CHAPTER V

TYPE III SEMIRINGS

SECTION 5.1 BASIC THEOREMS

There are many examples of nontrivial finite type I and type II semirings. This is not the case for type III semirings.

5.1.1 Proposition. Let S be a type III semiring. Then the order of S is 1 or the order of S is infinite. Also, the subsemiring generated by 1 is $\{1\}$ or is isomorphic to Z^+.

Proof: Suppose that the order of S is not 1. Since S is congruence-free, S has a quotient division semiring QS. If the order of S is finite then the order of QS is also finite. As mentioned in Chapter I this implies that $QS = \{1\}$. Thus $S = \{1\}$. This contradiction shows that the order of S must be infinite.

Now suppose that $R \neq \{1\}$. If $R \neq Z^+$ then the order of R is finite. R is MC so R can be embedded in its division semiring QR. But then again if R is finite then QR is finite which is a contradiction. Therefore $R = Z^+$. #

For type I and type II semirings we have proved that every element which is not a multiplicative zero, has a multiplicative inverse. As the next theorem shows this fails dramatically for type III semirings

5.1.2 Theorem. There exist type III semirings other than $\{1\}$ in which no element other than 1 has a multiplicative inverse.

Proof: Let $S = \left\{\frac{3m}{2^n} \middle| n,m \in Z^+\right\} \cup \{1\}$. For $x,y \in S$ define $x+y = \min(x,y)$ Let S have the usual multiplication. S is clearly a commutative semiring with 1 and no multiplicative zero. Choose $x \neq 1 \in S$. Then $x^{-1} \notin S$ since $x = \frac{3m_1}{2^{n_1}}$ for some $n_1, m_1 \in Z^+$. So if $yx = 1$ then writing $y = \frac{3m}{2^n}$ we get that $\frac{3^2 mm_1}{2^{n+n_1}} = 1$ which is impossible. thus $x^{-1} \notin S$.

It remains to show that S is congruence-free. Let $\sim \neq \Delta$ be a congruence on S. Then there exist $x \neq y \in S$ such that $x \sim y$. Without loss of generality assume that $x < y$. By multiplying by a large enough $k \in S$ if necessary we can assume that $x > 1$. Now choose any a in the open interval (x,y). $x+a \sim y+a$. Thus $x \sim s$, for all $s \in (x,y)$. We write this as $x \sim (x,y)$. Now S is dense in $\mathbb{R}^+$. Thus we can choose $k > 1 \in S$ such that $kx \in (x,y)$. Thus $x \sim kx$. But $kx \sim ky$. Thus $x \sim ky$ so by applying the arguement above one more time we get that

$x \sim (x,ky)$. Thus by induction for all $n \in Z^+$ we obtain $x \sim (x,k^n y)$. Thus for all $s \in S$ such that $s > x$, $s \sim x$. We write this as $x \sim (x,\infty)$. Now choose any $l \in S$ such that $l < x$. We can choose an $n \in Z^+$ so large that $\frac{3}{2^n}.x < l$. By the argument above showing that $x \sim (x,y)$ we see that $\frac{3}{2}n\ x \sim (\frac{3}{2}n\ x, \infty)$.

In particular $\frac{3}{2^n}x \sim x$, and $\frac{3}{2^n}x \sim l$. Thus $l \sim x$. Therefore, $\sim = S \times S$, so S is congruence-free . #

The next series of results describes the additive structure of type III semirings.

<u>5.1.3 Lemma</u>. Let S be a type III semiring. Then either S is AC or for all $x,y \in S$ there exists an $a \in S$ (a dependent on x and y) such that $x+a = y+a$.

Proof: Define a relation on S as follows. For $x,y \in S$ say that $x \sim y$ iff there exists a $z \in S$ such that $x+z = y+z$. Clearly $x \sim x$ for all $x \in S$ and $x \sim y$ implies that $y \sim x$ for all $x,y \in S$. Suppose $x \sim y$ and $y \sim z$. Then there exist $a,b \in S$ such that $x+a = y+a$ and $y+b = z+b$. Thus $x+a+b = y+a+b = z+a+b$, so $x \sim z$. Thus $\sim$ is an equivalence relation. Suppose that $x \sim y$. Then there exists an $a \in S$ such that $x+a = y+a$. Thus for all $k \in S$, $(x+k)+a = (y+k)+a$. Thus $x+k \sim y+k$. Similarly, $k(x+a) =$

$k(y+a)$ so $kx+ka = ky+ka$. Thus $kx \sim ky$, so we have that $\sim$ is a congruence on S. Thus $\sim = \Delta$ or $S\times S$. If $\sim = \Delta$ then S is AC. If $\sim = S\times S$ then for each $x,y \in S$ there exists an $a \in S$ such that $y+a = x+a$. #

5.1.4 Lemma. Let D be a division semiring. For $x \in D$ let $nc(x) = \{(a,b) \in (D\times D)\setminus\Delta \mid x+a = x+b\}$. Say that $x \sim y$ iff $nc(x) = nc(y)$. Then $\sim$ is a congruence on S.

