CHAPTER I

BASIC DEFINITIONS AND THEOREMS

Let R be a commutative ring and M an R-module. A mapping D of R into M is said to be a derivation of R into M if it satisfies the following conditions:

(1) $D(x+y) = D(x) + D(y)$ and

(2) $D(xy) = xD(y) + yD(x)$ for all x and y in R.

Immediate consequence is that for every x in R and for every positive integer n we have $D(x^n) = nx^{n-1}D(x)$. In particular, if R has an identity 1, then $D(1) = 0$. If A is a subset of R such that $D(x) = 0$ for every x in A, then we say that D is trivial on A or D is an A-derivation of R into M. A derivation of R is simply a derivation of R into R.

Lemma 1.1 ([10]). Let R be an integral domain and K the quotient field of R. Let D be a derivation of R into K. Then D can be extended, in a unique way,to a derivation D′of K. Furthermore, for x, y in R and $y \neq 0$ we have

$$D'(x/y) = (yD(x) - xD(y))/y^2. \tag{1.1}$$

Proof. We first show that (1.1) is well-defined. Let $x, x', y, y' \in R$ and both y and $y' \neq 0$ be such that $x/y = x'/y'$. So $xy' = x'y$ and hence

$$y'D(x) - x'D(y) = yD(x') - xD(y').$$

Dividing the above equality by yy' and then substitute $x/y = x'/y'$, we get

$$(yD(x) - xD(y))/y^2 = (y'D(x') - x'D(y'))/y'^2.$$

To show that D′ extends D, note that

$$D'(x) = D'((xy)/y) = (yD(xy) - (xy)D(y))/y^2 = D(x).$$

That the mapping D' is a derivation is straightforward. Finally the uniqueness of D' follows from

$$D(x) = D'(y(x/y)) = yD'(x/y) + (x/y)D(y)$$

which shows that relation (1.1) holds for every derivation D' of K which extends D. #

Example 1.1. Let R be a commutative ring with identity 1. For each $j = 1,\ldots,n$, define $D_j : R[X_1,\ldots,X_n] \rightarrow R[X_1,\ldots,X_n]$ as follows:

if $f = \sum a_{k_1,\ldots,k_n} X_1^{k_1} \cdots X_n^{k_n}$ in $R[X_1,\ldots,X_n]$, then

$$D_j(f) = \sum a_{k_1,\ldots,k_n} k_j X_1^{k_1} \cdots X_j^{k_j - 1} \cdots X_n^{k_n}.$$

It is easily checked that D_j is an R-derivation of $R[X_1,\ldots,X_n]$ and also trivial on $R[X_1,\ldots,X_{j-1},X_{j+1},\ldots,X_n]$. Note that these $D_1,\ldots,D_n$ are familiar <u>partial derivations.</u>

Example 1.2. Let R be a commutative ring with identity 1, and D a derivation of R. Define $D_0: R[X_1,\ldots,X_n] \rightarrow R[X_1,\ldots,X_n]$ as follows :

if $f = \sum a_{k_1,\ldots,k_n} X_1^{k_1} \cdots X_n^{k_n}$ in $R[X_1,\ldots,X_n]$, then

$$D_0(f) = \sum D(a_{k_1,\ldots,k_n}) X_1^{k_1} \cdots X_n^{k_n}.$$

Then D_0 is a derivation of $R[X_1,\ldots,X_n]$.

