CHAPTER II

THE DYNAMIC EQUATIONS

Before going into the actual object of this work concerning the ***Lorenz nonlinear dynamic model***, it may be useful to recall some of theoretical background underlying the theory to be developed. In this chapter we will first present the system of governing equations in a precise form. We will then introduce some of the frequently used approximations. The approximate systems have formed the basis for most of attempts to account for the ***global circulation***.

The Exact Equations

It is convenient to group the laws governing the atmosphere into two categories. First there are the basic hydrodynamic and thermodynamic laws which apply to all or large class of fluid systems. These include the laws of conservation of momentum, mass and energy. These laws are expressed mathematically in Newton's second law of motion, the continuity equation and the thermodynamic energy equation, which state that matter can neither be created nor destroyed , momentum can be altered only by a force and the internal energy can be altered by the performance of work or the addition or removal of heat. The ideal gas law also belongs in this category, although it is less general than the other laws.

The remaining laws are the ones needed to express the forces and the heating in terms of the current state of the atmosphere and its environment. This category includes the laws governing the absorption, emission and transfer of infra-red radiation, by the various atmospheric constituents, notably carbon dioxide, ozone and the various phase of water. It includes the *law of turbulent viscosity* and *conductivity*. In principle these laws could perhaps be derived from the basic laws of hydrodynamics and thermodynamics, but no one has succeeded in accomplishing this task. Finally, it includes the laws governing the evaporation and condensation of water, and the conversion of cloud droplets into raindrops and snow crystals. The list is by no means exhaustive.

The equations representing the basic hydrodynamic and thermodynamic laws may be written as

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega} \times \vec{V} - \alpha \vec{\nabla} p + \vec{g} + \vec{F}, \tag{2.1}$$

$$\frac{d\alpha}{dt} = \alpha \vec{\nabla} \cdot \vec{V}, \tag{2.2}$$

$$\frac{dT}{dt} = -(\gamma-1)T \vec{\nabla} \cdot \vec{V} + Q/c_V, \tag{2.3}$$

$$\frac{dp}{dt} = -\gamma p \vec{\nabla} \cdot \vec{V} + (\gamma-1) Q/\alpha, \tag{2.4}$$

$$p\alpha = RT, \tag{2.5}$$

where $\vec{r}$ is the position vector with respect to the Earth's center,

$\vec{V}$ is the velocity relative to rotating Earth,

p is the air pressure,

T is the air temperature,

α is the specific volume of air ($\alpha = 1/\rho$; where ρ is the density of air),

$\vec{\Omega}$ is the Earth's angular velocity,

$\vec{g}$ is the apparent acceleration of Earth's gravity,

c_V is the specific heat of air at constant volume ($J.kg^{-1}.K^{-1}$),

c_p is the specific heat of air at constant pressure ($J.kg^{-1}.K^{-1}$),

R is the gas constant for air ($R = c_p - c_V$),

$\gamma = c_p/c_V$ approximately 7/5,

$\vec{F}$ is the frictional force per unit mass,

Q is the rate change of heating per unit mass.

Note that the time derivatives in equations (2.1)-(2.4) are total time derivatives, referring to the rate of change at a point which moves with the flow.

Equations (2.1)-(2.5) are so-called the ***exact equations***, although even they contain a number of approximations. In a sense they are too exact. Examination reveals that they posses certain properties which render them somewhat awkward for a study of the *global circulation*.

Eq.(2.1) represents Newton's second law of motion. As written, it applies to a gas or a liquid. This equation is written for a frame of reference which rotates with angular velocity $\vec{\Omega}$. The absolute acceleration differs from the apparent acceleration by the Coriolis acceleration $2\vec{\Omega} \times \vec{V}$ and the centripetal acceleration $\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$. The rotation of the system is therefore fully taken into account by introducing the ***Coriolis acceleration*** $-2\vec{\Omega} \times \vec{V}$ and ***apparent gravity*** $\vec{g}$ which differs from the absolute gravity by $-\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$, and otherwise regarding the system as if it were not rotating. Once this has been accomplished, it is permissible for most purposes to treat the Earth (except for topographic features) as a sphere instead of an ellipsoid, with a gravitational force of constant magnitude directed toward the center.

Eq.(2.2) represents the continuity equation. As written, it applies equally well to a gas or a liquid. This equation describes the law of conservation of mass.

