CHAPTER III

Ginzburg - Landau Theory

In this chapter , we study the Ginzburg - Landau theory , both pure phenomenological and the microscopic theory.

One of the most fertile approaches to superconductivity has developed from the work of Ginzburg and Landau (15); the Ginzburg - Landau phenomenological theory of superconductivity is an extremely uselul tool to study spatially dependent effects. It was formulated before the microscopic BCS theory , and is independent of the detailed mechanisms responsible for the superconductivity. The Ginzburg - Landau theory has been shown to be very successful in describing macroscopic properties of superconductors , especially in inhomogeneous situations where the ordinary BCS theory is difficult to apply.

Because of the success and generality of Ginzburg - Landau theory,it is appealing to apply it to description of some of the macroscopic properties of the high - T_c superconductors (18 - 22). There is not yet a generally accepted microscopic theory of these high - T_c temperature superconductor but , regardless of the underlying mechanism , the Ginzburg-Landau theory (with its own resticted validity) is certain to be applicable

3.1 The Phenomenological Ginzburg - Landau Theory.

As originally , this theory was a triumph of physical intuition, in which a pseudo - wavefunction $\Psi(x)$ was introduced as a complex order parameter $|\Psi(x)|^2$ to represent the local density of superconducting electrons ,$n_s(\vec{x})$. The theory was developed by applying a variational method to an assumed expansion of the free - energy density in power of $|\Psi|^2$ and $|\nabla\Psi|^2$ leading to a pair of coupled differential equations for $\Psi(x)$ and the vector potential $\vec{A}(\vec{x})$.The result was a generalization of the London theory to deal with situations in which n_s varied in space , and also to deal with the nonlinear response to fields strong enough to change n_s. Athough quite successful in explaning intermediate - state phenomena, where the need for a theory capable of dealing with spatially inhomogeneous superconductivity was evident , this theory was generally given limited attention in the western literature becouse of its phenomenological foundation.

3.1.1 <u>The Ginzburg - Landau Free Energy</u>

The basic postulate of Ginzburg and Landau is that if Ψ is small and varies slowly in space , we can write down the following expression of the free energy density of a superconducting body , expand in a series of the form

$$F_s = F_{no} + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \ldots \tag{3.1}$$

where F_{no} is the free energy density in the normal state in zero magnetic field and where α and β are real temperature dependent phenomenological constants. Since the order parameter is uniform in absence of external fields , Ginzburg and Landau added a term proportional to $|\nabla\Psi|^2$, tending to suppress spatial variations in Ψ .In analogy with the Schrodinger equation , this term is written as $(2m^*)^{-1}\ |\nabla\Psi|^2$, When the magnetic fields are present , this term is assumed to take the gauge invariant form

$$(2m^*)^{-1} \quad (-i\hbar\nabla - \frac{e\vec{A}}{c}) \tag{3.2}$$

and $\quad \vec{h}(\vec{x}) = \vec{\nabla} \times \vec{A}$

determines the microscopis magnetic field. In this way , the total free energy density of the superconducting state in a magnetic field becomes

$$F_s = F_n + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \frac{1}{2m^*}\left|(-i\hbar\nabla - \frac{e^*\vec{A}}{c})\Psi\right|^2 + \frac{h^2}{8\pi} \tag{3.3}$$

where the last term represents the energy density of the magnetic field. It is conventional to choose a particular normalization of, Ψ

$$|\Psi|^2 = n_s^* = \frac{1}{2} n_s \tag{3.4}$$

where n_s^* defines an effective superelectron density , and n_s an electron density.

3.1.2 The Ginzburg - Landau Equations

The free energy of the sample is obtained by integrating Eq. (3.3) over the total volume v. In a uniform external field H , however , the relevent thermodynamic potential is the Gibbs free energy (compare Sec. 2.3) , and we must consider

$$\int d^3x\, [F_s - (4\pi)^{-1}\, \vec{h} \cdot \vec{H}\,] = \int d^3x\, G_s \tag{3.5}$$

where G_s is the microscopic Gibbs free energy density. To find the stable state at temperature T and field H , we must minimize the microscopic Gibbs free energy density. Now , since G_s depends on functions ψ and $\vec{A}$, we must use the Euler - Lagrange equations of the calculus of variations. Since Ψ is complex , we can minimize with respect to either ψ or ψ^*, the ψ^* equation is

$$\frac{\partial G_s}{\partial \Psi^*} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial G_s}{\partial(\nabla_j \Psi^*)} = 0 \qquad (3.6)$$

where $\nabla_j \Psi^*$ is the component of the gradient in the direction j . After some manipulation , and with the restriction that we use the gauge $\vec{\nabla}.\vec{A} = 0$, Eq.(3.6) becomes

$$\frac{1}{2m^*}(i\hbar\nabla - \frac{e^*\vec{A}}{c})^2\psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0 \qquad (3.7)$$

In carrying through the variational procedure,the boundary conditions must be provided.

$$\hat{n}.\frac{\partial G_s}{\partial(\nabla \Psi^*)} = 0$$

or $$\hat{n}.(i\hbar\nabla - \frac{e}{c}\vec{A}) = 0 \qquad (3.8)$$

where $\hat{n}$ is the unit vector normal to the surface. Similarly , we can minimize with respect to variational of the vector potential $\vec{A}$. The appropriate Euler - Lagrange equation is

$$\frac{\partial G_s}{\partial A_i} - \sum_j \frac{\partial}{\partial x_j} \frac{\partial G_s}{\partial(\partial A_i/\partial x_i)} = 0 \qquad (3.9)$$

where A_i is i component of $\vec{A}$ in the i th direction. Using the constraint of Eq. (3.2) , and we have assume that

$$\hat{n} \times (\vec{h} - \vec{H}) = 0 \qquad (3.10)$$

where $\hat{n}$ is the unit vector normal to the surface , Eq. (3.9) becomes

$$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{h} = \frac{-ic\hbar}{2m} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) - \frac{(e^*)^2}{mc} |\Psi|^2 \vec{A} \qquad (3.11)$$

Equations (3.7) and (3.11) form the complet of GL equations. Note that the current expression (3.11) has exactly the form of the usual quantum - mechanical expression for particles of mass m^*, charge e^* , and wavefunction $\Psi(x)$. Similarly , apart from the nonlinear term , Eq. (3.7) has the form of Schrodinger's equation for such particles , with energy eigenvalue $-\alpha$. The nonlinear term acts like a repulsive potential of Ψ on itself, tending to favor wave function $\Psi(x)$ which are spread out as uniformly as possible in space.

The boundary condition on Ψ is vary different from that of the usual Schrodinger theory , however , and may be understood as guaranteeing that $\hat{n}.\hat{j}$ vanishes at the surface of the body. Eq. (3.10) implies that the tangential component of the magnetic field is continuous across this surface.

3.1.3 Solution in Simple Cases

In the absence of fields and gradients , Eq.(3.3) , we have

$$F_s - F_{no} = \alpha |\Psi|^2 + \frac{1}{2} \beta |\Psi|^4 \qquad (3.12)$$

Eq. (3.7) has the spatially uniform solutions

$$|\Psi|^2 = 0 \quad \text{or} \quad |\Psi|^2 = -\frac{\alpha}{\beta} \tag{3.13}$$

The first solution clearly represents the normal state because F_s then equals F_{no}. The second solution is physically acceptable only if $|\Psi|^2 > 0$ or $\alpha/\beta < 0$, it represents the superconducting state with a corresponding free energy density, Eq. (3.12) becomes

$$F_s - F_{no} = -\frac{1}{2}\frac{\alpha^2}{\beta} \tag{3.14}$$

ane comparing with Eq. (2.50) yields

$$\frac{\alpha^2}{2\beta} = \frac{H_c(T)}{8\pi} \tag{3.15}$$

In the superconducting phase $F_s - F_{no} < 0$, Eq. (3.14) shows that B must be positive, which also ensure that F_s is bounded from below (see Eq. (3.31)). Inspection of Eq. (3.13) shows that α must be nagative. Following Ginzburg - Landau, we assume that β is inedpendent of temperature, while α can be expanded about T_c,

$$\alpha = \alpha_o (1-t), \ t = T/T_c \tag{3.16}$$

where α_o is positive constants. This is negative for $T < T_c$, vanishing linearly at T_c. Note that in view of Eq. (3.15), this assumption is consistent with the linear variation of $H_c(T)$ with (1-t). Putting these temperature variational of α and β into Eq. (13.13), gives

$$|\Psi|^2 \sim (1-t) \tag{3.17}$$

near T_c . This is consistent with correlating $|\Psi|^2$ with n_s, the density of superelectron in the London theory , since $n_s \sim \lambda^{-2} \sim (1-t)$ near T_c.

