CHAPTER II.

THEORETICAL BACKGROUND

Introduction.

Nonlinear theories in continuum mechanics have been in existence for many years with very limited applications. For most practical problems, nonlinear analyses can be performed only with the aid of numerical methods. In this chapter, the basic principles of nonlinear structural mechanics are reviewed , using the principle of virtual displacements, a variational form of the incremental equation of motion for nonlinear static analysis is derived for structures undergoing large displacements and large strains. Then the discretization of the equations of motion using the finte element displacement formulation is discussed. The derivation of the element matrices is performed for the particular case of a three dimensional isoparametric hexahedral finite element.

An Incremental Nonlinear Formulation of Equations of Motion for Finite Deformation.

1. Introduction to the Concept of the Incremental Nonlinear Formulation of the Equations of Motion.

Basically, two differnt approaches have been pursued in the incremental nonlinear finite element analysis. In the first, kinematic and kinetic variables are referred to the initial configuration. This procedure is generally called Lagrangian formulation. In the second approach which is generally called Eulerian, moving coordinate or update formulation, all the kinematic and kinetic variables are referred to an updated configuration in each load step. The Lagrangian and Eulerian formulation differ in the identification and transformation of kinematic and kinetic variables. Given consistent material laws, both descriptions are theoretically equivalent because they use the same balance pricinples. The choice between the two formulations, therefore, depends upon the ease and the relative numerical effectiveness of the method. In this research, nonlinear finite element formulation will be developed based on the Lagrangian description of motion due to the following advantages.

(a) Large displacement effects are implicit in the strain displacement relationships, so that the element property matrices need not be explicitly transformed to account for updating of the nodal coordinates resulting from changes in geometry.

(b) The material laws may be simpler to express because the stress are always referred to the undeformed configuration.

(c) The stresses are obtained through the process of simple additions whereas in the formulation using the Eulerian description, the stresses will be obtained by transformation and addition.

In the following section, consistent formulation of equations of motion for finite deformation response will be developed using the concept of the incremental theory in connection with the principle of virtual work. Kinematic and kinetic variables are derived from the basic theories in continuum mechanics. The formulation presented here closely follows those appearing in references (10,11).

Conceptually, the formulation of the incremental nonlinear equations of motion requires that the path of deformation of a body be divided into a number of equilibrium states ${}^{o}\Omega$, ${}^{1}\Omega,\ldots$, ${}^{n}\Omega$, ${}^{n+1}\Omega,\ldots$, ${}^{f}\Omega$ where ${}^{o}\Omega$ and ${}^{f}\Omega$ are the initial and final states of the deformation respectively, while ${}^{n}\Omega$ is an arbitrary intermediate state. It is assumed that all of state variables such as stresses, strains and displacements, together with the loading history, are known up to the ${}^{n}\Omega$ state and the state variables in the ${}^{n+1}\Omega$ state are required next. Then the equation of incremental virtual work between the state ${}^{n}\Omega$ and ${}^{n+1}\Omega$ is established to express the equilibrium of the body in the state ${}^{n+1}\Omega$. However, the configuration at ${}^{n+1}\Omega$ is unknown, and therefore all the state variables must be referred to a known or previously calculated equilibrium state. In principle, any one of the already calculated equilibrium states can be used. Basically, the state variables are referred to either the initial ${}^{o}\Omega$ state or the current equilibrium ${}^{n}\Omega$ state and the corresponding formulations are called Lagrangian formulation and Eulerian respectively. The stresses and strains in the first formulation are the second Piola-Kirchhoff stress and the Green-Lagrange strain, while those in the second formulation are the Cauchy stress and strain. The incremental process for the next required equilbrium state is typical and would be applied repetitively until the final state, ${}^{f}\Omega$, has been reached.

Consider the motion of the body as shown in Figure 2.1.

There are three configulations in its path of deformation that are of interest, i.e.,

(a) the undeformed configuration C_o,

(b) the current deformed configuration C_1, and

(c) a second deformed configuration C_2

taken as a neighboring configuration to the current deformed configuration C_1.

The state variables in configuration C_1 and C_2 are defined as follows:

$$^1S_{IJ} \ , \ ^1E_{IJ} \ , \ ^1U_I \ , \ ^1t_I \ , \ ^1f_I \qquad \text{in } C_1$$

$$^2S_{IJ} \ , \ ^2E_{IJ} \ , \ ^2U_I \ , \ ^2t_I \ , \ ^2f_I \qquad \text{in } C_2$$

where S_{IJ}, E_{IJ}, U_I, t_I and f_I are stresses, strains, displacements, surface tractions and body forces, respectively; a left superscript indicates the configuration of the body in which the quantity occurs.

The incremental decomposition of the state variables are given by

$$^2S_{IJ} = {}^1S_{IJ} + S_{IJ} \tag{2.1a}$$

$$^2E_{IJ} = {}^1E_{IJ} + E_{IJ} \tag{2.1b}$$

$$^2U_I = {}^1U_I + U_I \tag{2.1c}$$

where S_{IJ}, E_{IJ} and U_I are incremental stress, incremental strain and incremental displacement between C_1 and C_2.