Lemma 1.2 ([10]). Let F be a field, $K = F(x_1,\ldots,x_n)$ a finitely generated extension field of F, D a derivation of F into K, and $\{u_1,\ldots,u_n\}$ a set of n elements of K. For each $f \in F[X_1,\ldots,X_n]$, we define

$$H_f(X_1,\ldots,X_n) = D_o(f) + \sum_{i=1}^{n} u_i D_i(f),$$

where the D_i and D_o are derivations of $F[X_1,\ldots, X_n]$ defined by

$$D_i(aX_1^{k_1}\cdots X_i^{k_i}\cdots X_n^{k_n}) = aX_1^{k_1}\cdots k_iX_i^{k_i-1}\cdots X_n^{k_n} \quad \text{and}$$

$$D_0(aX_1^{k_1}\cdots X_n^{k_n}) = D(a)X_1^{k_1}\cdots X_n^{k_n}, \quad \text{respectively.}$$

Then there is a unique derivation D' of K extending D and such that $D'(x_i) = u_i$ for $i = 1,\ldots,n$ if and only if $H_f(x_1,\ldots,x_n) = 0$ for all $f \in F[X_1,\ldots,X_n]$ with $f(x_1,\ldots,x_n) = 0$.

Proof.(==>) Assume that there is a unique derivation D' of K extending D and such that $D'(x_i) = u_i$ for $i = 1,\ldots,n$. Then, by induction, we have that for every polynomial $g \in F[X_1,\ldots,X_n]$,

(1.2) $$D'(g(x_1,\ldots,x_n)) = H_g(x_1,\ldots,x_n),$$

since (1.2) is true if g is a monomial. By linearity, (1.2) is true for any polynomial g. For $f \in F[X_1,,,X_n]$ such that $f(x_1,\ldots,x_n) = 0$, we then have

$$H_f(x_1,\ldots,x_n) = D'(f(x_1,\ldots,x_n)) = D'(0) = 0.$$

(<==) Assume that $H_f(x_1,\ldots,x_n) = 0$ for all $f \in F[X_1,\ldots,X_n]$ such that $f(x_1,\ldots,x_n) = 0$. Define $D': F[x_1,\ldots,x_n] \to K$ by

$$D'(f(x_1,\ldots,x_n)) = H_f(x_1,\ldots,x_n) \quad \text{for all } f \text{ in } F[x_1,\ldots,x_n].$$

We will show that D' is well -defined. If $g(x_1,\ldots,x_n) = h(x_1,\ldots,x_n)$, then $(g\text{-}h)(x_1,\ldots,x_n) = 0$. So $H_{g-h}(x_1,\ldots,x_n) = 0$.

Now $0 = H_{g-h}(x_1,\ldots,x_n) = H_g(x_1,\ldots,x_n) - H_h(x_1,\ldots,x_n)$.

Therefore $H_g(x_1,\ldots,x_n) = H_h(x_1,\ldots,x_n)$. Hence D' is well- defined.

Also $D'(x_i) = u_i$ for $i = 1,\ldots,n$.

D' is a derivation since the mappings D_o and $f \to u_iD_i(f)$ $(i=1,\ldots,n)$ are derivations of $F[x_1,\ldots,x_n]$ into K. By Lemma 1.1, the derivation D' can be extended to K. #

Theorem 1.3 ([10]). Let F be a field and let $K = F(x)$ be a simple extension of F. Let D be a derivation of F into K.

(1) If x is transcendental over F and if u is any element of K, then there exists one and only one derivation D' of K extending D, such that $D'(x) = u$.

(2) If x is separable and algebraic over F, then there exists one and only one derivation D' of K extending D.

Proof. Referring to Lemma 1.2, (1) is obvious because 0 is the only polynomial f in F[X] such that $f(x) = 0$. For (2), observe that every polynomial $g \in F[X]$ such that $g(x) = 0$ is a multiple of the minimal polynomial f of x over F, that is, $g(X) = h(X)f(X)$ for some h in F[X]. Since x is separable over F, $D_1 f(x) \neq 0$. Let $u \in K$ be such that $D_o f(x) + uD_1 f(x) = 0$.

Hence $H_g(x) = (D_o g)(x) + u(D_1 g)(x) = 0$.

By Lemma 1.2, there is a unique derivation D' of K extending D and such that $D'(x) = u$. #

Corollary 1.4 ([10]). If a field K is a separable, algebraic extension of a field F then every derivation D of F into K can be extended, in one and only one way, to a derivation of K.