Proof: Clearly $\sim$ is an equivalence relation. Suppose that $x \sim y$. Let $(b_1, b_2) \in nc(x+a)$. Then $x+a+b_1 = x+a+b_2$ where $b_1 \neq b_2$. Suppose that $a+b_1 = a+b_2$. Then $y+a+b_1 = y+a+b_2$ so $(b_1,b_2) \in nc(y+a)$. Suppose that $a+b_1 \neq a+b_2$. Then $(a+b_1,a+b_2) \in nc(x)$. Thus $(a+b_1,a+b_2) \in nc(y)$ so $(b_1,b_2) \in nc(y+a)$. So in either case $nc(x+a) \subseteq nc(y+a)$. Similarly, $nc(y+a) \subseteq nc(x+a)$ so we have that $nc(x+a) = nc(y+a)$ or in other words that $x+a \sim y+a$. Now suppose that $(b_1,b_2) \in nc(ax)$. Then $ax+b_1 = ax+b_2$, so $x+\frac{b_1}{a} = x+\frac{b_2}{a}$. $b_1 \neq b_2$ so $\frac{b_1}{a} \neq \frac{b_2}{a}$. thus $(\frac{b_1}{a},\frac{b_2}{a}) \in nc(x)$. Thus $(\frac{b_1}{a},\frac{b_2}{a}) \in nc(y)$, so $\frac{b_1}{a}+y = \frac{b_2}{a}+y$. Thus $b_1+ay = b_2+ay$. Thus $(b_1,b_2) \in nc(ay)$ and $nc(ay) \subseteq nc(ax)$. So as above, $nc(ay)=nc(ax)$ and we have that $\sim$ is a congruence on D. #

5.1.5 Theorem. Let S be a type III semiring where $\|S\| > 1$. Then either $1+1=1$ and S is not AC or S is AC (and thus $1+1 \neq 1$).

Thus either S is AC or (S,+) is a band.

Proof: Consider QS, the quotient division semiring of S. QS is congruence-free. Define $\sim$ as in Lemma 5.1.4. By this Lemma $\sim$ is a congruence on QS. Therefore $\sim = \Delta$ or SxS. Let R denote the subsemiring generated by 1 in QS. We distinguish two cases:

Case A: $\sim = \Delta$. Thus $nc(x) \neq nc(y)$ for all $x,y \in QS$. thus QS (and S considered as a subsemiring of QS) is not AC since if it were AC $nc(s) = \emptyset$ for all $s \in QS$. Now suppose that $1+1 \neq 1$. Denote 1+1 as 2. We have that $nc(1) \neq nc(2)$. But clearly $nc(1) \subseteq nc(2)$. Thus $nc(1) \subset nc(2)$, so there exists $(x,y) \in nc(2)$ such that $x+2 = y+2$ but $x+1 \neq y+1$. Thus $\frac{x}{2} + 1 = \frac{y}{2} + 1$ so $\frac{y}{2}+\frac{x}{2}+1=\frac{y}{2}+\frac{y}{2}+1$

i.e. $\frac{x+y}{2}+1=y+1$. Reversing the role of x and y we get that $\frac{x+y}{2}+1 = x+1$. Thus $x+1=y+1$ which is a contradiction. Thus $1 \neq 1+1$ is impossible so $1+1=1$. Thus $x+x = x$ for all x in S.

Case B: $\sim = QSxQS$. Suppose that S is not AC. Then QS is not AC. Since $\sim = QSxQS$ if $x \neq y$ and there exists a $z \in QS$ such that $z+x=z+y$ then by applying Lemma 5.1.3 to QS we see that for all $x,y,z \in QS$, $x+y=x+z$. In particular, $(1+1)=1 \cdot (1+1)$. Thus R (the subsemiring generated by 1) is not isomorphic to Z^+. Thus R= 1 and (S,+) is a band. (But in fact this too is impossible since by the

the argument above $x+x=x+z$ for all $x,z \in QS$. Thus $(1+1)x= x+z$ so $1x=x+z$, i.e. $x=x+z$. Choose $x_1 \neq x_2 \in QS$. Then $x_1=x_1+x_2=x_2$ which is a contradiction.) #

As we have already proven that a non-AC type III semiring may fail to possess multiplicative inverses (the basic question of this thesis) from now on we shall primarily consider AC type III semirings. Let S be an AC type III semiring. Then S can be embedded in a difference ring. We have already shown that the quotient division semiring of S is congruence-free. As it happens the same result is true for the difference ring.

<u>5.1.6 Theorem.</u> Let S be an AC type III semiring. Then the difference ring of S is a field.

Proof: Let DS be the difference ring of S. Let $\sim$ be a nontrivial congruence on DS. Define a relation $\sim'$ on S by saying that $x \sim' y$ iff $x-y \sim 0$ in DS. Clearly $x \sim' x$ and $x \sim' y$ implies $y \sim' x$ for all $x,y \in S$. Suppose that $x \sim' y$ and $y \sim' z$. Then $x-y \sim 0$ and $y-z \sim 0$ so $z-y \sim 0$. Thus $x-y \sim z-y$ so $(x-y)-(z-y) \sim 0$. Thus $x-z \sim 0$ so $x \sim' z$. Thus $\sim'$ is an equivalence relation on S. But suppose $a \in S$ and $x \sim' y$. Then $x-y \sim 0$ so $(x+a)-(y+a) \sim 0$. Thus $x+a \sim' y+a$. Similarly considering a as an element in DS, $a(x-y) \sim 0$ so $ax-ay \sim 0$. Thus $ax \sim' ay$. Thus $\sim'$ is a congruence relation on S. But $\sim$ is nontrivial. Thus there exists $(x,y) \in S \times S$ such that

