibrium with its surroundings; thus, the flow is isothermal.

In the absence of useful work effects, such as a compressor or turbine, an energy balance on a differential length dx results in:

$$gdz + d\left(\frac{u^2}{2}\right) + \frac{dp}{\rho} + dF = 0 \tag{2.20}$$

In which frictional dissipation of energy per unit mass flowing is:

$$dF = 2f_F u^2 \frac{dx}{D} \tag{2.21}$$

Expansion of the differential $d\left(\frac{u^2}{2}\right)$, substitution of Eqn. (2.21) for dF, and division by u^2, transforms Eqn. (2.20) into:

$$\frac{gdz}{u^2} + \frac{du}{u} + \frac{dp}{\rho u^2} + 2f_F \frac{dx}{D} = 0 \tag{2.22}$$

The cross-sectional area of the pipeline is:

$$A = \frac{\pi D^2}{4} \tag{2.23}$$

Because of continuity, the mass velocity $G = \rho u$ does not vary, so that:

$$dG = 0 = \rho du + u d\rho \tag{2.24}$$

Assuming an ideal gas and noting that absolute pressures must be used:

$$\frac{du}{u} = -\frac{d\rho}{\rho} = -\frac{dp}{p} \tag{2.25}$$

Also note that:

$$\frac{1}{\rho u^2} = \frac{\rho}{G^2} \tag{2.26}$$

The following relation exists between the elevation and length differential increments:

$$dz = dx \sin\theta \tag{2.27}$$

Making the appropriate substitutions and collecting terms in Eqns. (2.22) results:

$$-\frac{dp}{p}+\frac{\alpha p dp}{G^2}+\frac{2f_F}{D}dx+gdx\frac{\alpha^2 p^2}{G^2}\sin\theta=0 \tag{2.28}$$

in which:

$$\alpha=\frac{M}{Z_{avg}RT}$$

Rearrangement gives:

$$\left(\frac{1}{p}-\frac{\alpha p}{G^2}\right)dp=\left(\frac{2f_F}{D}+g\frac{\alpha^2 p^2}{G^2}\sin\theta\right)dx \tag{2.29}$$

The variables are now separated and integration is performed from the inlet node j to the outlet node i as follows:

$$\int_{p_j}^{p_i}\frac{\frac{1}{p}-\frac{\alpha p}{G^2}}{\frac{2f_F}{D}+g\frac{\alpha^2 p^2}{G^2}\sin\theta}dp=\int_0^L dx=L \tag{2.30}$$

1/p in the numerator of the first integral is relatively small and can sometimes be neglected. In this case, the integration can be performed as follows. Let:

$$\beta=\frac{2f_F}{D} \qquad \text{and} \qquad \gamma=\frac{\alpha^2 g}{G^2}\sin\theta \tag{2.31}$$

so that:

$$\frac{-\alpha}{G^2}\int_{p_j}^{p_i}\frac{pdp}{\frac{2f_F}{D}+g\frac{\alpha^2 p^2}{G^2}\sin\theta}=\frac{-\alpha}{G^2}\int_{p_j}^{p_i}\frac{pdp}{\beta+\gamma p^2}=L \tag{2.32}$$

$$\frac{-\alpha}{G^2}\int_{p_j}^{p_i}\frac{pdp}{\beta+\gamma p^2}=\frac{1}{2\alpha g\sin\theta}\ln\left(\frac{\beta+\gamma p_j^2}{\beta+\gamma p_i^2}\right)=L \tag{2.33}$$

Therefore:

$$\frac{-\alpha}{G^2}\int_{p_j}^{p_i}\frac{pdp}{\frac{2f_F}{D}+g\frac{\alpha^2 p^2}{G^2}\sin\theta}=\frac{1}{2\alpha g\sin\theta}\ln\left(\frac{\beta+\gamma p_j^2}{\beta+\gamma p_i^2}\right)=L \tag{2.34}$$

