CHAPTER II

FORMULATION OF PHASE BOUNDARY EQUATION

In this chapter we will first calculate the spin magnetization and the pair-correlation function using the Hamiltonian for z + 1 spins in terms of the variables J_{oi} and H_i. The probability distribution $P_2(J_{ij})$ is assumed to be given. We will then determine what the probability distribution of the fields $P_1(H_i)$ should be. Knowing $P_1(H_i)$ and $P_2(J_{ij})$ we will be able to calculate the thermodynamic variables using the Bethe-Peierls-Weiss approximation.

The internal energy is also obtained. The free energy is then expressed in terms of the internal energy and a constant of integration S°. Thermodynamic quantities appropriate for the quenched system can be obtained by first obtaining the appropriate quantity from this free energy for a given configuration and then averaging it over all configurations. Finally we obtain the equation which determine the critical temperature.

2.1 Bethe-Peierls-Weiss (BPW) Approximation

To obtain the thermodynamic quantities in the BPW approximation (11,14,16) we write the Hamiltonian for outer z particles in the molecular field approximation and then treat the interaction between the z outer spins with the central spin S_o, placed at the origin, exactly. The interaction potentials J_{oi} between the outer spins and central spin and the z mean fields H_i are treated as independent random variables. We obtain the single spin magnetization $\langle S_i \rangle$ and the pair-correlation function $\langle S_o S_i \rangle$ using the Hamiltonian for the cluster of z + 1 spins in terms of the variables J_{oi} and H_i as follows.

The BPW Hamiltonian is

$$H_{BPW} = -\sum_{i=1}^{z} H_i S_i - \sum_{i=1}^{z} J_{oi} S_o S_i$$

Where H_i is the mean field at site i and J_{oi} is the interaction potential between the spins S_o and S_i, with the S being Ising spins.

It is somewhat difficult to handle with H_{BPW} in the last equation as it stands so we look instead at a single interaction

$$H_{BPW} = -H_1 S_1 - J_{o1} S_o S_1 \tag{2.1}$$

From this, we get $\langle S_o \rangle$

$$\langle S_o \rangle = \frac{\sum S_o e^{-\beta H}}{\sum e^{-\beta H}}$$

$$= \frac{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} S_o e^{\beta(H_1 S_1 + J_{o1} S_o S_1)}}{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} e^{\beta(H_1 S_1 + J_{o1} S_o S_1)}}$$

By taking the sum over S_o and then the sum over S_1 we get

$$\langle S_o \rangle = \tanh(\beta J_{o1}) \cdot \tanh(\beta H_1)$$

or $$\langle S_o \rangle = t_1 g_{o1}$$

where $t_1 = \tanh(\beta H_1)$, $g_{o1} = \tanh(\beta J_{o1})$, $\beta = \frac{1}{kT}$

$\langle S_1 \rangle$ can find in the same way

$$\langle S_1 \rangle = \frac{\sum S_1 e^{-\beta H}}{\sum e^{-\beta H}}$$

$$= \frac{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} S_1 \, e^{\beta(H_1 S_1 + J_{o1} S_o S_1)}}{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} e^{\beta(H_1 S_1 + J_{o1} S_o S_1)}}$$

We first perform the sum over S_o and then sum over of S_1, we get

$$\langle S_1 \rangle = \tanh(\beta H_1)$$

or $\langle S_1 \rangle = t_1$

where $t_1 = \tanh(\beta H_1)$

The spin correlation function $\langle S_o S_1 \rangle$ is given by

$$\langle S_o S_1 \rangle = \frac{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} S_o S_1 \, e^{-\beta H}}{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} e^{-\beta H}}$$

$$= \frac{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} S_o S_1 \, e^{\beta(H_1 S_1 + J_{o1} S_o S_1)}}{\sum_{S_o=-1}^{+1} \sum_{S_1=-1}^{+1} e^{\beta(H_1 S_1 + J_{o1} S_o S_1)}}$$