As a second simple case , consider a one - dimensional , where Ψ varies but $\vec{h}$ vanishes , Eq. (3.7) reduces to

$$-\frac{\hbar^2}{2m^*}\frac{d^2\Psi}{dz^2} + \alpha\Psi + |\Psi|^2\Psi = 0 \tag{3.18}$$

If we introduce a normalized wave function

$$f(z) = \Psi(z)/|\Psi_\infty| \tag{3.19}$$

where $$|\Psi_\infty| = \left(\frac{\alpha}{\beta}\right)^{1/2} \tag{3.20}$$

and the notation Ψ_∞ is conventionally used because Ψ approaches this value at infinitely deep in the interior of the superconductor , where it is screened from any surface fields or currents . A combination of Eqs. (3.18) and (3.19) given

$$\frac{\hbar^2}{2m^*|\alpha|} \cdot \frac{d^2f}{dz^2} - f - f^3 = 0 \tag{3.21}$$

This makes it natural to define characteristic length $\xi(T)$, for spatial variation of the order parameter

$$\xi(T) = \frac{\hbar^2}{2m^*|\alpha|} \sim \frac{1}{1-t} \tag{3.22}$$

This length is known as (Ginzburg - Landau) coherence length . Note that this $\xi(T)$ is certainly not same length as Pippard's ξ , which we used in our discussion of the nonlocal electrodymamies , since this $\xi(T)$ diverges at T_c . In term of $\xi(T)$, Eq. (3.12) becomes

$$\xi^2(T)\frac{d^2f}{dz^2} + f - f^3 = 0 \qquad (3.23)$$

multiplying Eq.(3.23) by df/dz and integrating with respect to z, give

$$\xi^2(T)\left(\frac{df}{dz}\right)^2 + f^2 - \frac{1}{2} f^4 = \text{const} \qquad (3.24)$$

with boundary conditions

$$f(0) = 0$$

$$\lim_{z\to\infty} f(z) = 1$$

thus , constant = 1/2 . This given

$$\xi^2(T)\left(\frac{df}{dz}\right)^2 = \frac{1}{2}(1 - f^2)^2 \qquad (3.25)$$

If f increases with increasing z , must take the positive square root must be taken Eq. (3.25) becomes

$$\frac{df}{dz} = \frac{1 - f^2}{\sqrt{2}\,\xi} \qquad (3.26)$$

The solution satisfying our boundary conditions is

$$f(z) = \tanh\left(\frac{z}{\sqrt{2}\,\xi}\right) \tag{3.27}$$

Just as in the electromagnetic penetration , the spatial variation of f is confined to a region $z \approx \xi$, because $1-|f|$ vanishes exponentially for $z \gg \xi$.

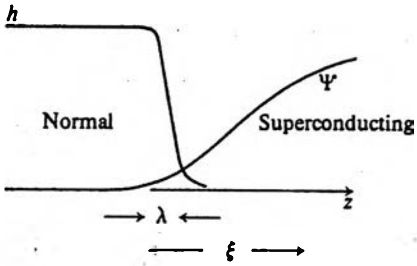
For a final example , consider and applied magnetic field with an essentially uniform order parameter $|\Psi| = |\Psi_\infty|$ The supercurrent Eq. (3.11) then reduces to

$$\vec{J}(\vec{x}) = -\frac{(e^*)^2}{mc} n_s^* \vec{A}(\vec{x}) = -\frac{e^2 n_s}{mc} \vec{A}(\vec{x}) \tag{3.28}$$

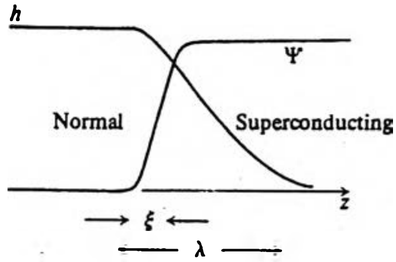
which takes precisely the form of the London equation (2.16). The penetration depth for magnetic fields follows immediately as

$$\lambda^2(T) = \frac{m^* c^2}{4\pi n^*(T)\,(e^*)^2} \sim \frac{1}{1-t} \tag{3.29}$$

In certain physical situations Ψ essentially vanishes at the boundary of the superconducting region (z = 0) , as illustrated in Fig 3.1 . The penetration depth λ provides the spatial variation in the electromagnetic effects .



(a) K << 1



(b) K >> 1

Figure 3.1 Surface region between normal and superconducting material for
(a) $\lambda \ll \xi$ ($K \ll 1$) and (b) $\lambda \gg \xi$ ($K \gg 1$).

It is important to emphasize that coherence length $\xi(T)$ and penetration length $\lambda(T)$ are both phenomenological quantities defined in terms of the constants α and β. It is conventional to introduce the Ginzburg - Landau parameter

$$K = \frac{\lambda(T)}{\xi(T)} \qquad (3.30)$$

which is independent of temperature near T_c. With the preceding definitions, a simple calculation yields.

$$K = \frac{\sqrt{2}\, e^*}{\hbar c} H_c(T)\, \lambda(T) \qquad (3.31)$$

$$K = \frac{m^* c}{\hbar c} \left(\frac{\beta}{2\pi}\right)^{1/2} \qquad (3.32)$$

each of which is useful in applications. In particular , Eq. (3.13) relates K to the measurable quantities H_c and λ , it may also be rewritten as

$$H_c = \frac{\phi_o}{2\pi\sqrt{2}\,\lambda\,\xi} \qquad (3.33)$$

where $\phi_o = hc/e^*$ is the flux quantum.

3.1.4 Flux Quantization

The Ginzburg - Landau expression Eq. (3.11) for the supercurrent allows us to verify Landau 's prediction of a quantized fluxoid in a superconductor . The order parameter may be written as

$$\Psi = |\Psi| e^{i\varphi} \tag{3.34}$$

where φ is real , and substitution into Eq. (3.11) gives

$$\vec{J} = \frac{e^{*}\hbar}{m^{*}} |\Psi|^{2} \vec{\nabla}\varphi - \frac{(e^{*})^{2} |\Psi|^{2}}{mc^{*}} \vec{A} \tag{3.35}$$

or

$$\vec{A} + \frac{\vec{J} m^{*} c}{(e^{*})^{2} |\Psi|^{2}} = \frac{\hbar c}{e^{*}} \vec{\nabla}\varphi \tag{3.36}$$

Integrate this equation around a closed path C lying wholly in the superconductor (Fig 2.6)

$$\oint_{c} \vec{A} . dl + \frac{m^{*}c}{(e^{*})^{2}} \oint_{c} \underline{\vec{J} . d\vec{l}} = \frac{\hbar c}{e^{*}} \oint_{c} \vec{\nabla}\varphi . d\vec{l} \tag{3.37}$$

The first term on the left may be rewritten with Stokes'theorem , if Ψ is assumed to be single valued , the integral on the right , it follows that

$$\int_{c} \vec{\nabla}\varphi . d\vec{l} = 2\pi n \tag{3.38}$$

where n is an integer. Hence Eq. (3.37) becomes

$$\int \vec{h} \cdot d\vec{s} + \frac{m^{*}c}{(e^{*})^{2}} \oint \vec{J} . d\vec{l} = \frac{nhc}{e^{*}} \tag{3.39}$$

Comparison which Eq. (2.34) shows that the left side is London's fluxiod ϕ generalized to nonunifrom systems , and we conclude that ϕ is quantized in units of ϕ_o = hc/e^* . If the path is closen such that $\vec{J}$ vanishes on C , or that $\vec{J}$ is perpendicular to $\vec{dl}$, then the middle term vanishes , and we obtain the quantization of magnetic flux,

$$\int \vec{h} . d\vec{s} = n \frac{hc}{e^*} \tag{3.40}$$

in units of $\phi_o = hc/e^* \simeq 2 \times 10^{-7}$ G - cm^2 where e^* has been taken as 2e , in agreement with experiment (See Sec. 2.2.3).

3.1.5 Surface Energy

The significance of the Ginzburg - Landau parameter K is most easily understood by studying the energy associated with a surface separating normal and superconducting material (See Fig 3.1) an applied field H_c parallel to the surface , since the Gibbs free energy deep in the normal region

$$\lim_{z \to -\infty} G(z) = G_{no} - \frac{H_c^2}{8\pi} \tag{3.41}$$

then equals that deep in the superconducting region

$$\lim_{z \to \infty} G(z) = G_{so} \tag{3.42}$$

(compare Eq.(2.49)). The possibility of a surface energy arises in the following way from the occurrence of the two lengths λ

and ξ . If the sample were entirely normal or entirely superconducting , the Gibbs free energy per unit area would be $\int_{-\infty}^{\infty} dz(-H_c^2 / 8\pi)$. In the surface region , however , the flux is expelled for $z \gg \lambda$, while the condensation energy builds up for $z \gg \xi$. Hence the true Gibbs free energy per unit area is given approximately as the sum of two terms

$$\simeq \int_{-\infty}^{\lambda} dz \left(\frac{-H_c^2}{8\pi}\right) + \int_{\xi}^{\infty} dz \left(\frac{-H_c^2}{8\pi}\right)$$