Noting that:

$$\frac{\beta}{\gamma} = \frac{2f_F G^2}{\alpha^2 g D \sin\theta} = \delta G^2 \tag{2.35}$$

in which:

$$\delta = \frac{2f_F}{\alpha^2 g D \sin\theta} = \frac{2f_F L}{\alpha^2 g D (z_i - z_j)} \tag{2.36}$$

Eqn. (2.34) gives:

$$\left(\frac{1}{2\alpha g \sin\theta}\right) \ln\left(\frac{\delta G^2 + p_j^2}{\delta G^2 + p_i^2}\right) = L \tag{2.37}$$

$$\ln\left(\frac{\delta G^2 + p_j^2}{\delta G^2 + p_i^2}\right) = 2\alpha g L \sin\theta \tag{2.38}$$

Rearrangement of Eqn. (2.38) gives:

$$\frac{\delta G^2 + p_j^2}{\delta G^2 + p_i^2} = \exp(2\alpha g L \sin\theta) \tag{2.39}$$

$$G^2 = \frac{p_j^2 - p_i^2 \exp(2\alpha g L \sin\theta)}{\delta \exp(2\alpha g L \sin\theta - 1)} \tag{2.40}$$

$$G^2 = \frac{p_j^2 - p_i^2 \exp[2\alpha g (z_i - z_j)]}{\delta \exp[(2\alpha g (z_i - z_j) - 1)]} \tag{2.41}$$

Thus, the square of the mass velocity in the pipeline with subscripts inserted to indicate the gas flow from node j to node i is:

$$G_{ji}^2 = \left(\frac{M}{Z_{avg} R T}\right)^2 \left(\frac{g D_{ji} (z_i - z_j)}{2 f_{F_{ji}} L_{ji}}\right) \frac{p_j^2 - \phi_{ji} p_i^2}{(\phi_{ji} - 1)} \tag{2.42}$$

in which:

$$\phi_{ji} = \exp\left(\frac{2Mg(z_i - z_j)}{Z_{avg} R T}\right)$$

The corresponding relation for flow from node i to node j would be:

$$G_{ij}^2 = \left(\frac{M}{Z_{avg}RT}\right)^2 \left(\frac{gD_{ji}(z_j - z_i)}{2f_{F_{ji}}L_{ji}}\right) \frac{p_i^2 - \phi_{ij}p_j^2}{(\phi_{ij} - 1)} \tag{2.43}$$

in which:

$$\phi_{ij} = \exp\left(\frac{2Mg(z_j - z_i)}{Z_{avg}RT}\right)$$

Since:

$$m = GA\,,$$

where m = mass flow rate and A = cross sectional area of pipeline, the mass flow rate from node j to node i is:

$$m_{ji} = A\left(\frac{M}{Z_{avg}RT}\right)\sqrt{\left(\frac{gD_{ji}(z_i - z_j)}{2f_{F_{ji}}L_{ji}}\right) \frac{p_j^2 - \phi_{ji}p_i^2}{(\phi_{ji} - 1)}} \tag{2.44}$$

in which:

$$\phi_{ji} = \exp\left(\frac{2Mg(z_i - z_j)}{Z_{avg}RT}\right)$$

Likewise, the mass flow rate from node i to node j is:

$$m_{ij} = -A_{ij}\left(\frac{M}{Z_{avg}RT}\right)\sqrt{\left(\frac{gD_{ji}(z_j - z_i)}{2f_{F_{ji}}L_{ji}}\right) \frac{p_i^2 - \phi_{ij}p_j^2}{(\phi_{ij} - 1)}} \tag{2.45}$$

in which:

$$\phi_{ij} = \exp\left(\frac{2Mg(z_j - z_i)}{Z_{avg}RT}\right)$$

Note that:

$$\phi_{ij} = \frac{1}{\phi_{ji}} \tag{2.46}$$

As usual, all constant quantities are in consistent units. Therefore, the mass flow rate from node j to node i is:

$$m_{ji} = \lambda_{ji}\sqrt{\frac{p_j^2 - \phi_{ji}p_i^2}{\delta_{ji}(\phi_{ji} - 1)}} \tag{2.47}$$

and the mass flow rate from node i to node j is:

$$m_{ij} = -\lambda_{ji}\sqrt{-\frac{p_j^2 - \phi_{ji}p_i^2}{\delta_{ji}(\phi_{ji} - 1)}} \tag{2.48}$$