There are three state variables describing the state of deformation that are stresses, strains and displacements. These state variables are related by the following natural relations:

(a) Kinematic Relations or Strain-displacement Relations

(b) Equilibrium Equations

(c) Constitutive Relations

2. Kinematic Relations.

The incremental strain E_{IJ} expressed in terms of displacements can be decomposed into a linear and nonlinear components as follow,

$$E_{IJ} = e_{IJ} + \eta_{IJ} \tag{2.2}$$

in which

$$2e_{IJ} = U_{I|J} + U_{J|I} + U_{K|I}\,{}^{1}U_{K|J} + {}^{1}U_{K|I}\,U_{K|J} \tag{2.3}$$

$$2\eta_{IJ} = U_{K|I}\,U_{K|J} \tag{2.4}$$

where a vertical bar indicates the covariant derivative in the undeformed configuration C_o.

3. Equilibrium Equations.

An incremental nonlinear equations of motion describing the deformation of the body between the two neighboring configuration C_1 and C_2 will be derived from the principle of virtual work.

Consider the body in its deformed equilibrium configuration C_1. The virtual work of the external forces in moving through an infinitesimal virtual displacement δU_I from the current state is given by

$$\delta\,{}^1W_{ext} = \int_{{}^1A} \delta U_I \; {}^1t_I \; da + \int_{{}^1V} \delta U_I \; {}^1f_I \; dv \tag{2.5}$$

in which ${}^1A$ is the part of the surface area in C_1 which has prescribed surface tractions, and ${}^1V$ is the volume of the body in C_1. da and dv are the differential area and volume of the body in C_1, respectively.

The virtual work done by the internal forces in configuration C_1 in an arbitrary virtual displacement δU_I can be expressed as

$$\delta\,{}^1W_{int} = \int_{{}^oV} {}^1S_{IJ} \; \delta e_{IJ} \; dV \tag{2.6}$$

in which ${}^oV$ is the volume of the body in C_o and dV is the differential volume of the body in C_o.

The equation of virtual work in configuration C_1 can be obtained by equating equation (2.6) with equation (2.5), i.e.,

$$\int_{{}^{o}V} {}^{1}S_{IJ}\ \delta e_{IJ}\ dV = \int_{{}^{1}A} \delta U_I\ {}^{1}t_I\ da + \int_{{}^{1}V} \delta U_I\ {}^{1}f_I\ dv \tag{2.7}$$

In order to develop the incremental virtual work equation between configuration C_1 and C_2, the virtual work equation in the deformed equilibrium configuration C_2 has to be established. This can be done by following the same procedures similar to those for the deformed configuration C_1. That is

$$\delta^2 W_{ext} = \int_{{}^{2}A} \delta U_I\ {}^{2}t_I\ da + \int_{{}^{2}V} \delta U_I\ {}^{2}f_I\ dv \tag{2.8}$$

in which ${}^{2}A$ is the part of the surface area of the body in C_2 which has prescribed surface tractions, ${}^{2}V$ is the volume of the body in C_2, and da and dv are the differential area and volume of the body in C_2, respectively.

The virtual work done by the internal forces in configuration C_2 in an arbitrary virtual displacement δU_I can be expressed as

$$\delta^2 W_{int} = \int_{{}^{o}V} {}^{2}S_{IJ}\ \delta E_{IJ}\ dV \tag{2.9}$$

Therefore, equation of virtual work in configuration C_2 is obtained by equating equation (2.9) with equation (2.8), i.e.,

$$\int_{{}^{o}V} {}^{2}S_{IJ}\ \delta E_{IJ}\ dV = \int_{{}^{2}A} \delta U_{I}\ {}^{2}t_{I}\ da + \int_{{}^{2}V} \delta U_{I}\ {}^{2}f_{I}\ dv \tag{2.10}$$

Since

$${}^{2}S_{IJ} = {}^{1}S_{IJ} + S_{IJ} \tag{2.1a}$$

$${}^{2}U_{I} = {}^{1}U_{I} + U_{I} \tag{2.1c}$$

$$E_{IJ} = e_{IJ} + \eta_{IJ} \tag{2.2}$$

Substituting the above relations into equation (2.10), resulting in

$$\int_{{}^{o}V} [({}^{1}S_{IJ} + S_{IJ})\ (\delta e_{IJ} + \delta\eta_{IJ})]\ dV = \int_{{}^{2}A} \delta U_{I}\ {}^{2}t_{I}\ da + \int_{{}^{2}V} \delta U_{I}\ {}^{2}f_{I}\ dv \tag{2.11}$$

The incremental virtual work between configuration C_1 and C_2 can be obtained by subtracting equation (2.7) from (2.11), i.e.,

$$\int_{{}^{o}V} [S_{IJ}(\delta e_{IJ} + \delta\eta_{IJ}) + {}^{1}S_{IJ}\ \delta\eta_{IJ}]\ dV$$

$$= \int_{{}^{2}A}\delta U_{I}\,{}^{2}t_{I}\,da + \int_{{}^{2}V}\delta U_{I}\,{}^{2}f_{I}\,dv - [\int_{{}^{1}A}\delta U_{I}\,{}^{1}t_{I}\,da + \int_{{}^{1}V}\delta U_{I}\,{}^{1}f_{I}\,dv\ ] \tag{2.12}$$

Equation (2.12) can be interpreted that the incremental virtual work of the body forces and surface tractions in the deformed equilibrium configuration C_1 and C_2 must equal the incremental virtual work of the state of stress in these configurations.

