CHAPTER IV

THE LORENZ EQUATIONS

Sets of coupled nonlinear differential equations can also yield solutions that are examples of ***deterministic chaos***. The classic example is the ***Lorenz equations***. In this chapter we will describe and interpret the Lorenz system of nonlinear differential equations.

Introduction

In 1963 E.N. Lorenz wrote a remarkable paper. In it he described a three-parameter family of three-dimensional ordinary differential equations which appeared, when integrated numerically on a computer, to have extremely complicated solutions. These equations, now know as the ***Lorenz equations***, have been studied by many authors in the year since 1963.

Lorenz's search for a three-dimensional set of ordinary differential equations which would model some of the unpredictable behaviour which we normally associate with the weather [Lorenz 1963]. The equations which he eventually hit upon were derived from a model of fluid convection. They are

$$\begin{aligned} \frac{dX}{dt} &= \sigma(Y - X), \\ \frac{dY}{dt} &= rX - Y - XZ, \\ \frac{dZ}{dt} &= XY - bZ, \end{aligned} \tag{4.1}$$

where X is proportional to the intensity of convective motion,

Y is proportional to the temperature difference between ascending and descending currents,

Z is proportional to the distortion (from linear) of the vertical temperature temperature profile,

σ is the so-called the ***Prandtl number***,

r $= R_a/R_c$; where

R_a is the ***Rayleigh number***,

R_c is a ***critical number***,

and r is the so-called ***normalized Rayleigh number*** (For details see Appendix B),

b is a constant related to the given space.

Note also that we only consider positive values of σ, r, and b. Briefly, the original derivation [Lorenz 1963] can be described as follows. A two-dimensional fluid all is warmed from below and cooled from above and the resulting convective motion is modelled by partial differential equations (See Appendix B). The variables in these equations are expanded into an infinite number of modes, all but three of which are then set identically to zero. The three remaining modes give the Eqs.(4.1).

For wide ranges of values of parameters, approximate solutions to the Eqs. (4.1), calculated on a computer, look extremely complicated Fig. 4.1 show the projection onto the XZ-plane of one such solution calculated when $\sigma = 10$, $b = 8/3$ and $r = 28$. Note that the trajectory shown does not intersect itself if we consider the full three-dimensional picture (See also Fig. 4.2).

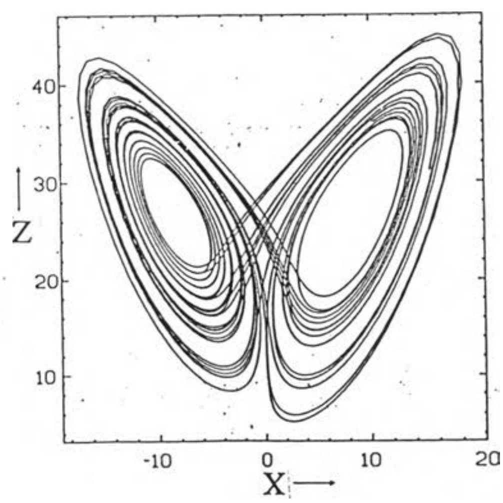


Figure 4.1 A numerically computed solution to the Lorenz equations projected onto the XZ - plane when $\sigma = 10$, $b = 8/3$ and $r = 28$. [Sparrow 1982]

We should note that the figure appears to have various 'turbulent' properties as follows:

1. The trajectory shown in Fig. 4.1 is not periodic.
2. The figure does not appear to shown a transient phenomenon. However long we continue the numerical integration the trajectory continues to wind around and around, first on one side, then on the other, without ever setting down to either periodic or stationary behaviour.
3. The general form of the figure does not depend at all on our choice of initial conditions or on our choice of integrating routine.
4. The exact sequence of loops which the trajectory makes is extremely sensitive to both changes in initial conditions and changes in the integrating routine. As a consequence of this, it is not possible to predict the detail of how the trajectory will develop over anything other than a very short time interval.

Lorenz's original paper [Lorenz 1963] was titled *"Deterministic Non-periodic Flows"*. Notice that the Lorenz equations are deterministic. They contain no random, noisy or stochastic terms and we know that they determine that a unique flow which is valid for all time. The suggestion, motivated partly by Lorenz's work, that complicated 'turbulent' behaviour in system with an infinite number of degrees of freedom (such as the atmosphere) might be modelled by simple deterministic finite-dimensional system (such as the Lorenz equations) is one of the reason why the Lorenz equations have attracted so much attention.