By definition , the surface energy σ_{ns} is the difference between the actual Gibbs free energy per unit area and the value that would occur if the sample were uniformly normal or superconducting

$$\sigma_{ns} \simeq -\frac{H_c^2}{8\pi}\left(\int_{-\infty}^{\lambda} dz + \int_{\xi}^{\infty} dz - \int_{-\infty}^{\infty} dz\right)$$

$$\simeq (\xi - \lambda)\frac{H_c^2}{8\pi} \qquad (3.43)$$

We see that σ_{ns} is positive for $K \ll 1$ but negative for $K \gg 1$

The Ginzbury - Landau theory permits us to study the surface energy in greater detail. Consider a one - dimensional geometry (Fig 3.1) with the magnetic field $\vec{h}(z) = h(z)\,\hat{x}$ and vector potential $\vec{A}(z) = A(z)\,\hat{y}$. In the present problem , Ψ may be chosen real in an appropriate gauge

$$-\frac{\hbar^2}{2m^*}\frac{d^2\Psi}{dz^2} - \alpha\Psi + \beta\Psi^3 - (e^*)^2\frac{A^2\Psi}{2m^*c^2} = 0 \qquad (3.44a)$$

$$-\frac{c}{4\pi}\frac{dh}{dz} = \frac{(e^*)^2\Psi^2 A}{m^* c} \qquad (3.44b)$$

$$\frac{dA}{dz} = -h \qquad (3.44c)$$

The corresponding boundary conditions

$$\begin{aligned} \Psi &= o \text{ and } h = H_c \text{ as } z \to -\infty \\ \Psi &= \Psi_\infty \text{ and } h = o \text{ as } z \to +\infty \end{aligned} \qquad (3.45)$$

guarantee that Eqs. (3.41) and (3.42) are satisfied. With the same definition of the surface energy , we fine

$$\begin{aligned} \sigma_{ns} &= \int_{-\infty}^{\infty} dz\,[\,G(z) - G_{so}\,] \\ &= \int_{-\infty}^{\infty} dz\,[\,G(z) - G_{no} + \frac{H_c^2}{8\pi}\,] \\ &= \int_{-\infty}^{\infty} dz\,[F(z) - \frac{h(z)\,H_c}{4\pi} - F_{no} + \frac{H_c^2}{8\pi}\,] \\ &= \int_{-\infty}^{\infty} dz\,[\,\alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \frac{1}{2m^*}\left|(-i\hbar\nabla - \frac{e^*\vec{A}}{c})\Psi\right|^2 + \frac{(H_c - h)^2}{8\pi}] \end{aligned} \qquad (3.46)$$

Here the third and fourth lines are obtained with Eqs (3.5) and (3.3) , respectivity. This can be further simplified by noting that if one multiplies the Ginzburg Landau differential equation (3.7)by Ψ^* and integrates over all z by parts , one obtains the identity

$$\int_{-\infty}^{\infty} dz\,[\,\alpha|\Psi|^2 + \beta|\Psi|^4 + \frac{1}{2m^*}\left|(-i\hbar\nabla - \frac{e^*\vec{A}}{c})\Psi\right|^2\,] = 0$$

Subtracting this from Eq. (3.46) , we obtain the concise from

$$\sigma_{ns} = \int_{-\infty}^{\infty} dz \, [-\frac{1}{2}\beta|\Psi|^4 + \frac{(H_c - h)^2}{8\pi}] \tag{3.47}$$

Both Eqs. (3.46) and (3.47) are exact , the former is also variationally correct while the latter use of the exact field equations , is considerably simpler.

It is conventional to characterize the surface energy with a length δ

$$\sigma = \frac{H_c^2}{8\pi}\delta \tag{3.48}$$

where

$$\delta = \int_{-\infty}^{\infty} dz \, [\left|\frac{\Psi}{\Psi_\infty}\right|^4 + (1-\frac{h}{H_c})^2] \tag{3.49}$$

has been rewritten with the aid of Eqs. (3.15) and (3.20). Although a numerical solution of Eqs. (3.44) and (3.45)is needed to evaluate δ for arbitrary K , spectial limiting cases, using Eqs. (3.19) , (3.27) and (3.45) . For these cases, the following exact results have been obtained as follows.

$$\delta = \frac{4\sqrt{2}}{3}\xi \simeq 1.89\,\xi \quad , \quad K \ll \frac{1}{\sqrt{2}} \tag{3.50a}$$

$$\delta = 0 \quad , \quad K = \frac{1}{\sqrt{2}} \tag{3.50b}$$

$$\delta = -\frac{3}{8}(\sqrt{2}-1)\lambda \simeq -1.104\lambda, \quad K \gg \frac{1}{\sqrt{2}} \tag{3.50c}$$

These results support our qualitative reasoning that δ should be of the order of $(\xi - \lambda)$ in Eq. (3.43).

The surface energy is important in determining the behavior of a superconductor in an applied magnetic field , and a material is conventionally classified as type I or type II according as is positive or negative. Comparison with Eq. (3.50) yields the following criterion

$$\text{Type I :} \qquad K < \frac{1}{\sqrt{2}}, \xi(T) > \sqrt{2}\,\lambda(T) \quad \sigma_{ns} > o \qquad (3.51a)$$

$$\text{Type II :} \qquad K > \frac{1}{\sqrt{2}}, \xi(T) < \sqrt{2}\,\lambda(T) \quad \sigma_{ns} < o \qquad (3.51b)$$

The positive surface energy of type - I materials (most elements) keeps the sample spatially homogeneous , and it exhibits a complete Meissner effect for all $H < H_c$. In contrast , type - II materials (most alloys and compounds) tend to break up into microscopic domains as soon as the magnetic field exceeds a lower critical field H_{c1} , which is always lass than H_c . For $H > H_{c1}$, magnetic flux penetrates the sample in the from of supercurrent vortices that surround a normal core containing one flux quantum (hc / 2e) , and the sample is said to be in the vortex state (or mixed state) . Structure of the vortex state illustrated in figure 3.2.

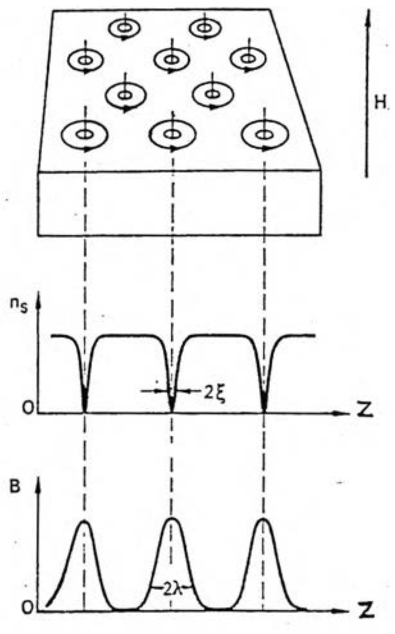


Figure 3.2 Vortex state in applied magnetic field of strength just $H_{c1} < H < H_{c2}$

(a) Lattice of cores and associated vortices.

(b) Variation with position of concentration of superelectrons.

(c) Variation of flux density

This state persists up to an upper critecal field $H_{c2} = \sqrt{2}\, KH_c$ above which the sample becomes the normal state. We sketched magnetization curves in figures 3.3 and 3.4.

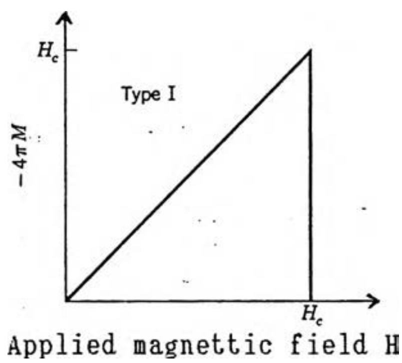


Figure 3.3 Magnetization versus applied magnetic field for a bulk superconductor exhibiting a complete Meissner effect (perfect diamagnetism).

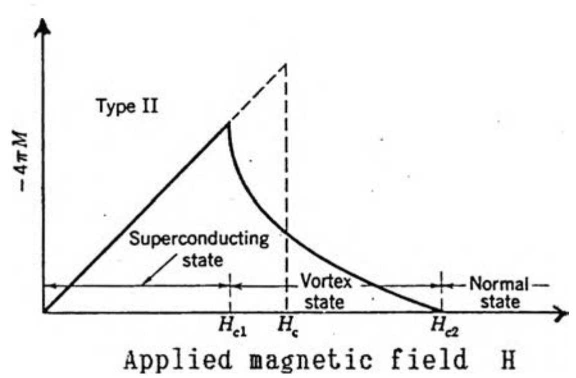


Figure 3.4 Superconducting magnetization curve of a type II superconductor.

The possibility of type - II superconductivity was first suggested by Abrikosov (50) , who used the Ginzburg - Landau theory to study the vortex state in detail. We regret that this vast subject cannot be included here, and we must refer the reader to other sources (51).

3.2 Microscopic Ginzburg - Landau Theory

For in this section , we now present Gorkov's microscopic derivation of the Ginzburg - Landau equations (16). This calcutation determines the phenomenological constant directly in terms of the microscopic parameter ; it also clarifies the range of validity of the equations and allows direct extensions to more complicated systems such as superconducting alloys. Indeed , many microscopic calculation now proceed by deriving approximate Ginzburg - Landau equations , whose solution is considerably simpler than that of the original equations.

3.2.1 General Formulation

We take the model BCS Hamiltonian of Eqs. (2.73) (2.74) and (2.83) to describe the electrons without the magnetic field. If we add the interaction with the magnetic field , discussed in Ref. 52 , the Hamiltonian operater can be rewritten in terms of these field operators as follows

$$\hat{K} = \hat{K}_o + \hat{V}$$

$$= \int d^3x\, \hat{\Psi}^{+}(\vec{x})\, T(\vec{x})\hat{\Psi}(\vec{x}) + \frac{1}{2}\int\int d^3x\, d^3x'\, \hat{\Psi}^{+}(\vec{x})\, \hat{\Psi}(\vec{x}')\, V(\vec{x},\vec{x}')\, \hat{\Psi}(\vec{x}')\hat{\Psi}(\vec{x})$$

$$= \int d^3x\, \hat{\Psi}^{+}_{\alpha}(\vec{x})\, \{\frac{1}{2m}[-i\hbar\nabla - \frac{e\vec{A}}{c}(\vec{x})]^2 - \mu\}\, \hat{\Psi}_{\alpha}(x)$$

$$-\frac{1}{2} V \int d^3x\, \hat{\Psi}^{\dagger}_{\alpha}(\vec{x})\, \hat{\Psi}^{\dagger}_{\beta}(\vec{x})\, \hat{\Psi}_{\beta}(\vec{x})\, \hat{\Psi}_{\alpha}(\vec{x}) \qquad (3.52)$$

where μ is the chemical potential , $\vec{A}(\vec{x})$ is the vector potential.