Proof. Let D be a derivation of F into K. To show the existence of a derivation of K we shall use Zorn's lemma. Let

$$I = \{(G,E) \mid G \text{ is a field with } F \subseteq G \subseteq K \text{ and } E \text{ is a derivation of } G \text{ into } K \text{ such that } E(x) = D(x) \text{ for all } x \in F\}.$$

The set I is non-empty since (F, D) belongs to I. Define a partial ordering on I as follows : Let $(G_1,E_1), (G_2, E_2) \in I$,

$(G_1,E_1) < (G_2,E_2)$ if and only if $G_1 \subseteq G_2$ and $E_2(a) = E_1(a)$ for all a in G_1.

Let $\{(G_\alpha, E_\alpha) \mid \alpha \in \Lambda\}$ be a chain in I. Let $N = \bigcup_{\alpha\in\Lambda} G_\alpha$.

Clearly $\bigcup_{\alpha\in\Lambda} G_\alpha$ is a subfield of K containing F. Define $E : N \to K$ as follows :

Let $x \in N$. There exists $\alpha \in \Lambda$ such that $x \in G_\alpha$. Define $E(x) = E_\alpha(x)$.

We shall show that E is well-defined. Suppose that there exists $\beta \in \Lambda$ such that $x \in G_\beta$; but $(G_\alpha, E_\alpha) < (G_\beta, E_\beta)$ or $(G_\beta, E_\beta) < (G_\alpha, E_\alpha)$. Without loss of generality, we assume that $(G_\alpha, E_\alpha) < (G_\beta, E_\beta)$. Hence $E_\beta(x) = E_\alpha(x)$.

Next, we show that E is a derivation, let $x, y \in N$. There exist α_1, α_2 such that $x \in G_{\alpha_1}$ and $y \in G_{\alpha_2}$. We may assume that $(G_{\alpha_1}, E_{\alpha_1}) < (G_{\alpha_2}, E_{\alpha_2})$.

Thus $x, y \in G_{\alpha_2}$. Hence

$$E(x+y) = E_{\alpha_2}(x+y) = E_{\alpha_2}(x) + E_{\alpha_2}(y) = E(x) + E(y),$$

$$E(xy) = E_{\alpha_2}(xy) = xE_{\alpha_2}(y) + yE_{\alpha_2}(x) = xE(y) + yE(x).$$

It is clear that (N, E) is an upper bound for $\{(G_\alpha, E_\alpha) \mid \alpha \in \Lambda\}$.

By Zorn's lemma, there exists a maximal element (F', D') in I.

If $F' \neq K$, then there exists an element y in K such that $y \notin F'$. Therefore F'(y) is a simple extension field of F'. Note that y is separable and algebraic over F'.

By Theorem 1.3, there exists a derivation D" of F'(y) which extends D'. This contradicts the maximality of (F', D') in I. Hence F'= K.

To prove the uniqueness, let D_1, D_2 be derivations of K that extend D. Let $b \in K$. There exists the minimal polynomial $g(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0$ over F such that $g(b) = 0$. It is easy to verify that, for j = 1,2,

$$D_j(g(b)) = [mb^{m-1} + a_{m-1}(m-1)b^{m-2} + \ldots + a_1]D_j(b) + \sum_{i=0}^{m-1} D_j(a_i)b^i.$$

Since D_1 and D_2 extend D, $[mb^{m-1} + \ldots + a_1](D_1 - D_2)(b) = (D_1 - D_2)(g(b)) = 0$.

Since b is separable, $mb^{m-1} + \ldots + a_1 \neq 0$. Then we get $D_1(b) = D_2(b)$. #

Lemma 1.5. Let F be a field of characteristic zero with derivation D. Let K be an algebraic extension field of F. Let σ be an F-automorphism of K. Then for every derivation D′ of K which extends D, $D'(\sigma(b)) = \sigma(D'(b))$ for all b in K.