$x-y \not\sim' 0$. Thus $x \not\sim' y$ so $\sim' \neq S \times S$. Since S is congruence-free $\sim' = \Delta$. But since $\sim \neq \Delta$, there exist $x,y,z,d \in S$ such that $x-y \neq z-d$ in DS and $x-y \sim z-d$. Thus $(x+d)-(y+z) \sim 0$. Thus $x+d \sim' y+z$ in S. But $x+d \neq y+z$. Thus $\sim' \neq \Delta$ which is a contradiction. Thus DS is a congruence-free commutative ring with 1, i.e. DS is a field. #

Thus every AC type III semiring can be embedded in a field. The converse of the theorem above is false. $S=\{x \in Q^+$ such that $x \geq 1\}$ with the usual addition and multiplication is not congruence-free since $\sim = \{x \geq 2$ such that $x \in S\} \times \{x \geq 2$ such that $x \in S\} \cup \Delta$ is a congruence on S. But DS=Q.

Definition: Let S be a commutative semiring with 1. Then S is said to be precise iff for all $x,y \in S$, $1+xy=x+y$ implies $x=1$ or $y=1$.

5.1.7 Proposition. Every AC type III semiring is precise.

Proof: Let S be an AC type III semiring. Then S can be embedded in its difference ring DS which is a field by Theorem 5.1.6. Suppose $1+xy=x+y$. Then considering 1, x and y as elements in DS we get that $x(y-1)=y-1$. Thus $x=1$ or $y=1$. #

The converse of this proposition is false. There exist precise AC division semirings which are not congruence free.

For example let $S=\mathbb{R}^+(X)$, the division semiring formed by nonzero polynomials with coefficients in $\mathbb{R}_0^+$. Give S the usual addition and multiplication. $DS=R(X)$ which is a field. However, S is not congruence free as can be shown by the following rather complex relation . For $\frac{f_1(x)}{f_2(x)}, \frac{g_1(x)}{g_2(x)} \in \mathbb{R}^+(x)$ say that

$\frac{f_1(x)}{f_2(x)} \sim \frac{g_1(x)}{g_2(x)}$ iff $\deg f_1 - \deg f_2 = \deg g_1 - \deg g_2$.

$\sim$ is clearly well defined, symmetric, reflexive and transitive. Claim that $\sim$ is a congruence relation. Suppose $\frac{f_1(x)}{f_2(x)} \sim \frac{g_1(x)}{g_2(x)}$ and let $\frac{h_1(x)}{h_2(x)} \in \mathbb{R}^+(X)$.

$\deg h_1f_1 - \deg h_2f_2 = \deg h_1 + \deg f_1 - \deg h_2 - \deg f_2 =$
$\deg h_1 + \deg g_1 - \deg h_2 - \deg g_2 = \deg h_1g_1 - \deg h_2g_2 =$
$\deg \frac{h_1g_1}{h_2g_2}$. Thus $\frac{h_1(x)}{h_2(x)} \cdot \frac{g_1(x)}{g_2(x)} \sim \frac{h_2(x)}{h_2(x)} \cdot \frac{f_1(x)}{f_2(x)}$

Now $\deg\left(\frac{h_1}{h_2} + \frac{f_1}{f_2} \right) = \deg\left(\frac{h_1f_2 + h_2f_1}{h_2f_2} \right) =$

$\max(\deg h_1f_2, \deg h_2f_1) - \deg h_2g_2$. Similarly

$\deg\left(\frac{h_1}{h_2} + \frac{g_1}{g_2} \right) = \max(\deg h_1g_2, g_1h_2) - \deg(h_2g_2)$ (1).

Now suppose that deg $h_1 f_2 \geq$ deg $h_2 f_1$. Then deg h_1 + deg $f_2 \geq$ def h_2 + deg f_1 (2). Then $\deg(\frac{h_1}{h_2} + \frac{f_1}{f_2}) =$ deg h_1 + deg f_2 - deg f_2 - deg h_2 = deg h_1 - deg h_2. by (2) deg h_1 - deg $h_2 \geq$ deg g_1 + deg h_2. Thus deg h_1 + deg $g_2 \geq$ deg g_1 - deg g_2. Thus deg $h_1 g_2 \geq$ deg $g_1 h_2$ so by (1) we get that $\deg(\frac{h_1}{h_2} + \frac{g_1}{g_2}) =$ deg h_1 + deg g_2 - deg h_2 - deg g_2 = deg h_1 - deg h_2= $\deg(\frac{h_1}{h_2} + \frac{f_1}{f_2})$. Thus $\frac{h_1}{h_2} + \frac{f_1}{f_2} \sim \frac{h_1}{h_2} + \frac{g_1}{g_2}$. Suppose that deg $f_1 h_2 \geq$ deg $h_1 f_2$ (3) . Then by the above argument $\deg(\frac{h_1}{h_2} + \frac{f_1}{f_2})$ = deg f_1+ deg h_2- deg h_2- deg f_2= deg f_1- deg f_2 = deg g_1 - deg g_2. But by (3) deg f_1 + deg $h_2 \geq$ deg f_2+ deg h_1. Thus deg f_1- deg $f_2 \geq$ deg h_1- deg h_2 so deg g_1 - deg $g_2 \geq$ deg h_1- deg h_2 i.e. deg $g_1 h_2 \geq$ deg $h_1 g_2$. Thus from (1) $\deg(\frac{h_1}{h_2} + \frac{g_1}{g_2})$ = deg g_1- deg g_2 so again $\frac{h_1}{h_2} + \frac{f_1}{f_2} \sim \frac{h_1}{h_2} + \frac{g_1}{g_2}$. Thus $\sim$ is indeed a nontrivial congruence on $\mathbb{R}^+(X)$ so $\mathbb{R}^+(X)$ is not congruence-free.