As an introduction to our subsequent development, we first review the Hartree - Fock theory , which may be obtained by approximating the exact interaction operator $\hat{V}$ in Eq. (3.52) as a bilinear form

$$\hat{V} \approx \hat{V}_{HF} = -V \int d^3x \, [\langle \hat{\psi}^{\dagger}_{\alpha}(\vec{x}) \hat{\psi}_{\alpha}(\vec{x}) \rangle_{HF} \hat{\psi}^{\dagger}_{\beta}(\vec{x}) \hat{\psi}_{\beta}(\vec{x}) - \langle \hat{\psi}^{\dagger}_{\alpha}(\vec{x}) \hat{\psi}_{\beta}(\vec{x}) \rangle_{HF} \hat{\psi}^{\dagger}_{\beta}(\vec{x}) \hat{\psi}_{\alpha}(\vec{x})] \tag{3.53}$$

Here the angular brackets denote an ensemble average with the density operator

$$\hat{\rho}_{HF} = \frac{e^{-\beta \hat{K}_{HF}}}{T_r e^{-\beta \hat{K}_{HF}}} \tag{3.54a}$$

and

$$\hat{K}_{HF} = \hat{K}_o + \hat{V}_{HF} \tag{3.54b}$$

In this approach , $\hat{K}_{HF}$ is used to define a finite - temperatrue Heisenberg operator

$$\hat{\psi}_{K\gamma}(\vec{x}\tau) = e^{\hat{K}_{HF}\tau/\hbar} \hat{\psi}_{\gamma}(\vec{x}) e^{-\hat{K}_{HF}\tau/\hbar}$$

with the equation of motion

$$\hbar \frac{\partial \hat{\psi}_{K\gamma}(\vec{x}\tau)}{\partial \tau} = -[\frac{1}{2m} (-i\hbar\nabla - \frac{e}{c}\vec{A})^2 - \mu] \hat{\psi}_{K\gamma}(\vec{x}\tau)$$

$$+ V \langle \hat{\psi}^{\dagger}_{\alpha}(\vec{x}) \hat{\psi}_{\alpha}(\vec{x}) \rangle_{HF} \hat{\psi}_{K\gamma}(\vec{x}\tau) - V \langle \hat{\psi}_{\alpha}(\vec{x}) \hat{\psi}_{\gamma}(\vec{x}) \rangle_{HF} \hat{\psi}_{K\alpha}(\vec{x}\tau)$$

The corresponding single - particle Green's function

$$G_{\alpha\beta}(\vec{x}\tau,\vec{x}'\tau') = -\langle T_\tau\left[\hat{\psi}_{K\alpha}(\vec{x}\tau)\,\hat{\psi}^{+}_{K\beta}(\vec{x}'\tau')\right]\rangle_{HF}$$

The self - consistency here appears through $\hat{V}_{HF}$, which both determines and is determined by Eq. (51.4).

The precise structure of $\hat{V}_{HF}$ can be understood by seeking a linear approximation to the exact equations of motion. Since the commutator $[\hat{V},\ \hat{\psi}_\alpha(\vec{x})\ ]$ contains three filed operators , it must be approximated by alinear from $[\hat{V},\ \hat{\psi}_\alpha(\vec{x})] \rightarrow f_{\alpha\beta}\,\hat{\psi}_\beta(\vec{x})$ where the $f_{\alpha\beta}$ are c-mumber coefficients. This replaecment has the consequence that $\{[\hat{V},\ \hat{\psi}_\alpha(\vec{x})],\ \ \hat{\psi}^{+}_\beta(\vec{y})\} \rightarrow f_{\alpha\beta}\,\delta(\vec{x}-\vec{y})$. In face , the left side of this relation is still quadratic in the field operators. We therefore replace it by its ensembly average to obtain the linearized theory , which provides a prescription for determinining $f_{\alpha\beta}$. The approximate form $\hat{V}_{HF}$ in Eq. (3.53) is chosen to reproduce the corresponding linear equations.

The foregoing discussion must now be generalized to include the one essentially new feature of a superconductor , namely the possibility that two electrons of opposite spins can form a self - bound Cooper pair. As a model for this phenomenon , we add two extra terms representing the pairing amplitudes to Eq. (3.53)

$$\hat{V} \approx \hat{V}_{HF} - \frac{1}{2} V \int d^3x \,[\langle \hat{\psi}^{+}_\alpha(\vec{x})\,\hat{\psi}^{+}_\beta(\vec{x})\rangle\,\hat{\psi}_\beta(\vec{x})\,\hat{\psi}_\alpha(\vec{x})$$

$$+\ \hat{\psi}^{+}_\alpha(\vec{x})\,\hat{\psi}^{+}_\beta(\vec{x})\,\langle\,\hat{\psi}_\beta(\vec{x})\,\hat{\psi}_\alpha(\vec{x})\,\rangle] \qquad (3.55)$$

Since the Cooper pair has spin zero , the indices α and β in Eq. (35.5) must refer to opposite spin projections , and the total effective hamiltonian becomes

$$\hat{K}_{eff} = \hat{K}_o - V \int d^3x [\langle \hat{\psi}^+_\downarrow(\vec{x}) \; \hat{\psi}^+_\uparrow(\vec{x}) \rangle \; \hat{\psi}_\uparrow(\vec{x}) \; \hat{\psi}_\downarrow(\vec{x}) + \hat{\psi}^+_\downarrow(\vec{x}) \; \hat{\psi}^+_\uparrow(\vec{x}) \; \langle \; \hat{\psi}_\uparrow(\vec{x}) \; \hat{\psi}_\downarrow(\vec{x}) \; \rangle] \tag{3.56}$$

which forms the basis for the BCS theory. The theory is self - consistent , because the angular brackets are interpreted as an ensemble average evaluabed with $\hat{K}_{eff}$, in particular , quantities such as

$$\langle \hat{\psi}^+_\downarrow(\vec{x}) \; \hat{\psi}^+_\uparrow(\vec{x}) \rangle = \frac{T_r [e^{-\beta \hat{K}eff} \; \hat{\psi}^+_\downarrow(\vec{x}) \; \hat{\psi}^+_\uparrow(\vec{x})]}{T_r \; e^{-\beta \hat{K}eff}}$$

do not vanish , because $[\hat{K}_{eff} \; , \; \hat{N}] \neq 0$

we now introduce Heisenberg operators

$$\hat{\psi}^+_{k\downarrow}(\vec{x}\tau) = e^{\hat{K}eff \; \tau/\hbar} \; \hat{\psi}_\uparrow(\vec{x}) \; e^{-\hat{K}eff \; \tau/\hbar} \tag{3.57a}$$

$$\hat{\psi}^+_{k\downarrow}(\vec{x}\tau) = e^{\hat{K}eff \; \tau/\hbar} \; \hat{\psi}^+_\downarrow(\vec{x}) \; e^{-\hat{K}eff \; \tau/\hbar} \tag{3.57b}$$

Satisfying the linear equations of motion

$$\frac{\hbar \partial \hat{\psi}_{k\uparrow}}{\partial \tau} = -[\frac{1}{2m} (-i\hbar\nabla - \frac{e\vec{A}}{c})^2 - \mu] \; \hat{\psi}_{k\uparrow} \; v \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle \; \hat{\psi}^+_{k\downarrow} \tag{3.58a}$$

$$\frac{\hbar \partial \hat{\psi}^+_{k\downarrow}}{\partial \tau} = [\frac{1}{2m} \; (i\hbar \; \nabla - \frac{e\vec{A}}{c})^2 - \mu] \; \hat{\psi}^+_{k\downarrow} - v \; \langle \hat{\psi}^+_\downarrow \hat{\psi}^+_\uparrow \rangle \; \hat{\psi}_{k\uparrow} \tag{3.58b}$$

As the final step in these , we define a single - particle Green's function

$$G(\vec{x}\tau,\ \vec{x}'\tau') = -\langle T_\tau [\ \hat{\psi}_{k\uparrow}(\vec{x}\tau)\ \hat{\psi}^{+}_{k\uparrow}(\vec{x}'\tau')]\ \rangle \tag{3.59}$$

where a particular choice of spin indices has been made to simplify the motion. Differentiate G with respect to τ. The derivative acts both on the field operator and on the step functions implicit in the "time" ordering , which yields

$$\begin{aligned}\hbar \frac{\partial}{\partial \tau} G\ (\vec{x}\tau,\ \vec{x}'\tau') &= -\hbar\delta(\tau-\tau')\langle\{\ \hat{\psi}_{k\uparrow}(\vec{x}\tau),\ \hat{\psi}^{+}_{k\uparrow}(\vec{x}'\tau')\}\rangle \\ &\quad - \langle\ T_\tau[\ \frac{\hbar\partial\hat{\psi}_{k\uparrow}(\vec{x}\tau)}{\partial\tau}\ \hat{\psi}^{+}_{k\uparrow}(\vec{x}'\tau'\ )]\rangle \\ &= -\hbar\ \delta(\vec{x}-\vec{x}')\ \delta(\tau-\tau') - [\frac{1}{2m}\ (-\ i\hbar\nabla - \frac{e\vec{A}}{c})^2 - \mu] \end{aligned}$$