Eq.(2.3) represents the first law of thermodynamics, while Eq.(2.5) is the equation of state. As written, they apply to an ideal gas. Certain modifications are needed to then apply to an atmosphere where water can appear in different phases or in varying amounts. In formulating Eq.(2.3) we have noted that the internal energy per unit mass is c_vT, and we have used the customary assumption that the work done upon a unit mass in compressing it is given by $-p\frac{d\alpha}{dt}$; we will presently consider the implications of this assumption. We have then used Eqs.(2.2) and (2.5) to express the work as $-RT\, \vec{\nabla} \cdot \vec{V}$, after which Eq.(2.3) follows.

Eq.(2.4) may be derived from Eqs.(2.2) and (2.3) with the aid of Eq.(2.5). It is often more convenient to use the density ρ as a dependent variable in place of its specific volume α.

A. Components Equations in Spherical Coordinates

For practical reasons it is often desirable to express the Eq.(2.1) in scalar form. The Earth is sufficiently spherical in shape to justify the use of a spherical coordinate system. Because of the curvature of the spherical coordinate system, the components of the acceleration $\frac{d\vec{V}}{dt}$ are not the time derivatives of the components of $\vec{V}$. Additional terms involving the time derivatives of $\hat{i}, \hat{j}$ and $\hat{k}$ occur. Thus the equations of motion become

$$\frac{du}{dt} = \frac{\tan\phi}{r} uv - \frac{1}{r} uw + (2\Omega \sin\phi) v - (2\Omega \cos\phi) w - \frac{\alpha}{r \cos\phi} \frac{\partial p}{\partial \lambda} + F_\lambda , \quad (2.6)$$

$$\frac{dv}{dt} = -\frac{\tan\phi}{r} u^2 - \frac{1}{r} vw - (2\Omega \sin\phi) u - \frac{\alpha}{r} \frac{\partial p}{\partial \phi} + F_\phi , \quad (2.7)$$

$$\frac{dw}{dt} = \frac{1}{r} u^2 + \frac{1}{r} v^2 + (2\Omega \cos\phi) u - g - \alpha \frac{\partial p}{\partial z} + F_z , \quad (2.8)$$

where λ is the longitude measured eastward,

ϕ is the latitude measured northward,

z is the elevation measured upward,

r is the magnitude of $\vec{r}$,

u is the eastward component of $\vec{V}$,

v is the northward component of $\vec{V}$,

w is the upward component of $\vec{V}$,

a is the Earth's mean radius,

Ω is the magnitude of $\vec{\Omega}$,

g is the magnitude of $\vec{g}$,

F_λ, F_ϕ, F_z are the components of $\vec{F}$.

We have noted that the velocity components u, v and w are the scalar products of $\vec{V}$ with the unit vectors;

$$\hat{i} = \frac{(\vec{\Omega} \times \vec{r})}{|\vec{\Omega} \times \vec{r}|}, \quad \hat{j} = \hat{k} \times \hat{i}, \quad \text{and} \quad \hat{k} = \frac{\vec{r}}{r}.$$

B. The Potential Temperature and the Specific Entropy

It is often advantageous to write the equations in terms of the potential temperature

$$\Theta = T\left(\frac{p_0}{p}\right)^{\kappa}, \tag{2.9}$$

or the related specific entropy (of an ideal gas)

$$s = c_p ln\, \Theta, \tag{2.10}$$

where $\kappa = R/c_p$ is about 2/7 and the constant p_0 = 1000 mb has been introduced to make Θ and T dimensionally similar. It follows from Eq.(2.3) and (2.4) that

$$\frac{d\Theta}{dt} = \left(\frac{p_0}{p}\right)^{\kappa} \frac{Q}{c_p}, \tag{2.11}$$

$$\frac{ds}{dt} = \frac{Q}{T}, \tag{2.12}$$

Eq.(2.12) reveals the nature of the thermodynamic assumptions which occurs in the usual formulation of the governing equations. According to Eq.(2.12), the rate change of entropy equals the ratio of the rate change of heating to the temperature. It is a fundamental principle of thermodynamics that this is so during a reversible process, but not necessarily during on an irreversible process.

C. The Angular Momentum and Energy Equations

Equations (2.1) - (2.8), together with suitable expressions for $\vec{F}$ and Q, are principle sufficient for a mathematical study of the circulation. Qualitative arguments are nevertheless often more readily presented in terms of angular momentum and energy.