Proof. Let $b \in K$. Let $f(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ be the minimal polynomial of b over F.

For derivation D′ of K which extends D, we have

$$0 = D'(f(b)) = nb^{n-1}D'(b) + \sum_{i=0}^{n-1} D(a_i)b^i + \sum_{i=0}^{n-1} ia_i b^{i-1}D'(b),$$

and so

$$(1.3)\quad 0 = \sigma(D'(f(b))) = (n\,\sigma(b)^{n-1} + \sum_{i=0}^{n-1} ia_i\,\sigma(b)^{i-1})\,\sigma(D'(b)) + \sum_{i=0}^{n-1} D(a_i)\sigma(b)^i.$$

Similarly,

$$(1.4)\quad 0 = D(\sigma(f(b))) = [\, n\sigma(b)^{n-1} + \sum_{i=0}^{n-1} ia_i\sigma(b)^{i-1}]\, D'(\sigma(b)) + \sum_{i=0}^{n-1} D(a_i)\sigma(b)^i.$$

So (1.3)-(1.4) yields, $[\, n\sigma(b)^{n-1} + \sum_{i=0}^{n-1} ia_i\sigma(b)^{i-1}]\,[\sigma(D'(b)) - D'(\sigma(b))] = 0.$

If $n\sigma(b)^{n-1} + \sum_{i=0}^{n-1} ia_i\sigma(b)^{i-1} = 0$, then $\sigma(nb^{n-1} + \sum_{i=0}^{n-1} ia_i\, b^{i-1}) = 0.$

Since σ is injective, $nb^{n-1} + \sum_{i=0}^{n-1} ia_i\, b^{i-1} = 0$ which is impossible.

Hence $n\sigma(b)^{n-1} + \sum_{i=0}^{n-1} ia_i\sigma(b)^{i-1} \neq 0$. Therefore $\sigma(D'(b)) = D'(\sigma(b))$. #

Theorem 1.6 ([6], [11]). Let R be a commutative ring and S a subring of R. Then there exists an R-module $\Omega_{R/S}$ and a S-derivation d of R into $\Omega_{R/S}$ such that for any S-derivation D of R into an R-module M there exists a unique R-homomorphism $f : \Omega_{R/S} \to M$ such that $D = f \circ d$. Moreover, the pair ($\Omega_{R/S}$, d) is unique up to isomorphism.

The R-module $\Omega_{R/S}$ is called <u>the module of differentials</u> of R over S.

The map $d : R \to \Omega_{R/S}$ is called <u>the canonical derivation</u> and is denoted by $d_{R/S}$ if necessary.

Proof. See [6], [11].

Theorem 1.7 ([6]). Let F be a field of characteristic zero and K an extension field of F. Let $u_1,\ldots,u_n$, v be elements of K, with $u_1,\ldots,u_n$ nonzero, and let $c_1,\ldots,c_n$ be elements of F that are linearly independent over **Q**. Then the element

$$c_1\frac{du_1}{u_1}+\cdots+c_n\frac{du_n}{u_n}+dv$$

of $\Omega_{K/F}$ is zero if and only if each $u_1,\ldots,u_n$, v is algebraic over F.

For the proof of this theorem we need the following facts from [6]:

(1) If $F \subseteq K \subseteq L$ are fields of characteristic zero, the natural K - homomorphism $h : \Omega_{K/F} \to \Omega_{L/F}$ is injective, where $h(d_{K/F}a) = d_{L/F}a$ for all $a \in K$.

(2) Let $F \subseteq K$ be fields of characteristic zero. If $\{x_1,\ldots,x_n\}$ is a transcendence basis of K over F, then $\{dx_1,\ldots,dx_n\}$ is a basis for $\Omega_{K/F}$ over K.

Proof. ($\Leftarrow$) We first show that if $w \in K$ is algebraic over F then $dw = 0$ in $\Omega_{K/F}$. Since w is algebraic over F, there exists the minimal polynomial $f(X) = X^m + a_{m-1}X^{m-1} + \cdots + a_0$ over F such that $f(w) = 0$.