4. Constitutive Relation.

The incremental equations of motion (2.12) are valid for any type of material irrespective of its constitutions. However, the application of these equations to physical nonlinear problems requires detailed knowledge of the material characterization, specifically the relationship between incremental stress and incremental strain.

For elastic materials, the incremental stress S_{IJ}, is linearly related to the incremental strain, E_{IJ}. That is

$$S_{IJ} = C_{IJMN} E_{MN} \tag{2.13}$$

in which C_{IJMN} are the components of the constitutive matrix.

5. Incremental Nonlinear Equations of Motion with Equilibrium Corrections.

The solution of the equations of motion (2.12) cannot be achieved directly since they are nonlinear in displacement increments. Therefore, it needs to be linearized for practical applications. However, the process of linearization must take account of three effects as follows:

(a) It is sufficient to assume the linear stress-strain relationship in the general form of equation (2.13).

(b) If the relationship (2.13) is substituted into equation (2.12) it would result in terms such as:

$$C_{IJMN}\,(e_{MN}\,\delta\eta_{IJ} + \eta_{MN}\,\delta e_{IJ})$$

and $$C_{IJMN}\,\eta_{MN}\,\delta\eta_{IJ}$$

which are nonlinear in the incremental displacements. The linearization process requires that these terms be omitted.

(c) If the prescribed surface tractions are deformation dependent, the external virtual work integrals in configuration C_2 can be evaluated approximately.

Due to the above linearization and computational inaccuracies, the current deformed configuration C_1 may not be in complete equilibrium, thus resulting in residual work. That is,

$$\delta\,{}^1W_{res} = \int_{{}^1A} \delta U_I\,{}^1t_I\,da + \int_{{}^1V} \delta U_I\,{}^1f_I\,dv - \int_{{}^oV} {}^1S_{IJ}\,\delta e_{IJ}\,dV \tag{2.14}$$

To prevent excessive departure of the solution from the true response, the corrective term (Equation 2.14) should be added to the right hand side of equation (2.12). Thus we have

$$\int_{{}^oV}[S_{IJ}(\delta e_{IJ}+\delta\eta_{IJ})+{}^1S_{IJ}\delta\eta_{IJ}]dV = \int_{{}^2A}\delta U_I\,{}^2t_I\,da + \int_{{}^2V}\delta U_I\,{}^2f_I\,dv - \int_{{}^oV}{}^1S_{IJ}\delta e_{IJ}dV \tag{2.15}$$

The incremental nonlinear equation (2.15) will be solved by using the linearized form and applying the load in small increments together with an iterative process for equilibrium correction. Furthermore, it will be assumed that the components of the surface tractions are always known in the reference system and are defined per unit of undeformed area and volume. Therefore, the integral expressions can be approximated by the following expression:

$$\int_{{}^{2}A} \delta U_I \, {}^{2}t_I \, da + \int_{{}^{2}V} \delta U_I \, {}^{2}f_I \, dv \sim \int_{{}^{0}A} \delta U_I \, {}^{2}t_I \, dA + \int_{{}^{0}V} \delta U_I \, {}^{2}f_I \, dV \qquad (2.16)$$

The equations of motion suitable for using as a basis for discretization by the finite element method then take the form:

$$\int_{{}^{0}V} [S_{IJ}(\delta e_{IJ} + \delta \eta_{IJ}) + {}^{1}S_{IJ}\delta \eta_{IJ}] \, dV$$

$$= \int_{{}^{0}A} \delta U_I \, {}^{2}t_I \, dA + \int_{{}^{0}V} \delta U_I \, {}^{2}f_I \, dV - \int_{{}^{0}V} {}^{1}S_{IJ} \delta e_{IJ} \, dV \qquad (2.17)$$

Finite Element Formulation of Equations of Motion for Finite Deformation.

1. Introduction.

In the previous section, the incremental equations of motion have been derived in a variational form. In order to develop finite element formulation, discretization techniques will be used to decompose the global form of equations of motion into a discrete type of equations. The concepts of the finite element method, its mathematical foundations and the discretization techniques are well established and a comprehensive study of these subjects can be found in many texts (e.g.,12,13,14).

A basic characteristic of the finite element method is that a typical element can be isolated from the element assemblage, and its behavior can be studied independently of the behavior of the other elements. Moreover, the assembly process is independent of the linearity or nonlinearity of the system, and complete mathematical model is established by a simple mapping. Therefore, in the following section, only a single finite element should be considered. Within the scope of this research, the typical element to be used for the discrete analysis is the isoparametric hexahedral finite element, and its element property matrices will be derived in detail.

2. Discretization of Equations of Motion by Finite Element Method.

Consider a single isoparametric element and introduce a local approximation of the displacement field within the element by

$$ {}^{\alpha}U_{K}(x) = \phi^{m}(x)\, {}^{\alpha}q_{mK} \qquad , \alpha = 1,2 \tag{2.18} $$

where ${}^{\alpha}U_{K}(x)$ are the components of the displacement of material coordinate x, in configuration C_{α}. $\phi^{m}(x)$ are the interpolation functions at node m. ${}^{\alpha}q_{mK}$ are the components of displacement at node m. The index m is assumed over all nodes of the element.