The absolute angular momentum (per unit mass) about the Earth's axes is given by the formula

$$M = \Omega r^2 cos^2\phi + r\, cos\, \phi\, u\,, \tag{2.13}$$

The first term on the right-hand side of Eq.(2.13) represents the *so-called Ω - momentum*, the absolute angular momentum which would be present if the atmosphere were in solid rotation with the Earth.

The second term is the relative angular momentum, associated with the motion relative to the Earth. The terms in Eq.(2.6) containing 1/r, depending upon the curvature of the coordinate system, and the terms containing Ω, depending upon the rotation, drop out in the angular-momentum equation

$$\frac{dM}{dt} = -\alpha \frac{dp}{d\lambda} + r\, cos\, \phi\, F_\lambda\,, \tag{2.14}$$

which state that the absolute angular momentum is altered only by a torque. An equivalent statement would be that relative angular momentum is altered only by a torque, provided the *Coriolis torque* is included.

Likewise, per unit mass, the kinetic energy, potential energy and internal energy (of an ideal gas) are given by

$$K = \frac{1}{2}\vec{V}\cdot\vec{V}, \quad \Phi = gz \quad \text{and} \quad I = c_V T\,,$$

where K is the kinetic energy per unit mass,

Φ is the potential energy per unit mass,

I is the internal energy per unit mass.

According to Eqs.(2.1) - (2.8), we can obtain the equation of total energy

$$\frac{d}{dt}\left(K + \Phi + I\right) = -\alpha \vec{\nabla} \cdot p\vec{V} + \vec{V} \cdot \vec{F} + Q \ , \qquad (2.15)$$

when integrated over any region with a fixed boundary, the term $-\alpha \vec{\nabla} \cdot p\vec{V}$ represents the work done on this region by the pressure force on the boundary; thus in general it describes a transfer of energy from one region to another.

The angular-momentum and energy principles are fundamental in any treatment of the circulation. If in some approximate formulation of the equations they are not retained, the results are likely to be unrealistic. A spurious energy source may cause the wind to increase without limit.

D. Basic Equations for the Real Atmosphere

For the real atmosphere the many equations governing radiation, turbulence, phase change of water, and other processes, are required. It is beyond the scope of this discussion to present all of the relevant equations. However, we will indicate the modifications of Eqs.(2.1) - (2.5) required by the presence of water.

The hydrodynamic equations (2.1) and (2.2) appear to remain virtually unaltered. In the equation of state (2.5), the gas constant R must be replaced by the slightly greater variable gas *"constant"* appropriate to a mixture of air and water. Thus Eq.(2.5) becomes

$$T_V = (1 - q)T + (R_w/R)\,qT \ , \qquad (2.16)$$

where R_w is the gas constant for water,

q is the specific humidity,

T_V is the virtual temperature of air.

Throughout much of the atmosphere T_V and T differ by less than 1°C, but near the surface of Earth the difference may exceed 4°C.

The more important effects of water vapour appear in the thermodynamic equations (2.3) and (2.4). The internal energy of dry air must be replaced by the internal energy of moist air, given by

$$I \;=\; c_V(1-q)T + (c-R_w)qT + Lq\,, \tag{2.17}$$

where c is the specific heat of water,

L is the latent heat of condensation at temperature T.

Alternatively, the release of latent heat, given approximately by $-L\frac{dq}{dt}$, may be included as part of the heating. In either event the specific humidity q must be included as an additional dependent variable.

A common simplification is the assumption that liquid water falls out immediately upon forming from condensation. In this case q may be considered to remain constant, except in ascending saturated air, where it retains its saturation value and near the Earth, where it may increase as a result of turbulent diffusion. Thus

$$\frac{dq}{dt} \;=\; \begin{cases} -\alpha\dfrac{\partial E}{\partial z} & if \;\; q < q_s \;\; or \;\; \dfrac{dq_s}{dt} \geq 0\,, \\ \dfrac{dq_s}{dt} & if \;\; q = q_s \;\; or \;\; \dfrac{dq_s}{dt} < 0\,, \end{cases} \tag{2.18}$$

where E is the upward turbulent transfer of water vapour per unit horizontal area,

$q_s(T, p)$ is the value of q which saturated air at temperature T and pressure p.

It would be more realistic to retain the liquid water content as another dependent variable, in which case q would retain its saturation value in descending air containing liquid water. If the solid water content is retained as still another variable, the possibility of supercooled water clouds in place of ice-crystal clouds must be recognized.