The summations are carried out as follows

$$\langle S_o S_1 \rangle = \frac{\sum_{S_1=-1}^{+1} [S_1 \, e^{\beta(H_1 S_1 + J_{o1} S_1)} - S_1 \, e^{\beta(H_1 S_1 - J_{o1} S_1)}]}{\sum_{S_1=-1}^{+1} [e^{\beta(H_1 S_1 + J_{o1} S_1)} + e^{\beta(H_1 S_1 - J_{o1} S_1)}]}$$

$$= \frac{\sum_{S_1=-1}^{+1} S_1 e^{\beta H_1 S_1} (e^{\beta J_{o1} S_1} - e^{-\beta J_{o1} S_1})}{\sum_{S_1=-1}^{+1} e^{\beta H_1 S_1} (e^{\beta J_{o1} S_1} + e^{-\beta J_{o1} S_1})}$$

$$= \frac{\sum_{S_1=-1}^{+1} S_1 e^{\beta H_1 S_1} \circ \sinh(\beta J_{o1} S_1)}{\sum_{S_1=-1}^{+1} e^{\beta H_1 S_1} \cdot \cosh(\beta J_{o1} S_1)}$$

$$= \frac{e^{\beta H_1} \circ \sinh(\beta J_{o1}) - e^{-\beta H_1} \circ \sinh(-\beta J_{o1})}{e^{\beta H_1} \circ \cosh(\beta J_{o1}) + e^{-\beta H_1} \cdot \cosh(-\beta J_{o1})}$$

$$= \tanh(\beta J_{o1})$$

or $\langle S_o S_1 \rangle = g_{o1}$

for $H_{BPW} = -H_1 S_1 - J_{o1} S_o S_1$

we get these set

$$\langle S_o \rangle = t_1 g_{o1}$$

$$\langle S_1 \rangle = t_1$$

$$\langle S_o S_1 \rangle = g_{o1}$$

Rewriting $\langle S_1 \rangle$ as follows

$$\langle S_1 \rangle = t_1$$

$$= (1 - t_1^2 g_{o1}^2)^{-1}\, t_1\, (1 - t_1^2 g_{o1}^2)$$

$$= (1 - t_1^2 g_{o1}^2)^{-1}\, t_1\, (1 - g_{o1}^2 + g_{o1}^2 - t_1^2 g_{o1}^2)$$

$$= (1 - t_1^2 g_{o1}^2)^{-1}[t_1(1 - g_{o1}^2) + g_{o1}(1 - t_1^2)\ t_1 g_{o1}]$$

we get

$$\langle S_1\rangle = (1 - t_1^2 g_{o1}^2)^{-1}[t_1\ (1 - g_{o1}^2) + g_{o1}(1 - t_1^2)\ \langle S_o\rangle\ ]$$

A new form for $\langle S_o S_1\rangle$ is similarly obtained, i.e.,

$$\begin{aligned}
\langle S_o S_1\rangle &= g_{o1} \\
&= (1 - t_1^2 g_{o1}^2)^{-1}\ g_{o1}(1 - t_1^2 g_{o1}^2) \\
&= (1 - t_1^2 g_{o1}^2)^{-1}\ [g_{o1}(1 - t_1^2) + t_1(1 - g_{o1}^2) t_1 g_{o1}] \\
&= (1 - t_1^2 g_{o1}^2)^{-1}[g_{o1}(1 - t_1^2) + t_1(1 - g_{o1}^2)\ \langle S_o\rangle\ ]
\end{aligned} \tag{2.2}$$

If we have z Ising spins ; $S_1, S_2, \ldots, S_z$

$$H_{BPW} = - \sum_{i=1}^{z} H_i S_i - \sum_{i=1}^{z} J_{oi} S_o S_i$$

then $\langle S_o\rangle$ is equal to the sum of $\langle S_o\rangle$ of each Ising spin.

$$\langle S_o\rangle = t_1 g_{o1} + t_2 g_{o2} + \ldots + t_z g_{oz}$$

or $$\langle S_o\rangle = \sum_{i=1}^{z} t_i g_{oi}$$

and so

$$\begin{aligned}
\tanh^{-1}\langle S_o\rangle &= \tanh^{-1} \sum_{i=1}^{z} t_i g_{oi} \\
&= \sum_{i=1}^{z} \tanh^{-1}\ (t_i g_{oi})
\end{aligned}$$

$$\langle S_o\rangle = \tanh\ [\sum_{i=1}^{z} \tanh^{-1}\ (t_i g_{oi})] \tag{2.3}$$

$\langle S_1\rangle$ is in the form

$$\langle S_1\rangle = (1 - t_1^2 g_{o1}^2)^{-1}[t_1(1 - g_{o1}^2) + g_{o1}(1 - t_1^2) \langle S_o\rangle]$$

where $\langle S_o\rangle$ is from Eq.(2.3), so we get

$$\langle S_i\rangle = (1 - t_i^2 g_{oi}^2)^{-1}[t_i(1 - g_{oi}^2) + g_{oi}(1 - t_i^2) \langle S_o\rangle] \tag{2.4}$$