Here:

$$\lambda_{ji} = \left(\frac{M}{Z_{avg}RT}\right)\left(\frac{\pi D_{ji}^2}{4}\right) \tag{2.49}$$

$$\delta_{ji} = \left(\frac{2f_{F_{ji}}L_{ji}}{gD_{ji}(z_i - z_j)}\right) \tag{2.50}$$

$$\phi_{ji} = \exp\left(\frac{2Mg(z_i - z_j)}{Z_{avg}RT}\right) \tag{2.51}$$

If we define:

$$w = p_j^2 - \phi_{ji}p_i^2 \tag{2.52}$$

the flow rate from node j to node i is given by:

$$m_{ji} = \lambda_{ji}\sqrt{\frac{w}{\delta_{ji}(\phi_{ji} - 1)}} \quad \text{for} \quad w > 0 \tag{2.53}$$

The flow rate from node j to node i is given by:

$$m_{ji} = -\lambda_{ji}\sqrt{\frac{-w}{\delta_{ji}(\phi_{ji} - 1)}} \quad \text{for} \quad w < 0 \tag{2.54}$$

The mass flow rate from node j to node i can be converted to a volumetric flow rate at standard conditions:

$$Q_{sc-ji} = \frac{Z_{sc}RT_{sc}}{p_{sc}M}\lambda_{ji}\sqrt{\frac{w}{\delta_{ji}(\phi_{ji} - 1)}} \quad \text{for} \quad w > 0 \tag{2.55}$$

In the same manner, the volumetric flow rate from node i to node j at standard conditions is:

$$Q_{sc-ji} = -\frac{Z_{sc} R T_{sc}}{p_{sc} M} \lambda_{ji} \sqrt{\frac{-w}{\delta_{ji}\left(\phi_{ji}-1\right)}} \quad \text{for} \quad w<0 \tag{2.56}$$

2.3.2 Horizontal Flow ($z_i = z_j$):

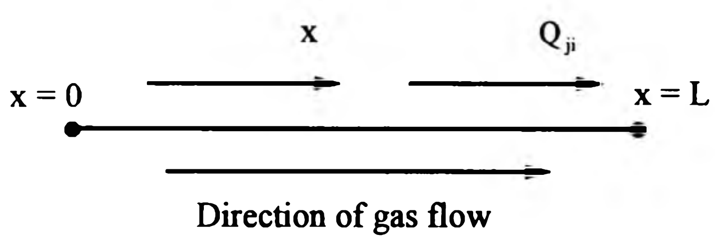


Fig. 2.4 Horizontal compressible gas flow from node j to node i.

For Eqn. (2.41), if the elevation change becomes zero or $\theta = 0$, there is no definite value, because there is no effect for hydrostatic pressure in the horizontal flow. Therefore, it is necessary to consider separately horizontal steady flow from node j to node i as shown in Fig. 2.4. Thus, for $\theta = 0$, Eqn. (2.29) gives:

$$\left(\frac{1}{p} - \frac{\alpha p}{G^2}\right) dp = \frac{2 f_F}{D} dx \tag{2.57}$$

in which:

$$\alpha = \frac{M}{Z_{avg} R T}$$

The variables are now separated and integration is performed from the inlet node j to the outlet node i as follows:

$$\int_{p_j}^{p_i} \left(\frac{1}{p} - \frac{\alpha p}{G^2}\right) dp = \int_0^L \frac{2 f_F}{D} dx = \frac{2 f_F L}{D} \tag{2.58}$$