$$\times G(\vec{x}\tau,\ \vec{x}'\tau') + v\ \langle\hat{\psi}_{\uparrow}(\vec{x})\ \hat{\psi}_{\downarrow}(\vec{x})\ \rangle\ \langle\ T_\tau\ [\ \hat{\psi}^{+}_{k\downarrow}(\vec{x}\tau)\ \ \hat{\psi}^{+}_{k\uparrow}(\vec{x}'\tau')]\rangle \tag{3.60}$$

we are thus led to consider two new functions

$$J\ (\vec{x}\tau,\ \vec{x}'\tau') = -\langle T_\tau [\ \hat{\psi}_{k\uparrow}(\vec{x}\tau)\ \hat{\psi}_{k\downarrow}(\vec{x}'\tau')\ ]\rangle \tag{3.61a}$$

$$J^{+}(\vec{x}\tau,\ \vec{x}'\tau') = -\langle T_\tau [\ \hat{\psi}^{+}_{k\downarrow}(\vec{x}\tau)\ \ \hat{\psi}^{+}_{k\uparrow}(\vec{x}'\tau')\ ]\rangle \tag{3.61b}$$

and Eq. (3.60) becomes

$$[\ -\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}\ (-i\hbar\nabla - \frac{e\vec{A}}{c}\ ) + \mu\ ]\ G\ (\vec{x}\tau,\ \ \vec{x}'\tau') - v\langle\hat{\psi}_{\uparrow}(\vec{x})\ \hat{\psi}_{\downarrow}(\vec{x})\ \rangle$$

$$\times J^{+}\ (\vec{x}\tau, \vec{x}'\tau') = \hbar\ \delta(\vec{x}-\vec{x}')\ \delta(\tau-\tau') \tag{3.62}$$

In the usual case of a time - independent hamiltonion, the functions G , J and J^{+} depend only on the difference $\tau-\tau'$, and it is convenient to introduce the abbreviation

$$\Delta(\vec{x}) \quad VJ\,(\vec{x}\tau,\ \vec{x}'\tau') = -V\langle \hat{\psi}_{\uparrow}(\vec{x})\ \hat{\psi}_{\downarrow}(\vec{x})\rangle = V\langle \hat{\psi}_{\downarrow}(\vec{x})\ \hat{\psi}_{\uparrow}(\vec{x})\rangle \tag{3.63}$$

which defines the gap function $\Delta(\vec{x})$. A combination of Eqs. (3.62)and (3.63) yields

$$[-\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}(-i\hbar\nabla - \frac{e\vec{A}}{c})^2 + \mu\,]\ G\,(\vec{x}\tau,\ \vec{x}'\tau') + \Delta(\vec{x})\ J^+\,(\vec{x}\tau\ ,\ \vec{x}'\tau')$$

$$= \quad \hbar\ \delta(\vec{x}-\vec{x}')\ \delta(\tau-\tau') \tag{3.64}$$

In a similar way , the functions J and J^+ are easily seen to obey the equations of motion

$$[-\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}(-i\hbar\nabla - \frac{e\vec{A}}{c})^2 + \mu]\ J\,(\vec{x}\tau,\ \vec{x}'\tau')$$

$$= \quad -V\langle \hat{\psi}_{\uparrow}(\vec{x})\ \hat{\psi}_{\downarrow}(\vec{x})\rangle\langle T_{\tau}\,[\,\hat{\psi}^{+}_{k\downarrow}(\vec{x}\tau)\ \hat{\psi}_{k\downarrow}(\vec{x}'\tau')]\,\rangle$$

$$= \quad \Delta\,(\vec{x})\ G\,(\vec{x}\tau,\ \vec{x}'\tau') \tag{3.65}$$

$$[\hbar\frac{\partial}{\partial\tau} - \frac{1}{2m}\ (i\hbar\nabla - \frac{e\vec{A}}{c})^2 + \mu\,]\ J^+\,(\vec{x}\tau,\ \vec{x}'\tau')$$

$$= \quad g\,\langle\ \hat{\psi}^{+}_{\downarrow}(\vec{x})\ \hat{\psi}^{+}_{\uparrow}(\vec{x})\rangle\langle T_{\tau}\,[\,\hat{\psi}_{k\uparrow}(\vec{x}\tau)\ \hat{\psi}^{+}_{k\uparrow}(\vec{x}'\tau')]\,\rangle$$

$$= \quad \Delta^{*}\,(\vec{x})\ G\,(\vec{x}\tau,\ \vec{x}'\tau') \tag{3.66}$$

For most purposes , it is sufficient to consider Eq. (3.64) and (3.66) as a pair of coupled equations for G and J^+. Nevertheless ,there are definite advantages to combining the three equations in a single matrix equation. Introduce a two - component field operator

$$\Psi_{K}(\vec{x}\tau) \equiv \begin{bmatrix} \hat{\psi}_{K\uparrow}(\vec{x}\,\tau) \\ \psi^{+}_{K\downarrow}(\vec{x}\,\tau) \end{bmatrix} \tag{3.67}$$

and a 2x2 matrix Green's function

$$G(\vec{x}\tau,\ \vec{x}'\tau') = -\langle T_{\tau}[\hat{\Psi}_{K}(\vec{x}\tau)\,\hat{\Psi}^{+}_{K}(\vec{x}'\tau')]\rangle \tag{3.68}$$
$$= \begin{bmatrix} G(\vec{x}\tau,\ \vec{x}'\tau') & J(\vec{x}\tau,\ \vec{x}'\tau') \\ J^{+}(\vec{x}\tau,\ \vec{x}'\tau') & -G^{+}(\vec{x}'\tau',\ \vec{x}\tau) \end{bmatrix}$$

The corresponding equation of motion becomes

$$D_{\vec{x}\tau}\, G(\vec{x}\tau,\vec{x}'\tau') = \hbar 1\, \delta(\vec{x}-\vec{x}')\, \delta(\tau-\tau') \tag{3.69}$$

where $D_{\vec{x}\tau}$ is a matrix differential operator

$$D_{\vec{x}\tau} = \begin{bmatrix} -\hbar\dfrac{\partial}{\partial\tau} - \dfrac{1}{2m}\left(-i\hbar\nabla - \dfrac{e\vec{A}}{c}\right)^{2} + \mu & \Delta(\vec{x}) \\ \Delta^{*}(\vec{x}) & -\hbar\dfrac{\partial}{\partial\tau} + \dfrac{1}{2m}\left(i\hbar\nabla - \dfrac{eA}{c}\right)^{2} - \mu \end{bmatrix} \tag{3.70}$$

This matrix formulation has bee used extensively in studies of the electron - phonon interaction in superconductor. Unfortunately , it is not possible to consider these questions here , and we generally rely on the original Gorkov equations (3.64) and (3.66).

As we now , in the temperature technique all quantities are expanded in Fourier series with respect to the frequency ω_n. In almost all case of interest the hamiltonian is time independent, and the corresponding Green's functions depend only on $\tau-\tau'$. It is then useful to introduce a Fourier representation.

$$G(\vec{x}\tau,\vec{x}\tau'') = (\beta\hbar)^{-1} \; e^{-i\omega_n(\tau-\tau')} \; G(\vec{x},\vec{x}',W_n) \tag{3.71a}$$

$$J^{+}(\vec{x}\tau,\vec{x}\tau') = (\beta\hbar)^{-1} \; e^{-i\omega(\tau-\tau')} \; J^{+}(\vec{x},\vec{x}',W_n) \tag{3.71b}$$

where the choice $W_n = (2n+1)\pi/\beta\hbar$ guarantees the proper Fermi statistics. The corresponding equations of motions are

$$\left[i\hbar w_n - \frac{1}{2m}\left(-i\hbar\nabla-\frac{e}{c}\vec{A}\right)^2 + \mu \right] G(\vec{x},\vec{x}' w_n) + \Delta(\vec{x}) J^{+}(\vec{x},\vec{x},w_n)$$

$$= \hbar\,\delta(\vec{x}-\vec{x}') \tag{3.72a}$$

$$\left[-i\hbar W_n - \frac{1}{2m}\left(i\hbar\nabla - \frac{e}{c}A\right)^2 + \mu \right] J^{+}(\vec{x},\vec{x}',w_n) - \Delta^{*}(\vec{x})\, G(\vec{x},\vec{x}',w_n) = 0 \tag{3.72b}$$

which must be solved along with the self - consistency conditions

$$\begin{aligned}\Delta^{*}(\vec{x}) &= -v\,\langle \hat{\psi}^{+}_{\downarrow}(\vec{x})\; \hat{\psi}^{+}_{\uparrow}(\vec{x})\rangle = V J^{+}(\vec{x}\tau^{+},\vec{x}\tau) \\ &= \frac{v}{\beta\hbar}\sum_n e^{-iw_n\eta}\; J^{+}(\vec{x},\vec{x}',w_n)\end{aligned} \tag{3.73}$$

3.2.2 <u>Microscopic Derivation of Ginzburg - Landau Equations</u>

The properties of superconductors near the critical temperature constitute a special case , since then the size of the gap $\Delta(\vec{x})$ is small enough to cause all the equations to become much simple. We start from the pair of coupled equations (3.72) for G and J^{+}. Here we are interested in the effect of arbitrary magnetic fields.