For the isoparametric finite element, the incremental displacement between configuration C_1 and C_2 and the coordinates of the material point are interpolated similarly using the above functions. That is

$$U_K(x) = \phi^m(x)\, q_{mK} \tag{2.19}$$

$$X_K = \phi^m(x)\, X_{mK} \tag{2.20}$$

$$^{\alpha}X_K = \phi^m(x)\, {}^{\alpha}X_{mK} \quad , \ \alpha = 1,2 \tag{2.21}$$

where q_{mK} are the components of the incremental displacement at node m. X_{mK} and $^{\alpha}X_{mK}$ are the components of nodal coordinates in the configurations C_o and C_α, respectively.

Substituting the stress-strain relation (2.13) into the incremental equations of motion (2.17) yields

$$\int_{^oV} [\, C_{IJMN}\, e_{MN}\, \delta e_{IJ} + C_{IJMN}\, (\, e_{MN}\, \delta\eta_{IJ} + \eta_{MN}\, \delta e_{IJ}\,)$$

$$+ C_{IJMN}\, \eta_{MN}\, \delta\eta_{IJ} + {}^1S_{IJ}\, \delta\eta_{IJ}\,]\, dV$$

$$= \int_{^oA} \delta U_I\, {}^2t_I\, dA + \int_{^oV} \delta U_I\, {}^2f_I\, dV - \int_{^oV} {}^1S_{IJ}\, \delta e_{IJ}\, dV \tag{2.22}$$

Thus, the following discretized equations of motion for a typical finite element are obtained.

$$\delta q.[\{K_L({}^1q) + K_1(q) + K_2(q.q) + K_G({}^1S)\}q] = \delta q.[{}^2P - {}^1R] \quad (2.23)$$

where

$$\delta q.K_L.q = \int_{{}^oV} C_{IJMN}\, e_{MN}\, \delta e_{IJ}\, dV \quad (2.24a)$$

$$\delta q.K_1.q = \int_{{}^oV} C_{IJMN}\, (\, e_{MN}\, \delta\eta_{IJ} + \eta_{MN}\, \delta e_{IJ}\,)\, dV \quad (2.24b)$$

$$\delta q.K_2.q = \int_{{}^oV} C_{IJMN}\, \eta_{MN}\, \delta\eta_{IJ}\, dV \quad (2.24c)$$

$$\delta q.K_G.q = \int_{{}^oV} {}^1S_{IJ}\, \delta\eta_{IJ}\, dV \quad (2.24d)$$

$$\delta q.{}^1R = \int_{{}^oV} {}^1S_{IJ}\, \delta e_{IJ}\, dV \quad (2.24e)$$

$$\delta q.{}^2P = \int_{{}^oA} \delta U_I\, {}^2t_I\, dA + \int_{{}^oV} \delta U_I\, {}^2f_I\, dV \quad (2.24f)$$

In the above arrays, K_L is the linear stiffness matrix, including initial displacement effect; K_1 and K_2 are nonlinear stiffness matrices, a linear and quadratic functions of the incremental displacement q respectively; K_G is the geometric stiffness matrix, a function of the initial stress ${}^1S$; ${}^2P$ is the generalized nodal loads due to the body forces and conservative surface tractions;

${}^{1}R$ is the consistent nodal load vector in equilibrium with the state of stress in configuration C_1.

By neglecting the nonlinear terms K_1 and K_2, a linearized form of the finite element formulation of the nonlinear equations of motion for finite deformation is given by

$$\delta q.[\{K_L({}^{1}q) + K_G({}^{1}S)\}q] = \delta q.[{}^{2}P - {}^{1}R] \tag{2.25}$$

Equation (2.25) represents a system of nonlinear equations in the unknown nodal displacement components, describing the incremental finite deformation of an element between the current deformed configuration C_1 and a neighboring deformed configuration C_2.

3. <u>Three Dimensional Isoparametric Finite Element Matrices.</u>

3.1 <u>Linear Isoparametric Hexahedral Element.</u> In this section, the element matrices for a general 8-node isoparametric hexahedral element are given in detail. Geometry and transformations (mappings) of coordinates are given in Figure 2.2.

3.1.1 Components of Stresses and Strains. For the three dimensional analysis , the component of stress and strain are as follow:

$$E_{IJ} = (E_{11}\ E_{22}\ E_{33}\ 2E_{12}\ 2E_{23}\ 2E_{13}) \tag{2.26a}$$

$$S_{IJ} = (S_{11}\ S_{22}\ S_{33}\ S_{12}\ S_{23}\ S_{13}) \tag{3.26b}$$

3.1.2 Interpolation Functions. For an 8-node isoparametric hexahedral element as shown in Figure 2.2, the interpolation functions for the corner nodes written in terms of the natural coordinates (r,s,t) are given by

$$\phi^m(r,s,t) = 1/8\ (1+r.r_m)(1+s.s_m)(1+t.t_m) \tag{2.27}$$

in which m = 1,...,8 and $(r_m, s_m, t_m) = +1\ ,\ -1$.