1/p in the first term of integration is often relatively small and can be neglected, in which case integration of Eqn. (2.58) results in:

$$\frac{\alpha}{2G^2}\left(p_j^2 - p_i^2\right) = \frac{2f_F L}{D} \tag{2.59}$$

$$G^2 = \frac{\alpha D}{4f_F L}\left(p_j^2 - p_i^2\right) \tag{2.60}$$

$$G^2 = \frac{M}{Z_{avg}RT}\left(\frac{D}{4f_F L}\right)\left(p_j^2 - p_i^2\right) \tag{2.61}$$

Thus, the square of the mass velocity in the pipeline with subscripts inserted to indicate the gas flow from node j to node i is:

$$G_{ji}^2 = \frac{M}{Z_{avg}RT}\left(\frac{D_{ji}}{4f_{F_{ji}} L_{ji}}\right)\left(p_j^2 - p_i^2\right) \tag{2.62}$$

The corresponding relation for flow from node i to node j would be:

$$G_{ij}^2 = \frac{M}{Z_{avg}RT}\left(\frac{D_{ji}}{4f_{F_{ji}} L_{ji}}\right)\left(p_i^2 - p_j^2\right) \tag{2.63}$$

In the same manner, since $m = GA$, the gas flow from node j to node i is:

$$m_{ji} = \left(\frac{\pi D_{ji}^2}{4}\right)\sqrt{\frac{M}{Z_{avg}RT}\left(\frac{D_{ji}}{4f_{F_{ji}} L_{ji}}\right)\left(p_j^2 - p_i^2\right)} \tag{2.64}$$

The gas flow from node i to node j is:

$$m_{ij} = -\left(\frac{\pi D_{ji}^2}{4}\right)\sqrt{-\frac{M}{Z_{avg}RT}\left(\frac{D_{ji}}{4f_{F_{ji}} L_{ji}}\right)\left(p_j^2 - p_i^2\right)} \tag{2.65}$$

As usual, all constant quantities are in consistent units. The mass flow rate from node j to node i is:

$$m_{ji} = A_{ji}\sqrt{\xi_{ji}\left(p_j^2 - p_i^2\right)} \qquad \text{for} \qquad p_j > p_i \tag{2.66}$$

and the mass flow rate from node i to node j is:

$$m_{ij} = -A_{ji}\sqrt{-\xi_{ji}\left(p_j^2 - p_i^2\right)} \qquad \text{for} \quad p_j < p_i \qquad (2.67)$$

Here:

$$A_{ji} = \frac{\pi D_{ji}^2}{4} \qquad (2.68)$$

$$\xi_{ji} = \frac{M}{Z_{avg}RT}\left(\frac{D_{ji}}{4f_{F_{ji}}L_{ji}}\right) \qquad (2.69)$$

The mass flow rate from node j to node i can be converted to a volumetric flow rate at standard conditions:

$$Q_{sc-ji} = \frac{Z_{sc}RT_{sc}}{p_{sc}M}A_{ji}\sqrt{\xi_{ji}\left(p_j^2 - p_i^2\right)} \qquad \text{for} \quad p_j > p_i \qquad (2.70)$$

and, the volumetric flow rate from node i to node j at standard conditions is:

$$Q_{sc-ij} = -\frac{Z_{sc}RT_{sc}}{p_{sc}M}A_{ji}\sqrt{-\xi_{ji}\left(p_j^2 - p_i^2\right)} \qquad \text{for} \quad p_j < p_i \qquad (2.71)$$