The Hydrostatic Equation

One of the most perminent features of the circulation is hydrostatic equilibrium which is the approximate balance between gravity and the vertical pressure gradient force. The familiar hydrostatic equation can be written as

$$\frac{\partial p}{\partial z} = -\rho g , \tag{2.19}$$

Since the hydrostatic equation is diagnostic, its introduction leaves the new system with no prognostic equation for *w*. There are two prognostic equations for p, namely the thermodynamic equation (2.4) and the hydrostatic equation obtained by integrating Eq.(2.19) with the upper boundary condition p = 0 at z = ∞. Thus

$$\frac{\partial p}{\partial t} = -g \int_z^{\infty} \vec{\nabla} \cdot \rho \vec{V} \, dz , \tag{2.20}$$

Elimination of $\frac{dp}{dt}$ and $\frac{\partial p}{\partial t}$ from Eqs.(2.4) and (2.20) yields an additional diagnostic equation, which may be solved for *w* in terms of the remaining variables using the lower boundary condition *w* = 0 at z = 0.

The Primitive Equations

Since the field of vertical motion is assumed to be that field required to maintain hydrostatic equilibrium by compensating for the effects of the horizontal motions. With *w* itself defined in terms of the other variables there is no need for an explicit expression for $\frac{dw}{dt}$, and with the aid of the diagnostic equations the system reduces to a closed system of three equations in the three dependent variables *u*, *v* and p.

We now present the new system of equations in two forms. The first form uses the coordinate system of Eqs.(2.6) - (2.8). In the second form we use the pressure p instead of elevation z as the vertical coordinate.

With z as the vertical coordinate, the new system may be written

$$\frac{d\vec{U}}{dt} = -f\hat{k} \times \vec{U} - \alpha\vec{\nabla}p + \vec{F}, \qquad (2.21)$$

$$\frac{d\vec{p}}{dt} = -\gamma p\,\vec{\nabla}\cdot\vec{U} - \gamma p\frac{\partial w}{\partial z} + (\gamma - 1)Q/\alpha, \qquad (2.22)$$

$$\gamma p\frac{\partial w}{\partial t} = -\gamma p\,\vec{\nabla}\cdot\vec{U} - \vec{U}\cdot\vec{\nabla}p + g\int_z^{\infty}\vec{\nabla}\cdot\rho\vec{V}\,dz + (\gamma - 1)Q/\alpha, \qquad (2.23)$$

$$\frac{1}{\alpha} = \frac{1}{g}\frac{\partial p}{\partial z}, \qquad (2.24)$$

where f is the Coriolis parameter ($f = 2\Omega\sin\phi$),

$\vec{U}$ is the horizontal velocity $\left(\vec{U} = u\hat{i} + v\hat{j}\right)$,

Here $\vec{\nabla}$ denotes a horizontal differential operator.Wherever 1/r would ordinarily occur it is to be replaced by 1/a, where a is the Earth's mean radius,; thus the components of the equation of motion become

$$\frac{du}{dt} = \frac{\tan\phi}{a}uv + fv - \frac{\alpha}{a\cos\phi}\frac{\partial p}{\partial\lambda} + F_\lambda, \qquad (2.25)$$

$$\frac{dv}{dt} = -\frac{\tan\phi}{a}u^2 - fu - \frac{\alpha}{a}\frac{\partial p}{\partial\phi} + F_\phi, \qquad (2.26)$$

The total and the local time derivatives of a scalar X are connected by the relation

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \vec{U}\cdot\vec{\nabla}X + w\frac{\partial X}{\partial z}, \qquad (2.27)$$

while the horizontal divergence is

$$\vec{\nabla} \cdot \vec{U} = \frac{1}{a \cos \phi} \left(\frac{\partial}{\partial \lambda} u + \frac{\partial}{\partial \phi} v \cos \varphi \right) , \tag{2.28}$$

with an analogous expression for $\vec{\nabla} \cdot \rho X \vec{U}$. An element of volume is assumed to be $a^2 \cos \phi \, d\lambda \, d\phi \, dz$.