In a matrix form we have

$$\phi = \frac{1}{8} \left\{ \begin{array}{c} (1+r)(1-s)(1-t) \\ (1+r)(1+s)(1-t) \\ (1-r)(1+s)(1-t) \\ (1-r)(1-s)(1-t) \\ (1+r)(1-s)(1+t) \\ (1+r)(1+s)(1+t) \\ (1-r)(1+s)(1+t) \\ (1-r)(1-s)(1+t) \end{array} \right\} \tag{2.28}$$

3.1.3 Strain-displacement Transformation. The decomposition of the incremental strain into a linear and nonlinear components which given by equation (2.2) becomes:

$$\left\{\begin{matrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{13} \end{matrix}\right\} = \left\{\begin{matrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{12} \\ 2e_{23} \\ 2e_{13} \end{matrix}\right\} + \left\{\begin{matrix} \eta_{11} \\ \eta_{22} \\ \eta_{33} \\ 2\eta_{12} \\ 2\eta_{23} \\ 2\eta_{13} \end{matrix}\right\} \qquad (2.29)$$

An explicit relation between the nonlinear strains and nodal displacements will be evident when the evaluation of the geometric stiffness is considered in the next section. The relation between linear strains and nodal displacements from equation (2.3) can be written in terms of deformation gradients as:

$$2e_{IJ} = (\delta_{KJ} + {}^{1}U_{K|J})\, U_{K|I} + (\delta_{KI} + {}^{1}U_{K|I})\, U_{K|J} \qquad (2.30a)$$

where δ_{KJ} = the Kronecker delta

That is

$$e = {}^{1}F\, U_{\partial} \qquad (2.30b)$$

where ${}^{1}F$ is the matrix of material deformation gradients in the current configuration C_1 and U_∂ is the vector of displacement gradients.

$$
{}^{1}F =
$$

$$
\begin{bmatrix}
1+f_{11} & 0 & 0 & f_{21} & 0 & 0 & f_{31} & 0 & 0 \\
0 & f_{12} & 0 & 0 & 1+f_{22} & 0 & 0 & f_{32} & 0 \\
0 & 0 & f_{13} & 0 & 0 & f_{23} & 0 & 0 & 1+f_{33} \\
f_{12} & 1+f_{11} & 0 & 1+f_{22} & f_{21} & 0 & f_{32} & f_{31} & 0 \\
0 & f_{13} & f_{12} & 0 & f_{23} & 1+f_{22} & 0 & 1+f_{33} & f_{32} \\
f_{13} & 0 & 1+f_{11} & f_{23} & 0 & f_{21} & 1+f_{33} & 0 & f_{31}
\end{bmatrix}
\tag{2.31}
$$

where

$$
f_{11} = \frac{\partial\,{}^{1}U_1}{\partial X_1}
$$

$$
f_{12} = \frac{\partial\,{}^{1}U_1}{\partial X_2}
$$

$$
f_{13} = \frac{\partial\,{}^{1}U_1}{\partial X_3}
$$

$$f_{21} = \frac{\partial\, {}^1U_2}{\partial X_1}$$

$$f_{22} = \frac{\partial\, {}^1U_2}{\partial X_2}$$

$$f_{23} = \frac{\partial\, {}^1U_2}{\partial X_3}$$

$$f_{31} = \frac{\partial\, {}^1U_3}{\partial X_1}$$

$$f_{32} = \frac{\partial\, {}^1U_3}{\partial X_2}$$

$$f_{33} = \frac{\partial\, {}^1U_3}{\partial X_3} \qquad (2.32)$$

and $$U^T_\partial = \left(\frac{\partial U_1}{\partial X_1}\ \frac{\partial U_1}{\partial X_2}\ \frac{\partial U_1}{\partial X_3}\ \frac{\partial U_2}{\partial X_1}\ \frac{\partial U_2}{\partial X_2}\ \frac{\partial U_2}{\partial X_3}\ \frac{\partial U_3}{\partial X_1}\ \frac{\partial U_3}{\partial X_2}\ \frac{\partial U_3}{\partial X_3} \right) \qquad (2.33)$$

where the superscript T denotes the transpose of a vector or a matrix.

The displacement gradients $U_{I|J}$ and ${}^1U_{I|J}$ are related to the nodal displacements through the local approximations of the displacement field, equations (2.18 and 2.19). That is

$$U_\partial = N.q \qquad (2.34a)$$

or

$$
\begin{Bmatrix}
\dfrac{\partial U_1}{\partial X_1} \\[2ex]
\dfrac{\partial U_1}{\partial X_2} \\[2ex]
\dfrac{\partial U_1}{\partial X_3} \\[2ex]
\dfrac{\partial U_2}{\partial X_1} \\[2ex]
\dfrac{\partial U_2}{\partial X_2} \\[2ex]
\dfrac{\partial U_2}{\partial X_3} \\[2ex]
\dfrac{\partial U_3}{\partial X_1} \\[2ex]
\dfrac{\partial U_3}{\partial X_2} \\[2ex]
\dfrac{\partial U_3}{\partial X_3}
\end{Bmatrix}
=
\begin{bmatrix}
\phi^T{}_{,X_1} & 0 & 0 \\
\phi^T{}_{,X_2} & 0 & 0 \\
\phi^T{}_{,X_3} & 0 & 0 \\
0 & \phi^T{}_{,X_1} & 0 \\
0 & \phi^T{}_{,X_2} & 0 \\
0 & \phi^T{}_{,X_3} & 0 \\
0 & 0 & \phi^T{}_{,X_1} \\
0 & 0 & \phi^T{}_{,X_2} \\
0 & 0 & \phi^T{}_{,X_3}
\end{bmatrix}
\begin{Bmatrix}
q_1 \\ q_2 \\ q_3
\end{Bmatrix}
\tag{2.34b}
$$