For $\langle S_o S_i\rangle$, we generalize Eq. (2.2) to get

$$\langle S_o S_i\rangle = (1 - t_i^2 g_{oi}^2)^{-1}[g_{oi}(1 - t_i^2) + t_i^2(1 - g_{oi}^2) \langle S_o\rangle] \tag{2.5}$$

where $\langle S_o\rangle$ is given by Eq. (2.3)

The internal energy U in the BPW approximation is

$$U = - \tfrac{1}{2} \sum_{ij} J_{ij} \langle S_i S_j\rangle \tag{2.6}$$

In the expression for U there are two independent random variables, the quantities J_{oi} and t_i.

The next task is to determine the probability distribution of the fields $P_1(H_i)$ when the probability distribution $P_2(J_{ij})$ is given. We start by letting $\langle S_o\rangle = \tanh(\beta H_o)$ in Eq.(2.3) gives for the field H_o at spin S_o

$$H_o = \beta^{-1} \sum_{i=1}^{z} \tanh^{-1}(t_i g_{oi})$$

The z variables H_i are assumed to be independent of each other. The interactions J_{oi} are a priori independent variables of the model. The H_i's are also assumed to be independent of the J_{oi} since the fields H_i arise from the spins and interactions existing outside the cluster of z + 1 spins.

This assumption neglects the fact that the spins outside the cluster are not totally independent of the spins in the cluster. With these assumptions the distribution of H_0, $P_0(H_0)$ can be written as (13,14)

$$P_0(H_0) = \prod_{i=1}^{z} \int_{-\infty}^{\infty} P_1(H_i)dH_i \prod_{i=1}^{z} \int_{-\infty}^{\infty} P_2(J_{0i})dJ_{0i} \cdot \delta\left[H_0 - \beta^{-1} \sum_{i=1}^{z} \tanh^{-1}(t_i g_{0i})\right] \quad (2.7)$$

We now require that the distribution $P_0(H)$ and $P_1(H)$ be identical. Rewriting Eq.(2.7) gives

$$P_1(H_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_0} d\rho \prod_{i=1}^{z} \left(\int_{-\infty}^{\infty} P_1(H_i)dH_i \int_{-\infty}^{\infty} P_2(J_{0i})\, dJ_{0i} \cdot e^{[-i\rho\beta^{-1}\tanh^{-1}(t_i g_{0i})]} \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_0}\, d\rho \left(\int_{-\infty}^{\infty} P_1(H_i)dH_i \int_{-\infty}^{\infty} P_2(J_{0i})dJ_{0i} \cdot e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{0i})} \right)^z \quad (2.8)$$

Given $P_2(J_{0i})$ we can find $P_1(H_0)$ which we do next.

2.2 Delta Function Distribution of Interaction Strengths

In this part $P_2(J_{0i})$ is assumed to be a delta function. We first let the number of neighbors $z \rightarrow \infty$ and let $\langle J_{0i} \rangle_c \propto z^{-1}$ and $\langle J_{0i}^2 \rangle_c \propto z^{-1}$, where $\langle\,\rangle_c$ denotes a configurational average. (Since we will be taking the limit $z \rightarrow \infty$, the average of the interaction strength, $\langle J_{0i} \rangle_c$ must go as $\frac{1}{z}$ in order that $\langle \Sigma J_{0j} S_0 S_j \rangle_c$ be finite in the limit $z \rightarrow \infty$. With this, we find that the configuration average of J_{0i}^2 must go as $\frac{1}{z}$. This is due to the way the configuration averages are calculated.) In this limit $\tanh^{-1}(t_i g_{0i}) = \beta t_i J_{0i} + O\left(\frac{1}{z}\right)$

We shall assume that

$$P_2(J_{oi}) = c\,\delta(J_{oi} - J_1) + (1 - c)\,\delta(J_{oi} - J_2) \qquad (2.9)$$

where c is the concentration of bonds having the interaction J_1. The distribution function for the interaction J is the average distribution function, i.e, it is the distribution function averaged over all the cluster in the system. If we are working with a particular cluster, with given numbers of J_1 and J_2 bonds, we must work with the distribution function for that particular cluster, which is

$$P_2(J_{oi}) = P\,\delta(J_{oi} - J_1) + (1 - P)\,\delta(J_{oi} - J_2)$$

where P is the projection operator which takes on the value 1 if a bond is a J_1 type and the value 0 if the bond is a J_2 type. The projection operator has the property that $P^2 = P$ and that the average of P over all the bonds in all clusters is c, i.e., $\overline{P} = c$.