It is convenient to introduce a new temperature Green's function $G(\vec{x},\vec{x}',W_n)$ that describes the normal state in the same magnetic field. It satisfies the equation

$$\left[i\hbar W_n + \frac{\hbar^2}{2m}\left(\nabla - \frac{ie\vec{A}(\vec{x})}{\hbar c}\right)^2 + \mu\right]G^\circ(\vec{x},\vec{x}',W_n) = \hbar\,\delta(\vec{x}-\vec{x}') \tag{3.74a}$$

$$\left[i\hbar W_n + \frac{\hbar^2}{2m}\left(\nabla + \frac{ie\vec{A}(\vec{x})}{\hbar c}\right)^2 + \mu\right]G^\circ(\vec{x},\vec{x}',W_n) = \hbar\,\delta(\vec{x}-\vec{x}') \tag{3.74b}$$

obtained from Eq. (3.72a) and its analog for $G(\vec{x},\vec{x}',w_n)$ with $\Delta = 0$. This auxiliary function enables us to rewrite Eqs. (3.72) as the following pair of coupled integral equations

$$G(\vec{x},\vec{x}',w_n) = \tilde{G}^\circ(\vec{x},\vec{x}',w_n) - \hbar^{-1}\int d^3y\, \tilde{G}^\circ(\vec{x},\vec{y},w_n)\,\Delta(\vec{y})\,J^+(\vec{y},\vec{x}',w_n) \tag{3.75a}$$

$$J^+(\vec{x},\vec{x}',w_n) = \hbar^{-1}\int d^3y\, \tilde{G}^\circ(\vec{y},\vec{x},-w_n)\,\Delta^*(\vec{y})\,G(\vec{y},\vec{x}',w_n) \tag{3.75b}$$

which are easily verified by direct substitution into the original differential equations. Note carefully the rather complicated arguments in Eq. (3.75b) , they are necessary to reproduce the structure of Eq. (3.72b) A simple manipulation of Eqs. (3.75) yields

$$\begin{aligned} G(\vec{x},\vec{x}',w_n) = {} & \tilde{G}^\circ(\vec{x},\vec{x}',w_n) - \hbar^{-2}\int d^3y\, d^3z\, \tilde{G}^\circ(\vec{x},\vec{y},W_n)\,\Delta(\vec{y}) \\ & \times\tilde{G}^\circ(\vec{z},\vec{y},-w_n)\,\Delta^*(\vec{z})\,G(\vec{z},\vec{x}',w_n) \end{aligned} \tag{3.76a}$$

$$\begin{aligned} J^+(\vec{x},\vec{x}',w_n) = {} & \hbar^{-1}\int d^3y\, \tilde{G}^\circ(\vec{y},\vec{x},-w_n)\,\Delta^*(\vec{y})\,G^\circ(\vec{y},\vec{x}',w_n) - \hbar^2\int d^3y\, d^3z \\ & \times G^\circ(\vec{y},\vec{x},-w_n)\,\Delta^*(\vec{y})\,\tilde{G}^\circ(\vec{y},\vec{z}\,w_n)\,\Delta(\vec{z})\,J^+(\vec{z},\vec{x},w_n) \end{aligned} \tag{3.76b}$$

which are exact integral equations for G and J^+ separately

Further progress depends on the assumption of small Δ , and we first concentrate on Eq. (3.75b). The second term on the right becomes a small perturbation in this limit , and an expansion gives

$$J^{+}(\vec{x},\vec{x}',w_n) = \hbar^{-1}\int d^3y\, \tilde{G}^{\circ}(\vec{y},\vec{x},-w_n)\, \Delta^{*}(\vec{y})\, \tilde{G}^{\circ}(\vec{y},\vec{x},w_n)$$

$$-\hbar^{3}\int d^3y\, d^3z\, d^3w\, \tilde{G}^{\circ}(\vec{y},\vec{x},-w_n)\, \Delta^{*}(\vec{y})\, \tilde{G}^{\circ}(\vec{y},\vec{z},w_n)\, \Delta(\vec{z})$$

$$\times\tilde{G}^{\circ}(\vec{w},\vec{z},-w_n)\, \Delta^{*}(\vec{w})\, G^{\circ}(\vec{w},\vec{x}',w_n)+\ldots \qquad (3.77)$$

Where Eq. (3.77) is combined with the self - consistent gap condition (3.73) , we obtain an integral equation for the gap function itself

$$V^{-1}\Delta^{*}(\vec{x}) = \int d^3y\, Q(\vec{x},\vec{y})\, \Delta^{*}(\vec{y})$$
$$+\int d^3y\, d^3z\, d^3w\, R(\vec{x},\vec{y},\vec{z},\vec{w})\, \Delta^{*}(\vec{y})\Delta(\vec{z})\Delta^{*}(\vec{w}) \qquad (3.78)$$

where higher - order terms have been neglected. Here the Kernels invole $\tilde{G}^{o}$ and thus depen only on the properties of the normal metal

$$Q(\vec{x},\vec{y}) = (\beta\hbar^{2})^{-1}\sum_{n} \tilde{G}^{\circ}(\vec{y},\vec{x},-w_n)\, \tilde{G}^{\circ}(\vec{y},\vec{x},w_n) \qquad (3.79)$$

$$R(x,y,z,w) = -(\beta\hbar^{4})^{-1}\sum_{n} \tilde{G}^{\circ}(\vec{y},\vec{x},-w_n)$$

$$\times\tilde{G}^{\circ}(\vec{y},\vec{z},-w_n)\, \tilde{G}^{\circ}(\vec{w},\vec{z},-w_n)\, \tilde{G}^{\circ}(\vec{w},\vec{x},w_n) \qquad (3.80)$$

We see that the assumption of small $|\Delta|$ only leads to a nonlinear integral equation. The simpler differential structure of the Ginzburg - Landau equations requires the additional and separate assumption $(T_c-T)/T_c \ll 1$, since Δ^{*} and $\vec{A}$ then very slowly with respect to the range of the kernels Q and R . These two conditions are physically quite distinct , for a sufficiently strong magnetic field can render Δ^{*} small, even at $T = 0$.

It is now neecssary to examine the Kernels Q and R. As a first step , we evaluate the normal - state Green'a function G° in the absence of a magnetic field. This function depends only on $\vec{x}-\vec{x}$ and is precisely that studied temperature Green's functions

$$\overset{\circ}{G}(\vec{x},W_n) = \hbar\,(2\pi)^{-3}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\,(i\,\hbar\,W_n - \xi_K)^{-1} \tag{3.81}$$

where

$$\xi_K = \frac{\hbar^2 k^2}{2m} - \mu$$

Although the complete spatial dependence is rather complicated, the relevant lengths in Eqs. (3.78) to (3.79) are all much longer than interatomic dimensions, and it is therefore permissible to assume $k_F x \gg 1$. In addidion if the discrete frequency satisfies the restriction $|\hbar\omega_n| \ll \mu$, then the dominant contribution arises from the vicinity of the Fermi surface , and we find

$$G^{\circ}(\vec{x}\,w_n) \approx \hbar\,N(o)\int \frac{d\xi}{(i\hbar\omega_n - \xi)}\,j_o\,(kx)$$

$$\approx \frac{\hbar\,N\,(o)}{2iK_F X}\int \frac{d\xi}{(i\hbar\,W_n - \xi)}\left\{\exp\left(i\,(K_F + \frac{\xi}{\hbar v_F})x\right) - \exp\left(-i\,(K_F + \frac{\xi}{\hbar v_F})x\right)\right\}$$

$$= -\frac{\pi\hbar N(o)}{K_F\,x}\,\exp\left(ik_F x\,\mathrm{sgn}w_n - x\,\frac{|w_n|}{v_F}\right) \tag{3.82}$$

This last restriction ($|hw_n| \ll \mu$) is fully justified in practice, because the terms omitted make a negligible contribution to the sum over n in Eqs.(3.79) and (3.80). Varies with the natural length $\lambda(T)$. In contrast, G° oscillates with a much shorter length K^{-1}_F. so the $\vec{A}$ can be considered locally constant over many wavelengths. Gorkov thus makes an eikonal (phase-integral) approximation, assuming that the dominant efffect of the magnetic field can be included in a slowly varying envelope function φ

$$\tilde{G}^{o}\ (\vec{x},\vec{x}'\ w_n)\quad e^{i\varphi(\vec{x},\vec{x}')}\ G^{o}\ (\vec{x},-\vec{x},w_n) \tag{3.83}$$

The contribution of $\vec{A}$ is negligible for $\vec{x}\approx\vec{x}'$; hence ϕ is chosen to satisfy

$$\phi\ (\vec{x},\vec{x}) = 0$$

Direct calculation with Eq. (3.83) gives

$$\left(\nabla-\frac{ie\vec{A}}{hc}\right)^2\tilde{G}^{o} = e^{i\phi}\left\{\nabla^2G^{o} + 2i\ (\nabla\phi-\frac{e\vec{A}}{\hbar c})\ .\ \nabla G^{o} + [i\nabla^2\phi\right.$$

$$\left. - \frac{ie(\nabla\cdot\vec{A}\ )}{\hbar c} - (\nabla\phi-\frac{e\vec{A}}{\hbar c})^2]\ \ G^{o}\right\} \tag{3.85}$$

Here the terms are grouped in approximate ascending powers of $eA/\hbar cK_F$ because G^{o} varies with the characteristic length K_F. Since $\vec{A}$ is of order $\lambda\vec{H}$, this parameter may be rewritten as $\lambda eH/\hbar cK_F$, which is small for all magnetic fields of interest. In consequence, we neglect the last term of Eq.(3.85) entirely, while ϕ is determined from the condition

$$[\ \nabla\cdot\phi\ (\ \vec{x},\vec{x}'\ ) - \frac{e\vec{A}\ (\vec{x})}{\hbar c}\ ]\cdot(\ \vec{x}-\vec{x}'\ ) = 0 \tag{3.86}$$

Given $\vec{A}(\vec{x})$, this first - order differential equation can always be integrated.