and $\quad {}^{1}U_{\partial} = N.{}^{1}q$ (2.35)

where $\quad {}^{1}U_{\partial}^{T} = \left(\frac{\partial\,{}^{1}U_1}{\partial X_1} \; \frac{\partial\,{}^{1}U_1}{\partial X_2} \; \frac{\partial\,{}^{1}U_1}{\partial X_3} \; \frac{\partial\,{}^{1}U_2}{\partial X_1} \; \frac{\partial\,{}^{1}U_2}{\partial X_2} \; \frac{\partial\,{}^{1}U_2}{\partial X_3} \; \frac{\partial\,{}^{1}U_3}{\partial X_1} \; \frac{\partial\,{}^{1}U_3}{\partial X_2} \; \frac{\partial\,{}^{1}U_3}{\partial X_3} \right)$

(2.36)

Commas in equation (2.34b) indicate "partial derivative with respect to". Finally, the linear strain-displacement transformation matrix B can be obtained from equations (2.30 and 2.34) as follows:

$$e = {}^{1}F.N.q = B.q \quad (2.37)$$

3.1.4 Jacobian Transformation. To calculate the derivative of the interpolation function $\phi^{m}(r,s,t)$ with respect to the global coordinates X_1 , X_2 and X_3, a Jacobian transformation is needed to relate derivatives with respect to the local (r,s,t) system to those with respect to the global axes. That is

$$\left\{ \begin{matrix} \dfrac{\partial}{\partial r} \\[2ex] \dfrac{\partial}{\partial s} \\[2ex] \dfrac{\partial}{\partial t} \end{matrix} \right\} = \begin{bmatrix} X_{1,r} & X_{2,r} & X_{3,r} \\[2ex] X_{1,s} & X_{2,s} & X_{3,s} \\[2ex] X_{1,t} & X_{2,t} & X_{3,t} \end{bmatrix} \left\{ \begin{matrix} \dfrac{\partial}{\partial X_1} \\[2ex] \dfrac{\partial}{\partial X_2} \\[2ex] \dfrac{\partial}{\partial X_3} \end{matrix} \right\} \quad (2.38)$$

where,

$$X_{1,r} = \phi^T_{,r} X_1 \quad , \quad X_{1,s} = \phi^T_{,s} X_1 \quad , \quad X_{1,t} = \phi^T_{,t} X_1$$

$$X_{2,r} = \phi^T_{,r} X_2 \quad , \quad X_{2,s} = \phi^T_{,s} X_2 \quad , \quad X_{2,t} = \phi^T_{,t} X_2$$

$$X_{3,r} = \phi^T_{,r} X_3 \quad , \quad X_{3,s} = \phi^T_{,s} X_3 \quad , \quad X_{3,t} = \phi^T_{,t} X_3 \tag{2.39}$$

Inversely,

$$\begin{Bmatrix} \dfrac{\partial}{\partial X_1} \\ \dfrac{\partial}{\partial X_2} \\ \dfrac{\partial}{\partial X_3} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{\partial}{\partial s} \\ \dfrac{\partial}{\partial t} \end{Bmatrix} \tag{2.40}$$

where,

$$A_{11} = X_{2,s} X_{3,t} - X_{3,s} X_{2,t}$$

$$A_{12} = X_{3,s} X_{1,t} - X_{1,s} X_{3,t}$$

$$A_{13} = X_{1,s} X_{2,t} - X_{2,s} X_{1,t}$$

$$A_{21} = X_{3,r} X_{2,t} - X_{2,r} X_{3,t}$$

$$A_{22} = X_{1,r}\, X_{3,t} - X_{3,r}\, X_{1,t}$$

$$A_{23} = X_{2,r}\, X_{1,t} - X_{1,r}\, X_{2,t}$$

$$A_{31} = X_{2,r}\, X_{3,s} - X_{3,r}\, X_{2,s}$$

$$A_{32} = X_{3,r}\, X_{1,s} - X_{1,r}\, X_{3,s}$$

$$A_{33} = X_{1,r}\, X_{2,s} - X_{2,r}\, X_{1,s} \tag{2.41}$$

where $\det J = X_{1,r}\, A_{11} + X_{2,r}\, A_{12} + X_{3,r}\, A_{13}$ (2.42)

Therefore, $\phi^T_{,X_1}$, $\phi^T_{,X_2}$ and $\phi^T_{,X_3}$ can be calculated from (2.40) as follows:

$$\phi_{,X_1} = \frac{1}{\det J} [\, A_{11}\, \phi^T_{,r} + A_{21}\, \phi^T_{,s} + A_{31}\, \phi^T_{,t} \,] \tag{2.43}$$

$$\phi_{,X_2} = \frac{1}{\det J} [\, A_{12}\, \phi^T_{,r} + A_{22}\, \phi^T_{,s} + A_{32}\, \phi^T_{,t} \,] \tag{2.44}$$

$$\phi_{,X_3} = \frac{1}{\det J} [\, A_{13}\, \phi^T_{,r} + A_{23}\, \phi^T_{,s} + A_{33}\, \phi^T_{,t} \,] \tag{2.45}$$

Also, the differential volume dV is given by

$$dV = (\det J)\, dr\, ds\, dt \tag{2.46}$$

3.2 Evaluation of Element Matrices. The element matrices required to solve the discrete finite element equations of motion (2.25) can be obtained by evaluating the virtual work integrals (2.24) using the Gaussian quadrature formulas for numerical integrations.