Substitute Eq. (2.9) into Eq. (2.8) give

$$P_1(H_o) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \left(\int_{-\infty}^{\infty} P_1(H_i)dH_i \int_{-\infty}^{\infty} [p\,\delta(J_{oi} - J_1) + (1 - p)\,\delta(J_{oi} - J_2)]dJ_{oi} . e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})} \right)^z$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \left[\int_{-\infty}^{\infty} P_1(H_i)dH_i \int_{-\infty}^{\infty} p\,\delta(J_{oi} - J_1)dJ_{oi} . e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})} \right]^z + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \left[\int_{-\infty}^{\infty} P_1(H_i)dH_i \int_{-\infty}^{\infty} (1 - p)\,\delta(J_{oi} - J_2)\, dJ_{oi} . e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})} \right]^z$$

$$= \frac{p}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \; [\int_{-\infty}^{\infty} P_1(H_i) dH_i \int_{-\infty}^{\infty} \delta(J_{oi} - J_1) dJ_{oi}$$

$$. e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})}]^z$$

$$+ \frac{(1-p)}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \, [\int_{-\infty}^{\infty} P_1(H_i) dH_i \int_{-\infty}^{\infty} \delta(J_{oi} - J_2) dJ_{oi}$$

$$. e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})}]^z$$

$$= \frac{p}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \, [1 - V_1(\rho)]^z$$

$$+ \frac{(1-p)}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \, [1 - V_2(\rho)]^z \qquad (2.10)$$

where
$$V_1(\rho) = \int_{-\infty}^{\infty} P_1(H_i) dH_i \int_{-\infty}^{\infty} \delta(J_{oi} - J_1) dJ_{oi} \, (1 - e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})})$$

$$V_2(\rho) = \int_{-\infty}^{\infty} P_1(H_i) dH_i \int_{-\infty}^{\infty} \delta(J_{oi} - J_2) \, dJ_{oi} \, (1 - e^{-i\rho\beta^{-1}\tanh^{-1}(t_i g_{oi})})$$

We now looking at the first term in the right hand side of Eq.(2.10).Now

$$A = \frac{p}{2\pi} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \, [1 - V_1(\rho)]^z$$

expanding the exponential in $V_1(\rho)$ and keeping only terms of lowest power in $\frac{1}{z}$ (note that $\langle J_{oi}^{2k} \rangle \propto z^{-k}$ and $\langle J_{oi}^{2k+1} \rangle \propto z^{-k-1}$) we get

$$V_1(\rho) = \int_{-\infty}^{\infty} P_1(H_i) dH_i \int_{-\infty}^{\infty} \delta(J_{oi} - J_1) dJ_{oi} (1 - e^{-i\rho t_i J_{oi}})$$

$$= \int_{-\infty}^{\infty} P_1(H_i) dH_i \int_{-\infty}^{\infty} \delta(J_{oi} - J_1) dJ_{oi} \, (i\rho\, t_i J_{oi} + \tfrac{1}{2} \rho^2 t_i^2 J_{oi}^2 + \ldots)$$

$$= i\rho m J_1 + \tfrac{1}{2}\rho^2 q J_1^2$$

where $$m = \int_{-\infty}^{\infty} P_1(H_i)\, t_i \, dH_i \qquad (2.11)$$

$$q = \int_{-\infty}^{\infty} P_1(H_i)\, t_i^2 \, dH_i \qquad (2.12)$$

In limit $z \rightarrow \infty$ we have

$$[1 - V_1(\rho)]^z = e^{-zV_1(\rho)}$$

$$A = \frac{p}{2\P} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \,.\, e^{-z(i\rho m J_1 + \frac{1}{2}\rho^2 q J_1^2)}$$

$$= \frac{p}{\sqrt{2\P}\,\sigma} \cdot e^{-\frac{1}{2}[H_o - mzJ_1]^2/\sigma^2}$$

where $\sigma = \sqrt{qz}$

Similarly, the second term in the right hand side of Eq.(2.10) is obtained as

$$B = \frac{(1 - p)}{2\P} \int_{-\infty}^{\infty} e^{i\rho H_o} d\rho \, [1 - V_2(\rho)]^z$$