SECTION 5.2 PARTIAL ORDERS ON TYPE III SEMIRINGS

In this section we investigate partial orders on type III semirings.

There exist nontrivial MC commutative semirings which have an additive identity. For example consider $(S,+,.)$ where $S=\{x \in \mathbb{R} \mid x \geq 1\}$, multiplication is as usual and for any $x,y \in S$, $x+y = \max(x,y)$. Then S is an MC commutative semiring with 1 and 1 is the additive identitiy in S. However, we shall now show that no nontrivial type III semiring S (i.e. $S \neq \{1\}$) has an additive identity. Suppose that S is AC and $\alpha \in S$ is an additive identity. Then $\alpha+\alpha=\alpha$ so $(1+1)\alpha = 1\alpha$ i.e. $1+1=1$ which contradicts Theorem 5.1.6. If S isn't AC, $S \neq \{1\} \Rightarrow \|S\| = \infty$ Since [illegible] is AC this contradicts Theorem 2.1.6. We have proved the following theorem.

<u>5.2.2 Theorem.</u> Let S be a type III semiring of order greater than 1. Then S has no additive identity.

<u>Definition:</u> Let S be a semiring which has no additive identity but with a multiplicative identity and which has a partial order $\geq$. We say that $\geq$ is compatible iff for all $x,y,a \in S$, $x \geq y \Rightarrow ax \geq ay$, $xa \geq ya$, $x+a \geq y+a$ and $a+x \geq a+y$. Additionally, $1+1 \geq 1$.

For example the usual ordering on Z^+ is compatible. We require that S not have an additive identity to avoid conflict with the usual definition of compatible partial orders on rings. By Theorem 4.2.2 "=" or the trivial partial order is compatible for every non AC type III semiring.

If S is type III and non AC then 1+1=1 so the stipulation that $1 + 1 \geq 1$ is redundant. If S is AC and type III and $\geq$ is a compatible partial order on S then $1+1\geq 1$ implies that for all $n \in Z^+$, and for all $a \in S$, $na \geq a$.

<u>52.3 Theorem.</u> Let S be an AC type III semiring and $\geq$ a compatible partial order on S. Assume also that $\|S\| > 1$. Then S has no minimal or maximal elements with respect to $\geq$.

Proof: Let K be the set of all minimal elements in S. $K \neq S$ since $\geq$ cannot be the trivial partial order = since $1+1\neq 1$ but $1+1\geq 1$. Claim that $S \setminus K$ is a double ideal. Suppose that $s \in S \setminus K$. Then there exists a $k \in S$ such that $s>k$. Therefore $as>ak$ for all $a \in S$ ($ak\neq as$ since S is MC). Thus $as \notin K$ so $as \in S\setminus K$. Similarly $a+s>a+k$ ($a+s\neq a+k$ since S is AC). so $a+s \in S \setminus K$. Thus $S \setminus K$ is a double ideal. Thus by Proposition 2.2.1 $S \setminus K = S$ i.e. $K=\emptyset$. #

Notation: Let S be an AC type III semiring. We can define a natural partial order $\geq_+$ on S which is compatible by saying that for $x,y \in S$, $x \geq_+ y$ iff $x=y$ or there exists an $a \in S$ such that $x=y+a$. First we verify that $\geq_+$ is indeed a partial order. Clearly $x \geq_+ x$ for all $x \in S$. Suppose that $x \geq_+ y$ and $y \geq_+ x$. Then either $x=y$ or there exist $a_1, a_2 \in S$ such that $x=y+a_1$ and $y=x+a_2$. Thus $y=y+a_1+a_2$. Thus $y+a_1+a_2=y+2(a_1+a_2)$. Since S is AC, $2(a_1+a_2)=a_1+a_2$. But S is also MC so 2=1 which contradicts Theorem 5.1.6.

Thus $x=y$. Now suppose $x>_+y$ and $y\geq_+z$. Since the case where $x=y$ or $y=z$ is trivial suppose $x>_+y$ and $y>_+z$. Then there exist $a_1, a_2 \in S$ such that $x= +a_1$ and $y=z+a_2$. Thus $x = z+a_1+a_2$. so $x\geq_+ z$ and $\geq_+$ is transitive. Thus $\geq_+$ is a partial order on S.

To show that $\geq_+$ is compatible, suppose that $x>_+y$. Then there exists an $a \in S$ such that $x=y+a$. Thus for all $k \in S$, $x+k=y+k+a$, so $x+k>_+y+k$. Similarly $kx=k(y+a)= ky+ka$ so $kx>_+ky$. $1+1\geq_+1$ by definition. Thus $\geq_+$ is compatible. Now a compatible partial order is just a congruence without symmetry so it is not surprising that the study of $\geq_+$ yeilds interesting results in the theory of congruence-free AC semirings. We call $\geq_+$ the natural additive partial order on S.