3.2.1 Linear Element Stiffness Matrix. The linear element stiffness matrix is given by the integral (2.24a) as

$$K_L = \int_{^{o}V} B^T . C . B \, dV \tag{2.47}$$

For the purpose of numerical integration, it is written in the natural coordinates as

$$K_L = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B^T . C . B \det J . dr \, ds \, dt \tag{2.48}$$

The direct application of one dimensional numerical integration formula yields

$$K_L = \sum_i \sum_j \sum_k W_i W_j W_k \, [B^T(r_i,s_j,t_k).C(r_i,s_j,t_k).B(r_i,s_j,t_k)] .\det J(r_i,s_j,t_k) \tag{2.49}$$

3.2.2 Geometric Stiffness Matrix. The geometric stiffness matrix can be obtained by evaluating the integral (2.24d) as follows:

$$\delta q.K_G.q \;=\; \int_{{}^oV} {}^1S_{IJ}\,\delta\eta_{IJ}\,dV \tag{2.24d}$$

where the nonlinear incremental strain η_{IJ} is given by

$$\begin{Bmatrix} \eta_{11} \\ \eta_{22} \\ \eta_{33} \\ 2\eta_{12} \\ 2\eta_{23} \\ 2\eta_{13} \end{Bmatrix} = \begin{Bmatrix} \tfrac{1}{2}\left[\left(\dfrac{\partial U_1}{\partial X_1}\right)^2 + \left(\dfrac{\partial U_2}{\partial X_1}\right)^2 + \left(\dfrac{\partial U_3}{\partial X_1}\right)^2 \right] \\ \tfrac{1}{2}\left[\left(\dfrac{\partial U_1}{\partial X_2}\right)^2 + \left(\dfrac{\partial U_2}{\partial X_2}\right)^2 + \left(\dfrac{\partial U_3}{\partial X_2}\right)^2 \right] \\ \tfrac{1}{2}\left[\left(\dfrac{\partial U_1}{\partial X_3}\right)^2 + \left(\dfrac{\partial U_2}{\partial X_3}\right)^2 + \left(\dfrac{\partial U_3}{\partial X_3}\right)^2 \right] \\ \left[\dfrac{\partial U_1}{\partial X_1}\dfrac{\partial U_1}{\partial X_2} + \dfrac{\partial U_2}{\partial X_1}\dfrac{\partial U_2}{\partial X_2} + \dfrac{\partial U_3}{\partial X_1}\dfrac{\partial U_3}{\partial X_2} \right] \\ \left[\dfrac{\partial U_1}{\partial X_2}\dfrac{\partial U_1}{\partial X_3} + \dfrac{\partial U_2}{\partial X_2}\dfrac{\partial U_2}{\partial X_3} + \dfrac{\partial U_3}{\partial X_2}\dfrac{\partial U_3}{\partial X_3} \right] \\ \left[\dfrac{\partial U_1}{\partial X_1}\dfrac{\partial U_1}{\partial X_3} + \dfrac{\partial U_2}{\partial X_1}\dfrac{\partial U_2}{\partial X_3} + \dfrac{\partial U_3}{\partial X_1}\dfrac{\partial U_3}{\partial X_3} \right] \end{Bmatrix} \tag{2.50}$$

To put η_{IJ} into a symmetric form with respect to ${}^1S_{IJ}$, we can write

$$2\,{}^{1}S_{IJ}\eta_{IJ} = \left(\frac{\partial U_1}{\partial X_1}\ \frac{\partial U_1}{\partial X_2}\ \frac{\partial U_1}{\partial X_3} \right) \begin{bmatrix} {}^{1}S_{11} & {}^{1}S_{12} & {}^{1}S_{13} \\ {}^{1}S_{21} & {}^{1}S_{22} & {}^{1}S_{23} \\ {}^{1}S_{31} & {}^{1}S_{32} & {}^{1}S_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_1}{\partial X_1} \\ \frac{\partial U_1}{\partial X_2} \\ \frac{\partial U_1}{\partial X_3} \end{Bmatrix}$$

$$+ \left(\frac{\partial U_2}{\partial X_1}\ \frac{\partial U_2}{\partial X_2}\ \frac{\partial U_2}{\partial X_3} \right) \begin{bmatrix} {}^{1}S_{11} & {}^{1}S_{12} & {}^{1}S_{13} \\ {}^{1}S_{21} & {}^{1}S_{22} & {}^{1}S_{23} \\ {}^{1}S_{31} & {}^{1}S_{32} & {}^{1}S_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_2}{\partial X_1} \\ \frac{\partial U_2}{\partial X_2} \\ \frac{\partial U_2}{\partial X_3} \end{Bmatrix}$$

$$+ \left(\frac{\partial U_3}{\partial X_1}\ \frac{\partial U_3}{\partial X_2}\ \frac{\partial U_3}{\partial X_3} \right) \begin{bmatrix} {}^{1}S_{11} & {}^{1}S_{12} & {}^{1}S_{13} \\ {}^{1}S_{21} & {}^{1}S_{22} & {}^{1}S_{23} \\ {}^{1}S_{31} & {}^{1}S_{32} & {}^{1}S_{33} \end{bmatrix} \begin{Bmatrix} \frac{\partial U_3}{\partial X_1} \\ \frac{\partial U_3}{\partial X_2} \\ \frac{\partial U_3}{\partial X_3} \end{Bmatrix}$$