Taking the F - derivation $d : K \to \Omega_{K/F}$ to the equation $f(w) = 0$, we get

$$(mw^{m-1} + (m-1)a_{m-1}w^{m-2} + \ldots + a_1)\, dw = 0.$$

Since $\quad mw^{m-1} + (m-1)a_{m-1}w^{m-2} + \ldots + a_1 \neq 0, \quad dw = 0.$

($\Rightarrow$) Assume that $c_1\frac{du_1}{u_1}+\cdots+c_n\frac{du_n}{u_n}+dv = 0$ in $\Omega_{K/F}$.

Claim that each $u_1,\ldots,u_n$ is algebraic over F. Suppose that there exists u_i which is not algebraic over F for some $i \le n$, say u_1. Let $F' = F(u_1,\ldots,u_n, v)$.

Let $x_1, \ldots, x_m$ be a transcendence basis for F' over F, with $x_1 = u_1$.

Thus F' is algebraic over $F(x_1, \ldots, x_m)$ and $[F' : F(x_1, \ldots, x_m)] < \infty$.

Let $G = F(x_2, \ldots, x_m)$. Hence F' is algebraic over $G(u_1)$. Let E be an extension field of F' such that E is Galois over $G(u_1)$ and $[E : G(u_1)] < \infty$. By the fact (1) above, the natural F'- homomorphism $\Omega_{F'/F} \to \Omega_{K/F}$ is injective, and so

$$c_1 \frac{du_1}{u_1} + \cdots + c_n \frac{du_n}{u_n} + dv = 0 \quad \text{in } \Omega_{F'/F}.$$

Since $F \subseteq F' \subseteq E$, by the fact above, the natural F'- homomorphism $\Omega_{F'/F} \to \Omega_{E/F}$ is injective. Thus

(1.5) $$c_1 \frac{du_1}{u_1} + \cdots + c_n \frac{du_n}{u_n} + dv = 0 \quad \text{in } \Omega_{E/F}.$$

By Theorem 1.3, let D be a derivation of $G(u_1)$ such that $D(u_1) = 1$ and $D(a) = 0$ for all $a \in G$.

By Corollary 1.4, D can be extended, in one and only one way, to a derivation of E (using the same notation D for the derivation of E). By Theorem 1.6, there exists a unique E-homomorphism $g : \Omega_{E/F} \to E$ such that $D = g \circ d$. Applying g to equation (1.5), we get

(1.6) $$c_1 D(u_1)/u_1 + \cdots + c_n D(u_n)/u_n + D(v) = 0 \quad \text{in } E.$$

We apply each $\sigma \in \mathrm{Aut}(E/G(u_1))$ to equation (1.6) and then sum over all σ in $\mathrm{Aut}(E/G(u_1))$ to get

(1.7) $$rc_1 D(u_1)/u_1 + c_2 D(Nu_2)/(Nu_2) + \cdots + c_n D(Nu_n)/(Nu_n) + D(Tv) = 0,$$

where N and T denote the norm and the trace respectively, and r is a positive integer. For each $i = 2, 3, \ldots, n$, we can write

$Nu_i = s_i u_1^{\alpha_i} p_{i1}^{\alpha_{i1}} \cdots p_{i\beta_i}^{\alpha_{i\beta_i}}$ where $s_i \in G$, the p_{ij} are monic irreducible elements distinct from u_1 of $G[u_1]$, the $\alpha_{ij} \in \mathbf{Z}\setminus\{0\}$ and the $\alpha_i \in \mathbf{Z}$.