<u>5.2.4 Theorem</u>. Suppose that S is an AC type III semiring which is totally ordered by $\geq_+$. Then S is a division semiring.

Proof: Consider S as a subsemiring ofits difference ring DS. We have already proven that DS is a field. Choose $x \in S$. then $1/x \in DS$. But $1/x = a - b$ for some $a,b \in S$. However, since $\geq_+$ is total either $a>_+b$ or $b>_+a$ (clearly $a\neq b$). Suppose that $a>_+b$. Then there exists an $s \in S$ such that $a=b+s$. Thus $1/x = s$, so $1/x \in S$. Claim that $b>_+a$ is impossible. Suppose that $b>_+a$. Then there exists $s \in S$ such that $b=a+s$ But then $1 = x(a-b) =$

$x(-s)$. Thus $1 + xs = x(-s) + xs = x(s-s) = 0$ so $0 \in S$ which is impossible. Thus $b >_+ a$ is impossible and $1/x \in S$. Since $x \in S$ was arbitrary, S is a division semiring. #

When $\geq_+$ is total $DS = S \cup -S \cup \{0\}$ where $-S = \{-1s \mid s \in S \subseteq DS\}$ where S is considered as a subset of DS. This is true since for all $x, y \in S$ $x \geq_+ y$ or $y \geq_+ x$ so there exists an a_1 or an $a_2 \in S$ such that $x + a_1 = y$ or $y + a_2 = x$. Thus $x - y = a_1$ or $y - x = a_2$. After the next few results we can prove a partial converse to Theorem 5.1.7. First we introduce the concept of Archimedean semirings.

<u>5.2.2 Definition</u>. Let S be a semiring without an additive identity with a compatible partial order $\geq$. Then S is said to be <u>Archimedean</u> iff for all $x, y \in S$ there exists an $n \in \mathbb{Z}^+$ such that $nx \not\leq y$.

For example $\mathbb{Z}^+$ with the usual order is Archimedean. Thus the definition of Archimedean in the context of semirings without an additive identity is the analogue of its classical definition for rings except that no consideration of negative integers or zero is necessary (or possible). Note that this concept retains virtually no meaning in the case of non AC type three semirings since they are bands with respect to addition. In fact, if we were to apply the definition above to non AC type III semirings then the only Archimedean partial order would be the trivial partial order. It turns out, that each AC type III semiring is Archimedean in a nice way.

<u>5.2.5 Theorem</u>. Let S be an AC type III semiring and $\geq$ a compatible order on S. Then given $x,y \in S$ there exists an $n \in Z^+$ such that $nx \geq y$.

Proof: Define a relation $\sim$ on S by saying that for $x,y \in S$, $x \sim y$ iff there exist $n_1, n_2 \in Z^+$ such that $n_1 x \geq y$ and $n_2 y \geq x$. Clearly for all $x,y \in S$ $x \sim x$ and $x \sim y$ implies $y \sim x$. To show transitivity suppose that $x \sim y$ and $y \sim z$. Then there exist $n_1, n_2, n_3, n_4 \in Z^+$ such that $n_1 x \geq y$, $n_2 y \geq x$, $n_3 y \geq z$ and $n_4 z \geq y$. Thus $n_3 n_1 x \geq n_3 y \geq z$ and $n_2 n_4 z \geq n_2 y \geq x$. Thus $x \sim z$ so $\sim$ is transitive. Now suppose that $x \sim y$. As above there exist $n_1, n_2 \in Z^+$ such that $n_1 x \geq y$ and $n_2 y \geq x$. Choose $a \in S$. Then $n_1 x + n_1 a \geq y + n_1 a \geq y + a$. Thus $n_1(x+a) \geq y+a$. Similarly $n_2(y+a) \geq x+a$. Thus $y+a \sim x+a$. Clearly $n_1 xa \geq ya$ and $n_2 ya \geq xa$ so $ax \sim ay$. Thus $\sim$ is a congruence on S so $\sim = \Delta$ or $\sim = S \times S$. But given $x \in S$, $1(2x) \geq x$ and $4x \geq 2x$. Thus $\sim = S \times S$ so given $x,y \in S$ there exists an $n \in Z^+$ such that $nx \geq y$.

EXAMPLE: Let $S = (a_n)_{n=1}^{\infty}$, $a_n \in \mathbb{R}^+$ for all $n \in Z^+$. Define $(a_n)_{n=1}^{\infty} + (b_n)_{n=1}^{\infty} = (a_n + b_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty} \cdot (b_n)_{n=1}^{\infty} = (a_n b_n)_{n=1}^{\infty}$. Then $(S,+,.)$ is an AC, MC commutative semiring with 1 . Say that $(a_n)_{n=1}^{\infty} \geq_1 (b_n)_{n=1}^{\infty}$ iff there exists an $N \in Z^+$ such that for all $n \geq N$, $a_n \geq b_n$. ($\geq$ is the usual order on on $\mathbb{R}^+$). Then $\geq_1'$ is a compatible partial order on S but for any $k \in \mathbb{Z}^+$ $(k)_{n=1}^{\infty}(1)_{n=1}^{\infty} \leq_1 (n)_{n=1}^{\infty}$. Thus S is not Archimedean and thus not congruence-free.