(2.51a)

or $$2\,{}^{1}S_{IJ}\eta_{IJ} = U^{T}_{a} \cdot {}^{1}S \cdot U_{a} \tag{2.51b}$$

in which U_a is given by equation (2.33) and ${}^{1}S$ is given by

$$
{}^{1}S = \begin{bmatrix}
{}^{1}S_{11} & {}^{1}S_{12} & {}^{1}S_{13} & & & & & & \\
{}^{1}S_{21} & {}^{1}S_{22} & {}^{1}S_{23} & & 0 & & & 0 & \\
{}^{1}S_{31} & {}^{1}S_{32} & {}^{1}S_{33} & & & & & & \\
 & & & {}^{1}S_{11} & {}^{1}S_{12} & {}^{1}S_{13} & & & \\
 & 0 & & {}^{1}S_{21} & {}^{1}S_{22} & {}^{1}S_{23} & & 0 & \\
 & & & {}^{1}S_{31} & {}^{1}S_{32} & {}^{1}S_{33} & & & \\
 & & & & & & {}^{1}S_{11} & {}^{1}S_{12} & {}^{1}S_{13} \\
 & 0 & & & 0 & & {}^{1}S_{21} & {}^{1}S_{22} & {}^{1}S_{23} \\
 & & & & & & {}^{1}S_{31} & {}^{1}S_{32} & {}^{1}S_{33}
\end{bmatrix} \tag{2.52}
$$

Substituting for U_a from equation (2.34) into equation (2.51) yields

$$ {}^{1}S_{IJ}\ \eta_{IJ} = \tfrac{1}{2}\, q^{T}\ [N]^{T}.[{}^{1}S].[N].q \tag{2.53} $$

and by taking the variation with respect to nonlinear strain as

$$ {}^{1}S_{IJ}\delta\eta_{IJ} = \delta q^{T}\ [N]^{T}.[{}^{1}S].[N].q \tag{2.54} $$

Therefore, the geometric stiffness matrix in a symmetric form is

$$ K_{G} = \int_{{}^{o}V} [N]^{T}.[{}^{1}S].[N]\ dV \tag{2.55} $$

or in the natural coordinates as

$$ K_{G} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [N]^{T}.[{}^{1}S].[N]\ \det J\ .\ dr\ ds\ dt \tag{2.56} $$

numerically

$$ K_{G} = \sum_{i}\sum_{j}\sum_{k} W_{i}W_{j}W_{k}\ [N^{T}(r_{i},s_{j},t_{k}).{}^{1}S(r_{i},s_{j},t_{k}).N(r_{i},s_{j},t_{k})]\ .\det J(r_{i},s_{j},t_{k}) \tag{2.57} $$

3.2.3 Equivalent Nodal Load Vector ${}^{1}R$ for Equilibrium Correction. This is given by evaluating in the integral (2.24e) as follows:

$$ \delta q^{T}.{}^{1}R = \int_{{}^{o}V} {}^{1}S_{IJ}\ \delta e_{IJ}\ dV \tag{2.58} $$

That is

$$^1R = \int_{^oV} B^T . {}^1S \; dV \tag{2.59}$$

where

$$({}^1S)^T = ({}^1S_{11} \;\; {}^1S_{22} \;\; {}^1S_{33} \;\; {}^1S_{12} \;\; {}^1S_{23} \;\; {}^1S_{13}) \tag{2.60}$$

In the (r,s,t) system we then have

$$^1R = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} B^T . {}^1S \;\; \det J \; . \; dr \; ds \; dt \tag{2.61}$$

or numerically

$$^1R = \sum_i \sum_j \sum_k W_i W_j W_k \; [B^T(r_i,s_j,t_k).{}^1S(r_i,s_j,t_k)].\det J(r_i,s_j,t_k) \tag{2.62}$$

3.2.4 Consistent Nodal Load Vector ^2P. Consistent nodal load vector, ^2P, is obtained from the evaluation of the body forces and surface tractions by the conventional finite element method. That is

$$^2P = \int_{^oA} [\hat{N}]^T.\{^2t\}.dA + \int_{^oV} [\phi]^T.\{^2f\}.dV \tag{2.63}$$

in which $[\hat{N}]$ is the displacement transformation which relates displacements of the loaded surface to the nodal displacements; $\{{}^2t\}$ is the vector of surface tractions; $[\phi]$ is the matrix of interpolation functions given by equations (2.19,2.27); and $\{{}^2f\}$ is the vector of body forces. The intregrals in the equation (2.63) are calculated numerically.