We now return to kernel $Q(\vec{x},\vec{y})$. A combinition of Eqs. (3.79) and (3.83) gives

$$Q(\vec{x},\vec{y}) = (\beta\hbar^2)^{-1}\sum_n e^{2i\varphi(\vec{y},\vec{x})}\, G^{\circ}(\vec{y}-\vec{x},-W_n)\, G^{\circ}(\vec{y}-\vec{x},\omega_n)$$

$$= e^{2i\varphi(\vec{y},\vec{x})}\, Q^{\circ}(\vec{x}-\vec{y}) \qquad (3.87)$$

which defines the kernal Q° in the absence of a magnetic field. This function is readily evaluated with Eq. (3.82) and the relation

$$\sum_n f(|\omega_n|) = 2\sum_{n=0}^{\infty} f(\omega_n)$$

$$Q^{\circ}(\vec{x}) = \left[\frac{\pi N(o)}{K_F x}\right]^2 \frac{1}{\beta}\sum_n \exp\left(-\frac{2x|W_n|}{V_F}\right)$$

$$= \left[\frac{\pi N(o)}{K_F x}\right] \frac{1}{\beta \sinh(2\pi x/\beta\hbar V_F)} \qquad (3.88)$$

Near T_c, the kernel Q° vanishes exponentially for $x \gg \hbar v_F/k_B T_c \approx O(\xi_0)$, and it thus has a range comparable with the (temperature - indepent) Pippard coherence length. Since ξ_0 is much shorter than the scale of variation of either the vector potential or the gap function, it is permissible to treat $\vec{A}$ and Δ^* as slowly vary functions. In particular , Eq.(3.86) can be integrated explicitly

$$\phi(\vec{x},\vec{x}') = \frac{e}{2\hbar c}[\vec{A}(\vec{x}) + \vec{A}(\vec{x}')].(\vec{x}-\vec{x}')$$

where the symetrized form represents a compromise between the two forms of Eq.(3.74). Furthermore, $\vec{A}$ is of order $H_c(T)\lambda(T) \propto (T_c-T)^{1/2}$

the restricted range of Q° then mean that ϕ $(\vec{x},\vec{x}')$ itself becomes small as $T \to T_c$, permitting an expansion of $e^{i\phi}$ in power of ϕ.

The above condition allow us to evaluate the first term on the right side of Eq. (3.78). With the definition $\vec{z} = \vec{y} - \vec{x}$ we have

$$\int d^3y\, Q\,(\vec{x},\vec{y})\,\Delta^*(\vec{y}) = \int d^3z\, Q^{\circ}\,(z)\exp\left\{\frac{ie}{\hbar c}[\vec{A}(\vec{x})+\vec{A}(\vec{x}+\vec{z})].\vec{z}\right\}\Delta^*(\vec{x}+\vec{z})$$

$$= \int d^3z\, Q^{\circ}\,(z)\left\{1+\frac{ie}{\hbar c}[\vec{A}(\vec{x})+\vec{A}(\vec{x}+\vec{z})].\vec{z}-\frac{1}{2}\left(\frac{e}{\hbar c}\right)^2\right.$$

$$\left.\times\{[\vec{A}(\vec{x})+\vec{A}(\vec{x}+\vec{z})].\vec{z}\}^2+\ldots\right\}\Delta^*(\vec{x}+\vec{z}) \quad (3.90)$$

The short range of Q° requires that $z \lesssim \xi_0$, and the remaining functions may be expanded in a Taylor series about z=0 . Retaining the leading correction term, we find

$$\int d^3y\quad Q\quad(\vec{x},\vec{y})\quad\Delta^*(\vec{y}) \simeq \Delta^*\quad(\vec{x})\quad\int\ d^3z\quad Q^{\circ}\quad(\vec{z})$$

$$+\frac{1}{6}\left[\nabla+\frac{2ie\,\vec{A}(\vec{x})}{\hbar c}\right]^2\Delta^*(\vec{x})\int d^3z\ z^2Q^{\circ}(\vec{z}) \qquad (3.91)$$

where ∇ denotes the gradient with respect to x and acts only on the vector potential $\vec{A}(\vec{x})$ and the gap function Δ^* $(\vec{x})$. Note that we have now reduced the original integral operator to simpler differential one. The properties of the normal state appear only in the numerical coefficients of Eq.(3.91), and we first consider $\int d^3z$ $Q^{\circ}(\vec{z})$, which diverges logarithmically at the origin. this singular behavior reflects the unphysical approximation of short- range potential in Eq. (3.52). It must therefore be cut off in momentum space of $|\xi_K| = \hbar w_D$,

$$\int d^2z\, Q_o(\vec{z}) = (\beta\hbar^2)^{-1}(2\pi)^{-2} \int d^2k \sum_n G^o(k\, W_n)\, G^o(-K,-W_n)$$

$$= \beta^{-1}(2\pi)^{-2} \int d^2K \sum_n (\hbar^2 W_n^2 + \xi_K^2)^{-1}$$

we now introduce the approximation $(2\pi)^{-2} \int d^2k \ldots = N(o) \int d\xi \ldots$

$$\int d^2z\, Q^o(\vec{z}) = N(o) \int_0^{\hbar w_D} d\xi\, \xi^{-1} \tan(\tfrac{1}{2}\beta\xi)$$

$$= N(o) \ln(\hbar w_D \beta) - \int_0^\infty d\xi \ln\xi \frac{d}{d\xi} \tan h(\xi/2)$$

$$= N(o) \ln(2\hbar\, w_D \beta e^{\gamma} \pi^{-1})$$

and comparison with Eq. (2.90) gives

$$\int d^2z\, Q^o(z) = N(o) \ln\left(\frac{Tc}{T}\right) + \frac{1}{V} \approx N(o)\left(1-\frac{T}{T_c}\right) + v^{-1} \qquad (3.92)$$

The other integral in Eq.(3.91) can be evaluated directly in coordinate space using Eq. (3.88). Since this term is already the coefficient of a small correction . we set $T=T_c$ and obtain

$$\int d^3z\, z^2 Q^o(\vec{z}) \quad \frac{1}{\beta_c}\left(\frac{\pi N(o)}{K_F}\right) \int d^3z\, \mathrm{csch}\left(\frac{2\pi z}{\beta_c \hbar v_F}\right)$$

$$= \frac{1}{4} N(o) \left(\frac{\beta_c \hbar v_F}{\pi} \right)^2 \int_0^\infty dy \frac{y^2}{\sinh y}$$

$$= \frac{7}{8} \zeta(3) \, N(o) \left(\frac{\hbar v_F}{\pi k_B T_c} \right)^2 \tag{3.93}$$

where

$$K_F = \frac{(2\pi^2 \hbar^2)}{m} N(o)$$

$$\zeta(3) = \frac{2}{7} \int_0^\infty dy \left(\frac{y}{\sinh y} \right)^2$$

The only other calculation is the small nonlinear correction in Eq. (3.78), which may be evaluated in lowest order ϕ by setting $\tilde{G}^\circ \approx G^\circ$ and taking all the factor of Δ at the same point

$$\int d^3y \, d^3z \, d^3w \, R(\vec{x},\vec{y},\vec{z},\vec{w}) \, \Delta^*(\vec{y}) \, \Delta(\vec{z}) \, \Delta^*(\vec{w})$$
$$\approx \Delta^*(\vec{x}) \, |\Delta(\vec{x})|^2 \int d^3y \, d^3z \, d^3w \, R(\vec{x},\vec{y},\vec{z},\vec{w})$$

A straightforward calculation in momentum space gives

$$\int d^3y \, d^3z \, d^3w \, R(\vec{x},\vec{y},\vec{z},\vec{w}) = -\beta_c \sum_n (2\pi)^{-3} \int d^3k \left(\hbar^2 w_n^2 + \xi^2 \right)^{-2}$$

$$= -\frac{\pi N(o)}{2\beta_c} \sum_n |\hbar W_n|^{-3}$$

$$= -N(o) \frac{7\zeta(3)}{8} (\pi k_B T_c)^{-2} \tag{3.94}$$

where T has again been set equal T_c.