Next we prove a partial converse to Theorem 5.1.7.

5.2.6 Theorem. Let S be an AC division semiring which is Archimedean with respect to the total order $\geq_+$. Then S is congruence-free

Proof: Let $\sim$ be a congruence on S which is not Δ. Let $C = \{ s \in S \mid s \sim 1\}$. There exist $x \neq y \in S$ such that $x \sim y$. Thus $x/y \in C$. Since $x/y \sim 1$ implies $y/x \sim 1$ also, we get that $y/x \in C$. Now if $x/y <_+ 1$ then $1 <_+ y/x$. so we can assume that there exists an $a \in C$ such that $a >_+ 1$. Now for all $z \in S$ and $x \in C$, $1 + z \sim x + z$. Thus $(x+z)/(1+z) \sim 1$ so $(x+z/(1+z) \in C$. Suppose $y \in S$ and $1 <_+ y <_+ a$. Then there exist $b,d \in S$ such that $1+d=y$ and $y+b=a$. Set $z=b/d \in S$. Then $(z+a)/(z+1)=y$. Thus $y \in C$. Thus for all $y \in S$ such that $a \geq_+ y \geq_+ 1$, $y \in C$. We write this as $1 \sim [1,a]$ .But $1 \sim a$ implies that $a \sim a^2$ so $1 \sim a^2$. Thus $1 \sim [1,a^2]$ and by induction for arbitrary $n \in Z^+$, $1 \sim [1,a^n]$.

Now $a=1+b+d$. Thus $a^n=(1+b+d)^n \geq_+ 1 + n(b+d)$. Since S is Archimedean and totally ordered for each $m \geq_+ 1$ we can choose an $n \in Z^+$ such that $n(b+d) \geq_+ m$. Thus C contains $\{x \in S \mid x \geq_+ 1\}$.

Now suppose that $y <_+ 1$. Then $1/y >_+ 1$. Thus $1/y \in C$, so $y \in C$. Thus for all $s \in S$, $s \in C$. Thus $\sim$ is $S\times S$. Since $\sim \neq \Delta$ was an arbitrary congruence on S, S is congruence-free. (Note: This proof is a generalization of the proof that $\mathbb{Q}^+$ is congruence-free on page 127 of ref. 2.)

Let S be an AC commutative semiring with 1 which has no additive identity. Then $\geq_+$ is well defined on S, i.e. $\geq_+$ is a compatible partial order on S. DS, the difference ring of S, is a ring. We define a partial order $\geq$ on DS by saying that for $x,y,a,b \in S$, $x - y \geq a - b$ iff $x+b \geq_+ a+y$. It is easy to verify that $\geq$ is a well-defined partial order on DS. Claim that $\geq$ is compatible as a partial order on a ring.

We must show that: 1) $\alpha \geq \beta$ implies $\alpha + \gamma \geq \beta + \gamma$ for all α, β and $\gamma \in DS$

and 2) $\alpha \geq 0$ and $\beta \geq 0$ imply that $\alpha\beta \geq 0$ for all $\alpha, \beta \in DS$.

To prove 1) suppose $\alpha \geq \beta$. Write $\alpha = x_1 - y_1$, $\beta = x_2 - y_2$ and $\gamma = x_3 - y_3$ where x_1, x_2, x_3, y_1, y_2, and $y_3 \in S$. Then $x_1+y_2+x_3+y_3 \geq_+ x_2+y_1+x_3+y_3$. Thus $(x_1-y_1)+(x_3-y_3) \geq (x_2-y_1)+(x_3-y_3)$ so $\alpha + \gamma \geq \beta + \gamma$ . .

To prove 2) using the notation above assume that $\alpha > 0$ and $\beta > 0$. Thus for any $s \in S$, $s-s = 0$ in DS so $x_1+s \geq_+ y_1+s$ and $x_2+s \geq_+ y_2+s$. Thus $x_1 \geq_+ y_1$ and $x_2 \geq_+ y_2$. Therefore there exist $a_1, a_2 \in S$ such that

$x_1 = y_1 + a_1$ and $x_2 = y_2 + a_2$. We get that $x_1a_2 \geq_+ y_1a_2$. Thus choosing an $s \in S$, we have $x_1a_2 + s \geq_+ s + y_1a_2$. Thus $(x_1a_2 - y_1a_2) \geq s - s = 0$. Thus $(x_1 - y_1)a_2 \geq 0$ considering x_1, y_1, and a_2 as elements in DS. Thus $(x_1 - y_1)(x_2 - y_2) \geq 0$ since in DS $a_2 = x_2 - y_2$. This prooves the claim.

If $\geq_+$ is total on S then as mentioned earlier DS= $S \cup -S \cup \{0\}$ since each nonzero element in DS is either in the image of the canonical embedding of S in DS or is the additive inverse of an element in that image. Now we prove a theorem which is related to Theorem 5.2.6.

<u>5.2.7 Theorem</u>. Let F be a field and S a semiring with 1 without an additive identity such that $F = S \cup -S \cup \{0\}$. Furthermore assume that F is Archimedeam with respect to $\geq$, the order inherited from $\geq_+$ on S (i.e. $\geq$ as described above). Then S is congruence-free.