This question of the relationship between 'turbulent' behaviour in the real world and 'turbulent' behaviour in finite-dimensional systems is still unresolved. Consequently we should state at once that Lorenz himself admits [Lorenz 1979] that the Eqs.(4.1) are not the realistic model of his original fluid dynamical problem if the parameter r is far from unity. In addition, if we examine higher-dimensional problem then it seems that the type of behaviour observed in Fig. 4.1 does not always occur in the same way. However, these two observations do not imply that there is no relationship between infinite-dimensional and finite-dimensional 'turbulent', nor that the Lorenz equations are irrelevant to the debate about this relationship. Because of the amount of interest generated by the Lorenz equations, other authors have sought to discover, or stumble upon, other real world problems for which the Eqs.(4.1) are an accurate model when r is much larger than one. They have had some success.

Haken [Haken 1975] derives the Lorenz equations from a problem of irregular spiking in lasers, Malkus [Malkus 1972] and Yorke & Yorke [Yorke and Yorke 1978] both studied a problem of convection in a toriodal region, and Knobloch [Knobloch 1981] discusses a derivation from a disc dynamo. Malkus has constructed a laboratory water wheel which behave in a similar fashion to the Eqs.(4.1) and from whose equations of motion the Eqs.(4.1) can be derived [Lorenz 1963]. This derivation is described in Appendix B. Using asymptotic methods, Pedlosky [Pedlosky 1972] and Pedlosky & Frenzen [Pedlosky and Frenzen 1980] have derived the Lorenz equations from a study of the dynamics of a weakly unstable, finite amplitude, boroclinic wave (two-layer model), Brindley & Moroz [Brindley and Moroz 1980] obtain the equations in a similar problem (continuously stratified model), and Gibbon & McGuinness [Gibbon and McGuinness 1980] discuss both the two-layer baroclinic model and a laser problem.

Chaotic Ordinary Differential Equations

we now know of a great number of set s of ordinary differential equations derived, convincingly or not, from an even greater number of real world problems, which have solutions that look like Fig. 4.1. These equations are general called ***"Chaotic"***, as are their numerically calculated solutions. [Helleman 1980, Lichtenberg and Lieberman 1982]

When a system is bounded, as well as dissipative, we can deduce that all trajectories eventually tend towards some bounded set of zero volume lying in the phase space. Though there are technical distinctions which allow us to define this set in various different ways, we can state that we are especially interested in the bound set of zero volume called the ***non-wandering set.*** This set contains all the recurrent behaviour of the flow and we expect that all true trajectories will tend towards it. The non-wandering set may have several components. A component might be a stationary point, a periodic orbit, or some more complicate set of zero volume. If we know the structure of the non-wandering set, the way that the flow behaves on the non-wandering set, and the parts of the non-wandering set which are attracting, then we can sensibly claim that we know all the important things about our differential equations. If, for some practical application, we need to know how a particular trajectory with a particular set of initial conditions behaves, we can attempt to discover this by experiment. Eventually, though, the trajectory will move close to the non-wandering set and its behaviour will be governing by the motion of the flow on this set.

We can use the term "chaotic" to describe the numerical phenomenon seen in figures such as Fig. 4.1. We will not use expressions like ***"chaotic attractor"*** or ***"strange attractor"*** unless we have a special reason to believe that we really are seeing an approximate trajectory which lying close to a single (strange) attracting piece of the non-wandering set. In this way we will avoid confusing what we see with what we understand.

Approach to the Lorenz Equations

The actual techniques, numerical and mathematical, which we will use to study the Lorenz equations will be introduced. Our general aim will be to discover as much as possible about the behaviour of the system for a wide range of parameter values. In particular, we will hope to understand the many different kinds of chaotic behaviour that have been observed by other authors.