$$Tv = q(u_1) + \sum_{i=1}^{\lambda}\sum_{j=1}^{r_i}\frac{b_{ij}(u_1)}{(h_i(u_1))^j},$$

where $q(u_1) \in G[u_1]$, the $b_{ij}(u_1) \in G[u_1]$, the $h_i(u_1)$ are monic irreducible elements in $G[u_1]$ with $\deg b_{ij} < \deg h_i$. From (1.7),

$$rc_1\left(\frac{1}{u_1}\right) + \sum_{i=2}^{n} c_i\left(\alpha_i\left(\frac{1}{u_1}\right) + \alpha_{i1}\frac{D(p_{i1})}{p_{i1}} + \cdots + \alpha_{i\beta_i}\frac{D(p_{i\beta_i})}{p_{i\beta_i}}\right)$$
$$+ D(q(u_1)) + \sum_{i=1}^{\lambda}\sum_{j=1}^{r_i}\left(\frac{D(b_{ij})}{h_i} - jb_{ij}\frac{D(h_i)}{h_i^{j+1}}\right) = 0.$$

Hence

$$(1.8) \qquad \left(rc_1 + \sum_{i=2}^{n} c_i\alpha_i\right)\left(\frac{1}{u_1}\right) = -\sum_{i=2}^{n} c_i\left(\alpha_{i1}\frac{D(p_{i1})}{p_{i1}} + \cdots + \alpha_{i\beta_i}\frac{D(p_{i\beta_i})}{p_{i\beta_i}}\right)$$
$$- D(q(u_1)) - \sum_{i=1}^{\lambda}\sum_{j=1}^{r_i}\left(\frac{D(b_{ij})}{h_i} - jb_{ij}\frac{D(h_i)}{h_i^{j+1}}\right).$$

We may assume that $h_i \neq u_1$ for all $i \leq \lambda$ (for if $h_i = u_1$, then $b_{ij} \in G$ and hence $D(b_{ij}) = 0$). From (1.8), comparing the coefficient of $(1/u_1)$,

$$rc_1 + \sum_{i=2}^{n} c_i\alpha_i = 0.$$

Thus $c_1,\ldots,c_n$ are linearly dependent over $\mathbb{Q}$, a contradiction. So we have the claim. Since $u_1,\ldots,u_n$ are algebraic over F, $du_i = 0$ for $i = 1,\ldots,n$. Hence $dv = 0$.

Claim that v is algebraic over F. Suppose not. By the fact (2), the element dv of $\Omega_{F(v)/F}$ is a basis for $\Omega_{F(v)/F}$ over F(v). Hence $dv \neq 0$ in $\Omega_{F(v)/F}$. By the fact (1), the natural F(v)- homomorphism $\Omega_{F(v)/F} \to \Omega_{K/F}$ is injective, hence $dv \neq 0$ in $\Omega_{K/F}$, which is impossible. #

By a differential field we mean a field F together with an indexed family $\{D_i \mid i \in I\}$ of derivations of F. For brevity, the term "differential field F" is used exclusively, without further notice, for the differential field (F, $\{D_i \mid i \in I\}$), and the term "the given derivations of F" refers to the set $\{D_i \mid i \in I\}$.

A differential extension field of F is an extension field K of F together with a family of derivations $\{D_i' \mid i \in I\}$ of K indexed by the same set such that the restriction of each D_i' to F is D_i. With no loss of generality, we use the same symbol D_i for the derivation D_i'.

An element c in F is said to be a constant if $D_i(c) = 0$ for all $i \in I$. The set of all constants of a differential field F is $\bigcap_{i \in I} \ker D_i$, which is a subfield of F, and also called the subfield of constants.

Theorem 1.8 ([6]). Let F be a differential field of characteristic zero, K a differential extension field of F with the same subfield of constants C. For each $i = 1,\ldots,n$ and $j = 1,\ldots,m$ let $c_{ij} \in C$ and let v_i be an element of K, u_j a nonzero element of K. Suppose that for each $i = 1,\ldots,n$ and each given derivation D of K,

$$D(v_i) + \sum_{j=1}^{m} c_{ij} D(u_j) / u_j \in F.$$

Then either $\text{tr.deg.}F(u_1,\ldots,u_m, v_1,\ldots,v_n)/F \geq n$ or the n elements of $\Omega_{K/F}$ given by $dv_i + \sum_{j=1}^{m} c_{ij} \frac{du_j}{u_j}$, $i = 1,\ldots,n$, are linearly dependent over C.