Proof: First note that since $F = S \cup -S \cup \{0\}$, S is AC so $\geq_+$ can be defined on S. Note also that F = DS. Next claim that S is totally ordered by $\geq_+$. Choose any $x \neq y \in S$. Then $x-y$ and $y-x \in F$. Now both $x-y$ and $y-x$ cannot be in $-S$ since otherwise $0 \in -S$ so $0 \in S$. Without loss of generality suppose that $y-x \in S$. Then $y = x+s$ for some $s \in S$, is $y \geq_+ x$. Thus $\geq_+$ is a total order on S.

Claim that S is a division semiring. Choose $x \in S$. Then $x^{-1} \in F$. But suppose $x^{-1} \in -S$. Then there exists an $s \in S$ such that $-sx = 1$. Thus $sx = -1$ so S is not closed with respect to multiplication which is a contradiction. Thus S is a division semiring.

Finally claim that S is Archimedean with respect to $\geq_+$. Choose $a, b \in S$. Since F is Archimedean with respect to $\geq$ there exists an $n \in Z$ such that $b \not> na$. Thus $b \not>_+ na$ if $n \in Z^+$. Otherwise na is not defined in S. But if $n = 0$ then $b \not> 0$ i.e. $2b - 2b \not< 2b - b$. which contradicts the fact that $3b \leq_+ 4b$. Thus $n \neq 0$. Note that since $F = S \cup -S \cup \{0\}$, $\geq_+$ totally orders S and thus $\geq$ totally orders F. Thus if $0 > n$ (in Z) and $nb \not< a$ then $nb \geq a$. Thus $0 \geq (-n)b + a$. But $(-n)b + a \in S$ and each $s \in S \geq 0$. Thus $n < 0$ is impossible. Thus S is Archimedean, so by Theorem 5.2.3 S is congruence-free.

<u>5.2.8 Theorem</u>. Let S be an AC type III semiring which is totally ordered by $\geq_+$. Then there exists a monomorphism ϕ from S into R^+ such that ϕ is isotonic with respect to the orders $\geq_+$ on S and the usual order $\geq$ on R^+.

Proof: Let $\geq_1$ be the order induced by $\geq_+$ on DS. Since $\geq_+$ is total on S, $\geq_1$ is total on DS. Moreover, DS is Archimedean with respect to the total order $\geq_1$

since S is Archimedean with respect to the total order $\geq_+$. Thus by Theorem 1.1.3 there exists an isotonic monomorphism α : DS $\rightarrow \mathbb{R}$ (isotonic with respect to $\geq_1$ and $\geq$). Let ι be the natural embedding of S into DS. Then setting $\phi = \alpha \circ \iota$, we see that ϕ is isotonic with respect to $\geq_+$ and $\geq$. For all $s \in S$ $\iota(s) >_1 0$. Thus $(\alpha \circ \iota)(s) \in \mathbb{R}^+$. Thus $\phi : S \rightarrow \mathbb{R}^+$. #

In less precise terms the theorem above shows that each AC type III semiring which is totally ordered by $\geq_+$ (i.e. each AC congruence-free division semiring which is totally ordered by $\geq_+$) is a subsemiring of R^+. Thus we see, for example, that C (the complex field) cannot equal $S \cup -S \cup \{0\}$ where S is a semiring with 1 and without an additive identity since C cannot be embedded in R. The following corollary is an immediate consequence of Theorem 5.2.8.

<u>5.2.9 Corollary</u>. Let S be a semiring which can be embedded in an AC type III semiring S' which is totally ordered by $\geq_+$ ($\geq_+$ is the additive partial order on S'). Then there exists an embedding $\phi : S \rightarrow \mathbb{R}^+$ which is isotonic with respect to the natural additive partial order on S and the usual ordering on $\mathbb{R}^+$.

To sum up the results above. Every AC type III semiring which is totally ordered by $\geq_+$ is a sub division semiring of $\mathbb{R}^+$. Thus the natural question is:

"Which AC type III semirings S are totally ordered by $\geq_+$?" The next theorem shows that either S is totally ordered by S is S is extremely pathological with respect to $\geq_+$.

5.2.10 Theorem. Let S be an AC type III semiring. Then S is totally ordered by $\geq_+$ or for each $x, y \in S$, there exists an $\alpha \in DS$ such that considering x and y as elements of DS α is related to $\underline{x}$ by $\geq$ (the partial order on DS inherited from S) but not to y or vice versa.

Proof: Let $\geq$ be the partial order induced by $\geq_+$ on DS, the difference ring of S. For $x \in S$, define $cor(x) = \{s \in DS$ such that s is related to x by $\geq\}$. For $x, y \in S$ define $x \sim y$ iff $cor(x) = cor(y)$. $\sim$ is an equivalence relation on S.

To show that $\sim$ preserves multiplication suppose that $x \sim y$ and $a \in S$. considering x,y and a as elements in DS suppose that $\alpha \in DS$ and $\alpha \geq ax$. $a > 0$ so $\alpha/a \geq x$. Thus $\alpha/a \in cor(x)$ so $\alpha/a \in cor\ (y)$. Thus $\alpha/a \geq y$ or $y \geq \alpha/a$. Thus $\alpha \geq ay$ or $ay \geq \alpha$ so $\alpha \in cor(ay)$. After similar arguments we see that $cor(ax) = cor(ay)$ so $ax \sim ay$.