The final equation for $\Delta^*(\vec{x})$ is obtained by combining Eqs. (3.78), (3.91), and (3.92) to (3.94). After manipulation, the term $v^{-1}\Delta^*(\vec{x})$ cancels indentically , and we find

$$\frac{\hbar^2}{4m}\left[\nabla + \frac{2ie\vec{A}(\vec{x})}{\hbar c}\right]^2 \Delta^*(\vec{x}) + \frac{6\pi^2(k_BT_c)^2}{7\zeta(3)\epsilon_F^\circ}\left[\frac{T_c-T}{T_c}\Delta^*(\vec{x}) - \frac{7\zeta(3)\Delta^*(\vec{x})|\Delta(\vec{x})|^2}{8(\pi k_BT_c)^2}\right]$$

$$= 0 \qquad (3.95)$$

The relation with the Gingburg-Landau equation can be made explicitly by defining a wave function

$$\Psi(\vec{x}) = \left[\frac{7\zeta(3)n}{8(\pi k_BT_c)^2}\right]^{1/2}\Delta(x) = \left[\frac{7\zeta(3)}{8\pi^2}\right]^{1/2}\frac{\Delta(\vec{x})n^{1/2}}{k_BT_c} \qquad (3/96)$$

that satisties the following equation (note the complex conjugation)

$$\frac{1}{4m}\left[-i\hbar\nabla - \frac{2e\vec{A}(\vec{x})}{c}\right]^2\Psi(\vec{x}) + \frac{6\pi^2(k_BT_c)^2}{7\zeta(3)\epsilon_F^\circ}\left[-\frac{(T_c-T)}{T_c}\Psi(\vec{x})\right.$$

$$\left. + n^{-1}\Psi(\vec{x})\left|\Psi(\vec{x})\right|^2\right] = 0 \qquad (3.97)$$

Here n is the total electron density, and comparison with Eq. (3.7) identifies the phenomenological parameters

$$m^* = 2m \quad , \quad e^* = 2e^*$$

$$\alpha = -\frac{6\pi^2(k_BT_c)^2}{7\zeta(3)\epsilon_F^\circ}\frac{(T_c-T)}{T_c}, \quad \beta = \frac{6\pi^2(k_BT_c)^2}{7\zeta(3)\epsilon_F^\circ n} \qquad (3.98)$$

The preceding derivation shows how the first Ginzburg-Landau equation emerges as an expansion of the self-consistent , gap equation near T_c. We now consider the supercurrent, from the quantum-mechanical expression, which can be related to spatial derivatives of the single-particle Green's function G

$$\vec{J}(\vec{x}) = \frac{-ie\hbar}{m} (\nabla-\nabla') G(\vec{x}T, \vec{x}'T^{+})\Big|_{\vec{x}=\vec{x}'} \frac{-2e^2}{mc} \vec{A}(\vec{x}) G(\vec{x}T, \vec{x}T^{+}) \quad (3.99)$$

Here the factor2 arises from the spin sums. If Eq.(3.76a) is expanded as

$$G(\vec{x},\vec{x}',W_n) = \tilde{G}^{\circ}(\vec{x},\vec{x}',W_n) + \delta G(\vec{x},\vec{x}',W_n)$$

with

$$G(\vec{x},\vec{x}' W_n) = -\hbar^{-2} \int d^3y\, d^3z\, \tilde{G}^{\circ}(\vec{x},\vec{y},W_n)$$
$$\times \tilde{G}^{\circ}(\vec{z},\vec{y},-W_n)\, G^{\circ}(\vec{z},\vec{x}',W_n)\, \Delta(\vec{y})\, \Delta^{*}(\vec{z}) \quad (3.100)$$

then a simple calculation with Eqs. (3.83) and (3.89) shows that the zero-order contribution vanishes

$$-\frac{ie\hbar}{m} (\nabla-\nabla') \tilde{G}^{\circ}(\vec{x}\tau,\vec{x}'\tau^{+})\Big|_{\vec{x}'=\vec{x}} - \frac{2e^2}{mc} \vec{A}(\vec{x})\tilde{G}^{\circ}(\vec{x}\tau,\vec{x}\tau^{+}) = 0$$

In this way , the total supercurrent reduces to

$$\vec{J}(\vec{x}) = \frac{-ie}{m\beta} \sum_n [(\nabla-\nabla') G(\vec{x},\vec{x}',W_n)] \quad \frac{2e^2}{mc\beta\hbar} \vec{A}(\vec{x}) \sum_n \delta G(\vec{x},\vec{x}.W_n)$$

The remaining calculation depends on the explicit form of G, and substitution of Eq. (3.100) gives

$$\vec{J}(\vec{x}) + \frac{2e^2}{mc\beta\hbar} \vec{A}(\vec{x}) \sum_n \delta G(x,x,w_n) = \frac{e}{mi\beta\hbar^2} \sum_n \int d^3y\, d^3z\, \Delta(\vec{y})\, \Delta^*(\vec{z})$$

$$\times \tilde{G}^o(\vec{z},\vec{y},-w_n)\, [\tilde{G}^o(\vec{z},\vec{x},w_n)\, \nabla_x \tilde{G}^o(\vec{x},\vec{y},w_n) - \tilde{G}^o(\vec{x},\vec{y},w_n)\, \nabla_x \tilde{G}^o(\vec{z},\vec{x},w_n)] \quad (3.101)$$

The spatial derivatives can be evaluated with Eqs. (3.83) and (3.89), and the result can be simplified with the slow variation of $\vec{A}$

$$\vec{J}(\vec{x}) + \frac{2e^2}{mc\beta\hbar} \vec{A}(\vec{x}) \sum_n \delta G(\vec{x},\vec{x},W_n) = \frac{ie}{m\beta\hbar^2} \sum_n \int d^3y\, d^3z\, \Delta(\vec{y})\, \Delta^*(\vec{z})$$

$$\times G^\circ(\vec{z},\vec{y},-W_n)\{\frac{2ie\, \vec{A}(\vec{x})}{\hbar c}\, \tilde{G}^\circ(\vec{x},\vec{y},W_n)\, \tilde{G}^\circ(\vec{z},\vec{x},W_n) + e^{i\alpha(\vec{x},\vec{y})}\, e^{i\alpha(\vec{z},\vec{x})}$$

$$\times [G^\circ(\vec{z}-\vec{x},W_n)\, \nabla_x G^\circ(\vec{x}-\vec{y},W_n) - G^\circ(\vec{x}-\vec{y},W_n)\, \nabla_x G^\circ(\vec{z}-\vec{x},W_n)]\}$$

The first term on the right side now cancels the left. Which the same approximation as Eq. (3.90), the supercurrent near T_c becomes

$$\vec{J}(\vec{x}) = \frac{ie}{m\hbar^2\beta} \sum_n \int d^3y\, d^3z\, G^\circ(\vec{z}-\vec{y},-W_n)\, [G^\circ(\vec{z}-\vec{x},W_n) \nabla_x G^\circ(\vec{x}-\vec{y},W_n)$$

$$- G^\circ(\vec{x}-\vec{y},W_n)\, \nabla_x G^\circ(\vec{z}-\vec{x},W_n)][|\Delta(\vec{x})|^2 + |\Delta(\vec{x})|^2 \frac{2ie}{\hbar c}$$

$$\times \vec{A}(\vec{x}).(\vec{z}-\vec{y}) + \Delta^*(\vec{x})(\vec{y}-\vec{x}).\nabla\Delta(\vec{x}) + \Delta(\vec{x})(\vec{z}-\vec{x}).\nabla\Delta^*(\vec{x})] \quad (3.102)$$

several terms vanish identically owing to the spherical symmetry ,

$$\vec{J}(\vec{x}) = \frac{ie}{m\beta\hbar^2} \sum_n \int d^3y\, d^3z\, G^\circ(\vec{z}-\vec{y},-W_n)\, [G^\circ(\vec{z}-\vec{x},W_n)\, \nabla_x G^\circ(\vec{x}-\vec{y},W_n)$$

$$-G^\circ(\vec{x}-\vec{y},W_n)\, \nabla_x G^\circ(\vec{z}-\vec{x},W_n)] \times [\frac{2ie}{\hbar c} |\Delta(\vec{x})|^2\, \vec{A}(\vec{x}).(\vec{z}-\vec{y}) + \Delta(\vec{x}) \nabla_x \Delta^*(\vec{x}).\vec{z}$$

$$+ \Delta^*(\vec{x})\, \nabla_x \Delta(\vec{x}).\vec{y}] \quad (3.103)$$

The remaining integration is most easily performed with the Fourier representation of G° from eq.(3.81), and a lengthy but straightforward calculation gives

$$\vec{J}(\vec{x}) = \frac{7\zeta(3)n}{16(\pi k_B T_c)^2}\left\{\frac{-2ie\hbar}{2m}[\Delta(\vec{x})^*\nabla\Delta(\vec{x})-\Delta(\vec{x})\nabla\Delta^*(\vec{x}) - \frac{4e^2}{mc}\vec{A}(\vec{x})|\Delta(\vec{x})|^2\right\} \quad (3.104)$$

With the wave function in Eq. (3.96), we finally obtain

$$\vec{J}(\vec{x}) = \frac{-ie\hbar}{2m}[\psi^*(\vec{x})\nabla\psi(\vec{x})-\psi(\vec{x})\nabla\psi^*(\vec{x})] - \frac{2e^2}{mc}\vec{A}(\vec{x})|\psi(x)|^2 \quad (3.105)$$

or

$$\vec{J}(\vec{x}) = \frac{-ie^*\hbar}{2m^*}[\psi^*(\vec{x})\nabla\psi(\vec{x})-\psi(\vec{x})\nabla\psi^*(x)] - \frac{(e^*)^2}{m^*c}|\psi(x)|^2\vec{A}(\vec{x}) \quad (3.106)$$

in complelte agreement with Eq. (3.11).

In summary, we have shown how the Ginzburg - Landau equations (3.7) and (3.11) can be obtained from the Gorkov equations under the following set of assumptions

1. The order parameter $\Delta(\vec{x})$ and the vector potential $\vec{A}(\vec{x})$ are small.

2. The range of the kernels in Eqs. (3.) and (3.) is small compared to the characteristic length for spatial variations of $\Delta(\vec{x})$ (i.e. , the coherence length ξ) and $\vec{A}(\vec{x})$ (i.e., the penetration length λ).

3. The eikonal approximation also requies $\lambda K_F \gg 1$ (which is are always satisfied sufficiently close to the transition temparature T_c.