For many purposes this new system is suitable. For other purposes it is far more convenient to introduce pressure p as a new vertical coordinate; thus p becomes an independent variable while z becomes a dependent variable, and $\omega = \frac{dp}{dt}$ replaces w as a further dependent variable. In this system the continuity equation becomes the diagnostic equation, and the complete system may be written

$$\frac{d\vec{U}}{dt} = -f \hat{k} \times \vec{U} - g\vec{\nabla} z + \vec{F} , \tag{2.29}$$

$$\frac{dT}{dt} = \kappa T \omega / p + Q / c_p , \tag{2.30}$$

$$\vec{\nabla} \cdot \vec{U} + \frac{\partial \omega}{\partial p} = 0 , \tag{2.31}$$

$$\frac{\partial z}{\partial p} = - \frac{RT}{gp} , \tag{2.32}$$

Likewise, the components of the equation of motion become

$$\frac{du}{dt} = \frac{\tan \phi}{a} uv + fv - \frac{g}{a \cos \phi} \frac{\partial z}{\partial \lambda} + F_\lambda , \tag{2.33}$$

$$\frac{dv}{dt} = - \frac{\tan \phi}{a} u^2 - fu - \frac{g}{a} \frac{\partial z}{\partial \phi} + F_\phi , \tag{2.34}$$

The total and the local time derivatives of a scalar X are connected by the relation

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \vec{U} \cdot \vec{\nabla} X + \omega \frac{\partial X}{\partial p} , \tag{2.35}$$

or with the aid of Eq.(2.31)

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \vec{\nabla}\cdot\left(X\,\vec{U}\right) + \frac{\partial}{\partial p}(\omega X)\ , \tag{2.36}$$

while the horizontal divergence is

$$\vec{\nabla}\cdot\vec{U} = \frac{1}{a\cos\phi}\left(\frac{\partial}{\partial\lambda}u + \frac{\partial}{\partial\phi}v\cos\varphi\right)\ , \tag{2.37}$$

with an analogous expression for $\vec{\nabla}\cdot\rho X\vec{U}$. It is understood that the partial derivatives $\partial/\partial t$, $\partial/\partial\lambda$, $\partial/\partial\phi$ and $\vec{\nabla}$ are now to be interpreted as derivatives with p held constant, so that their meaning is not the same as in equations (2.21)-(2.24). Formally Eq.(2.37) is identical with Eq.(2.28), but the partial derivatives have their altered meaning. An element of mass is assumed to be $(1/g)a^2\cos\phi\ d\lambda\ d\phi\ dp$.

For some purposes a satisfactory approximation is obtained by assuming as a lower boundary the coordinate surface $p = p_0 =$ constant, with $\omega = 0$ as a lower boundary condition. The height of the lower boundary is then considered variable. In particular this approximation does not introduce spurious sources of angular momentum and energy. It has the effect of eliminating the so-called *external gravity wave*, whose propagation involves oscillations of the total mass within a vertical column.

Equations (2.21) - (2.24) and their equivalent forms (2.29) - (2.32) are the so-called ***primitive equations***. This designation has arisen from their use in numerical weather prediction, where they have been taken as the starting point for the derivation of the simpler *geostrophic model* which we will presently consider. Apparently it was thought improbable that anyone would attempt to use the ***exact equations***, which are more primitive than the ***primitive equations***.

Vorticity and Divergence Equations

For many purposes it is advantageous to express the horizontal wind field $\vec{U}$ in terms of its vorticity ζ and its divergence δ :

$$\zeta = \vec{\nabla}\cdot\vec{U} \times \hat{k} , \quad (2.38)$$

$$\delta = \vec{\nabla}\cdot\vec{U} , \quad (2.39)$$

Here $\vec{\nabla}$ denotes differentiation along an isobaric surface, although the slightly different vorticity and divergence fields defined by the same formulas with $\vec{\nabla}$ denoting differentiation along a horizontal surface, have also been used.

If the stream function ψ and the velocity potential χ are defined by the equations

$$\nabla^2\psi = \zeta , \quad (2.40)$$

$$\nabla^2\chi = -\delta , \quad (2.41)$$

the rotational non-divergence wind fields $\vec{U}_r$ and the divergence irrotational wind field $\vec{U}_d$ defined

$$\vec{U}_r = \hat{k} \times \vec{\nabla}\psi , \quad (2.42)$$

$$\vec{U}_d = -\vec{\nabla}\chi , \quad (2.43)$$

satisfy the relation

$$\vec{U} = \vec{U}_r + \vec{U}_d . \quad (2.44)$$

It should be observed that in a circulation which is symmetric with respect to the Earth's axis, such as Hadley's circulation, the zonal motion u is completely determined by $\vec{U}_r$, while the meridional motion v is completely determined by $\vec{U}_d$. In the more general case, the eastward and northward motion are determined respectively by $\vec{U}_r$ and $\vec{U}_d$.