Suppose that $\alpha \geq x+a$. Then $\alpha - a \geq x$ so $\alpha - a \in cor(x)$. Thus $\alpha - a \in cor(y)$ so $\alpha \in cor(y+a)$. Again after similar arguments we see that $cor(y+a) = cor(x+a)$, so

$x+a \sim y+a$.

Thus $\sim$ is a congruence on S. Thus $\sim = \Delta$ or SxS. If $\sim = \Delta$ then $\geq_+$ is a total order on S. If $\sim =$ SxS then for each $x,y \in S$ there exists an $\alpha \in DS$ such that $\alpha \in cor(x)$ but $\alpha \notin cor(y)$ or vice versa. This completes the proof. #

Section 5.3 Congruence free subsemirings of $\mathbb{Q}^+$

In this section we briefly study congruence-free subsemirings of $\mathbb{Q}^+$ where $\mathbb{Q}^+$ has the usual additive and multiplicative structure. Thus $\mathbb{Q}^+$ is AC. We have already shown that if S is a type III semiring then if $\gtrsim_+$ is total, S is a devsion-semiring. We have also shown that if $\gtrsim_+$ is not total then S has an improbably pathological structure with respect to $\gtrsim_+$. Thus we suspect that $\gtrsim_+$ must be total.

Every AC semidivision ring of order > 1 must contain an isomorphic copy of $\mathbb{Q}^+$. If the hypothesis above is true we should be able to prove that every non-trivial congruence free subsemiring of $\mathbb{Q}^+$ is $\mathbb{Q}^+$. Unfortunately we have not been able to prove this but have derived some interesting results concerning congruence-free subsemirings of $\mathbb{Q}^+$.

5.3.1 Theorem : Let S be an AC type III semiring which can be embedded in an AC type III semiring S' which is totally ordered by $\gtrsim_+$. Let $\phi : S \to \mathbb{R}^+$ be the isotonic monomorphism described in Corollary 5.2.9. Then $\phi(S)$ is dense in $\mathbb{R}^+$. Moreover if $\phi(S) \neq \mathbb{R}^+$ then $\mathbb{R}^+ \setminus \phi(S)$ is dense in $\mathbb{R}^+$. ($\mathbb{R}^+$ has the usual topology.)

Proof : Note first that $\phi(S)$ is congruence-free. Let $\gtrsim$ be the usual ordering on $\mathbb{R}^+$. By Theorem 5.2.1 $\phi(S)$ has no minimal element with respect to $\gtrsim$. Choose an open interval (a,b) in $\mathbb{R}^+$. We can choose a $k \in \phi(S)$ such that $k < \frac{b-a}{2}$. Thus there exists an

$n \in \mathbb{Z}^+$ such that $nk \in (a,b)$. Since the open intervals are a basis for the topology on $\mathbb{R}^+$, $\phi(S)$ is dense in $\mathbb{R}^+$.

Now suppose that $\phi(S) \neq \mathbb{R}^+$. Suppose that there exists an open interval $(a,b) \subseteq \mathbb{R}^+$ such that $(a,b) \subseteq \phi(S)$. Choose any $x \in \mathbb{R}^+$. Since $\Phi(S)$ is dense in $\mathbb{R}^+$ there exists a $k \in \phi(S)$ such that $x \in (ka,kb)$ (Since we can choose a monotonic increasing sequence (k_n) in $\phi(S)$ such that $k_n a < x$ and $(k_n a)$ approaches x from the left. $k_n \to \frac{x}{a}$. For som n_o large enough $x \in (k_{n_o} a, k_{n_o} b)$). Thus $x \in \phi(S)$. Thus $\phi(S) = \mathbb{R}^+$ which is a contradiction. Thus $(a,b) \cap (\mathbb{R}^+ \setminus \phi(S)) \neq \phi$. Since (a,b) was arbitrary $\mathbb{R}^+ \setminus \phi(S)$ is dense in $\mathbb{R}^+$. #

From the proof of the theorem above the following corollary is immediate.

5.3.2 Corollary. Let S be a congruence-free subsemiring of $\mathbb{Q}^+$ with the usual algebraic structure, Then

1) S is dense in $\mathbb{Q}^+$

2) If $S \neq \mathbb{Q}^+$ then $\mathbb{Q}^+ \setminus S$ is dense in $\mathbb{Q}^+$.

We conclude with the following theorem.

5.3.3 Corollary. Let S be as described above and let p be a prime number. Then p divides the denominator of some element in S reduced to lowest terms. Moreover S is infinitely generated as a semiring.

Proof : Let p be a prime number. Let $(Si)_{i \in I}$ be a set of generators of S. Then DS, the difference ring of S, is a field so DS = $\mathbb{Q}$. Thus $\frac{1}{p} \in DS$. Thus $\frac{1}{p} = f_1(Si_1,\dots,Si_k) - f_2(s\alpha_1,\dots, S\alpha_n)$ where f_1 and f_n are multivariable polynomials with coefficients in $\mathbb{Z}^+$

Thus if we reduce $Si_1, \ldots, Si_k, S\alpha_1, \ldots, S\alpha_n$ to lowest terms p must divide the denominator of at least one of the $Si_1, \ldots, Si_k, S\alpha_1, \ldots, S\alpha_n$.

Thus $(Si)_{i \in I}$ when reduced to lowest terms must contain an element Si_o such that p divides the denominator of Si_o. Thus $(Si)_{i \in I}$ must be infinite. #