Before proving the theorem, we quote two facts from [12] and [11].

1) ([12]). Let $f : F \to K$ be a homomorphism of commutative rings with identity, D a derivation of K such that there exists a map $D_F : F \to F$ satisfying $f \circ D_F = D \circ f$. Then there exists a unique map $L_D : \Omega_{K/F} \to \Omega_{K/F}$ such that for all $w, v \in \Omega_{K/F}$ and all $a \in K$ we have

$$L_D(w+v) = L_D(w)+L_D(v),$$
$$L_D(aw) = (Da)w + a(L_D w) \quad \text{and}$$
$$L_D(da) = d(Da).$$

2) ([11]). Let F be a field of characteristic zero and L a finitely generated extension of F. Then dim $\Omega_{L/F}$ = tr.deg. L/F.

Proof. If $u_1,\dots,u_m$, $v_1,\dots,v_n$ are algebraic over F, then, by Theorem 1.7, $du_j = 0$ for all $j = 1,\dots,m$ and $dv_i = 0$ for all $i = 1,\dots,n$. Hence

$$dv_i + \sum_{j=1}^{m} c_{ij}\frac{du_j}{u_j} = 0 \text{ for all } i = 1,\dots,n,$$

and the result is trivially true.

Assume that there exists one u_j or v_i which are not algebraic over F for some $j \le m$ and $i \le n$.

Suppose that $dv_i + \sum_{j=1}^{m} c_{ij}\frac{du_j}{u_j}$, $i = 1,\dots,n$, are linearly independent over C.

For each given derivation D of K and each $i=1,\dots,n$ we obtain

$$L_D\left(dv_i + \sum_{j=1}^{m} c_{ij}\frac{du_j}{u_j}\right) = d(Dv_i) + \sum_{j=1}^{m} d\left(c_{ij}\frac{Du_j}{u_j}\right)$$
$$= d(Dv_i + \sum_{j=1}^{m} c_{ij}\frac{Du_j}{u_j}) = 0.$$

Write $w_i = \left(dv_i + \sum_{j=1}^{m} c_{ij}\frac{du_j}{u_j}\right)$ for $i = 1,\dots,n$.

Claim that $w_1,\dots,w_n$ are linearly independent over K. Suppose not. There are $a_1,\dots,a_n$ in K, not all zero, such that $a_1w_1 + \cdots + a_nw_n = 0$. Choose $a_1,\dots,a_n$ so that the number of nonzero a_i's is minimal, and that one of them, say a_1, is 1. For each derivation D of K we get

$$0 = L_D(a_1w_1 + \cdots + a_nw_n) = (Da_1)w_1 + \cdots + (Da_n)w_n = (Da_2)w_2 + \cdots + (Da_n)w_n.$$

Since the number of nonzero a_i's was minimal, we get $Da_2 = \ldots = Da_n = 0$.

Hence each $a_i \in C$. Therefore $w_1,\ldots,w_n$ are linearly dependent over C, a contradiction.

So we have the claim.

Let $F' = F(u_1,\ldots,u_m, v_1,\ldots, v_n)$. Hence $F \subseteq F' \subseteq K$.

Claim that the n elements $dv_i + \sum_{j=1}^{m} c_{ij} \frac{du_j}{u_j}$ of $\Omega_{F'/F}$ are linearly independent over F'.

Let $a_1,\ldots,a_n \in F'$ be such that

$$\sum_{i=1}^{n} a_i \left(dv_i + \sum_{j=1}^{m} c_{ij} \frac{du_j}{u_j} \right) = 0 \qquad (\text{in } \Omega_{F'/F}) \tag{1.9}$$

Now consider $\Omega_{K/F}$ as F'- module. By Theorem 1.6, there exists F'- homomorphism $g : \Omega_{F'/F} \to \Omega_{K/F}$ such that $g \circ d_{F'/F} = d_{K/F}$.