A form of Eq.(2.21) which is exactly equivalent but more convenient for many purposes is

$$\frac{\partial \vec{U}}{\partial t} = -(\zeta + f)\hat{k} \times \vec{U} - \omega \frac{\partial \vec{U}}{\partial p} - \vec{\nabla}\left(gz + \frac{U^2}{2}\right) + \vec{F} , \qquad (2.45)$$

From Eq.(2.44) we may derive the ***vorticity equation***

$$\frac{\partial \zeta}{\partial t} = -\vec{U} \cdot \vec{\nabla}(\zeta + f) - \omega \frac{\partial \zeta}{\partial z} - (\zeta + f)\delta - \vec{\nabla}\omega \cdot \frac{\partial \vec{U}}{\partial p} \times \hat{k} + \vec{\nabla} \cdot \vec{F} \times \hat{k} , \qquad (2.46)$$

and the ***divergence equation***

$$\frac{\partial \delta}{\partial t} = -\vec{U} \cdot \vec{\nabla}(\zeta + f) \times \hat{k} - \omega \frac{\partial \delta}{\partial z} - (\zeta + f)\zeta - \vec{\nabla}\omega \cdot \frac{\partial \vec{U}}{\partial p} + \nabla^2\left(gz + \frac{U^2}{2}\right) + \vec{\nabla} \cdot \vec{F} , \qquad (2.47)$$

In dealing with certain features of the circulation, rather than the total circulation, we may neglect the weaker field $\vec{U}_d$ and hence δ and ω together. The vorticity equation by itself then becomes a closed system, provided that the friction $\vec{F}$ can be expressed in terms of $\vec{U}_r$. If $\vec{F}$ also neglected, the vorticity equation reduces to

$$\frac{\partial \zeta}{\partial t} = -\vec{\nabla}\psi \cdot \vec{\nabla}(\zeta + f) \times \hat{k} , \qquad (2.48)$$

or equivalently,

$$\frac{d}{dt}(\zeta + f) = 0 , \qquad (2.49)$$

The sum ζ + f is the absolute vorticity. Eq.(2.49) expresses the conservation of absolute vorticity.

Eq.(2.48) contains neither sources nor sinks for absolute vorticity. It strictly conserves the total kinetic energy and also the total absolute angular momentum, at each level, and hence allows no conversion between the kinetic energy and other forms of energy. Inclusion of friction would merely lead to a dissipation of the kinetic energy. In dealing with the total circulation it is therefore necessary to retain the divergence.

The Geostrophic Equation

A feature of the circulation in middle and higher latitudes is geostrophic equilibrium which is the approximate balance between the Coriolis force and the horizontal pressure gradient force. The familiar geostrophic equation is obtained by

$$\vec{U}_g = (g/f)\hat{k} \times \vec{\nabla} z\,, \tag{2.50}$$

where $\vec{U}_g$ is the *geostrophic wind.*

However, since the wind $\vec{U}$ is expressible as the sum of $\vec{U}_r$ and a smaller residual $\vec{U}_d$, and also as the sum of $\vec{U}_g$ and the smaller residual $\vec{U} - \vec{U}_g$, it follows that $\vec{U}_r$ is the sum of $\vec{U}_g$ and a reasonably small residual $\left(\vec{U} - \vec{U}_g\right) - \vec{U}_d$. The geostrophic vorticity $\vec{\nabla}.\vec{U}_g \times \hat{k}$ is generally a fair approximation to the vorticity ζ, although it is a considerable overestimate in intense cyclones. The geostrophic divergence $\vec{\nabla} \cdot \vec{U}_g$ bears little resemblance to the divergence δ as observed in the atmosphere.

Further modifications are now needed to retain the energy principle. The kinetic energy must be redefined as $K = \frac{U_r^2}{2}$, and all the quadratic terms in the vorticity equation except those involving $\vec{U}_r$ only must be discarded. The vorticity equation and the thermodynamic equation then assume the form

$$\frac{\partial \zeta}{\partial t} = -\vec{\nabla}\psi \cdot \vec{\nabla}(\zeta + f) \times \hat{k} + \vec{\nabla}\cdot\left(f\vec{\nabla}\chi\right) + \vec{\nabla}\cdot\vec{F} \times \hat{k}\,, \tag{2.51}$$

$$\frac{\partial T}{\partial t} = -\vec{\nabla}\psi \cdot \vec{\nabla}T \times \hat{k} + \vec{\nabla}T \cdot \vec{\nabla}\chi + \sigma\omega + Q/c_p\,, \tag{2.52}$$

where $\sigma = -\left(\frac{\partial T}{\partial p} - \kappa(T/p)\right) \equiv$ a measure of the static stability.