$$= \frac{(1 - p)}{\sqrt{2\P}\,\sigma} \circ e^{-\frac{1}{2}[H_o - mzJ_2]^2/\sigma^2}$$

where $\sigma = \sqrt{qz}$

Finally the distribution function $P_1(H_o)$ is obtained as

$$P_1(H_o) = \frac{p}{\sqrt{2\P}\,\sigma} e^{-\frac{1}{2}[H_o - mzJ_1]^2/\sigma^2} + \frac{(1 - p)}{\sqrt{2\P}\,\sigma} \cdot e^{-\frac{1}{2}[H_o - mzJ_2]^2/\sigma^2}$$

$$\lim_{\sigma^2 \longrightarrow 0} P_1(H_o) = c\,\delta\,(H_o - mzJ_1) + (1 - c)\,\delta\,(H_o - mzJ_2) \qquad (2.13)$$

2.3 Free Energy

We next discuss the expression for the internal energy and the free energy of the random system. Solving Eq.(2.3) for $\langle S_o \rangle$ and substituting into Eq.(2.5) we get

$$U = -\tfrac{1}{2} \sum_{i,j} J_{ij} \langle S_i S_j \rangle$$

$$= -\tfrac{1}{2} \sum_{i,j} J_{ij} [\langle S_i \rangle \langle S_j \rangle$$

$$+ \beta J_{ij} (1 - \langle S_i \rangle^2)(1 - t_j^2)] \qquad (2.14)$$

The free energy is obtained in terms of the internal energy $U(\beta)$ and a constant of integration $S°$ as follows (14):

$$F = \frac{1}{\beta} \left(\int_0^\beta U(\beta')d\beta' + S° \right) \qquad (2.15)$$

In Eq. (2.15) only the explicit dependence of $U(\beta')$ upon β' is taken into account. The phenomenological expression for $S°$ is(14)

$$S° = \sum_i \left[\left(\frac{1 + \langle S_i \rangle}{2}\right) \ln \left(\frac{1 + \langle S_i \rangle}{2}\right) + \left(\frac{1 - \langle S_i \rangle}{2}\right) \ln \left(\frac{1 - \langle S_i \rangle}{2}\right) \right] \qquad (2.16)$$

which is the entropy of a set of independent spins constrained to have a value $\langle S_i \rangle$. Substituing for Eq. (2.14) into F, we get

$$F = \frac{1}{\beta} \left(\int_0^\beta U(\beta')d\beta' + S° \right)$$

$$= \frac{1}{\beta}\left(\int_0^\beta - \tfrac{1}{2}\sum_{i,j} J_{ij}\left[\langle S_i\rangle\langle S_j\rangle\right.\right.$$

$$\left.\left. + \beta' J_{ij}(1 - \langle S_i\rangle^2)(1 - t_j^2)\right]d\beta' + S^\circ\right)$$

or $\quad F$

$$= \frac{1}{\beta}\left(\int_0^\beta - \sum_{j>i} J_{ij}\left[\langle S_i\rangle\langle S_j\rangle\right.\right.$$

$$\left.\left. + \beta' J_{ij}(1 - \langle S_i\rangle^2)(1 - t_j^2)\right]d\beta' + S^\circ\right)$$

$$= \frac{1}{\beta}\left(-\sum_{j>i} J_{ij}\langle S_i\rangle\langle S_j\rangle\beta\right)$$

$$+ \frac{1}{\beta}\left(-\tfrac{1}{2}\sum_{j>i} J_{ij}^2(1 - \langle S_i\rangle^2)(1 - t_j^2)\beta^2\right) + \frac{1}{\beta}.S^\circ$$

$$= -\sum_{j>i} J_{ij}\langle S_i\rangle\langle S_j\rangle$$

$$- \frac{\beta}{2}\sum_{j>i} J_{ij}^2(1 - \langle S_i\rangle^2)(1 - t_j^2) + \frac{1}{\beta}S^\circ$$

Because $\langle S_j\rangle = \tanh(\beta H_j)$

$$= t_j$$

$$\langle S_j\rangle^2 = t_j^2$$

$$F = -\sum_{j>i} J_{ij}\langle S_i\rangle\langle S_j\rangle - \tfrac{1}{2}\beta\sum_{j>i} J_{ij}^2$$

$$(1 - \langle S_i\rangle^2)(1 - \langle S_j\rangle^2) + \frac{1}{\beta}S^\circ \qquad (2.17)$$

The free energy in Eq. (2.17) is the free energy for a given configuration of the random interactions J_{ij}. This is a variational free energy with respect to the variable $\langle S_i\rangle$ and β. Thermodynamic quantities appropriate for the quenched system can be obtained by first obtaining the appropriate quantity from Eq. (2.17) for a given configuration and then averaging it over all configurations.