The notion of bifurcation is central to this approach. As we change the parameters, the behaviour of the flow will only change in an important way when the topology of the non-wandering set changes. Each time this occurs, we say there is a ***bifurcation***. Many bifurcations can be dealt with theoretically at a local level; our problem will be to fit them all together into a global picture. If we can build a global picture, however tentative, that allows us to explain the observed changes in behaviour via a theoretically acceptable sequence of bifurcations, our understanding of the behaviour at particular parameter values will be enhanced. If we cannot, then the problems involved in building the global picture may suggest to us where we should look to find as yet unobserved bifurcations.

It must be remembered that any global statements we make will be tentative. Non-linear systems of ordinary differential equations are not well understood, and the

Lorenz equations are no exception. There are two levels (at least) at which any global picture will be uncertain.

The first is the numerical and observational level. We can only ever do a finite number of numerical experiments at a finite number parameter values, each with only finite accuracy. It is always possible that, for some parameter values we have not examined, the behaviour is completely different, or that there are strange things going on in some region of phase space that we have not investigated. Whilst we will take as much care with our numerical experiments as seems appropriate, we can never completely dispel doubts of this kind. However, until numerical experiment indicate where we have gone wrong, the picture we have will be the best available description of the Lorenz equations. Providing the picture is self-consistent, it may remain of interest even if it eventually transpires that it is not an accurate description of the Lorenz system.

The second level of uncertainty is the theoretical one. Even if we assume that our numerical experiments are not misleading, our theoretical knowledge may not be adequate for us to be able to answer all the questions about the flow that we would like.

Simple Properties of the Lorenz Equations

We now examine some of properties of the Eqs.(4.1) [Lorenz 1963, Marsden 1977] :

A. Symmetry

The Lorenz equations, Eqs.(4.1), have a natural symmetry $(X, Y, Z) \rightarrow (-X, -Y, Z)$. This symmetry persists for all values of the parameters.

B. The Z-axis

The Z-axis, $X = Y = 0$, is invariant. All trajectories which start on the Z-axis remain on it and tend towards the origin (0, 0, 0) only if $b > 0$. Furthermore, all trajectories which rotate around the Z-axis do so in a clockwise direction when view from above the plane Z = 0. This follows from the fact that if X = 0 then $\frac{dX}{dt} > 0$ when Y > 0 and $\frac{dX}{dt} < 0$ when Y < 0. We can give a partial description of periodic orbits in the system by counting the number of times they wind around the Z-axis. This description will not change as we alter the the parameters, providing the same periodic orbit continues in existence.

C. Divergence of the Flow

The divergence of the flow, D;

$$D \equiv \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(b + \sigma + 1) \ , \tag{4.2}$$

which is negative since b and σ are positive. Denoting a typical element of phase-space volume by Γ(t), we thus have a contraction of the form

$$\Gamma(t) = \Gamma(0) \exp[-(b + \sigma + 1)t] \ , \tag{4.3}$$

Hence all trajectories will ultimately become confined to some form of limiting manifold of volume zero.

D. Critical Points

The points satisfying the condition $\dot{X} = \dot{Y} = \dot{Z} = 0$ are

1. X = Y = Z = 0, corresponding to the state of ***no convection***, that is, pure conduction.

2. $X = Y = \sqrt{b(r-1)}$, $Z = r - 1$ and $X = Y = -\sqrt{b(r-1)}$, $Z = r - 1$, corresponding to the state of *steady convection*. Note that the steady convective states only exist for r > 1.

E. Stability Properties

The linearized transformation near a critical point is the form

$$\frac{d}{dt}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix} = \begin{pmatrix}-\sigma & \sigma & 0\\ (\sigma - Z) & -1 & -X\\ Y & X & -b\end{pmatrix}\begin{pmatrix}\delta X\\ \delta Y\\ \delta Z\end{pmatrix}, \tag{4.4}$$

From which we can deduce the stability properties of the critical points.

1. (X, Y, Z) = (0, 0, 0) :

   For r < 1 this is stable, that is, all eigenvalues have negative real parts; for r < 1, one eigenvalue acquires a positive real part. The critical point is unstable and hence convection will start on infinitesimal perturbation. Note that the stability of the critical point depends only on the value of the (normalized) Rayleigh number.