2.4 Compressible Gas Flow across a Compressor with Elevation change

According to the energy equation, the theoretical work required to compress a unit mass of gas from node j to node i is given by:

$$dw_c = vdp + d\left(\frac{u^2}{2}\right) + gdz + dF \qquad (2.72)$$

Neglecting the friction losses and the change in kinetic energy, performing the integration from the inlet node j to the outlet node i gives:

$$w_c = \int_{p_j}^{p_i} vdp + g\left(z_i - z_j\right) \qquad (2.73)$$

For isentropic compression:

$$pv^{k} = c = \text{constant} \quad \text{and} \quad k = \frac{c_p}{c_v} \tag{2.74}$$

Therefore:

$$w_c = c^{1/k}\int_{p_j}^{p_i} p^{-1/p}dp + g(z_i - z_j) \tag{2.75}$$

where c is a constant. Upon integration, Eqn. (2.75) becomes:

$$w_c = \frac{k}{k-1}c^{1/k}\left[p_i^{(k-1)/k} - p_j^{(k-1)/k}\right] + g(z_i - z_j) \tag{2.76}$$

$$w_c = \frac{k}{k-1}p_j\left(\frac{c}{p_j}\right)^{1/k}\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j) \tag{2.77}$$

$$w_c = \frac{k}{k-1}p_j v_j\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j) \tag{2.78}$$

Since:

$$p_j v_j = \frac{Z_{avg}RT_j}{M} \tag{2.79}$$

Eqn. (2.78) becomes:

$$w_c = \frac{k}{k-1}\frac{Z_{avg}RT_j}{M}\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j) \tag{2.80}$$

Conversion of the theoretical work required per unit mass to compression power gives:

$$W_c = m_{ji}\left\{\frac{k}{k-1}\frac{Z_{avg}RT_j}{M}\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j)\right\} \tag{2.81}$$

Thus, the compression power with subscripts inserted to emphasize that the power is required to compress gas from node j to node i is:

$$W_{c-ji} = m_{ji}\left\{\frac{k}{k-1}\frac{Z_{avg}RT_j}{M}\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j)\right\} \tag{2.82}$$

The mass flow rate across the compressor from node j to node i is given by:

$$m_{ji} = \frac{W_{c-ji}}{\left\{\frac{k}{k-1}\frac{Z_{avg}RT_j}{M}\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j)\right\}} \tag{2.83}$$

The corresponding relation for the mass flow rate across the compressor from node i to node j would be:

$$m_{ji} = \frac{-W_{c-ij}}{\left\{\frac{k}{k-1}\frac{Z_{avg}RT_i}{M}\left[\left(\frac{p_j}{p_i}\right)^{(k-1)/k} - 1\right] + g(z_j - z_i)\right\}} \tag{2.84}$$

The mass flow rate across the compressor can be converted to a volumetric flow rate at standard conditions as follows:

$$m = \frac{p_{sc}M}{Z_{sc}RT_{sc}}Q_{sc} \tag{2.85}$$

Thus, the volumetric flow rate across the compressor at standard conditions for flow from node j to node i is:

$$Q_{sc-ji} = \frac{Z_{sc}RT_{sc}}{p_{sc}M}\frac{W_{c-ji}}{\left\{\frac{k}{k-1}\frac{Z_{avg}RT_j}{M}\left[\left(\frac{p_i}{p_j}\right)^{(k-1)/k} - 1\right] + g(z_i - z_j)\right\}} \tag{2.86}$$

and the volumetric flow rate at standard conditions across the compressor from node i to node j is:

$$Q_{sc-ij} = \frac{Z_{sc}RT_{sc}}{p_{sc}M}\frac{-W_{c-ij}}{\left\{\frac{k}{k-1}\frac{Z_{avg}RT_i}{M}\left[\left(\frac{p_j}{p_i}\right)^{(k-1)/k} - 1\right] + g(z_j - z_i)\right\}} \tag{2.87}$$

Here:

W_{c-ij} is the compression power for flow from node i to node j

W_{c-ji} is the compression power for flow from node j to node i