Applying g to equation (1.9), we get

$$\sum_{i=1}^{n} a_i \left(dv_i + \sum_{j=1}^{m} c_{ij} \frac{du_j}{u_j} \right) = 0 \qquad (\text{in } \Omega_{K/F})$$

By the linear independence of $w_i = dv_i + \sum_{j=1}^{m} c_{ij} \frac{du_j}{u_j}$ $(i = 1,\ldots,n)$, we must have $a_i = 0$ for $i = 1,\ldots,n$. So we have the claim.

By the fact 2) tr.deg. $F'/F = \dim. \Omega_{F'/F} \geq n$. #

Theorem 1.9 ([6]). Let F be a differential field of characteristic zero, K a differential extension field of F with the same subfield of constants, with K algebraic over F(t) for some given $t \in K$. Suppose that $c_1,\ldots,c_n$ are constants of F that are linearly independent over **Q**, that $u_1,\ldots,u_n$, v are elements of K, with $u_1,\ldots,u_n$ nonzero, and that for each given derivation D of K we have

$$\sum_{i=1}^{n} c_i D(u_i)/u_i + D(v) \in F.$$

If for each given derivation D of K we have $D(t) \in F$, then $u_1,\dots,u_n$ are algebraic over F and there exists a constant c of F such that $v + ct$ is algebraic over F. If for each given derivation D of K we have $D(t)/t \in F$, then v is algebraic over F and there are integers $m_0, m_1, \dots, m_n$, with $m_0 \neq 0$, such that each $u_i^{m_0} t^{m_i}$ $(i = 1,\dots,n)$ is algebraic over F.

Proof. If t is algebraic over F, then K is algebraic over F. So the result is trivially true. Assume that t is transcendental over F. By Theorem 1.7, $dt \neq 0$ in $\Omega_{K/F}$.

Case 1. For each given derivation D of K, $D(t) \in F$.

By Theorem 1.8, $\sum_{i=1}^{n} c_i \frac{du_i}{u_i} + dv$ and dt are linearly dependent over C. There exists $c \in C$ such that $(\sum_{i=1}^{n} c_i \frac{du_i}{u_i} + dv) + cdt = 0$. Thus $\sum_{i=1}^{n} c_i \frac{du_i}{u_i} + d(v + ct) = 0$. By Theorem 1.7, $u_1,\dots,u_n$, $v+ct$ are algebraic over F.

Case 2. For each given derivation D of K, $D(t)/t \in F$.

By Theorem 1.8, $\sum_{i=1}^{n} c_i \frac{du_i}{u_i} + dv$ and $\frac{dt}{t}$ are linearly dependent over C. There exist $c \in C$ such that $(\sum_{i=1}^{n} c_i \frac{du_i}{u_i} + dv) + c\frac{dt}{t} = 0$. We have that $c, c_1,\dots,c_n$ are linearly dependent over **Q** (for if $c, c_1,\dots,c_n$ are linearly independent over **Q**, then by Theorem 1.7, $u_1,\dots,u_n$, v, t are algebraic over F).

So we can write $c = (\sum_{i=1}^{n} m_i c_i)/m_o$ for some integer $m_o, m_1,\dots,m_n$ with $m_o \neq 0$.

Then

$$c_1 \frac{d\left(u_1^{m_0} t^{m_1}\right)}{u_1^{m_0} t^{m_1}} + \cdots + c_n \frac{d\left(u_n^{m_0} t^{m_n}\right)}{u_n^{m_0} t^{m_n}} + d(m_o v) = 0.$$

By Theorem 1.7, each $u_i^{m_0} t^{m_i}$ is algebraic over F and v is algebraic over F. #