2. (X, Y, Z) = $(\pm\sqrt{b\,(r-1)}, \pm\sqrt{b\,(r-1)}, r-1)$

   For r > 1, the eigenvalues consist of one real negative root and a pair of complex conjugate roots. This pair of critical points can be shown to become unstable if

   $$r = \frac{\sigma\,(\sigma + b + 3)}{(\sigma - b - 1)}$$

   a condition that can only be satisfied for positive r if $\sigma > b + 1$. In Lorenz's study [Lorenz 1963], Lorenz chose the parameter values b = 8/3 and $\sigma = 10$. With this choice, the steady (or convective) states become unstable at r = 470/19 ≅ 24.74..... and the contraction rate is D = -13.67, which is, in fact, extremely fast.

We now summarize what happens to the solutions of the Lorenz equations, Eqs.(4.1), as r is gradually increased for $b = 8/3$ and $\sigma = 10$.

(i) $0 < r < 1$:

The origin is a globally attracting stationary solution and all trajectories (i,e., all different initial conditions) eventually spiral into it.\

(ii) $1 < r < 24.74$:

The origin becomes unstable and bifurcates into a pair of locally attracting stationary solutions

$$C = (\sqrt{b\,(r - 1)},\ \sqrt{b\,(r - 1)},\ r - 1) \text{ and } C' = (-\sqrt{b\,(r - 1)},\ -\sqrt{b\,(r - 1)},\ r - 1)$$

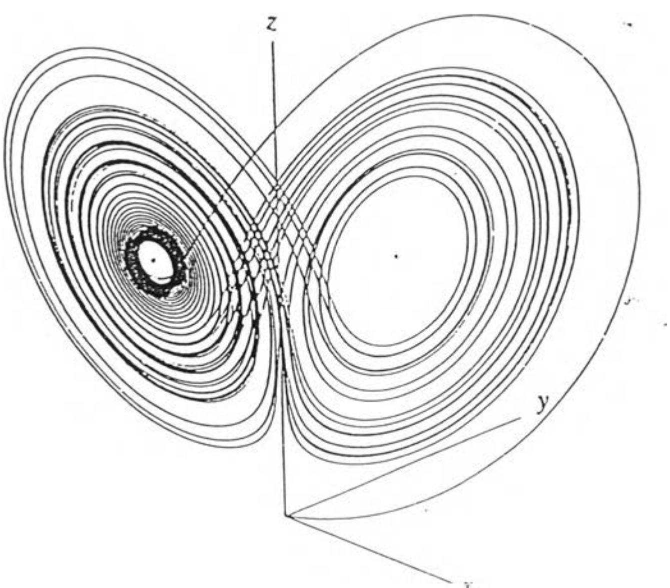
Virtually all trajectories converge to either C or C′. The exceptions are the set of trajectories that stay in the vicinity of the origin. At around r is about 13.926, the origin develops into a homoclinic point. Beyond this r value, the*"basins of attraction"* around C and C′ are no longer distinct and trajectories ca cross backward and forward between the two before settling down.

(iii) $r \cong 24.74$:

As mentioned, this is the critical value at which the steady states C and C′ become unstable. However, the Hopf analysis shows that as this critical values of r is passed, there is an inverted bifurcation; that is, the limit point C and C′ do not become stable limit cycles.

(iv) $r > 24.74$:

Trajectories integrated in this regime show remarkable behaviour. In Lorenz's original paper [Lorenz 1963], he studied the trajectory with the initial condition $(x, y, z) = (0, 1, 0)$ at a value of $r = 28$. For this value of r, the unstable steady states are $C = (6\sqrt{2}, 6\sqrt{2}, 27)$ and $C' = (-6\sqrt{2}, -6\sqrt{2}, 27)$. His computations showed that once certain transient oscillations had died away, the motion became ***highly erratic***. This is a result of the solution spiraling around one of the fixed points, C and C$'$, for some arbitrary period and then jumping to the vicinity of the other fixed point, spiraling around that for a while and then jumping back to the other, and so on. This combination of spiraling out and returning gives rise to the stretching and folding mechanism discussed earlier and results in a highly complex manifold, namely, a form of *strange attractor*. A typical orbit on this attractor is shown in Fig. 4.2. The apparent regularity of this structure in the figure is deceptive - the attractor is highly complex. The power spectrum of trajectory is essentially continuous, indicating highly ***chaotic motion***.



Lorenz attractor

Figure 4.2 Solution of the Lorenz equations computed at r = 28. The horizontal plane is at Z = 27. [Gulick 1992]