The atmosphere is said to be statically stable or unstable according to wether Θ increases or decreases with elevation. Eqs.(2.51) and (2.52) describe the so-called ***geostrophic model***, used extensively in numerical weather prediction, usually with additional simplifications. Although it is convenient to be rid of most of the quadratic terms in the vorticity equation, this simplification is too extreme for studying many aspects of *the general circulation*.

The Beta Plane

The beta-plane approximation was first introduced by Rossby [Rossby 1939] in the previously cite paper. In this approximation, the spherical surface of the Earth is replaced by a plane in which rectangular Cartesian coordinates (x, y) are introduced. In Rossby's original work the plane was of infinite horizontal extent, but in many subsequent applications it has been restricted to the area between two parallel lines, which are identified with latitude circles. In the x-direction all dependent variables are commonly assumed to vary periodically, acquiring their original values after a distance which is identified with the circumference of the Earth. In the divergence equation, or in the geostrophic equation , the Coriolis parameter f assigned a constant value. It is also taken as a constant in the vorticity equation, except in the term $-\vec{\nabla}\psi \cdot \vec{\nabla} f \times \hat{k}$ where its northward derivative $\frac{\partial f}{\partial y}$ is assigned a second constant value β. Thus the term reduces to $-\beta \frac{\partial \psi}{\partial x}$.

The remaining awkward feature of the ω-equation result from the variability of σ and the presence of the term $\vec{\nabla} T \cdot \vec{\nabla} \chi$ in Eq.(2.52).

The latter term represents the advection of temperature by $\vec{U}_d$, and in practice it is usually discarded. The static stability σ is also frequently replaced by $\tilde{\sigma}$, where the tilde ($\tilde{}$) denotes an average over an isobaric surface. Both of these approximations upset the energy principle, but this may restored by adding a suitable term depending upon p and t alone in the thermodynamic equation. The system of equations may be written as

$$\frac{\partial \zeta}{\partial t} = -\vec{\nabla}\psi \cdot \vec{\nabla}\zeta \times \hat{k} - \beta\frac{\partial \psi}{\partial \chi} - f\delta + \vec{\nabla}\cdot\vec{F} \times \hat{k}, \qquad (2.53)$$

$$\frac{\partial T}{\partial t} = -\vec{\nabla}\psi \cdot \vec{\nabla}T \times \hat{k} + \tilde{\sigma}\omega + \kappa\left(\widetilde{\omega T}/p\right) + Q/c_p, \qquad (2.54)$$

$$f\frac{\partial \psi}{\partial p} = -RT/p, \qquad (2.55)$$

$$f\frac{\partial^2 \omega}{\partial p^2} + (R\tilde{\sigma}/fp)\nabla^2\omega = \frac{\partial}{\partial p}\left(\vec{\nabla}\psi \cdot \vec{\nabla}\zeta \times \hat{k}\right) - \nabla^2\left(\vec{\nabla}\psi \cdot \vec{\nabla}\frac{\partial \psi}{\partial p} \times \hat{k}\right)$$

$$+ \beta\frac{\partial^2 \psi}{\partial \chi \partial p} - \vec{\nabla}\frac{\partial \vec{F}}{\partial p} \times \hat{k} - (\kappa/fp)\nabla^2 Q. \qquad (2.56)$$

The term containing $\widetilde{\omega T}$ may be omitted in applications where time variations of $\tilde{T}$ are irrelevant. Usually the variations of $\tilde{\sigma}$ are also suppressed; if they are to be included, the appropriate equation is obtained from Eq.(2.54). The greatly simplified ω-equation is now seen to be an elliptic differential equation in ω, since σ is almost invariably positive. In applications involving specific features of the circulation, the terms containing $\vec{F}$ and Q are often omitted.

Much effort has been devoted to justifying the use of the beta-plane. The general conclusion is that it should yield qualitatively realistic results if its application is restricted to middle and higher latitudes. Certainly it has rendered some problems tractable when they could not otherwise have been handled by analytic procedures.