Differentiating Eq.(2.17) with respect to $\langle S_i \rangle$ we get

$$\Sigma J_{oj} \langle S_j \rangle - \langle S_o \rangle \beta \Sigma J_{oj}^2 (1 - \langle S_j \rangle^2) = T \tanh^{-1} \langle S_o \rangle \tag{2.18}$$

Eq (2.18) can be obtain in another way by using Eq.(2.3) and (2.4)

From Eq.(2.3) ; $\langle S_o \rangle = \tanh[\sum_{j=1}^{z} \tanh^{-1}(t_j g_{oj})]$

$$= \tanh(\sum_{j=1}^{z} t_j g_{oj})$$

$$= \tanh [\sum_{j=1}^{z} \tanh(\beta H_j). \tanh(\beta J_{oj})]$$

$$= \tanh [\sum_{j=1}^{z} (\beta J_{oj}). \tanh(\beta H_j)]$$

$$= \tanh [\beta \sum_{j=1}^{z} J_{oj} .\tanh(\beta H_j)]$$

From Eq.(2.4); $\langle S_j \rangle = (1 - t_j^2 g_{oj}^2)^{-1}[t_j(1 - g_{oj}^2) + g_{oj}(1 - t_j^2) \langle S_o \rangle]$

$$= t_j + g_{oj}(1 - t_j^2) \langle S_o \rangle$$

$$= \tanh (\beta H_j) + \beta J_{oj} \langle S_o \rangle . [1 - \tanh^2 (\beta H_j)]$$

$$\Sigma J_{oj} \langle S_j \rangle = \Sigma J_{oj} \tanh(\beta H_j) + \Sigma \beta J_{oj}^2 . \langle S_o \rangle . [1 - \tanh^2 (\beta H_j)]$$

From Eq.(2.3); $\tanh^{-1}\langle S_o\rangle = \tanh^{-1}[\tanh(\beta\sum_{j=1}^{z} J_{oj}\tanh(\beta H_j))]$

$$= \beta\sum_{j=1}^{z} J_{oj}\tanh(\beta H_j)$$

$$KT.\tanh^{-1}\langle S_o\rangle = \sum_{j=1}^{z} J_{oj}\tanh(\beta H_j)$$

$$KT.\tanh^{-1}\langle S_o\rangle = \Sigma J_{oj}\langle S_j\rangle - \Sigma\beta J_{oj}^2\langle S_o\rangle.$$

$$.[1 - \tanh^2(\beta H_j)]$$

$$\langle S_j\rangle = \tanh(\beta H_j)$$

$$T\tanh^{-1}\langle S_o\rangle = \Sigma J_{oj}\langle S_j\rangle - \beta\langle S_o\rangle\Sigma J_{oj}^2(1 - \langle S_j\rangle^2)$$

; (we set $K = 1$)

which is the same as Eq. (2.18)

Since the transition to the spin glass phase in our prototype model occurs when non-zero values of q [the order parameter given by Eq. (2.12)] are possible, Eq.(2.18) must be rewritten in terms of the order parameter q. This is done by multiplying Eq.(2.18) by $\langle S_o\rangle$ and then expand everything in powers of J_{ij} and then keep only those terms which are proportional to z^{-1}. The equation for the order parameter is then obtained by averaging the expanded Eq.(2.18) over the distribution of the J_{ij}'s and H_i's. The resulting equation is

$$[cq_1^4 + (1 - c)q_2^4] - [1 + \tfrac{1}{2}(c + (1 - c)a^2)x][cq_1^2 + (1 - c)q_2^2]$$
$$+ [8(c + (1 - c)a^2)x + 5(c + (1-c)a^2)^2x^2][cq_1^2 + (1 - c)q_2^2]^2$$
$$= 0 \qquad (2.19)$$

From Eq. (2.11) we get

$$m - cq_1 - (1 - c)q_2 = 0 \qquad (2.19')$$

where

$$q_1 = \tanh (mzx)$$

$$q_2 = \tanh (mzxa)$$

The details of the calculation leading to Eq. (2.19) and (2.19′) are given in the Appendix A.